TIGHT BOUNDS FOR THE DIHEDRAL ANGLE SUMS OF A PYRAMID

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Abstract. We prove that eight dihedral angles in a pyramid with an arbitrary quadrilateral base always sum up to a number in the interval $(3\pi, 5\pi)$. Moreover, for any number in $(3\pi, 5\pi)$ there exists a pyramid whose dihedral angle sum is equal to this number, which means that the lower and upper bounds are tight. Furthermore, the improved (and tight) upper bound 4π is derived for the class of pyramids with parallelogramic bases. This includes pyramids with rectangular bases, often used in finite element mesh generation and analysis.

Keywords: pyramid; dihedral angle sum; tight angle bounds

MSC 2020: 51M20, 52B10

1. Introduction

An estimation of angles and their sums for simplices (see, e.g., [4], [5], [1], [6], and references therein) has been addressed for many decades by geometers. However, this issue has also important applications in numerics; for example, in finite element mesh generation and analysis. Thus, in order to guarantee a priori convergence, or to preserve qualitative properties of models by finite element approximations, one has to use simplicial meshes with certain geometric properties, which are often described in terms of various angles, e.g., minimum/maximum angle conditions and acuteness/nonobtuseness [9], [8], [2].

Besides simplices, some other shapes of solids are often considered in finite element analysis and modelling—say prisms, blocks and pyramids [3], [11]. While computing the sums of angles is a trivial issue for blocks, since all angles are right, the case of pyramids was an open problem. Here we derive tight bounds for the sum of the

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eight dihedral angles of pyramids in general, and we narrow these bounds for some special practical cases. We believe that our results will be important later on in deriving an analogue of the maximum angle condition (constructed for simplices [8] and prisms [7] so far) for pyramidal finite elements [10], thus allowing the use of various degenerating element shapes for meshing complicated 3D geometries.

It is worth mentioning that it is possible to construct pure pyramidal meshes; however, pyramidal elements are mostly used in hybrid meshes for a transition from hexahedral to tetrahedral elements, as depicted in Figure 1.

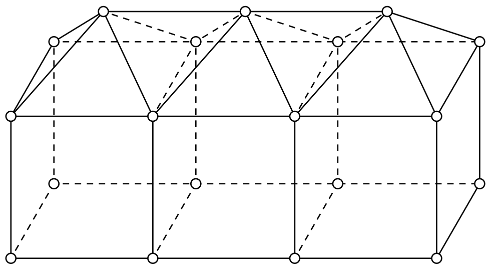

Figure 1. Hybrid mesh with pyramids connecting hexahedra and tetrahedra, see [10].

2. Bounds for a general pyramid

In the following, we will consider the class of pyramids with a quadrilateral base.

Definition 1. A (convex) *pyramid* is defined as the convex hull of five points, four of which form a planar, convex quadrilateral, with the fifth point (called the *apex*) being outside the plane of this quadrilateral.

A non-convex pyramid allows a non-convex quadrilateral base, see Section 4.

Definition 2. A *dihedral angle* in a three-dimensional polyhedron is the internal angle between two faces intersecting along an edge, such as depicted in Figure 2. The dihedral angle along an edge, say AB, is denoted by α_{AB}, etc. Also, in what follows we denote by σ_P the sum of all the dihedral angles of a polyhedron P.

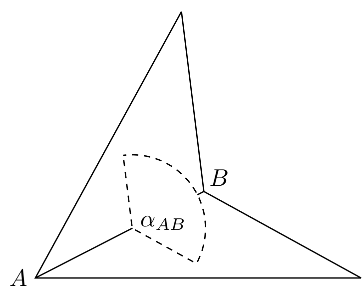

Figure 2. The dihedral angle α_{AB} along the edge AB.

The dihedral angles can be computed in terms of the outward unit normals of the faces. This is further explained in Section 3.

In contrast to the two-dimensional case, where the angle sum of a triangle is always constant and equal to π, it may vary in the case of tetrahedra; see the following result.

Theorem 1. *Consider a tetrahedron T with its dihedral angle sum σ_T. Then*

(a) *σ_T satisfies $2\pi < \sigma_T < 3\pi$,*

(b) *for any number $\sigma \in (2\pi, 3\pi)$ there is a tetrahedron T whose dihedral angle sum $\sigma_T = \sigma$.*

For the proof, see [4]. Part (b) of Theorem 1 can also be proved by the following example (which uses a different argumentation than [4]).

