TYKHONOV WELL-POSEDNESS OF A HEAT TRANSFER PROBLEM WITH UNILATERAL CONSTRAINTS

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Abstract. We consider an elliptic boundary value problem with unilateral constraints and subdifferential boundary conditions. The problem describes the heat transfer in a domain $D \subset \mathbb{R}^d$ and its weak formulation is in the form of a hemivariational inequality for the temperature field, denoted by $\mathcal{P}$. We associate to Problem $\mathcal{P}$ an optimal control problem, denoted by $\mathcal{Q}$. Then, using appropriate Tykhonov triples, governed by a nonlinear operator G and a convex $\widetilde{K}$, we provide results concerning the well-posedness of problems $\mathcal{P}$ and $\mathcal{Q}$. Our main results are Theorems 4.2 and 5.2, together with their corollaries. Their proofs are based on arguments of compactness, lower semicontinuity and pseudomonotonicity. Moreover, we consider three relevant perturbations of the heat transfer boundary valued problem which lead to penalty versions of Problem $\mathcal{P}$, constructed with particular choices of G and $\widetilde{K}$. We prove that Theorems 4.2 and 5.2 as well as their corollaries can be applied in the study of these problems, in order to obtain various convergence results.

Keywords: heat transfer problem; unilateral constraint; subdifferential boundary condition; hemivariational inequality; optimal control; Tykhonov well-posedness; approximating sequence; convergence results

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1. Introduction

A large number of mathematical models in physics, mechanics and engineering sciences are expressed in terms of nonlinear boundary value problems which involve inequalities. Their analysis and optimal control was made the object of many books and papers and, therefore, the literature in the field is extensive. Here we restrict ourselves to mention the books [1], [9], [11], [13], [14], [17], [18], [28], [30], [31] and

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more recently, [5], [27], [37]. The results presented in [1], [5], [9], [11], [14], [17], [18], [30] concern the analysis of various classes of variational inequalities and are based on arguments of monotonicity and convexity, including properties of the subdifferential of a convex function. The results in [13], [27], [28], [31], [37] concern the analysis of hemivariational inequalities and are based on properties of the subdifferential in the sense of Clarke, defined for locally Lipschitz functions, which may be nonconvex. Results in the study of optimal control for variational and hemivariational inequalities have been discussed in several works including [2], [3], [4], [8], [19], [23], [25], [26], [29], [32], [33], [35], [34], and more recently, [36], [38].

The main task in the analysis and control of inequality problems is to provide their well-posedness. The concepts of well-posedness vary from problem to problem and from author to author. A few examples are the concept of well-posedness in the sense of Hadamard for partial differential equations, the concept of well-posedness in the sense of Tykhonov for a minimization problem, the concept of well-posedness in the sense of Levitin-Polyak for a constrainted optimization problem, among others. The concept of Tykhonov well-posedness (well-posedness, for short) was introduced in [42] for a minimization problem and then it has been generalized for different optimization problems, see for instance [6], [7], [15], [16], [20], [45]. It has been extended in the recent years to various mathematical problems like inequalities, inclusions, fixed point and saddle point problems. Thus, the well-posedness of variational inequalities was studied for the first time in [21], [22] and the study of well-posedness of hemivariational inequalities was initiated in [10]. References in the field include [41], [43], among others. A general framework which unifies the view on the well-posedness for abstract problems in metric spaces was recently introduced in [40]. Moreover, the Tykhonov well-posedness of an antiplane shear problem was studied in our recent paper [39].

In this paper we consider a stationary boundary value problem which describes the heat transfer in a domain $D \subset \mathbb{R}^d$, $d = 1, 2, 3$. In particular cases, the problem was already considered in [3], [4]. There, the heat flux was assumed to be given on the part Γ_3 of the boundary of D and therefore the weak formulation of the problem was in a form of the variational inequality for the temperature field. In contrast, in this current paper we model the heat transfer on Γ_3 by using a subdifferential boundary condition, governed by a locally Lipschitz potential function. As a consequence, the weak formulation of the problem leads to an elliptic hemivariational inequality, denoted by $\mathcal{P}$. Moreover, we associate to this inequality an optimal control problem, denoted by $\mathcal{Q}$.

Our aim in this paper is twofold. The first one is to study the well-posedness of the hemivariational inequality $\mathcal{P}$. The second one is to study the well-posedness of the associated optimal control problem $\mathcal{Q}$. To this end, for both problems, we use specific

Tykhonov triples, constructed with approximating sets governed by a penalty-type operator and a set of constraints. Proving the well-posedness of the corresponding hemivariational inequality and the associated optimal control problem in this functional setting is nonstandard and represents the main trait of novelty of our paper. Moreover, the well-posedness of these problems implies general convergence results which allows us to deduce the weak-strong dependence of the solution to Problem $\mathcal{P}$ with respect to the data, its approach by a penalty-like method as well as the weak compactness of the set of solutions to Problem $\mathcal{Q}$. All these results represent a continuation of our previous results obtained in [3], [4].

The rest of the manuscript is structured as follows. In Section 2 we recall some preliminary material on hemivariational inequalities and well-posedness in the sense of Tykhonov. In Section 3 we introduce the heat transfer problem, list the assumptions on the data and state its variational formulation $\mathcal{P}$. Then, we introduce the associate optimal control problem $\mathcal{Q}$. In Section 4 we prove our first result, Theorem 4.2, which states the well-posedness of Problem $\mathcal{P}$. In Section 5 we prove our second result, Theorem 5.2, which states the weakly generalized well-posedness of Problem $\mathcal{Q}$. Both theorems are completed by several corollaries. Their proofs are based on arguments of pseudomonotonicity, compacteness, and lower semicontinuity. Finally, in Section 6 we consider three relevant versions of the heat transfer problem with penalty conditions. We prove that Theorems 4.2, 5.2 and their corollaries can be applied in the study of these problems in order to deduce various convergence results.

2. Preliminaries

We start with some notation and preliminaries on hemivariational inequalities and send the reader to [27], [28], [37] for more details on the material presented below. Let $(X, \|\cdot\|_X)$ be a reflexive Banach space and let X^* denote its dual. We use the symbol $\langle\cdot,\cdot\rangle$ for the duality pairing between X^* and X. The limits, upper and lower limits below are considered as $n \to \infty$, even if we do not mention it explicitely. The symbols "$\rightharpoonup$" and "$\to$" denote, respectively, the weak and the strong convergence in X or X^*.

For real valued functions defined on X we recall the following definitions.

Definition 2.1. A function $j\colon X \to \mathbb{R}$ is said to be *locally Lipschitz* if for every $x \in X$ there exists a neighborhood U_x of x and a constant $L_x > 0$ such that $|j(y) - j(z)| \leqslant L_x\|y - z\|_X$ for all $y, z \in U_x$. For such functions the *generalized (Clarke) directional derivative* of j at the point $x \in X$ in the direction $v \in X$ is defined by

$$j^0(x; v) = \limsup_{y\to x,\ \lambda\downarrow 0} \frac{j(y + \lambda v) - j(y)}{\lambda}.$$

The *generalized gradient (subdifferential)* of j at x is a subset of the dual space X^* given by

$$\partial j(x) = \{\zeta \in X^* \colon\ j^0(x;v) \geqslant \langle \zeta, v\rangle\ \forall\, v \in X\}.$$

The function j is said to be regular (in the sense of Clarke) at the point $x \in X$ if for all $v \in X$ the one-sided directional derivative $j'(x;v)$ exists and $j^0(x;v) = j'(x;v)$.

We shall use the following properties of the generalized directional derivative and the generalized gradient.

Proposition 2.2. *Assume that $j\colon X \to \mathbb{R}$ is a locally Lipschitz function. Then the following hold:*

(a) *For every $x \in X$, the function $X \ni v \mapsto j^0(x;v) \in \mathbb{R}$ is positively homogeneous, i.e. $j^0(x;\lambda v) = \lambda j^0(x;v)$ for all $\lambda \geqslant 0$, $v \in X$.*
(b) *For every $v \in X$ we have $j^0(x;v) = \max\{\langle \xi, v\rangle \colon\ \xi \in \partial j(x)\}$.*

We shall use these definitions and properties both in the case when $X = \mathbb{R}$ and $X = V$, where V is a Sobolev-type space which will defined in Section 3. Next, we proceed with the definition of some classes of nonlinear operators.

Definition 2.3. An operator $A\colon X \to X^*$ is said to be:

a) monotone if for all u, $v \in X$, we have $\langle Au - Av, u - v\rangle \geqslant 0$;
b) strongly monotone if there exists $m_A > 0$ such that

$$\langle Au - Av, u - v\rangle \geqslant m_A \|u - v\|_X^2 \quad \forall\, u, v \in X;$$

c) bounded if A maps bounded sets of X into bounded sets of X^*;
d) pseudomonotone if it is bounded and $u_n \rightharpoonup u$ in X with

$$\limsup \langle Au_n, u_n - u\rangle \leqslant 0$$

imply $\liminf \langle Au_n, u_n - v\rangle \geqslant \langle Au, u - v\rangle$ for all $v \in X$;
e) demicontinuous if $u_n \to u$ in X implies $Au_n \rightharpoonup Au$ in X^*.

We shall use the following result related to the pseudomonotonicity of operators.

Proposition 2.4. *Assume that the operator $A\colon X \to X^*$ is bounded, demicontinuous and monotone. Then A is pseudomonotone.*

We turn now to the study of hemivariational inequalities of the form

$$u \in K, \quad \langle Au, v-u\rangle + j^0(u; v-u) \geqslant \langle f, v-u\rangle \quad \forall\, v \in K, \tag{2.1}$$

and to this end we consider the following hypotheses on the data.

(2.2) K is a nonempty, closed and convex subset of X.

$$\begin{cases} A\colon\, X \to X^* \text{ is pseudomonotone and} \\ \text{strongly monotone with constant } m_A > 0. \end{cases} \tag{2.3}$$

$$\begin{cases} j\colon\, X \to \mathbb{R} \text{ is such that} \\ \text{(a) } j \text{ is locally Lipschitz,} \\ \text{(b) } \|\xi\|_{X^*} \leqslant \tilde{c}_0 + \tilde{c}_1\|v\|_X \qquad \forall\, v \in X,\ \xi \in \partial j(v),\ \text{with } \tilde{c}_0, \tilde{c}_1 \geqslant 0, \\ \text{(c) } j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \leqslant \alpha_j \|v_1 - v_2\|_X^2 \\ \qquad\qquad\qquad\qquad\qquad \forall\, v_1, v_2 \in X,\ \text{with } \alpha_j > 0. \end{cases} \tag{2.4}$$

(2.5) $\alpha_j < m_A$.

