CONTINUOUS DEPENDENCE ON PARAMETERS AND BOUNDEDNESS OF SOLUTIONS TO A HYSTERESIS SYSTEM

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Abstract. We analyze an ordinary differential system with a hysteresis-relay nonlinearity in two cases when the system is autonomous or nonautonomous. Sufficient conditions for both the continuous dependence on the system parameters and the boundedness of the solutions to the system are obtained. We give a supporting example for the autonomous system.

Keywords: ODE system; hysteresis relay; external disturbance; bounded solution; periodic solution

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1. Introduction

Proofs of the existence of periodic modes in nonlinear control systems, studies of their properties and configurations in phase spaces are the main problems of nonlinear oscillation theory [1]. Concurrently, one can solve the problem of constructing the periodic modes with given properties due to the choice of system parameters. In the paper, we discuss some aspects of solutions to the problem for the ODE system with a hysteresis-relay nonlinearity and an external disturbance. Apart from the periodic solutions, we study the bounded solutions to such systems, i.e. the solutions located in some bounded domain of the phase space.

Systems with hysteresis have been explored for a long time (see, e.g., [5], [8], [18], [20], [21], [31]). Nevertheless, the interest in research of these systems does not fade away nowadays. From the latest papers in this direction, we should mention [2], [4], [6], [7], [11], [14]–[17], [19], [22], [24], [27], [28], [32]–[35]. In general, hysteresis

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occurs almost everywhere in nature, namely, in several phenomena in hydrology [6], biology, engineering, physics, and so on [32]. The dynamic behaviour of hysteresis systems depends on the parameters that are set with some accuracy and can change in physical systems over time. For this reason, the problems for the existence of bounded modes as well as their stability and bifurcation while the parameters change are studied. The most important researches of global dynamics and bifurcation are based on using the smooth and non-smooth Preisach operators, searching for fixed points, and studying periodic solutions and their properties by different methods [2], [6], [22], [24], [27], including topological methods, for instance, the equivariant degree method [4].

For applications, scientists use widely the models of hysteresis coupled, as a rule, with PDEs [33]. Here we consider a non-ideal scalar relay as a hysteresis model coupled with ODEs. First, the non-ideal relay is the simplest model of discontinuous hysteresis [32]. Second, it is one of the few known nonlinearities that can be described by the Preisach model [11], involving its vector extension [33]. We develop the results discussed in [14]–[16], [34], [35] and investigate the dynamics of the control system governed by an N-dimensional system of ODEs. As the control, we consider the hysteresis-relay nonlinearity that is often used in automatic control systems [29]. In nonautonomous systems, periodic (see [12], [14]–[17], [34], [35]) or nonperiodic functions (see [26]) can stand for external disturbances. The studied mathematical model describes numerous automatic devices with a relay installed, for example, on river or sea vessels [9], [23].

At first, we consider the transformation of a closed set into itself via the Schauder fixed-point theorem. This transformation is given by the set of solutions to the system of ODEs. Assuming that there exists at least one periodic solution with $2n$ ($n \in \mathbb{N}$) switch points, we construct a system of transcendental equations with respect to the switch instants of the representative point of this periodic solution. Then, using the implicit function theorem, we obtain the conditions under which the move times between switch points depend locally continuously on the system parameters. Also, we find sufficient conditions for both the boundedness of steady-state modes and the continuous dependence of the periodic solutions on the system parameters, including the parameters of the feedback control and the external disturbance. These sufficient conditions provide a configuration robustness to the steady-state modes, give the property similar to a dissipativity one for the system and the property similar to the property of the Lagrange stability for the solutions. The above-mentioned dissipativity and stability properties are known for systems with continuous right-hand sides. For example, in [10], the decay rates for the energy are intrinsically described by the solution to a dissipative ODE. This differs from the present paper, in which the system of ODEs with a discontinuous nonlinearity is investigated.

