LOCAL-IN-TIME EXISTENCE FOR THE NON-RESISTIVE INCOMPRESSIBLE MAGNETO-MICROPOLAR FLUIDS

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Abstract. We establish the local-in-time existence of a solution to the non-resistive magneto-micropolar fluids with the initial data $u_0 \in H^{s-1+\varepsilon}$, $w_0 \in H^{s-1}$ and $b_0 \in H^s$ for $s > \frac{3}{2}$ and any $0 < \varepsilon < 1$. The initial regularity of the micro-rotational velocity w is weaker than velocity of the fluid u.

Keywords: non-resistive magneto-micropolar fluid; local existence

MSC 2020: 35A01, 35Q30, 76B03

1. Introduction

The non-resistive incompressible magneto-micropolar equations in three dimensions can be written as follows:

$$
\begin{cases}
u_t + u \cdot \nabla u + \nabla(p + \frac{1}{2}|b|^2) - (\mu + \zeta)\Delta u - (b \cdot \nabla)b - \zeta \nabla \times w = 0,\\
w_t + u \cdot \nabla w + 2\zeta w - \nu \Delta w - \lambda \nabla \operatorname{div} w - \zeta \nabla \times u = 0,\\
b_t + u \cdot \nabla b - b \cdot \nabla u = 0,\\
\operatorname{div} u = 0, \ \operatorname{div} b = 0,
\end{cases}
\tag{1.1}
$$

with the initial data:

$$
(u, w, b)(x, 0) = (u_0, w_0, b_0)(x),
\tag{1.2}
$$

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where $x \in \mathbb{R}^3$ and $t \geqslant 0$. The functions $u = u(x,t)$, $w = w(x,t)$, $b = b(x,t)$, and $p = p(x,t)$ denote the velocity of the fluid, the micro-rotational velocity, the magnetic field, and the pressure, respectively. The constants μ, ζ are the coefficients of kinematic viscosity, vortex viscosity, and ν, λ are angular viscosities.

The magneto-micropolar fluid equations model was first proposed in [1], which describes the motion of electrically conducting micropolar fluids in the presence of a magnetic field. Micropolar fluids represent a class of fluids with nonsymmetric stress tensor (called polar fluids) such as fluids consisting of suspending particles, dumbbell molecules, etc., see [6], [7], [13]. By using the spectral Galerkin method, Rojas-Medar [16] proved the existence and uniqueness of strong solutions for system (1.1) with resistive viscosity in bounded domain. Then, Yuan [18] proved the local existence of the strong solution with initial data in H^s with $s > \frac{3}{2}$ in the whole $\mathbb{R}^3$. The weak solution and global regularity were studied in [14], [17], [15], [16], [19], [20], [5], [4].

When the micro-rotational velocity disappears ($w = 0$), system (1.1) reduces to a non-resistive magneto-hydro-dynamic (MHD) system. There is a great deal of literature on the MHD system. Here we only introduce some studies related to this paper. Jiu and Niu [10] proved the local existence for 2D with the initial data in H^s, for integer $s > 3$. Fefferman et al. [8] established a local existence of non-resistive MHD when $u_0, b_0 \in H^s$ with $s > d/2$ for $d = 2, 3$. Recently, Fefferman et al. [9] extended the local existence result in [8] to the non-resistive MHD when $u_0 \in H^{s-1+\varepsilon}$, $b_0 \in H^s$ with $s > d/2$ for $d = 2, 3$ and any $0 < \varepsilon < 1$. Chemin et al. [3] established the local existence of weak solutions to 2D and 3D Cauchy problem for non-resistive MHD equations with divergence-free initial data in the Besov space $B^{n/2}_{2,1}$ and they also proved the uniqueness of solutions in 3D case.