(2.6) $f \in X^*$.

The unique solvability of the variational-hemivariational inequality (2.1) is provided by the following result proved in [37].

Theorem 2.5. *Assume* (2.2)–(2.6). *Then there exists a unique solution to inequality* (2.1).

We end this section by recalling some preliminaries concerning the concept of well-posedness in the sense of Tykhonov. For more details in the matter we send the reader to [40].

Assume that $\mathcal{M}$ is an abstract mathematical object called generic "problem", associated to a metric space X. Problem $\mathcal{M}$ could be an equation, a minimization problem, a fixed-point problem, an optimal control problem, an inclusion or an inequality problem. Its rigorous statement varies from example to example. We associate to $\mathcal{M}$ the concept of "solution" which follows from the context and which will be clearly defined in each example we present below in this paper. We now introduce the following definitions.

Definition 2.6. A Tykhonov triple for Problem $\mathcal{M}$ is a mathematical object of the form $\mathcal{T} = (I, \Omega, \mathcal{C})$, where I is a given nonempty set, $\Omega\colon\, I \to 2^X - \{\emptyset\}$ and $\mathcal{C} \subset \mathcal{S}(I)$, $\mathcal{C} \neq \emptyset$.

Note that in this definition and below in this paper, $\mathcal{S}(I)$ represents the set of sequences of elements of I and 2^X denotes the set of parts of the space X. A typical element of I will be denoted by θ and a typical element of $\mathcal{S}(I)$ will be denoted by $\{\theta_n\}$. We refer to the set I as the set of indices. Moreover, for any $\theta \in I$ we refer to the set $\Omega(\theta) \subset X$ as an approximating set and $\mathcal{C}$ will represent the so-called convergence criterion.

Definition 2.7. Given a Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$, a sequence $\{u_n\} \subset X$ is called an approximating sequence if there exists a sequence $\{\theta_n\} \subset \mathcal{C}$ such that $u_n \in \Omega(\theta_n)$ for each $n \in \mathbb{N}$.

Note that approximating sequences always exist since by assumption, $\mathcal{C} \neq \emptyset$ and moreover, for any sequence $\{\theta_n\} \subset \mathcal{C}$ and any $n \in \mathbb{N}$, the set $\Omega(\theta_n)$ is not empty. Therefore, Definition 2.7 above makes sense.

Definition 2.8. Given a Tykhonov triple $\mathcal{T} = (I, \Omega, \mathcal{C})$, Problem $\mathcal{M}$ is said to be:

a) (strongly) well-posedness if it has a unique solution and every approximating sequence converges in X to this solution;
b) weakly well-posedness if it has a unique solution and every approximating sequence converges weakly in X to this solution;
c) weakly generalized well-posedness if it has at least one solution and every approximating sequence contains a subsequence which converges weakly in X to a point of the solution set.

We remark that the concept of approximating sequence above depends on the Tykhonov triple $\mathcal{T}$ and for this reason, we sometimes refer to approximating sequences with respect to $\mathcal{T}$. As a consequence, the concepts of strongly, weakly and weakly generalized well-posedness depend on the Tykhonov triple $\mathcal{T}$ and therefore we refer to them as strongly, weakly and weakly generalized well-posedness with respect to $\mathcal{T}$, respectively.

3. The heat transfer problem

In this section we introduce the heat transfer problem, list the assumptions on the data, derive its variational formulation and state our optimal control problem. The problem under consideration is the following.

Problem $\mathcal{H}$. *Find a temperature field $u \colon D \to \mathbb{R}$ such that*

$$u \geqslant 0, \quad -\Delta u - f \geqslant 0, \quad u(\Delta u + f) = 0 \quad \text{a.e. in } D, \tag{3.1}$$

$$u = 0 \quad \text{a.e. on } \Gamma_1, \tag{3.2}$$

$$u = b \quad \text{a.e. on } \Gamma_2, \tag{3.3}$$

$$-\frac{\partial u}{\partial \nu} \in \partial j_\nu(u) \quad \text{a.e. on } \Gamma_3. \tag{3.4}$$

In (3.1)–(3.4) and below, D is a bounded domain in $\mathbb{R}^d$ ($d = 1, 2, 3$ in applications) with smooth boundary $\partial D = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and outer normal unit $\boldsymbol{\nu}$. We assume that Γ_1, Γ_2, Γ_3 are measurable sets such that $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$ and moreover, $\mathrm{meas}(\Gamma_1) > 0$. In addition, in (3.1)–(3.4) we do not mention the dependence of the different functions on the spatial variable $\boldsymbol{x} \in D \cup \partial D$. The functions f, b and j_ν are given and will be described below. Here we restrict ourselves to mention that f represents the internal energy, b is the prescribed temperature field on Γ_2, $\partial u/\partial\nu$ denotes the normal derivative of u on Γ_3 and ∂j_ν denotes the Clarke subdifferential of the potential function j_ν, assumed to be locally Lipschitz. Note that Problem $\mathcal{H}$ represents an extension of the problem considered in [3], where the boundary condition (3.4) was of the form

$$-\frac{\partial u}{\partial \nu} = q \quad \text{a.e. on } \Gamma_3 \tag{3.5}$$

with a given function $q\colon \Gamma_3 \to \mathbb{R}$. It is obvious to see that condition (3.5) can be recovered by condition (3.4) with an appropriate choice of j_ν.

For the variational analysis of Problem $\mathcal{H}$ we use standard notation for C^1, Lebesque and Sobolev spaces. We use the symbols "$\to$" and "$\rightharpoonup$" to indicate the strong and weak convergence in various spaces, respetively, which will be indicated below. We also use "$\to$" for the convergence in $\mathbb{R}$. Moreover, we shall use the space

$$V = \{v \in H^1(D)\colon\ v = 0 \text{ on } \Gamma_1\},$$

endowed with the inner product of the space $H^1(D)$, denoted by $(\cdot,\cdot)_V$, and the associated norm, denoted by $\|\cdot\|_V$. Since $\mathrm{meas}(\Gamma_1) > 0$, it is well known that $(V, (\cdot,\cdot)_V)$ is a real Hilbert space. In addition, by the Friedrichs-Poincaré inequality and the Sobolev trace theorem we have

$$\|v\|_V \leqslant c_0 \|\nabla v\|_{L^2(D)^d}, \tag{3.6}$$

$$\|v\|_{L^2(\Gamma_3)} \leqslant c_3 \|v\|_V \tag{3.7}$$

for all $v \in V$, respectively. Here and below in this paper, c_0 and c_3 are positive constants which depend on D, Γ_1 and Γ_3. In what follows we denote, by V^* the dual of V, by $\langle\cdot,\cdot\rangle$ the duality paring between V^* and V and by 0_V the zero element of V.

We now list the assumptions on the data of Problem $\mathcal{H}$. First, for the functions f and b we assume that

(3.8) $f \in L^2(D)$,

(3.9) $b \in L^2(\Gamma_2)$ and there exists $u_b \in V$ such that $u_b \geqslant 0$ in D and $u_b = b$ on Γ_2.

Moreover, for the potential function j_ν we assume the following:

$$
(3.10) \qquad \begin{cases} j_\nu\colon\ \Gamma_3 \times \mathbb{R} \to \mathbb{R} \text{ is such that} \\ \text{(a) } j_\nu(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R} \text{ and there} \\ \qquad \text{exists } \bar{e} \in L^2(\Gamma_3) \text{ such that } j_\nu(\cdot, \bar{e}(\cdot)) \in L^1(\Gamma_3). \\ \text{(b) } j_\nu(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) } |\xi| \leqslant \bar{c}_0 + \bar{c}_1 |r| \text{ for a.e. } \boldsymbol{x} \in \Gamma_3, \\ \qquad \text{for all } r \in \mathbb{R},\ \xi \in \partial j_\nu(\boldsymbol{x}, r),\ \text{with } \bar{c}_0, \bar{c}_1 \geqslant 0. \\ \text{(d) } j_\nu^0(\boldsymbol{x}, r_1; r_2 - r_1) + j_\nu^0(\boldsymbol{x}, r_2; r_1 - r_2) \leqslant \alpha_{j_\nu} |r_1 - r_2|^2 \\ \qquad \text{for a.e. } \boldsymbol{x} \in \Gamma_3,\ \text{all } r_1, r_2 \in \mathbb{R},\ \text{with } \alpha_{j_\nu} \geqslant 0. \\ \text{(e) either } j_\nu(\boldsymbol{x}, \cdot) \text{ or } -j_\nu(\boldsymbol{x}, \cdot) \text{ is regular on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases}
$$

Finally, we assume that the following smallness condition holds:

$$
(3.11) \qquad \alpha_{j_\nu} c_0^2 c_3^2 < 1.
$$

Examples of sets D, Γ_1, Γ_2 and functions b for which such assumption is satisfied can be easily constructed. Below, we restrict ourselves to the following ones.

Example 3.1. Let $\alpha > 0$, $\beta > 0$, and let

$$
\begin{aligned}
D &= \{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2\colon\ 0 < x_1 < \alpha,\ 0 < x_2 < \beta\}, \\
\Gamma_1 &= \{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2\colon\ x_1 = 0,\ 0 \leqslant x_2 \leqslant \beta\}, \\
\Gamma_2 &= \{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2\colon\ x_1 = \alpha,\ 0 \leqslant x_2 \leqslant \beta\}.
\end{aligned}
$$

Assume that φ is a positive function with regularity $\varphi \in C^1([0, \beta])$ and consider the functions $b\colon \Gamma_2 \to \mathbb{R}$, $u_b\colon D \to \mathbb{R}$ defined by $b(x_1, x_2) = \varphi(x_2)$, $u_b(x_1, x_2) = x_1 \alpha^{-1} \varphi(x_2)$. Then it is easy to see that condition (3.9) is satisfied.