Example 1. Consider the family of tetrahedra depicted in Figure 3, where α is an arbitrary angle between 0 and π. As $h \to 0$, we obtain the following values and limits of dihedral angles:

$$(1) \qquad \alpha_{AB} = \frac{\pi}{2}, \quad \alpha_{AC} = \frac{\pi}{2}, \quad \alpha_{AD} = \alpha, \quad \alpha_{BC} \to 0, \quad \alpha_{BD} \to \frac{\pi}{2}, \quad \alpha_{CD} \to \frac{\pi}{2}.$$

Thus, the sum of the dihedral angles in this family of tetrahedra is getting arbitrarily close to $2\pi + \alpha$. Varying $\alpha \in (0, \pi)$, we obtain any value of $\sigma_T \in (2\pi, 3\pi)$, thus proving Theorem 1 (b).

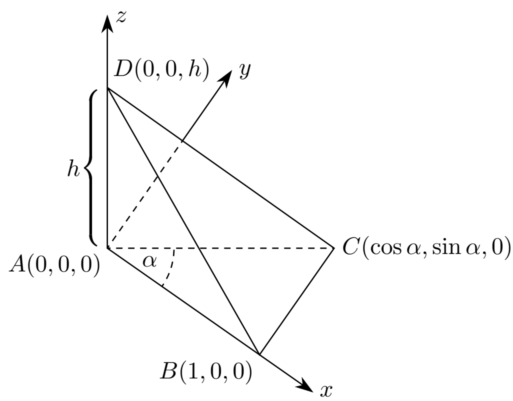

Figure 3. As $h \to 0$, the dihedral angle sum approaches $2\pi + \alpha$.

Theorem 2. *Consider a pyramid P with its dihedral angle sum σ_P. Then*

(a) *σ_P satisfies $3\pi < \sigma_P < 5\pi$,*

(b) *for any number $\sigma \in (3\pi, 5\pi)$ there is a pyramid whose dihedral angle sum $\sigma_P = \sigma$.*

P r o o f. (a) Let $ABCDE$ be any pyramid, as depicted in Figure 4. This pyramid is also the union of the two tetrahedra $T_1 = ABCE$ and $T_2 = ACDE$. From Theorem 1, their dihedral angle sums satisfy $2\pi < \sigma_{T_i} < 3\pi$ for $i = 1, 2$. Now, subtracting the two dihedral angles at the interface ACE between the two tetrahedra, whose sum always equals π, we obtain the following estimation for the dihedral angle sum of a pyramid:

$$3\pi < \sigma_{T_1} + \sigma_{T_2} - \pi = \sigma_T = \sigma_{T_1} + \sigma_{T_2} - \pi < 5\pi. \tag{2}$$

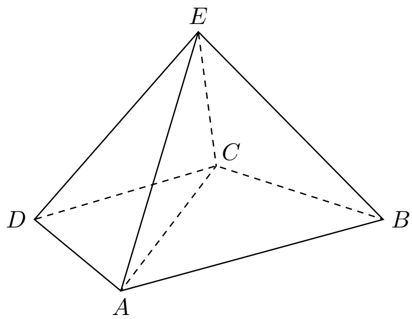

Figure 4. The pyramid $ABCDE$ split into two tetrahedra $ABCE$ and $ACDE$.

(b) Starting from the tetrahedron from Figure 3, we introduce a new vertex E at $(0, 0, h/2)$, thus defining the degenerate pyramid $ABCDE$ with a degenerated (to a triangle) base $ABDE$ (see Figure 5). In this construction, we obtain the new degenerated dihedral angle $\alpha_{CE} = \pi$, and $\alpha_{AE} = \alpha_{DE} = \alpha$.