Motivated by Fefferman-McCormick-Robinson-Rodrigo's approach (see [9]), we study the local-in-time existence of system (1.1)–(1.2). More precisely, we have the following result:

Theorem 1.1. *Suppose that the initial data* (u_0, w_0, b_0) *satisfy* $u_0 \in H^{s-1+\varepsilon}$, $w_0 \in H^{s-1}$, *and* $b_0 \in H^s$ *for* $s > \frac{3}{2}$ *and any* $0 < \varepsilon < 1$. *Then there exists* $T^* > 0$ *such that system* (1.1)–(1.2) *has a solution* (u, w, b) *satisfying*

$$u \in L^\infty(0, T^*; H^{s-1+\varepsilon}) \cap L^2(0, T^*; H^{s+\varepsilon}),$$
$$w \in L^\infty(0, T^*; H^{s-1}) \cap L^2(0, T^*; H^s),$$

and

$$b \in L^\infty(0, T^*; H^s).$$

R e m a r k 1.1. Comparing with the MHD system, the existence of micro-rotational velocity makes it more difficult. It is worth to point out that the initial regularity of w is weaker than u. When $w = 0$, Theorem 1.1 reduces to the result in [9]. Theorem 1.1 is an improvement of the result in [18].

2. Preliminaries

In this section, we will give some elementary facts which will be used later. Throughout the paper we use the notation Λ^s to denote the fractional derivative of order s, given in terms of the Fourier transform by $\widehat{\Lambda^s f} = |\xi|^s \hat{f}$. We write

$$\|u\|_{H^s}^2 = \|\Lambda^s u\|^2 + \|u\|^2, \quad s > 0,$$

which is equivalent to the standard H^s norm when s is a positive integer, where $\|\cdot\| \triangleq \|\cdot\|_{L^2}$. Before the proof of Theorem 1.1, we present the maximum regularity-type result of heat equation.

Lemma 2.1 ([9]). *If $f \in L^r(0,T;H^{s-1})$, $1 < r < \infty$, $s > 1$, and*

$$\partial_t u - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0, \quad u(0) = u_0 \in H^{s-1+\varepsilon},$$

where u_0 is divergence free, then for $T \leqslant 1$

$$\int_0^T \|u\|_{H^{s+1}} \, \mathrm{d}t \leqslant C_\varepsilon T^{\varepsilon/2} \|u_0\|_{H^{s-1+\varepsilon}} + C_r T^{1-1/r} \|f\|_{L^r(0,T;H^{s-1})}. \tag{2.1}$$

Lemma 2.2 ([8]). *Given $s > \frac{3}{2}$, there is a constant $C = C(s)$ such that for all u, b with $\nabla u, b \in H^s$,*

$$\|\Lambda^s[(u\cdot\nabla)b] - (u\cdot\nabla)(\Lambda^s b)\|_{L^2} \leqslant c\|\nabla u\|_{H^s}\|b\|_{H^s}.$$

In [11], [12], the following useful inequalities are proved in the Sobolev spaces.

Lemma 2.3 (Kato-Ponce inequality [11], [12]). *Let $s > 0$, $1 < p < \infty$. If $f \in W^{1,p_1} \cap W^{s,q_2}$, $g \in L_{p_2} \cap W^{s,q_1}$, then*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leqslant C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1} g\|_{L^{q_1}} + \|g\|_{L^{p_2}}\|\Lambda^s f\|_{L^{q_2}})$$

and

$$\|\Lambda^s(fg)\|_{L^p} \leqslant C(\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{q_1}} + \|g\|_{L^{p_2}}\|\Lambda^s f\|_{L^{q_2}})$$

with $1/p = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$.

3. Proof of Theorem 1.1

Multiplying $(1.1)_1$, $(1.1)_2$ and $(1.1)_3$ by u, w and b, respectively, integrating by parts yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + (\mu+\zeta)\|\nabla u\|^2 = \langle (b\cdot\nabla)b, u\rangle + \zeta\langle \nabla\times w, u\rangle, \tag{3.1}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \nu\|\nabla w\|^2 + \lambda\|\operatorname{div} w\|^2 + 2\zeta\|w\|^2 = \zeta\langle \nabla\times u, w\rangle, \tag{3.2}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|b\|^2 = \langle (b\cdot\nabla)u, b\rangle. \tag{3.3}$$