Example 3.2. Let $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and let

$$
\begin{aligned}
D &= \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3\colon\ 0 < x_1 < \alpha,\ 0 < x_2 < \beta,\ 0 < x_3 < \gamma\}, \\
\Gamma_1 &= \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3\colon\ x_1 = 0,\ 0 \leqslant x_2 \leqslant \beta,\ 0 \leqslant x_3 \leqslant \gamma\}, \\
\Gamma_2 &= \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3\colon\ x_1 = \alpha,\ 0 \leqslant x_2 \leqslant \beta,\ 0 \leqslant x_3 \leqslant \gamma\}.
\end{aligned}
$$

Assume that φ is a positive function with regularity $\varphi \in C^1([0, \beta] \times [0, \gamma])$ and consider the functions $b\colon \Gamma_2 \to \mathbb{R}$, $u_b\colon D \to \mathbb{R}$ defined by $b(x_1, x_2, x_3) = \varphi(x_2, x_3)$, $u(x_1, x_2, x_3) = x_1 \alpha^{-1} \varphi(x_2, x_3)$. Then it is easy to see that condition (3.9) is satisfied.

Note also that various examples of functions j_ν which satisfy condition (3.10) can be found in [12], [27].

Next, we define the set $K \subset V$, the bilinear form $a\colon V \times V \to \mathbb{R}$ and the function $j\colon V \to \mathbb{R}$ by equalities

$$K = \{v \in V\colon\ v \geqslant 0 \text{ in } D,\ v = b \text{ on } \Gamma_2\}, \tag{3.12}$$

$$a(u,v) = \int_D \nabla u \cdot \nabla v \,\mathrm{d}x \quad \forall\, u, v \in V, \tag{3.13}$$

$$j(v) = \int_{\Gamma_3} j_\nu(v) \,\mathrm{d}a \quad \forall\, v \in V. \tag{3.14}$$

It follows from assumption (3.9) that the set K is not empty. Moreover, Lemma 8 in [37] provides the following result.

Proposition 3.3. *Under assumption* (3.10) *the function* $j\colon V \to \mathbb{R}$ *defined by* (3.14) *satisfies conditions* (2.4) *on the space* $X = V$ *with* $\tilde{c}_0 = \sqrt{2\,\mathrm{meas}(\Gamma_3)}\bar{c}_0 c_3$, $\tilde{c}_1 = \sqrt{2}\bar{c}_1 c_3^2$ *and* $\alpha_j = \alpha_{j_\nu} c_3^2$. *In addition,*

$$j^0(u;v) = \int_{\Gamma_3} j_\nu^0(u,v) \,\mathrm{d}a \quad \forall\, u, v \in V. \tag{3.15}$$

Next, we turn to the variational formulation of Problem $\mathcal{H}$. To this end we assume that u is a regular function which satisfies (3.1)–(3.4) and we consider an arbitrary element $v \in K$. Then, using an integration by parts if follows that

$$\int_D \nabla u \cdot \nabla(v-u) \,\mathrm{d}x + \int_D \Delta u \cdot (v-u) \,\mathrm{d}x = \int_{\partial D} \frac{\partial u}{\partial \nu}(v-u) \,\mathrm{d}a.$$

Therefore, by (3.1)–(3.3) we find that

$$\int_D \nabla u \cdot \nabla(v-u) \,\mathrm{d}x \geqslant \int_D f(v-u) \,\mathrm{d}x + \int_{\Gamma_3} \frac{\partial u}{\partial \nu}(v-u) \,\mathrm{d}a \tag{3.16}$$

and using the boundary condition (3.4) combined with definition of the Clarke subdifferential, we obtain that

$$\int_{\Gamma_3} \frac{\partial u}{\partial \nu}(v-u) \,\mathrm{d}a \geqslant -\int_{\Gamma_3} j_\nu^0(u; v-u) \,\mathrm{d}a. \tag{3.17}$$

We now combine inequalities (3.16) and (3.17), then use notation (3.13) and equality (3.15) to see that

$$a(u, v-u) + j^0(u; v-u) \geqslant (f, v-u)_{L^2(D)}.$$

Finally, using this inequality and the regularity $u \in K$, guaranteed by (3.1)–(3.3), we deduce the following variational formulation of Problem $\mathcal{H}$.

Problem $\mathcal{P}$. *Find $u \in V$ such that*

$$u \in K, \quad a(u, v-u) + j^0(u; v-u) \geqslant (f, v-u)_{L^2(\Omega)} \quad \forall\, v \in K. \tag{3.18}$$

We refer to inequality (3.18) as a hemivariational inequality with unilateral constraint. A function $u \in V$ which satisfies (3.18) is called a *weak solution* to the heat transfer problem $\mathcal{H}$.

We now introduce the set of admissible pairs for inequality Problem $\mathcal{P}$ defined by

$$\mathcal{V}_{ad} = \{(u, f) \in K \times L^2(D) \text{ such that (3.18) holds}\}. \tag{3.19}$$

We also assume that $\mathcal{L}\colon\ V \times L^2(D) \to \mathbb{R}$ is a given cost functional which satisfies the following conditions.

$$\begin{cases} \text{For all sequences } \{u_n\} \subset V \text{ and } \{f_n\} \subset L^2(D) \text{ such that} \\ u_n \to u \text{ in } V,\ f_n \rightharpoonup f \text{in } L^2(D), \text{ we have } \liminf\limits_{n\to\infty} \mathcal{L}(u_n, f_n) \geqslant \mathcal{L}(u, f). \end{cases} \tag{3.20}$$

$$\begin{cases} \text{There exists } h\colon\ L^2(D) \to \mathbb{R} \text{ such that} \\ \text{(a) } \mathcal{L}(u, f) \geqslant h(f) \quad \forall\, u \in V,\ f \in L^2(D). \\ \text{(b) } \|f_n\|_{L^2(D)} \to \infty \Rightarrow h(f_n) \to \infty. \end{cases} \tag{3.21}$$

Example 3.4. A typical example of function $\mathcal{L}$ which satisfies conditions (3.20) and (3.21) is obtained by taking

$$\mathcal{L}(u, f) = g(u) + h(f) \quad \forall\, u \in V,\ f \in L^2(D),$$

where $g\colon\ V \to \mathbb{R}$ is a continuous positive function and $h\colon\ L^2(D) \to \mathbb{R}$ is a weakly lower semicontinuous coercive function, i.e. it satisfies condition (3.21)(b).

We also assume that $\mathcal{L}\colon\ V \times L^2(D) \to \mathbb{R}$ is a given cost function and we associate to Problem $\mathcal{P}$ the following optimal control problem.

Problem $\mathcal{Q}$. *Find $(u^*, f^*) \in \mathcal{V}_{ad}$ such that*

$$\mathcal{L}(u^*, f^*) = \min_{(u,f)\in\mathcal{V}_{ad}} \mathcal{L}(u, f). \tag{3.22}$$

Under the previous assumptions, the unique solvability of Problem $\mathcal{P}$ as well as the solvability of Problem $\mathcal{Q}$ can be obtained by using standard arguments, already used in [37] and [23], [24], respectively. Our aim in what follows is to study the well-posedness of these problems and to derive some consequences. To this end, as already mentionned in Section 2, for each problem we need to prescribe a Tykhonov triple, based on three ingredients: a set of indices, a familly of approximating sets and a convergence criterion for the sequences of indices.

4. A well-posedness result

This section is devoted to the well-posedness of the Problem $\mathcal{P}$, in the sense precised in Section 2, with $X = V$. In order to construct an appropriate Tykhonov triple for this problem, we consider a set $\widetilde{K}$, a penalty operator $G\colon V \to V^*$ and a penalty parameter λ such that the following conditions hold.

$$(4.1)\quad \begin{cases} \text{(a) } \widetilde{K} \text{ is a closed convex subset of } V. \\ \text{(b) } K \subset \widetilde{K}. \end{cases}$$

$$(4.2)\quad \begin{cases} \text{(a) } G\colon V \to V^* \text{ is a bounded, demicontinuous and monotone operator.} \\ \text{(b) } \langle Gu, v - u \rangle \leqslant 0 \quad \forall\, u \in \widetilde{K},\ v \in K. \\ \text{(c) } u \in \widetilde{K}, \quad \langle Gu, v - u \rangle = 0 \quad \forall\, v \in K \Rightarrow u \in K. \end{cases}$$

We start with the following result.

Proposition 4.1. *Assume that* (3.9)–(3.11), (4.1) *and* (4.2)(a) *hold. Then for each* $\lambda > 0$ *and* $f \in L^2(D)$ *there exists a unique element* $u = u(\lambda, f)$ *such that*

$$(4.3)\quad u \in \widetilde{K}, \quad a(u, v-u) + \frac{1}{\lambda}\langle Gu, v-u \rangle + j^0(u; v-u) \geqslant (f, v-u)_{L^2(D)} \quad \forall\, v \in \widetilde{K}.$$

Proof. Define the operator $A\colon V \to V^*$ by equality

$$\langle Au, v \rangle = a(u, v) + \frac{1}{\lambda}\langle Gu, v \rangle \quad \forall\, u, v \in V.$$

Then, using definition (3.13) of the form a, inequality (3.6) and properties (4.2)(a) of the operator G, it is easy to see that the operator A is bounded, demicontinuous and moreover,

$$(4.4)\qquad \langle Au - Av, u - v \rangle \geqslant \frac{1}{c_0^2}\|u - v\|_V^2 \quad \forall\, u, v \in V.$$

Therefore, it follows from Proposition 2.4 that A is pseudomonotone. In addition, inequality (4.4) shows that A is strongly monotone with constant $m_A = 1/c_0^2$. On the other hand, the functional $v \mapsto (f, v)_{L^2(D)}$ is linear and continuous on V and therefore, it defines an element in V^*. We note that assumption (3.9) implies that $K \neq \emptyset$, hence inclusion (4.1)(b) guarantees that $\widetilde{K} \neq \emptyset$. Moreover, recall assumption (4.1)(a), Proposition 3.3 and the smallness assumption (3.11). All these ingredients allow us to apply Theorem 2.5 in order to deduce the unique solvabilty of inequality (4.3), which concludes the proof. $\square$

We now take $\mathcal{T} = (I, \Omega, \mathcal{C})$, where

(4.5) $I = \{\theta = (\lambda, f)\colon \ \lambda > 0,\ f \in L^2(\Omega)\}$,

(4.6) $\Omega(\theta) = \{u \in \widetilde{K}$ such that (4.3) holds$\}$ $\quad \forall\, \theta = (\lambda, f) \in I$,

(4.7) $\mathcal{C} = \{\{\theta_n\}\colon \ \theta_n = (\lambda_n, f_n) \in I\ \forall\, n \in \mathbb{N},\ \lambda_n \to 0,\ f_n \rightharpoonup f$ in $L^2(D)$ as $n \to \infty\}$.