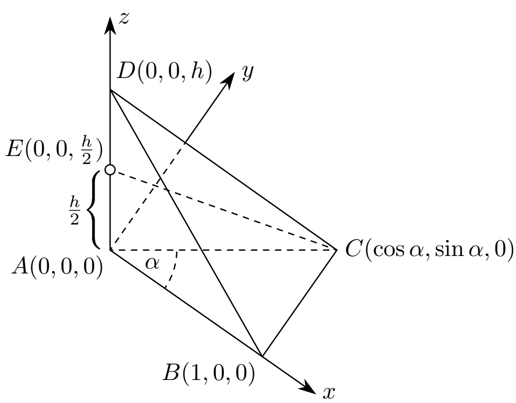

Figure 5. A degenerated pyramid with a degenerate quadrilateral base $ABDE$.

Now, to construct a proper, non-degenerate pyramid, we move the point E a small distance ε in the negative x-direction, as depicted in Figure 6, thus making $ABDE$ a real, convex, quadrilateral base for the pyramid.

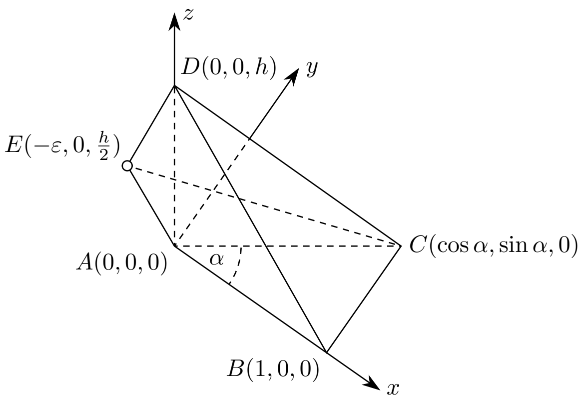

Figure 6. A non-degenerate pyramid with a quadrilateral base $ABDE$ in the xz-plane.

Consider α fixed. Taking the limits $\varepsilon \to 0$ and $h \to 0$, with the condition $\varepsilon \ll h$, we obtain the following values and limits of dihedral angles:

$$\begin{aligned}\alpha_{AB} = \frac{\pi}{2}, \quad \alpha_{AC} \to \frac{\pi}{2}, \quad \alpha_{BC} \to 0, \quad \alpha_{BD} \to \frac{\pi}{2},\\ \alpha_{CD} \to \frac{\pi}{2}, \quad \alpha_{AE} \to \alpha, \quad \alpha_{DE} \to \alpha, \quad \alpha_{CE} \to \pi.\end{aligned} \tag{3}$$

Thus, the sum of the dihedral angles of this pyramid is getting arbitrarily close to $3\pi + 2\alpha$. By varying $\alpha \in (0, \pi)$, we see that σ_P can take every value of $\sigma \in (3\pi, 5\pi)$. □

3. Pyramids with a parallelogramic base

Consider now a pyramid P with a parallelogramic base. Let n_1 be the outward unit normal vector for the face AED, and similarly we define n_2, n_3, n_4, n_5 for the other four faces (see Figure 7). Looking at the edge AE, the dihedral angle α_{AE} for the pyramid along this edge has the relationship $\cos\alpha_{AE} = -n_1 \cdot n_2$. We write n_in_j for the angle between n_i and n_j, which is also equal to the distance between the two vectors considered as points on the unit sphere. We obtain the relationship $\alpha_{AE} + n_1n_2 = \pi$. Now, using this connection and summing over the eight edges in P, one gets

$$\sigma_P + \sum n_in_j = 8\pi. \tag{4}$$

Here and later, we suppress the summation range. In detail, it is

$$\sum n_in_j := n_1n_2 + n_1n_4 + n_1n_5 + n_2n_3 + n_2n_5 + n_3n_4 + n_3n_5 + n_4n_5.$$

We have the following theorem for the bounds of the sum of dihedral angles in a pyramid with parallelogramic base.

Theorem 3. *Consider a pyramid P with a parallelogramic base, whose dihedral angle sum is denoted σ_P. Then*

(a) *σ_P satisfies $3\pi < \sigma_P < 4\pi$,*

(b) *for any number $\sigma \in (3\pi, 4\pi)$ there is a pyramid with a parallelogramic base whose dihedral angle sum $\sigma_P = \sigma$.*

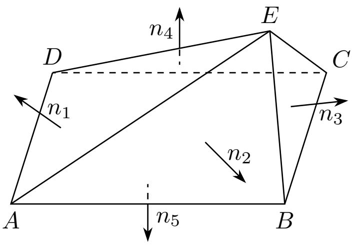

Figure 7. A pyramid $ABCDE$ with a parallelogramic base and outward unit normals.