Combining (3.1)–(3.3) together, noting that $\langle (b\cdot\nabla)b, u\rangle = -\langle (b\cdot\nabla)u, b\rangle$ and $\langle \nabla\times w, u\rangle = \langle \nabla\times u, w\rangle$, using the Cauchy inequality $|2\zeta\langle \nabla\times u, w\rangle| \leqslant \zeta\|\nabla u\|^2 + \zeta\|w\|^2$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2 + \|w\|^2 + \|b\|^2) + \mu\|\nabla u\|^2 + \nu\|\nabla w\|^2 + \lambda\|\operatorname{div} w\|^2 + \zeta\|w\|^2 \leqslant 0.$$

It follows that

$$\begin{aligned}(\|u\|^2 + \|w\|^2 + \|b\|^2) + 2\int_0^t (\mu\|\nabla u\|^2 + \nu\|\nabla w\|^2 + \lambda\|\operatorname{div} w\|^2 + \zeta\|w\|^2)\,\mathrm{d}s \\ \leqslant (\|u_0\|^2 + \|w_0\|^2 + \|b_0\|^2) \triangleq M_0.\end{aligned} \tag{3.4}$$

We also can obtain from (3.1)–(3.3)

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2 + \|w\|^2) + (\mu+\zeta)\|\nabla u\|^2 + \nu\|\nabla w\|^2 + \lambda\|\nabla w\|^2 + 2\zeta\|w\|^2 \\ &\quad = \langle (b\cdot\nabla)b, u\rangle + 2\zeta\langle \nabla\times u, w\rangle \leqslant C\|b\|\|\nabla u\|\|b\|_{L^\infty} + C\|w\|\|\nabla u\| \\ &\quad \leqslant \frac{\mu}{2}\|\nabla u\|^2 + C\|b\|_{H^s}^4 + CM_0\end{aligned} \tag{3.5}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|b\|^2 \leqslant \|b\|^2\|\nabla u\|_{L^\infty} \leqslant C\|b\|^2\|\nabla u\|_{H^s}. \tag{3.6}$$

We apply the operator Λ^s to $(1.1)_2$ and take the inner product with $\Lambda^s b$ in L^2. By Lemma 2.2 and the same argument as in [9], one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s b\|^2 \leqslant |\langle \Lambda^s[(b\cdot\nabla)u], \Lambda^s b\rangle| + |\langle \Lambda^s[(u\cdot\nabla)b], \Lambda^s b\rangle| \leqslant C\|b\|_{H^s}^2\|\nabla u\|_{H^s}. \tag{3.7}$$

Combining (3.6) and (3.7) together, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|b\|_{H^s}^2 \leqslant C_1\|b\|_{H^s}^2\|\nabla u\|_{H^s},$$

which implies that

$$\|b(t)\|_{H^s}^2 \leqslant \|b_0\|_{H^s}^2 \exp\Big\{C_1\int_0^t \|\nabla u\|_{H^s}\,\mathrm{d}\tau\Big\}. \tag{3.8}$$

Next, we get the estimates on u in the space $L^1(0,T;H^{s+1})$, using Lemma 2.1. We rewrite $(1.1)_1$ as follows:

$$u_t-(\mu+\zeta)\Delta u+\nabla\Big(p+\frac{1}{2}|b|^2\Big) = f \triangleq -(u\cdot\nabla)u+(b\cdot\nabla)b+\zeta\nabla\times w,\ \operatorname{div} u = 0,\ u(0) = u_0.$$

By (2.1), we have

$$\int_0^T \|u\|_{H^{s+1}}\,\mathrm{d}t \leqslant C_\varepsilon T^{\varepsilon/2}\|u_0\|_{H^{s-1+\varepsilon}} + C_r T^{1-1/r}\|f\|_{L^r(0,T;H^{s-1})}. \tag{3.9}$$