Then, using Proposition 4.1 we see that for each $\theta = (\lambda, f)$ the set $\Omega(\theta)$ defined by (4.6) is not empty and therefore, the triple (4.5)–(4.7) is a Tykhonov triple in the sense of Definition 2.6.

Our main result in this section is the following.

Theorem 4.2. *Assume that* (3.8)–(3.11), (4.1) *and* (4.2) *hold. Then Problem* $\mathcal{P}$ *is well-posed with respect to the Tykhonov triple* (4.5)–(4.7).

Proof. Following Definition 2.8 a), the proof is carried out in two main steps.

i) *Unique solvability of Problem* $\mathcal{P}$. Note that assumptions (4.1) and (4.2)(a) are satisfied if $\widetilde{K} = K$ and $Gv = 0_V$ for all $v \in V$, respectively. Moreover, with this particular choice, inequality (4.3) reduces to inequality (3.18) for any $\lambda > 0$. Therefore, the existence of a unique solution $u \in K$ to Problem $\mathcal{P}$ is a direct consequence of Proposition 4.1.

ii) *Convergence of approximating sequences.* To proceed, we consider an approximating sequence for Problem $\mathcal{P}$, denoted by $\{u_n\}$. Then, according to Definition 2.7 it follows that there exists a sequence $\{\theta_n\}$ of elements of I with $\theta_n = (\lambda_n, f_n)$ such that $u_n \in \Omega(\theta_n)$ for each $n \in \mathbb{N}$ and, moreover,

$$\lambda_n \to 0, \tag{4.8}$$

$$f_n \rightharpoonup f \quad \text{in } L^2(D). \tag{4.9}$$

Note that the inclusion $u_n \in \Omega(\theta_n)$ combined with definition (4.6) implies that for each $n \in \mathbb{N}$ the following inequality holds:

$$\begin{aligned} u_n \in \widetilde{K}, \quad a(u_n, v - u_n) + \frac{1}{\lambda_n}\langle Gu_n, v - u_n\rangle + j^0(u_n; v - u_n) \\ \geqslant (f_n, v - u_n)_{L^2(D)} \quad \forall\, v \in \widetilde{K}. \end{aligned} \tag{4.10}$$

Our aim in what follows is to prove the convergence

$$u_n \to u \quad \text{in } V \text{ as } n \to \infty. \tag{4.11}$$

To this end we proceed in three intermediate steps that we present below.

ii–a) *A first weak convergence result.* We claim that there is an element $\tilde{u} \in \widetilde{K}$ and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $u_n \rightharpoonup \tilde{u}$ in X, as $n \to \infty$.

To prove the claim, we establish the boundedness of the sequence $\{u_n\}$ in V. Let $n \in \mathbb{N}$ and let u be the solution of Problem $\mathcal{P}$. We use assumption (4.1)(b) and take $v = u$ in (4.10) to see that

$$a(u_n, u_n - u) \leqslant \frac{1}{\lambda_n}\langle G u_n, u - u_n\rangle + j^0(u_n, u - u_n) + (f_n, u_n - u)_{L^2(D)}.$$

Then, using inequalities (3.6) and (4.2)(b) we obtain that

$$\frac{1}{c_0^2}\|u_n - u\|_V^2 \leqslant a(u, u - u_n) + j^0(u_n; u - u_n) + (f_n, u_n - u)_{L^2(D)}. \tag{4.12}$$

On the other hand, by Proposition 2.2(b) we have

$$\begin{aligned} j^0(u_n; u - u_n) &= j^0(u_n; u - u_n) + j^0(u; u_n - u) - j^0(u; u_n - u) \\ &\leqslant j^0(u_n; u - u_n) + j^0(u; u_n - u) + |j^0(u; u_n - u)| \\ &= j^0(u_n; u - u_n) + j^0(u; u_n - u) + |\max\{\langle \xi, u_n - u\rangle :\ \xi \in \partial j(u)\}| \end{aligned}$$

and using Proposition 3.3, we deduce that

$$j^0(u_n; u - u_n) \leqslant \alpha_{j_\nu} c_3^2 \|u_n - u\|_V^2 + (\tilde{c}_0 + \tilde{c}_1\|u\|_V)\|u_n - u\|_V. \tag{4.13}$$

Finally, note that

$$a(u, u - u_n) + (f_n, u_n - u)_{L^2(D)} \leqslant (\|u\|_V + \|f_n\|_{L^2(D)})\|u_n - u\|_V. \tag{4.14}$$

We now combine inequalities (4.12)–(4.14) to see that

$$\begin{aligned} \frac{1}{c_0^2}\|u_n - u\|_V^2 \leqslant{}& (\|u\|_V + \|f_n\|_{L^2(D)}))\|u_n - u\|_V \\ &+ \alpha_{j_\nu} c_3^2\|u_n - u\|_V^2 + (\tilde{c}_0 + \tilde{c}_1\|u\|_V)\|u_n - u\|_V. \end{aligned} \tag{4.15}$$

Note that by (4.9) we know that the sequence $\{f_n\}$ is bounded in $L^2(D)$. Therefore, using inequality (4.15) and the smallness assumption (3.11), we deduce that there exists a constant $C > 0$ independent of n such that $\|u_n - u\|_X \leqslant C$. This implies that the sequence $\{u_n\}$ is bounded in V. Thus, from the reflexivity of V, by passing to a subsequence, if necessary, we deduce that

$$u_n \rightharpoonup \tilde{u} \quad \text{in } V \text{ as } n \to \infty \tag{4.16}$$

with some $\tilde{u} \in V$. Moreover, assumption (4.1)(a) and the convergence (4.16) imply that $\tilde{u} \in \widetilde{K}$, which completes the proof of the claim.

ii–b) *A property of the weak limit.* Next, we show that $\tilde{u}$ is a solution to Problem $\mathcal{P}$. Let v be a given element in $\widetilde{K}$ and let $n \in \mathbb{N}$. We use (4.10) to obtain that

$$\frac{1}{\lambda_n}\langle Gu_n, u_n - v\rangle \leqslant a(u_n, v - u_n) + j^0(u_n, v - u_n) + (f_n, u_n - v)_{L^2(D)}. \tag{4.17}$$

Then, using arguments similar to those used in the proof of (4.13), (4.14) and the boundedness of the sequence $\{u_n\}$, we deduce that each term on the right-hand side of inequality (4.17) is bounded. This implies that there exists a constant $M_0 > 0$, which does not depend on n, such that

$$\langle Gu_n, u_n - v\rangle \leqslant \lambda_n M_0.$$

We now pass to the upper limit in this inequality and use the convergence (4.8) to deduce that

$$\limsup \langle Gu_n, u_n - v\rangle \leqslant 0. \tag{4.18}$$

We now take $v = \tilde{u}$ in (4.18) and find that

$$\limsup \langle Gu_n, u_n - \tilde{u}\rangle \leqslant 0. \tag{4.19}$$

Therefore, using the pseudomonotonicity of the operator G, guaranted by assumption (4.2)(a) and Proposition 2.4, we obtain that

$$\liminf \langle Gu_n, u_n - v\rangle \geqslant \langle G\tilde{u}, \tilde{u} - v\rangle. \tag{4.20}$$

We now combine inequalities (4.20) and (4.18) to find that $\langle G\tilde{u}, \tilde{u} - v\rangle \leqslant 0$. On the other hand, by (4.2)(b) we deduce that $\langle G\tilde{u}, \tilde{u} - v\rangle \geqslant 0$. We conclude from above that

$$\langle G\tilde{u}, \tilde{u} - v\rangle = 0$$

and we recall that this equality holds for all $v \in \widetilde{K}$. Then, using assumptions (4.1)(b), (4.2)(c), we find that $\tilde{u} \in K$.

Next, we use (4.10), again to obtain that

$$a(u_n, u_n - v) \leqslant \frac{1}{\lambda_n}\langle Gu_n, v - u_n\rangle + j^0(u_n, v - u_n) + (f_n, u_n - v)_{L^2(D)}.$$

Using assumption (4.2)(b) we find that

$$a(u_n, u_n - v) \leqslant j^0(u_n, v - u_n) + (f_n, u_n - v)_{L^2(D)}. \tag{4.21}$$

Moreover, from (4.16), (3.15), the compactness of the trace and the properties of the integral it follows that

$$\limsup j^0(u_n; v - u_n) \leqslant j^0(\tilde{u}, v - \tilde{u}) \tag{4.22}$$

and in addition,

$$(f_n, u_n - v)_{L^2(D)} \to (f, \tilde{u} - v)_{L^2(D)}. \tag{4.23}$$

We now gather relations (4.21)–(4.23) to see that

$$\limsup a(u_n, u_n - v) \leqslant j^0(\tilde{u}, v - \tilde{u}) + (f, \tilde{u} - v)_{L^2(D)}. \tag{4.24}$$

Now, taking $v = \tilde{u} \in K$ in (4.24) and using Proposition 2.2(a), we obtain that

$$\limsup a(u_n, u_n - \tilde{u}) \leqslant 0. \tag{4.25}$$

On the other hand, using the properties of the form a, we have that

$$a(u_n, v) \to a(\tilde{u}, v) \quad \text{as } n \to \infty \tag{4.26}$$

and since $a(u_n - \tilde{u}, u_n - \tilde{u}) \geqslant 0$, we deduce that

$$a(u_n, u_n) \geqslant a(\tilde{u}, u_n) + a(u_n, \tilde{u}) - a(\tilde{u}, \tilde{u})$$

for each $n \in \mathbb{N}$. Using now (4.26) and convergence (4.16) we find that

$$\liminf_{n\to\infty} a(u_n, u_n) \geqslant a(\tilde{u}, \tilde{u}). \tag{4.27}$$

Therefore, combining (4.26), (4.27), we obtain that

$$a(\tilde{u}, \tilde{u} - v) \leqslant \liminf a(u_n, u_n - v) \tag{4.28}$$

and using (4.28) and (4.24), we have

$$a(\tilde{u}, \tilde{u} - v) \leqslant j^0(\tilde{u}, v - \tilde{u}) + (f, \tilde{u} - v)_{L^2(D)}.$$

It follows from here that $\tilde{u} \in K$ is a solution to Problem $\mathcal{P}$, as claimed.

ii–c) *A second weak convergence result.* We now prove the weak convergence of the whole sequence $\{u_n\}$.