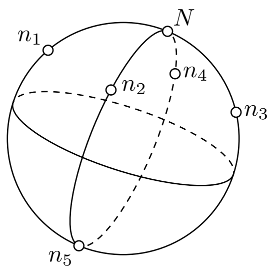

Figure 8. The normals of the pyramid depicted on the unit sphere.

P r o o f. Refer to Figure 7 for the notation used.

(a) The lower bound $3\pi < \sigma_P$ follows from Theorem 2. The base is a parallelogram, $AD \parallel BC$, which implies

$$AD \perp n_1, \quad AD \perp n_3, \quad AD \perp n_5. \tag{5}$$

Similarly, $AB \parallel DC$ implies

$$AB \perp n_2, \quad AB \perp n_4, \quad AB \perp n_5. \tag{6}$$

We can depict the normals on the unit sphere as in Figure 8. Here, n_5 is located on the south pole, and let N be the north pole of the unit sphere. From (5), we observe that n_1, n_3 and n_5 lie on a great circle, and from (6) that n_2, n_4 and n_5 also lie on a great circle. The vectors n_1 and n_3 are normals to planes intersecting E in a line parallel to AD. Also, n_2 and n_4 are normals to planes intersecting E in a line parallel to AB. Therefore, it is obvious that $n_1Nn_3 < n_1n_5n_3$, where n_1Nn_3 is the arc along the unit sphere from n_1 to N to n_3, and likewise for $n_2Nn_4 < n_2n_5n_4$. Additionally, we have the identity

$$n_5n_1 + n_1N + n_3N + n_3n_5 + n_5n_2 + n_2N + n_4N + n_4n_5 = 4\pi. \tag{7}$$

To prove the bounds for the dihedral angles of a pyramid with a parallelogramic base, we first introduce the following lemma.

Lemma 1. *With the notation introduced above,*

$$n_1N + n_2N + n_3N + n_4N < n_1n_2 + n_2n_3 + n_3n_4 + n_4n_1. \tag{8}$$

P r o o f. By the triangle inequality, we observe that

$$\begin{aligned} n_1N + n_3N &= n_1n_3 \leqslant n_1n_2 + n_2n_3, \quad & n_1N + n_3N &= n_1n_3 \leqslant n_1n_4 + n_4n_3, \\ n_2N + n_4N &= n_2n_4 \leqslant n_2n_1 + n_1n_4, \quad & n_2N + n_4N &= n_2n_4 \leqslant n_2n_3 + n_3n_4. \end{aligned} \tag{9}$$

This yields

$$2(n_1N + n_2N + n_3N + n_4N) \leqslant 2(n_1n_2 + n_2n_3 + n_3n_4 + n_4n_1). \tag{10}$$

To show that the inequality is strict, we note that if $n_1n_3 = n_1n_2 + n_2n_3$, n_2 has to lie on a great circle arc between n_1 and n_3. Since N also has this property, it follows that n_1 and n_3 are antipodal. This in turn means that the two triangular faces ADE and BCE are parallel, which is a contradiction. Therefore, (9) and (10) are strict inequalities. □

To conclude the proof of Theorem 3 (a), we have from (7) and (8) that

$$4\pi = n_5n_1 + n_1N + n_3N + n_3n_5 + n_5n_2 + n_2N + n_4N + n_4n_5 < \sum n_in_j. \tag{11}$$

Thus, from (11) and (4), it follows that

$$\sigma_P = 8\pi - \sum n_in_j < 4\pi. \tag{12}$$

For part (b), we give a constructive proof. Consider a pyramid with a parallelogramic base $ABCD$, in the plane $z = 0$, and apex E. Now, let the projection $x = \mathrm{proj}_{ABCD}\, E$ of the apex point E onto the plane $ABCD$ lie to the right of the edge BC, while also lying between the extended edges AB and CD, as depicted in Figure 9.