Since

$$\begin{aligned}\|f\|_{H^{s-1}} &= \|\operatorname{div}(b\otimes b)-\operatorname{div}(u\otimes u)+\zeta\nabla\times w\|_{H^{s-1}}\\ &\leqslant \|b\otimes b\|_{H^s}+\|u\otimes u\|_{H^s}+C\|w\|_{H^s}\\ &\leqslant C\|b\|_{H^s}^2+C\|u\|_{H^s}^2+C\|w\|_{H^s}\\ &\leqslant C\|b\|_{H^s}^2+C\|u\|^{2\varepsilon/(s+\varepsilon)}\|u\|_{H^{s+\varepsilon}}^{2s/(s+\varepsilon)}+C\|w\|_{H^s}\\ &\leqslant C\|b\|_{H^s}^2+CM_0^{2\varepsilon/(s+\varepsilon)}\|u\|_{H^{s+\varepsilon}}^{2s/(s+\varepsilon)}+C\|w\|_{H^s},\end{aligned}$$

combining (3.4) and choosing $r=(s+\varepsilon)/s>1$, from (3.9) we have

$$\begin{aligned}\int_0^T \|u\|_{H^{s+1}}\,\mathrm{d}t &\leqslant C_\varepsilon T^{\varepsilon/2}\|u_0\|_{H^{s-1+\varepsilon}}+C_\varepsilon T^{(s+\varepsilon)/s}\\ &\quad\times\Big(\int_0^T(\|b\|_{H^s}^{2(s+\varepsilon)/s}+M_0^{\varepsilon/s}\|u\|_{H^{s+\varepsilon}}^2+\|w\|_{H^s}^{(s+\varepsilon)/s})\,\mathrm{d}t\Big)^{s/(s+\varepsilon)}.\end{aligned} \tag{3.10}$$

We now estimate the norm of u,w in $H^{s-1+\varepsilon}$ and $H^{s+\varepsilon}$. We apply the operator $\Lambda^{s-1+\varepsilon}$ to $(1.1)_1$ and take the inner product with $\Lambda^{s-1+\varepsilon}u$. Then

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^{s-1+\varepsilon}u\|^2)+\|\Lambda^{s+\varepsilon}u\|^2 &\leqslant C|\langle\Lambda^{s-1+\varepsilon}[(u\cdot\nabla)u],\Lambda^{s-1+\varepsilon}u\rangle|\\ &\quad+C|\langle\Lambda^{s-1+\varepsilon}[(b\cdot\nabla)b],\Lambda^{s-1+\varepsilon}u\rangle|\\ &\quad+C|\langle\Lambda^{s-1+\varepsilon}(\nabla\times w),\Lambda^{s-1+\varepsilon}u\rangle|\\ &\triangleq\sum_{i=1}^3 I_i.\end{aligned} \tag{3.11}$$

By the Sobolev interpolation and the Young inequality, we have

$$\begin{aligned}
I_1 + I_2 &= C|\langle \Lambda^{s-1}\mathrm{div}(u\otimes u), \Lambda^{s-1+2\varepsilon}u\rangle| \\
&\quad + C|\langle \Lambda^{s-1}\mathrm{div}(b\otimes b), \Lambda^{s-1+2\varepsilon}u\rangle| \\
&\leqslant C\|u\|_{H^s}^2\|u\|_{H^{s-1+2\varepsilon}} + C\|b\|_{H^s}^2\|u\|_{H^{s-1+2\varepsilon}} \\
&\leqslant C(\|u\|_{H^{s-1+\varepsilon}}^{\varepsilon}\|u\|_{H^{s+\varepsilon}}^{1-\varepsilon})^2\|u\|_{H^{s-1+\varepsilon}}^{1-\varepsilon}\|u\|_{H^{s+\varepsilon}}^{\varepsilon} \\
&\quad + C\|b\|_{H^s}^2\|u\|_{H^{s-1+\varepsilon}}^{1-\varepsilon}\|u\|_{H^{s+\varepsilon}}^{\varepsilon} \\
&\leqslant C\|u\|_{H^{s-1+\varepsilon}}^{2(1+\varepsilon)/\varepsilon} + C\|b\|_{H^s}^{2(1+\varepsilon)} + \frac{1}{6}\|u\|_{H^{s+\varepsilon}}^2
\end{aligned}$$

and

$$I_3 = \frac{1}{6}\|\Lambda^{s+\varepsilon}u\|_{L^2}^2 + C\|\Lambda^{s-1+\varepsilon}w\|_{L^2}^2.$$