Since Problem $\mathcal{P}$ has a unique solution $u \in K$, we deduce from the previous step that $\tilde{u} = u$. Moreover, a careful analysis of the proof in step ii–b) reveals that every subsequence of $\{u_n\}$ which converges weakly in V has the weak limit u. In addition, we recall that the sequence $\{u_n\}$ is bounded in V. Therefore, using a standard argument we deduce that the whole sequence $\{u_n\}$ converges weakly in V to u as $n \to \infty$, i.e.

$$u_n \rightharpoonup u \quad \text{in } V \text{ as } n \to \infty. \tag{4.29}$$

ii–d) *Strong convergence.* In the final step of the proof, we prove that $u_n \to u$ in V as $n \to \infty$.

We take $v = \tilde{u} \in K$ in (4.28) and use (4.25) to obtain

$$0 \leqslant \liminf a(u_n, u_n - \tilde{u}) \leqslant \limsup a(u_n, u_n - \tilde{u}) \leqslant 0,$$

which shows that $a(u_n, u_n - \tilde{u}) \to 0$ as $n \to \infty$. Therefore, using equality $\tilde{u} = u$, the coercivity of a and the convergence $u_n \rightharpoonup u$ in V, we have

$$\frac{1}{c_0^2}\|u_n - u\|_V^2 \leqslant a(u_n - u, u_n - u) = a(u_n, u_n - u) - a(u, u_n - u) \to 0$$

as $n \to \infty$. Hence, it follows that $u_n \to u$ in V, which completes the proof. □

We now reformulate the convergence result obtained above. To this end, we denote by $u(\lambda_n, f_n)$ and $u(f)$ the solution of the hemivariational inequality (4.10) and (3.18), respectively. Then, if (4.8) and (4.9) hold, step ii) of the proof of Theorem 5.2 implies that

$$u(\lambda_n, f_n) \to u(f) \quad \text{in } V. \tag{4.30}$$

We shall use this convergence result in various places below in this paper.

We end this section with the following consequences of Theorem 4.2.

Corollary 4.3. *Assume that* (3.9)–(3.11), (4.1) *and* (4.2) *hold. Then the operator $f \mapsto u$ which associates to every $f \in L^2(D)$ the solution $u = u(f) \in V$ of inequality* (3.18) *is weakly-strongly continuous, i.e.*

$$f_n \rightharpoonup f \text{ in } L^2(D) \Rightarrow u(f_n) \to u(f) \text{ in } V. \tag{4.31}$$

P r o o f. Let $\{f_n\} \subset L^2(D)$ be a sequence such that $f_n \rightharpoonup f$ in $L^2(D)$. We take $\widetilde{K} = K$, $Gv = 0_V$ for all $v \in V$ and let $\{\lambda_n\} \subset \mathbb{R}$ be a sequence such that $\lambda_n > 0$ for each $n \in \mathbb{N}$ and $\lambda_n \to 0$. Then it follows that $u(\lambda_n, f_n) = u(f_n)$ for each $n \in \mathbb{N}$ and therefore convergence (4.31) is a direct consequence of convergence (4.30). □

Corollary 4.4. *Assume that* (3.9)–(3.11), (4.1) *and* (4.2) *hold. Then for any* $f \in L^2(D)$, *the solution* $u(\lambda, f) \in V$ *of inequality* (4.3) *converges to the solution* $u(f)$ *of inequality* (3.18) *as* $\lambda \to 0$, *that is,*

$$\lambda_n > 0,\ \lambda_n \to 0 \Rightarrow u(\lambda_n, f) \to u(f) \quad \text{in } V. \tag{4.32}$$

Proof. Convergence (4.32) is a direct consequence of convergence (4.30). □

Note that in the case when $\widetilde{K} = V$, inequality (4.3) represents a hemivariational inequality without constraints. In this particular case, Corollary 4.4 can be recovered by using convergence results for penalty variational-hemivariational inequalities obtained in [37], [44]. Nevertheless, we stress that the functional framework used in the above-mentioned papers does not allow the use of these results in the case when $\widetilde{K} \neq K$.

5. Weakly generalized well-posedness results

This section is devoted to the weakly generalized well-posedness of Problem $\mathcal{Q}$ in the sense precised in Section 2, with $X = V \times L^2(D)$. We shall construct a relevant example of Tykhonov triple, which will play a crucial role in the study of Problem $\mathcal{Q}$. To this end, everywhere in this section we assume that (3.9)–(3.11), (3.20), (3.21), (4.1) and (4.2) hold. Then for any $\lambda > 0$ and $f \in L^2(\Omega)$ we consider the hemivariatinal inequality (4.3), which represents a perturbation of the hemivariational inequality (3.18). Therefore, by analogy with (3.19), we introduce the set of perturbed admissible pairs for inequality Problem $\mathcal{P}$ defined by

$$\mathcal{V}_{ad}^{\lambda} = \{(u, f) \in \widetilde{K} \times L^2(D) \text{ such that (4.3) holds}\}. \tag{5.1}$$

Moreover, for each $\mu \geqslant 0$ let $\mathcal{L}_\mu\colon V \times L^2(D) \to \mathbb{R}$ be a perturbation of the cost functional, assumed to satisfy conditions (3.20) and (3.21), i.e.:

$$\begin{cases} \text{For all sequences } \{u_n\} \subset V \text{ and } \{f_n\} \subset L^2(D) \text{ such that} \\ u_n \to u \text{ in } V,\ f_n \rightharpoonup f \text{ in } L^2(D), \text{ we have } \liminf\limits_{n\to\infty} \mathcal{L}_\mu(u_n, f_n) \geqslant \mathcal{L}_\mu(u, f). \end{cases} \tag{5.2}$$

$$\begin{cases} \text{(a) } \mathcal{L}_\mu(u, f) \geqslant h(f) \quad \forall\, u \in V,\ f \in L^2(D). \\ \text{(b) } \|f_n\|_{L^2(D)} \to \infty \Rightarrow h(f_n) \to \infty. \end{cases} \tag{5.3}$$

In addition, we assume that the following properties hold:

$$(5.4)\qquad \begin{cases} \text{For all sequences } \{u_n\} \subset V, \ \ \{f_n\} \subset L^2(D) \text{ and } \{\mu_n\} \subset \mathbb{R}_+ \text{ such that} \\ u_n \to u \text{ in } V,\ f_n \rightharpoonup f \text{ in } L^2(D),\ \mu_n \to 0 \text{ and we have} \\ \liminf\limits_{n\to\infty} \mathcal{L}_{\mu_n}(u_n, f_n) \geqslant \mathcal{L}(u, f). \end{cases}$$

$$(5.5)\qquad \begin{cases} \text{For all sequences } \{u_n\} \subset V \text{ and } \{\mu_n\} \subset \mathbb{R}_+ \text{ such that} \\ u_n \to u \text{ in } V,\ \mu_n \to 0 \ \text{ and for any } f \in L^2(D) \text{ we have} \\ \lim\limits_{n\to\infty} \mathcal{L}_{\mu_n}(u_n, f) = \mathcal{L}(u, f). \end{cases}$$

An example of functionals $\mathcal{L}_\mu$ and $\mathcal{L}$ which satisfy conditions (5.4) and (5.5) will be provided in the next section. Here we restrict ourselves to remark that condition (5.5) implies that for $\mu = 0$ the functionals $\mathcal{L}_\mu$ and $\mathcal{L}$ are the same, i.e.

$$(5.6)\qquad \mathcal{L}_0(u, f) = \mathcal{L}(u, f) \quad \forall\, u \in V,\ f \in L^2(D).$$

It follows from here that assuming conditions (5.5), (5.2), and (5.3) for each $\mu \geqslant 0$ implies that conditions (3.20) and (3.21) hold, too.

We start with the following result.

Proposition 5.1. *Assume* (3.9)–(3.11), (4.1) *and* (4.2). *Then for each* $\lambda > 0$ *and each* $\mu \geqslant 0$ *such that* (5.2) *and* (5.3) *hold, there exists a pair* (u^*, f^*) *such that*

$$(5.7)\qquad (u^*, f^*) \in \mathcal{V}_{ad}^\lambda, \quad \mathcal{L}_\mu(u^*, f^*) = \min_{(u,f)\in\mathcal{V}_{ad}^\lambda} \mathcal{L}_\mu(u, f).$$

P r o o f. We use standard arguments that we resume in the following. Let $\lambda > 0$, $\mu \geqslant 0$, let

$$(5.8)\qquad \omega = \inf_{(u,f)\in\mathcal{V}_{ad}^\lambda} \mathcal{L}_\mu(u, f) \in \mathbb{R}$$

and let $\{(u_n, f_n)\} \subset \mathcal{V}_{ad}^\lambda$ be a minimizing sequence for the functional $\mathcal{L}_\mu$, i.e.

$$(5.9)\qquad \lim_{n\to\infty} \mathcal{L}_\mu(u_n, f_n) = \omega.$$

We claim that the sequence $\{f_n\}$ is bounded in $L^2(D)$. Arguing by contradiction, assume that $\{f_n\}$ is not bounded in $L^2(D)$. Then passing to a subsequence still denoted by $\{f_n\}$, we have

$$(5.10)\qquad \|f_n\|_{L^2(D)} \to \infty \quad \text{as } n \to \infty$$

and using (5.3) it turns that

$$(5.11)\qquad \lim_{n\to\infty} \mathcal{L}_\mu(u_n, f_n) = \infty.$$

Equalities (5.9) and (5.11) imply that $\omega = \infty$, which is in contradiction with (5.8).