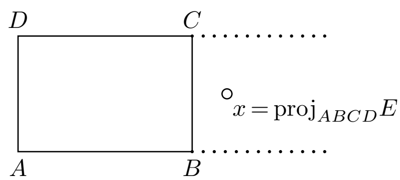

Figure 9. x lies to the right of BC and between the extended edges AB and CD.

As $h \to 0$, the apex $E(x_1, y_1, h)$ approaches the plane $ABCD$, as depicted in Figure 10. This yields the following limits of dihedral angles:

$$\begin{aligned} &\alpha_{AB} \to 0, \quad \alpha_{AD} \to 0, \quad \alpha_{AE} \to \pi, \quad \alpha_{BC} \to \pi, \\ &\alpha_{BE} \to 0, \quad \alpha_{CD} \to 0, \quad \alpha_{CE} \to 0, \quad \alpha_{DE} \to \pi. \end{aligned} \tag{13}$$

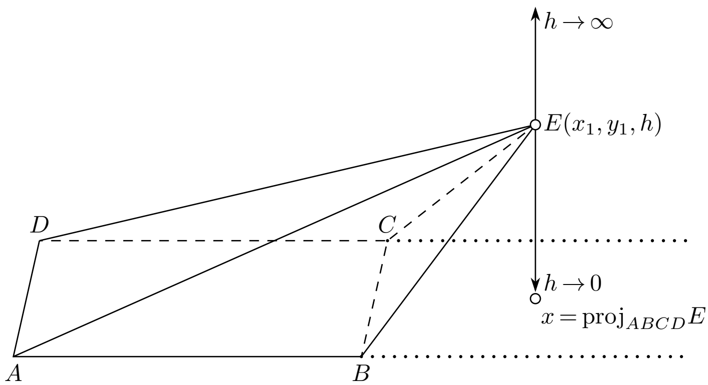

Figure 10. A representation of the diagram in Figure 9, including a depiction of the two limit cases, where the apex E approaches the plane $ABCD$, and ∞.

Thus, the dihedral angle sum approaches 3π. As $h \to \infty$, this yields the following limits of dihedral angles

$$\begin{gathered} \alpha_{AE} + \alpha_{BE} + \alpha_{CE} + \alpha_{DE} \to 2\pi, \\ \alpha_{AB} \to \frac{\pi}{2}, \quad \alpha_{BC} \to \frac{\pi}{2}, \quad \alpha_{CD} \to \frac{\pi}{2}, \quad \alpha_{AD} \to \frac{\pi}{2}. \end{gathered} \tag{14}$$

Now the dihedral angle sum approaches 4π. Since the dihedral angle sum varies continuously between these two limit values, there always exists a pyramid with a parallelogramic base whose angle sum $\sigma_P = \sigma$ for any $\sigma \in (3\pi, 4\pi)$, proving (b). $\square$

4. Final remarks

Theorem 2 also holds for non-convex pyramids with the same proof (however, this is less applicable e.g., for the finite element method).

Consider five points in $\mathbb{R}^3$ such that no four of them are in the same plane, and such that no one of them is in the convex hull of the others. Then the convex hull of the five points is a polyhedron that can be split into two tetrahedra sharing a common triangle. Compared to the case of pyramids, there is one additional edge, so a total of nine dihedral angles to consider. By arguing as in the proof of Theorem 2 part (a), we see that the sum of the dihedral angles is in the interval $(4\pi, 6\pi)$. In this case, the argument is slightly simpler, as we do not have the two angles adding up to π along a diagonal in the quadrilateral base. The arguments of Theorem 2 part (b) can be adapted by one small change: add a small value to the y-coordinate of the point E. The resulting family will then show that all values in $(4\pi, 6\pi)$ appear as the dihedral angle sum of a polyhedron of this form.

By using the known results for tetrahedra (as in Theorem 1), one can give naïve estimates for the dihedral angle sums of any polyhedron in a class of polyhedra with the same combinatorial triangulation. There are a number of questions one can ask in this generality, including whether these bounds are tight. Another problem would be to compute these bounds in terms of easily understood basic characteristics of the polyhedra one considers. For example, are the bounds for the dihedral angles sums if we consider (convex) polyhedra with N_3 triangular faces, N_4 quadrilateral faces, and so on, computable in terms of the finite sequence $\{N_k\}_k$?

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