Operating with Λ^{s-1} on $(1.1)_2$ and taking the inner product with $\Lambda^{s-1}w$,

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^{s-1}w\|^2) + \|\Lambda^s w\|^2 + \|\Lambda^{s-1}w\|^2 &\leqslant C|\langle \Lambda^{s-1}[(u\cdot\nabla)w], \Lambda^{s-1}w\rangle| \\
&\quad + C|\langle \Lambda^{s-1}(\nabla\times u), \Lambda^{s-1}w\rangle| \\
&\triangleq \sum_{i=4}^{5} I_i.
\end{aligned}\tag{3.12}$$

Now, we estimate the two terms on the right-hand side by using the Kato-Ponce inequality [11], [12]. Next, we estimate I_4 in three cases. For the case $s+\varepsilon = 5/2$ we have

$$\begin{aligned}
I_4 &= C|\langle \Lambda^{s-1}\mathrm{div}(u\otimes w), \Lambda^{s-1}w\rangle| \\
&= C|\langle \Lambda^{s-1}(u\cdot\nabla w), \Lambda^{s-1}w\rangle - \langle (u\cdot\nabla\Lambda^{s-1}w), \Lambda^{s-1}w\rangle| \\
&\leqslant C(\|\nabla u\|_{L^3}\|\Lambda^{s-1}w\|_{L^6} + \|\Lambda^{s-1}u\|_{L^{6/(3-2\varepsilon)}}\|\nabla w\|_{L^{3/\varepsilon}})\|\Lambda^{s-1}w\| \\
&\leqslant C\|u\|_{H^{s-1+\varepsilon}}^{4/(2s+2\varepsilon-3)}\|\Lambda^{s-1}w\|^2 + \frac{1}{4}\|\Lambda^s w\|^2.
\end{aligned}$$

For the case $3/2 < s+\varepsilon < 5/2$,

$$\begin{aligned}
I_4 &= C|\langle \Lambda^{s-1}\mathrm{div}(u\otimes w), \Lambda^{s-1}w\rangle| \\
&\leqslant C\|\Lambda^{s-1}(u\otimes w)\|\|\Lambda^s w\| \\
&\leqslant C(\|u\|_{L^{6/(5-2s-2\varepsilon)}}\|\Lambda^{s-1}w\|_{L^{6/(2s+2\varepsilon-2)}} \\
&\quad + \|\Lambda^{s-1}u\|_{L^{6/(3-2\varepsilon)}}\|w\|_{L^{3/\varepsilon}})\|\Lambda^s w\| \\
&\leqslant C\|u\|_{H^{s-1+\varepsilon}}\|\Lambda^{s-1}w\|^{s+\varepsilon-3/2}\|\Lambda^s w\|^{7/2-s-\varepsilon} \\
&\leqslant C\|u\|_{H^{s-1+\varepsilon}}^{4/(2s+2\varepsilon-3)}\|\Lambda^{s-1}w\|^2 + \frac{1}{4}\|\Lambda^s w\|^2.
\end{aligned}$$

For the case $s+\varepsilon > 5/2$,

$$\begin{aligned}
I_4 &= C|\langle \Lambda^{s-1}\mathrm{div}(u\otimes w), \Lambda^{s-1}w\rangle| \\
&\leqslant C\|\Lambda^{s-1}(u\otimes w)\|\|\Lambda^s w\| \\
&\leqslant C(\|u\|_{L^\infty}\|\Lambda^{s-1}w\| + \|\Lambda^{s-1}u\|_{L^{6/(3-2\varepsilon)}}\|w\|_{L^{3/\varepsilon}})\|\Lambda^s w\| \\
&\leqslant C\|u\|_{H^{s-1+\varepsilon}}^{4/(2s+2\varepsilon-3)}\|\Lambda^{s-1}w\|^2 + \frac{1}{4}\|\Lambda^s w\|^2 + \|u\|_{H^{s-1+\varepsilon}}^2\|\Lambda^{s-1}w\|^2. \\
I_5 &= C|\langle \Lambda^{s-1}(\nabla\times u), \Lambda^{s-1}w\rangle| \\
&\leqslant \frac{1}{6}\|\Lambda^{s+\varepsilon}u\|^2 + C\|\Lambda^{s-1-\varepsilon}w\|^2.
\end{aligned}$$