We conclude from above that the sequence $\{f_n\}$ is bounded in $L^2(D)$, as clamed. Therefore, we deduce that there exists $f^* \in L^2(D)$ such that passing to a subsequence still denoted by $\{f_n\}$, we have

$$f_n \rightharpoonup f^* \quad \text{in } L^2(D) \text{ as } n \to \infty. \tag{5.12}$$

Let u^* be the solution of the hemivariational inequality (4.3) for $f = f^*$. Then, by definition (5.1) of the set $\mathcal{V}_{ad}^{\lambda}$ we have

$$(u^*, f^*) \in \mathcal{V}_{ad}^{\lambda}. \tag{5.13}$$

Moreover, using convergence (5.12) and Corollary 4.3 it follows that

$$u_n \to u^* \quad \text{in } V \text{ as } n \to \infty. \tag{5.14}$$

We now use convergences (5.12), (5.14) and assumption (5.2) to deduce that

$$\liminf_{n\to\infty} \mathcal{L}_\mu(u_n, f_n) \geqslant \mathcal{L}_\mu(u^*, f^*). \tag{5.15}$$

It follows from (5.9) and (5.15) that

$$\omega \geqslant \mathcal{L}_\mu(u^*, f^*). \tag{5.16}$$

In addition, (5.13) and (5.8) yield

$$\omega \leqslant \mathcal{L}_\mu(u^*, f^*). \tag{5.17}$$

We combine inequalities (5.16), (5.17), then we use (5.13) to see that (5.7) holds, which concludes the proof. $\square$

We now take $\mathcal{T} = (I, \Omega, \mathcal{C})$, where

$$I = \{\theta = (\lambda, \mu) \colon \ \lambda > 0, \ \mu \geqslant 0\}, \tag{5.18}$$

$$\Omega(\theta) = \{(u^*, f^*) \in V \times L^2(D) \text{ such that (3.22) holds}\} \quad \forall\, \theta = (\lambda, \mu) \in I, \tag{5.19}$$

$$\mathcal{C} = \{\{\theta_n\} \colon \ \theta_n = (\lambda_n, \mu_n) \in I \ \forall\, n \in \mathbb{N}, \ \lambda_n \to 0, \ \mu_n \to 0 \text{ as } n \to \infty\}. \tag{5.20}$$

Then, using Proposition 5.1 we see that for each $\theta = (\lambda, \mu)$ the set $\Omega(\theta)$ defined by (3.22) is not empty and therefore, the triple (5.18)–(5.20) is a Tykhonov triple in the sense of Definition 2.6.

Our main result in this section is the following.

Theorem 5.2. *Assume* (3.9)–(3.11), (4.1), (4.2), (5.4), (5.5) *and for all* $\mu \geqslant 0$, *assume that* (5.2), (5.3) *hold. Then Problem* $\mathcal{Q}$ *is weakly generalized well-posed with respect to the Tykhonov triple* (5.18)–(5.20).

P r o o f. Following Definition 2.8, the proof is carried out in two main steps.

i) *Solvability of Problem $\mathcal{Q}$.* Recall that for $\widetilde{K} = K$ and $Gv = 0_V$, for all $v \in V$ inequality (4.3) reduces to inequality (3.18). Moreover, (5.6) shows that for $\mu = 0$ we have $\mathcal{L}_\mu = \mathcal{L}$. Therefore the existence of at least one solution to Problem $\mathcal{Q}$ is a direct consequence of Proposition 5.1.

ii) *Convergence of approximating sequences.* To proceed, we consider an approximating sequence for Problem $\mathcal{Q}$, denoted by $\{(u_n^*, f_n^*)\}$. Then, according to Definition 2.7 it follows that there exists a sequence $\{\theta_n\}$ of elements of I with $\theta_n = (\lambda_n, \mu_n)$, such that $(u_n, f_n) \in \Omega(\theta_n)$ for each $n \in \mathbb{N}$, (4.8) holds and moreover,

$$\mu_n \to 0. \tag{5.21}$$

In order to simplify the notation, for each $n \in \mathbb{N}$ we write $\mathcal{V}_{ad}^n$ and $\mathcal{L}_n$ instead of $\mathcal{V}_{ad}^{\lambda_n}$ and $\mathcal{L}_{\mu_n}$, respectively. Then, exploiting definition (5.19) we deduce that $(u_n^*, f_n^*) \in \mathcal{V}_{ad}^n$ and

$$\mathcal{L}_n(u_n^*, f_n^*) \leqslant \mathcal{L}_n(u_n, f_n) \tag{5.22}$$

for each couple of functions $(u_n, f_n) \in \mathcal{V}_{ad}^n$, i.e. for each couple of functions $(u_n, f_n) \in V \times L^2(D)$ which satisfies inequality (4.3) in which f is replaced by f_n, for each $n \in \mathbb{N}$.

We shall prove that there exists a subsequence of the sequence $\{(u_n^*, f_n^*)\}$, again denoted by $\{(u_n^*, f_n^*)\}$, and an element $(u^*, f^*) \in V \times L^2(D)$ such that

$$f_n^* \rightharpoonup f^* \quad \text{in } L^2(D) \text{ as } n \to \infty, \tag{5.23}$$

$$u_n^* \to u^* \quad \text{in } V \text{ as } n \to \infty, \tag{5.24}$$

$$(u^*, f^*) \quad \text{is a solution of Problem } \mathcal{Q}. \tag{5.25}$$

To this end we proceed in four intermediate steps that we present below.

ii–a) *A boundedness result.* We claim that the sequence $\{f_n^*\}$ is bounded in $L^2(D)$. Arguing by contradiction, assume that $\{f_n^*\}$ is not bounded in $L^2(D)$. Then, passing to a subsequence still denoted by $\{f_n^*\}$, we have

$$\|f_n^*\|_{L^2(D)} \to \infty \quad \text{as } n \to \infty. \tag{5.26}$$

We use assumption $(3.21)_\mu$ of the cost function to deduce that

$$\lim_{n\to\infty} \mathcal{L}_n(u_n^*, f_n^*) = \infty. \tag{5.27}$$

Next, let $f \in L^2(D)$ be given and let $\tilde{u}_n$ be the solution of the hemivariational inequality

$$\begin{aligned} \tilde{u}_n \in \widetilde{K}\colon\ a(\tilde{u}_n, v - \tilde{u}_n) + \frac{1}{\lambda_n}\langle G\tilde{u}_n, v - \tilde{u}_n\rangle + j^0(\tilde{u}_n; v - \tilde{u}_n) \\ \geqslant (f, v - \tilde{u}_n)_V \quad \forall\, v \in \widetilde{K}, \end{aligned} \tag{5.28}$$

i.e. $\tilde{u}_n = u(\lambda_n, f)$. Note that Proposition 4.1 guarantees that this solution exists and is unique for each $n \in \mathbb{N}$. Moreover, using (4.8) and (4.30) it follows that

$$\tilde{u}_n \to u \quad \text{in } V \text{ as } n \to \infty, \tag{5.29}$$

where $u = u(f)$ and by (5.21) and assumption (5.5), we deduce that

$$\mathcal{L}_n(\tilde{u}_n, f) \to \mathcal{L}(u, f) \quad \text{as } n \to \infty. \tag{5.30}$$

On the other hand, (5.28) implies that the pair $(\tilde{u}_n, f)$ satisfies inequality (4.3), i.e. $(\tilde{u}_n, f) \in \mathcal{V}_{ad}^n$. Therefore, (5.22) implies that

$$\mathcal{L}_n(u_n^*, f_n^*) \leqslant \mathcal{L}_n(\tilde{u}_n, f) \quad \forall\, n \in \mathbb{N}. \tag{5.31}$$

We now pass to the limit in (5.31) as $n \to \infty$ and use (5.27) and (5.30) to obtain a contradiction, which proves the claim.

ii–b) *Two convergence results.* We conclude from step ii-a) that the sequence $\{f_n^*\}$ is bounded in $L^2(D)$ and therefore we can find a subsequence, again denoted by $\{f_n^*\}$, and an element $f^* \in L^2(D)$ such that (5.23) holds. Denote by u^* the solution of inequality (3.18) for $f = f^*$ and note that definition (3.19) implies that

$$(u^*, f^*) \in \mathcal{V}_{ad}. \tag{5.32}$$

Moreover, since $u_n^* = u(\lambda_n, f_n)$, $u^* = u(f)$, assumption (4.8) and convergences (5.23), (4.31) imply that (5.24) holds, too.

ii–c) *Optimality of the limit.* We now prove that (u^*, f^*) is a solution to the optimal control problem $\mathcal{Q}$. To this end we use the convergences (5.23), (5.24), (5.21) and assumption (5.4) to see that

$$\mathcal{L}(u^*, f^*) \leqslant \liminf_{n\to\infty} \mathcal{L}_n(u_n^*, f_n^*). \tag{5.33}$$

Next, we fix a solution (u_0^*, f_0^*) of Problem $\mathcal{Q}$ and in addition, for each $n \in \mathbb{N}$ we denote by u_n^0 the unique element of $\widetilde{K}$ which satisfies inequality (4.3) with $\lambda = \lambda_n$ and $f = f_0^*$, i.e. $u_n^* = u(\lambda_n, f_0^*)$. Therefore, $(u_n^0, f_0^*) \in \mathcal{V}_{ad}^n$ and using the optimality of the pair (u_n^*, f_n^*), (5.22), we find that

$$\mathcal{L}_n(u_n^*, f_n^*) \leqslant \mathcal{L}_n(u_n^0, f_0^*) \quad \forall\, n \in \mathbb{N}.$$

We pass to the upper limit in this inequality to see that

$$\limsup_{n\to\infty} \mathcal{L}_n(u_n^*, f_n^*) \leqslant \limsup_{n\to\infty} \mathcal{L}_n(u_n^0, f_0^*). \tag{5.34}$$

Next, since $\lambda_n \to 0$ and $u_0^* = u(f_0^*)$, it follows from (4.30) that

$$u_n^0 \to u_0^* \quad \text{in } V \text{ as } n \to \infty.$$

Using now this convergence and assumption (5.5) yields

$$\lim_{n\to\infty} \mathcal{L}_n(u_n^0, f_0^*) = \mathcal{L}(u_0^*, f_0^*). \tag{5.35}$$

We now use (5.33)–(5.35) to see that

$$\mathcal{L}(u^*, f^*) \leqslant \mathcal{L}(u_0^*, f_0^*). \tag{5.36}$$

On the other hand, since (u_0^*, f_0^*) is a solution of Problem $\mathcal{Q}$, we have

$$\mathcal{L}(u_0^*, f_0^*) = \min_{(u,f)\in\mathcal{V}_{ad}} \mathcal{L}(u, f) \tag{5.37}$$

and therefore, inclusion (5.32) implies that

$$\mathcal{L}(u_0^*, f_0^*) \leqslant \mathcal{L}(u^*, f^*). \tag{5.38}$$

We now combine inequalities (5.36) and (5.38) to see that

$$\mathcal{L}(u^*, f^*) = \mathcal{L}(u_0^*, f_0^*). \tag{5.39}$$

Next, we use relations (5.32), (5.39) and (5.37) to see that (5.25) holds.

ii–d) *End of proof.* We remark that the convergences (5.23) and (5.24) imply the weak convergence (in the product Hilbert space $V \times L^2(D)$) of the sequence $\{(u_n^*, f_n^*)\}$ to the element (u^*, f^*). Theorem 5.2 is now a direct consequence of Definition 2.8 c). □

We turn now to some direct consequence of Theorem 5.2.

Corollary 5.3. *Assume that* (3.9)–(3.11), (3.20) *and* (3.21) *hold. Then the set of solutions of Problem $\mathcal{Q}$ is weakly sequentially compact.*

P r o o f. Assume that $\{(u_n^*, f_n^*)\}$ is a sequence of solutions to Problem $\mathcal{Q}$. Let $\{\lambda_n\} \subset \mathbb{R}$ be such that $\lambda_n > 0$ for each $n \in \mathbb{N}$, $\lambda_n \to 0$, and let $\mu_n = 0$ for each $n \in \mathbb{N}$. Also, consider the particular case when $\widetilde{K} = K$, $Gv = 0_V$ for any $v \in V$, and note that in this case we have $\mathcal{V}_{ad}^n = \mathcal{V}_{ad}$, $\mathcal{L}_n = \mathcal{L}$ for each $n \in \mathbb{N}$. It follows from above that $\{(u_n^*, f_n^*)\}$ is an approximating sequence for Problem $\mathcal{Q}$. Therefore, step (ii) in the proof of Theorem 5.2 implies that there exists a subsequence of the sequence $\{(u_n^*, f_n^*)\}$, again denoted by $\{(u_n^*, f_n^*)\}$, and an element $(u^*, f^*) \in V \times L^2(D)$ such that (5.23)–(5.25) hold, which concludes the proof. □

Corollary 5.4. *Assume* (3.9)–(3.11), (5.4), (5.5) *and* (5.2), (5.3) *for all* $\mu \geqslant 0$. *Then for each* $\mu \geqslant 0$ *there exists a pair* (u^*_μ, f^*_μ) *such that*

$$(u^*_\mu, f^*_\mu) \in \mathcal{V}_{ad}, \quad \mathcal{L}_\mu(u^*, f^*) = \min_{(u,f)\in\mathcal{V}_{ad}} \mathcal{L}_\mu(u, f). \tag{5.40}$$

Moreover, if $\{(u^*_n, f^*_n)\}$ *represents a sequence pairs such that* (u^*_n, f^*_n) *is a solution of Problem* $\mathcal{Q}$ *corresponding to* $\mu_n \geqslant 0$ *for each* $n \in \mathbb{N}$, *then there exists a subsequence of the sequence* $\{(u^*_n, f^*_n)\}$, *again denoted by* $\{(u^*_n, f^*_n)\}$, *and an element* $(u^*, f^*) \in V \times L^2(D)$ *such that* (5.23)–(5.25) *hold.*

The proof of Corollary 5.4 follows from arguments similar to those used in the proof of Corollary 5.3 and therefore we skip it.

Corollary 5.5. *Assume* (3.9)–(3.11), (4.1), (4.2), (5.4), (5.5) *and* (5.2), (5.3) *for all* $\mu \geqslant 0$. *Moreover, assume that Problem* $\mathcal{Q}$ *has a unique solution. Then Problem* $\mathcal{Q}$ *is weakly well-posed with respect to the Tykhonov triple* (5.18)–(5.20).

Proof. Let (u^*, f^*) be the unique solution to Problem $\mathcal{Q}$ and let $\{u^*_n, f^*_n\}$ be an approximating sequence for Problem $\mathcal{Q}$ with respect to the Tykhonov triple (5.18)–(5.20). First, it follows from the proof of Theorem 5.2 that the sequence $\{f^*_n\}$ is bounded in $L^2(D)$. Therefore, using arguments similar to those used in step i) of the proof of Theorem 4.2 we deduce that the sequence $\{u^*_n\}$ is bounded in V. We conclude from here that the sequence $\{(u^*_n, f^*_n)\}$ is bounded in the product space $V \times L^2(D)$. Second, a careful analysis of the proof of Theorem 5.2 reveals that (u^*, f^*) is the weak limit (in $V \times L^2(D)$) of any weakly convergent subsequence of the sequence $\{(u^*_n, f^*_n)\}$. The two properties above allow us to use a standard argument in order to deduce that the whole sequence $\{(u^*_n, f^*_n)\}$ converges weakly in $V \times L^2(D)$ to (u^*, f^*) as $n \to \infty$. Corollary 5.5 is now a direct consequence of Definition 2.8 b). $\square$

We end this section with the following remarks. First, Corollary 5.3 provides a topological property of the set of solutions of Problem $\mathcal{Q}$. Moreover, Corollary 5.4 provides a continuity result of the solutions of this problem with respect to the parameter μ and, implicitely, with respect to the cost functional. Finally, Corollary 5.5 provides a weakly well-posedness result for Problem $\mathcal{Q}$ in the particular case when this problem has a unique solution. Such situations arise for specific boundary condition and cost functionals, as explained in [3], [4].

We start this section with examples of perturbed heat transfer problems which lead to hemivariational inequalities of the form (4.3) for which conditions (4.1) and (4.2) are satisfied.

A unilateral problem with penalty conditions on D. The first problem we introduce in this section is obtained by considering a penalty version of the unilateral condition in (3.1). The problem is formulated as follows.

Problem $\mathcal{H}_0$. *Find a temperature field $u\colon D \to \mathbb{R}$ such that*

$$\Delta u + f = \frac{1}{\lambda} p_0(u) \quad \text{a.e. in } D, \tag{6.1}$$

$$u = 0 \quad \text{a.e. on } \Gamma_1, \tag{6.2}$$

$$u = b \quad \text{a.e. on } \Gamma_2, \tag{6.3}$$

$$-\frac{\partial u}{\partial \nu} \in \partial j_\nu(u) \quad \text{a.e. on } \Gamma_3. \tag{6.4}$$

Here $\lambda > 0$ is a penalty parameter and p_0 is a function which is assumed to satisfy the following condition.

$$\begin{cases} p_0\colon\ D \times \mathbb{R} \to \mathbb{R} \text{ is such that} \\ \text{(a) } |p_0(\boldsymbol{x}, r) - p_0(\boldsymbol{x}, s)| \leqslant L_0 |r - s| \ \forall\, r, s \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in D, \text{ with } L_0 > 0, \\ \text{(b) } (p_0(\boldsymbol{x}, r) - p_0(\boldsymbol{x}, s))(r - s) \geqslant 0 \ \forall\, r, s \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in D, \\ \text{(c) } \boldsymbol{x} \mapsto p_0(\boldsymbol{x}, r) \text{ is measurable on } D \ \forall\, r \in \mathbb{R}, \\ \text{(d) } p_0(\boldsymbol{x}, r) = 0 \text{ if and only if } r \geqslant 0, \text{ a.e. } \boldsymbol{x} \in D. \end{cases} \tag{6.5}$$

A typical example of such function is given by

$$p_0(x, r) = -c r^- \quad \forall\, r \in \mathbb{R}, \ \boldsymbol{x} \in D,$$

where $c > 0$ and r^- represents the negative part of r, i.e. $r^- = \max\{-r, 0\}$. Note that in this case the term $p_0(u)$ in (6.1) vanishes in the points of D, where $u \geqslant 0$ and equals cu/λ in the points of D where $u < 0$.

Using standard arguments it is easy to see that Problem $\mathcal{H}_0$ leads to a variational formulation of the form (4.3), in which

$$\widetilde{K} = \{v \in V\colon\ v = b \text{ on } \Gamma_2\}, \tag{6.6}$$

$$\langle G u, v\rangle = \int_D p_0(u) v \,\mathrm{d}a \quad \forall\, u, v \in V. \tag{6.7}$$

We have the following result.

Proposition 6.1. *Assume* (3.9) *and* (6.5)*. Then the set* (6.6) *and the operator* (6.7) *satisfy conditions* (4.1) *and* (4.2)*, respectively.*

P r o o f. Conditions (4.1) and (4.2)(a) are clearly satisfied. Assume now that $u \in \widetilde{K}$ and $v \in K$, where, recall, the set K is defined by (3.12). Then using (6.5) we see that

$$p_0(u)v \leqslant 0 \quad \text{and} \quad p_0(u)u \geqslant 0 \quad \text{a.e. in } D, \tag{6.8}$$

which imply that $p_0(u)(v-u) \leqslant 0$ a.e. in D. We conclude from here that

$$\int_D p_0(u)(v-u)\,\mathrm{d}x \leqslant 0$$

and therefore, condition (4.2)(b) holds.

Next, we assume that $u \in \widetilde{K}$ and $\langle Gu, v-u\rangle = 0$ for all $v \in K$, which implies that

$$\int_D p_0(u)u\,\mathrm{d}x = \int_D p_0(u)v\,\mathrm{d}x \quad \forall\, v \in K. \tag{6.9}$$

We now use inequalities (6.8) to deduce that

$$\int_D p_0(u)u\,\mathrm{d}x = 0. \tag{6.10}$$

Therefore (6.8), (6.9) combined with the implication

$$h \geqslant 0, \quad \int_\omega h = 0 \Rightarrow h = 0 \quad \text{a.e. on } \omega \tag{6.11}$$

show that $p_0(u)u = 0$ a.e. on D. This equality together with condition (6.5)(d) implies that $u \geqslant 0$ a.e. on D. Recall now that $u \in \widetilde{K}$. We deduce from here that $u = b$ a.e. on Γ_2. Therefore $u \in K$, which concludes the proof. □

A unilateral problem with penalty conditions on Γ_2. The second problem we introduce in this section is obtained by considering a penalty version of the boundary condition (3.3) on Γ_2. The problem is formulated as follows.