From the above estimates and (3.11)–(3.12) it follows

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^{s-1+\varepsilon}u\|^2 + \|\Lambda^{s-1}w\|^2) &+ \|\Lambda^{s+\varepsilon}u\|^2 + \|\Lambda^s w\|^2 + \|\Lambda^{s-1}w\|^2 \\
&\leqslant C\|u\|_{H^{s-1+\varepsilon}}^{2(1+\varepsilon)/\varepsilon} + C\|u\|_{H^{s-1+\varepsilon}}^{4/(2s+2\varepsilon-3)}\|\Lambda^{s-1}w\|^2 \\
&\quad + \frac{1}{2}\|u\|_{H^{s+\varepsilon}}^2 + \frac{1}{4}\|\Lambda^s w\|^2 \\
&\quad + C\|\Lambda^{s-1+\varepsilon}w\|^2 + C\|\Lambda^{s-1-\varepsilon}w\|^2 \\
&\leqslant C\|u\|_{H^{s-1+\varepsilon}}^{2(1+\varepsilon)/\varepsilon} + C\|u\|_{H^{s-1+\varepsilon}}^{8/(2s+2\varepsilon-3)} + \|w\|_{H^{s-1}}^4 \\
&\quad + \frac{1}{2}\|u\|_{H^{s+\varepsilon}}^2 + C\|w\|^2 + \frac{1}{2}\|\Lambda^s w\|^2.
\end{aligned} \tag{3.13}$$

Combining (3.5) and (3.13), we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{H^{s-1+\varepsilon}}^2 + \|w\|_{H^{s-1}}^2) &+ \|u\|_{H^{s+\varepsilon}}^2 + \|w\|_{H^s}^2 \\
&\leqslant C\|u\|_{H^{s-1+\varepsilon}}^{2(1+\varepsilon)/\varepsilon} + C\|u\|_{H^{s-1+\varepsilon}}^{8/(2s+2\varepsilon-3)} + \|w\|_{H^{s-1}}^4 \\
&\quad + C\|b\|_{H^s}^{2(1+\varepsilon)} + C\|b\|_{H^s}^4 + C(\|u\|^2 + \|w\|_{L^2}^2 + M_0) \\
&\leqslant C_2(\|u\|_{H^{s-1+\varepsilon}}^2 + \|w\|_{H^{s-1}}^2)^{\max\{(1+\varepsilon)/\varepsilon, 4/(2s+2\varepsilon-3), 2\}} \\
&\quad + C_3\|b\|_{H^s}^{2(1+\varepsilon)} + C_4\|b\|_{H^s}^4 + C_5 M_0.
\end{aligned} \tag{3.14}$$

We will choose $T^* > 0$ such that $\|b\|_{H^s} \leqslant 2\|b_0\|_{H^s}$ for all $t\in[0,T^*]$. Set

$$\begin{aligned}
M_1 &\triangleq \|u_0\|_{H^{s-1+\varepsilon}} + \|w_0\|_{H^{s-1}}, \\
M_2 &\triangleq 2^{2(1+\varepsilon)}C_3\|b_0\|_{H^s}^{2(1+\varepsilon)} + 2^4 C_4\|b_0\|_{H^s}^4 + C_5 M_0, \\
\chi &= \max\Bigl\{\frac{1+\varepsilon}{\varepsilon}, \frac{4}{2s+2\varepsilon-3}, 2\Bigr\}
\end{aligned}$$