Problem $\mathcal{H}_2$. *Find a temperature field $u\colon\ D\to\mathbb{R}$ such that*

$$u\geqslant 0,\quad -\Delta u-f\geqslant 0,\quad u(\Delta u+f)=0\quad \text{a.e. in } D, \tag{6.12}$$

$$u=0 \quad \text{a.e. on } \Gamma_1, \tag{6.13}$$

$$-\frac{\partial u}{\partial \nu}=\frac{1}{\lambda}p_2(u-b) \quad \text{a.e. on } \Gamma_2, \tag{6.14}$$

$$-\frac{\partial u}{\partial \nu}\in \partial j_\nu(u) \quad \text{a.e. on } \Gamma_3. \tag{6.15}$$

Here again, $\lambda>0$ is a penalty parameter and p_2 is a function which is assumed to satisfy the following condition.

$$\left\{\begin{array}{l} p_2\colon\ \Gamma_2\times\mathbb{R}\to\mathbb{R} \text{ is such that}\\ \text{(a) } |p_2(\boldsymbol{x},r)-p_2(\boldsymbol{x},s)|\leqslant L_2|r-s|\ \forall\, r,s\in\mathbb{R}, \text{a.e. } \boldsymbol{x}\in\Gamma_2,\ \text{with } L_2>0,\\ \text{(b) } (p_2(\boldsymbol{x},r)-p_2(\boldsymbol{x},s))(r-s)\geqslant 0\ \forall\, r,s\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_2,\\ \text{(c) } \boldsymbol{x}\mapsto p_2(\boldsymbol{x},r) \text{ is measurable on } \Gamma_2\ \forall\, r\in\mathbb{R},\\ \text{(d) } p_2(\boldsymbol{x},r)=0 \text{ if and only if } r=0,\ \text{a.e. } \boldsymbol{x}\in\Gamma_2. \end{array}\right. \tag{6.16}$$

A typical example of such function is given by

$$p_2(\boldsymbol{x},r)=cr\quad \forall\, r\in\mathbb{R},\ \boldsymbol{x}\in\Gamma_2,$$

where $c>0$. Using standard arguments it is easy to see that Problem $\mathcal{H}_2$ leads to a variational formulation of the form (4.3) in which

$$\widetilde{K}=\ \{v\in V\colon\ v\geqslant 0 \text{ in } D\}, \tag{6.17}$$

$$\langle Gu,v\rangle=\int_{\Gamma_2}p_2(u-b)v\,\mathrm{d}a\quad \forall\, u,v\in V. \tag{6.18}$$

We have the following result.

Proposition 6.2. *Assume* (3.9) *and* (6.16)*. Then the set* (6.17) *and the operator* (6.18) *satisfy conditions* (4.1) *and* (4.2)*, respectively.*

P r o o f. Conditions (4.1) and (4.2)(a) are clearly satisfied. Assume now that $u\in\widetilde{K}$ and $v\in K$. Then using (6.16) we see that $p_2(u-b)(v-u)=p_2(u-b)\times(b-u)$ a.e. on Γ_2 and using properties (6.16) of the function p_2 we deduce that $p_2(u-b)(v-u)\leqslant 0$ a.e. on Γ_3. We conclude from here that

$$\int_{\Gamma_2}p_2(u-b)(v-u)\,\mathrm{d}a\leqslant 0$$

and therefore, condition (4.2)(b) holds.

Assume now that $u \in \widetilde{K}$ and $\langle Gu, v-u\rangle = 0$ for all $v \in K$. This implies that

$$\int_{\Gamma_3} p_2(u-b)(b-u)\,\mathrm{d}a = 0.$$

Now, since (6.16) guarantees that $p_2(u-b)(b-u) \leqslant 0$ a.e. on Γ_2, we use implication (6.11), again to deduce that $p_2(u-b)(b-u) = 0$ a.e. on Γ_2. It now follows from condition (6.16)(d) that $u = b$ a.e. on Γ_3 and therefore, $u \in K$. □

A fully penalty problem. The third problem we introduce in this section is obtained by considering a penalty version of both conditions (3.1) and (3.3). The problem is formulated as follows.

Problem $\mathcal{H}_{02}$. *Find a temperature field $u\colon D \to \mathbb{R}$ such that*

$$\Delta u + f = \frac{1}{\lambda} p_0(u) \qquad \text{a.e. in } D, \tag{6.19}$$

$$u = 0 \qquad \text{a.e. on } \Gamma_1, \tag{6.20}$$

$$-\frac{\partial u}{\partial \nu} = \frac{1}{\lambda} p_2(u-b) \qquad \text{a.e. on } \Gamma_2, \tag{6.21}$$

$$-\frac{\partial u}{\partial \nu} \in \partial j_\nu(u) \qquad \text{a.e. on } \Gamma_3. \tag{6.22}$$

Here $\lambda > 0$ is a penalty parameter and p_0 and p_2 are functions which satisfy conditions (6.5) and (6.16), respectively. Using standard arguments it is easy to see that Problem $\mathcal{H}_2$ leads to a variational formulation of the form (4.3) in which

$$\widetilde{K} = V, \tag{6.23}$$

$$\langle Gu, v\rangle = \int_D p_0(u)v\,\mathrm{d}a + \int_{\Gamma_2} p_2(u-b)v\,\mathrm{d}a \quad \forall\, u, v \in V. \tag{6.24}$$

We have the following result.

Proposition 6.3. *Assume* (3.9), (6.5) *and* (6.16)*. Then the set* (6.23) *and the operator* (6.24) *satisfy conditions* (4.1) *and* (4.2), *respectively.*

P r o o f. Conditions (4.1) and (4.2)(a) are clearly satisfied. Assume now that $u \in \widetilde{K}$ and $v \in K$. Then using (6.5) and (6.16) it is easy to see that $p_0(u)(v-u) \leqslant 0$ a.e. in D and $p_2(u-b)(v-u) \leqslant 0$ a.e. on Γ_3. We conclude from here that

$$\int_D p_0(u)(v-u)\,\mathrm{d}x + \int_{\Gamma_2} p_2(u-b)(v-u)\,\mathrm{d}a \leqslant 0$$

and therefore, condition (4.2)(b) holds.

Assume now that $u \in \widetilde{K}$ and $\langle Gu, v - u\rangle = 0$ for all $v \in K$. This implies that

$$\int_D p_0(u)v\,\mathrm{d}x + \int_{\Gamma_3} p_2(u-b)(b-u)\,\mathrm{d}a = \int_D p_0(u)u\,\mathrm{d}x. \tag{6.25}$$

Now, recall that (6.5) and (6.16) guarantee that

$$p_0(u)v \leqslant 0 \quad \text{a.e. in } D, \quad p_2(u-b)(b-u) \leqslant 0 \quad \text{a.e. on } \Gamma_2, \tag{6.26}$$

$$p_0(u)u \geqslant 0 \quad \text{a.e. in } D. \tag{6.27}$$

We now combine equality (6.25) with inequalities (6.26) and (6.27) to find that

$$\int_D p_0(u)u\,\mathrm{d}x = 0. \tag{6.28}$$

Next, (6.27), (6.28) and (6.11) imply that $p_0(u)u = 0$ a.e. in D and using condition (6.5) we find that

$$u \geqslant 0 \quad \text{a.e. in } D. \tag{6.29}$$

We conclude from here that $p_0(u) = 0$ a.e. on D and therefore, (6.25) yields

$$\int_{\Gamma_3} p_2(u-b)(b-u)\,\mathrm{d}a = 0. \tag{6.30}$$

Next, (6.26), (6.30) and (6.11) imply that $p_2(u-b)(b-u) = 0$ a.e. on Γ_2. It now follows from condition (6.16)(d) that

$$u = b \quad \text{a.e. on } \Gamma_3. \tag{6.31}$$

We now use (6.29) and (6.31) to deduce that $u \in K$, which concludes the proof. $\square$

An example of cost functional. A large number of cost functionals for which our results in Section 5 hold can be considered. Here we restrict ourselves to provide the following example. Assume that

$$a_0 > 0, \quad a_2 > 0, \quad \varphi \in L^2(\Gamma_2), \tag{6.32}$$

$$\omega = [0,\infty) \to \mathbb{R} \text{ is a continuous function such that } \omega(0) = 1. \tag{6.33}$$

Consider the cost functional $\mathcal{L}\colon V \times L^2(D) \to \mathbb{R}$ defined by

$$\mathcal{L}(u,f) = a_0 \int_D f^2\,\mathrm{d}x + a_2 \int_{\Gamma_2} (u-\varphi)^2\,\mathrm{d}a \tag{6.34}$$

and for each $\mu \geqslant 0$ let $\mathcal{L}_\mu\colon V \times L^2(D) \to \mathbb{R}$ be defined by

$$\mathcal{L}_\mu(u,f) = a_0 \int_D f^2\,\mathrm{d}x + a_2 \int_{\Gamma_2} (u - \omega(\mu)\varphi)^2\,\mathrm{d}a. \tag{6.35}$$

Then it is easy to see that the following result holds.

Proposition 6.4. *Under assumptions* (6.32) *and* (6.33) *the functionals* $\mathcal{L}$ *and* $\mathcal{L}_\mu$ *satisfy conditions* (3.20), (3.21), (5.4), (5.5) *and* (5.2), (5.3) *for each* $\mu \geqslant 0$.

Final remarks. We end this section with the remark that Propositions 6.1–6.3 allow us to apply Theorem 4.2 and Corollaries 4.3, 4.4 in the study of the corresponding boundary value problems. In this way we obtain the unique solvability of Problems $\mathcal{H}_0$, $\mathcal{H}_2$ and $\mathcal{H}_{02}$, the weak-strong continuous dependence of their weak solutions with respect to $f \in L^2(D)$ and the convergence of these solutions to the weak solution of Problem $\mathcal{H}$, as $\lambda \to 0$. Moreover, Proposition 6.4 allows us to apply Theorem 5.2 and Corollaries 5.3–5.5 in the study of the corresponding optimal control problems. This allows to obtain the solvability of these optimal control problems, the sequential compactness of the sets of their solutions and their continuous dependence with respect to the parameter μ.

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