and choose T^* sufficiently small so that

$$0 < (1-(\chi-1)C_2T(M_1^2+TM_2)^{\chi-1})^{-1/(\chi-1)} < 2 \quad \forall T\in(0,T^*) \tag{3.15}$$

and

$$C_\varepsilon T^{\varepsilon/2}M_1 + C_\varepsilon T^{(s+\varepsilon)/s}[2^{2(s+\varepsilon)/s}T\|b_0\|_{H^s}^{2(s+\varepsilon)/s} + (M_0^{\varepsilon/s}+1)(2^\chi C_2T(M_1^2+TM_2)^\chi + M_2T) + T]^{s/(s+\varepsilon)} < \frac{\ln 2}{C_1} \tag{3.16}$$

for all $0 < T < T^*$.

Denote

$$T = \sup\{T_0 \in [0,T^*];\ \|b\|_{H^s} \leqslant 2\|b_0\|_{H^s}\ \forall t \in [0,T_0]\}.$$

Suppose that $T < T^*$. Then from (3.4) and (3.14) we have for all $t \in [0,T]$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{H^{s-1+\varepsilon}}^2 + \|w\|_{H^{s-1}}^2) + \|u\|_{H^{s+\varepsilon}}^2 + \|w\|_{H^s}^2 &\leqslant C_2(\|u\|_{H^{s-1+\varepsilon}}^2 + \|w\|_{H^{s-1}}^2)^\chi + 2^{2(1+\varepsilon)}C_3\|b_0\|_{H^s}^{2(1+\varepsilon)} \\ &\quad + 2^4C_4\|b_0\|_{H^s}^4 + C_5M_0 \\ &\leqslant C_2(\|u\|_{H^{s-1+\varepsilon}}^2 + \|w\|_{H^{s-1}}^2)^\chi + M_2.\end{aligned} \tag{3.17}$$

Using standard ODE comparison techniques (see Theorem 6 in [2]) and (3.15), we obtain

$$\begin{aligned}(\|u\|_{H^{s-1+\varepsilon}}^2 + \|w\|_{H^{s-1+\varepsilon}}^2) &\leqslant (M_1^2+TM_2)(1-(\chi-1)C_2T(M_1^2+TM_2)^{\chi-1})^{-1/(\chi-1)} \\ &\leqslant 2(M_1^2+TM_2).\end{aligned} \tag{3.18}$$

Inserting (3.18) into (3.17) and integrating over $[0,T]$ yields

$$\int_0^T(\|u\|_{H^{s+\varepsilon}}^2 + \|w\|_{H^s}^2)\,\mathrm{d}t \leqslant 2^\chi C_2T(M_1^2+TM_2)^\chi + M_2T. \tag{3.19}$$

Substituting (3.19) and $\|b\|_{H^s} \leqslant 2\|b_0\|_{H^s}$ into (3.10), by (3.16), we get

$$\begin{aligned}\int_0^T(\|u\|_{H^{s+1}})\,\mathrm{d}t &\leqslant C_\varepsilon T^{\varepsilon/2}M_1 + C_\varepsilon T^{(s+\varepsilon)/s} + C_\varepsilon T^{(s+\varepsilon)/s}[2^{2(s+\varepsilon)/s}T\|b_0\|_{H^s}^{2(s+\varepsilon)/s} \\ &\quad + (M_0^{\varepsilon/s}+1)(2^\chi C_2T(M_1^2+TM_2)^\chi + M_2T) + T]^{s/(s+\varepsilon)} \\ &< \frac{\ln 2}{C_1}.\end{aligned} \tag{3.20}$$

Substituting this into (3.8) ensures that $\|b\|_{H^s} \leqslant 2\|b_0\|_{H^s}$ for all $t \in [0,T]$, contradicting the maximality of T. It follows that $T = T^*$ and hence,

$$\|b\|_{H^s} \leqslant 2\|b_0\|_{H^s}, \quad t \in [0,T^*].$$

Theorem 1.1 now follows from (3.18), (3.19) and (3.20). □

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