2. Statement of the problem

We describe the mathematical model by the system of ODEs

$$\dot{X} = AX + BF(\sigma) + Kf(t), \quad \sigma = \langle \Gamma, X \rangle. \tag{2.1}$$

Here X is the state vector such that $X \in \mathbb{R}^N$, where $\mathbb{R}^N$ is an N-dimensional Euclidean space. The $(N \times N)$ matrix A and the $(N \times 1)$ vectors B, K, Γ are real and constant. We denote the scalar product of the vectors Γ and X by $\langle \Gamma, X \rangle$.

The function $F(\sigma)$ stands for the non-ideal relay with the thresholds l_1, l_2, the outputs m_1, m_2 and positive spin unlike [28]. Without restricting the generality, we put $l_1 < l_2$ and $m_1 < m_2$. The function $F(\sigma(t))$ is defined for $t \geqslant 0$ in the class of continuous functions and given as in [25]: $F(\sigma) = m_1$ follows from $\sigma(t) \leqslant l_1$, $F(\sigma) = m_2$ follows from $\sigma(t) \geqslant l_2$, and $F(\sigma(t_1)) = F(\sigma(t_2))$ follows from $l_1 < \sigma(t) < l_2$ $(t_1 < t \leqslant t_2)$. Thus, if $\sigma(0) \leqslant l_1$ or $\sigma(0) \geqslant l_2$, then one value of $F(\sigma(t))$ corresponds to $\sigma(t)$, and if $l_1 < \sigma(0) < l_2$, then two values of $F(\sigma(t))$ correspond to $\sigma(t)$. The hysteresis loop presented in the coordinates (σ, F) by $\sigma = \sigma(t)$, $F = F(\sigma(t))$ is followed counter-clockwise. In general, if at the initial instant $t = t_0$ we have $\sigma(t_0) \in (l_1, l_2)$, then it is necessary to specify $F(\sigma(t_0)) = m_1$ or $F(\sigma(t_0)) = m_2$ and follow the positive spin in the plane (σ, F): the value of $F(\sigma(t))$ is kept constant for all $t > t_0$ until $\sigma(t)$ crosses the threshold value l_2 from below or the threshold value l_1 from above, respectively; at these instants (when $\sigma(t) = l_i$, $i = 1, 2$) the value of $F(\sigma(t))$ is changed to m_1 or m_2, respectively. According to [19], the set of possible states of the non-ideal relay is the set of points $\{(\sigma, F)\}$ that belong to the two half-lines $F = m_1$ for $\sigma < l_2$ and $F = m_2$ for $\sigma > l_1$.

The function $F(\sigma)$ characterizes a relay with hysteresis and is used to describe mechanical, electromagnetic, chemical, biological phenomena, for example, a spatial delay of control mechanisms in the model of a ship autopilot or the friction in mechanical systems. Relays provide the basic bricks to build Preisach models, which are well known in the field of magnetic, ferromagnetic, and smart (shape-memory) materials.

Note that, in automatic control systems, the elements of the matrix A characterize the plant, the vector B consists of gain coefficients in the subsystem of final controlling units, the vector Γ defines the feedback and is called the relay-feedback vector. The vector Γ indicates the way in which the hyperplanes $\langle \Gamma, X \rangle = l_i$ $(i = 1, 2)$ and, therefore, the solutions to this system are arranged in the phase space. Further, these hyperplanes are called switch surfaces. It is certain that the analysis of systems with a relay feedback is quite a task [3], [13], [36].

The function $f(t)$ stands for an external or testing input disturbance imposed on the automatic system. In engineering practice, as a rule, the disturbance is carried out in the form of a jump, an exponential curve, a polynomial or sine. Such simplifications are often not quite adequate to actual input disturbances. That is why we consider $f(t)$ from the class of continuous functions bounded at least on a finite time interval, i.e. $|f(t)| \leqslant M = \text{const}$. The function $f(t)$ can be periodic or nonperiodic.

Usually the systems with real roots of characteristic equations are studied (see, for example, [3], [15], [16], [30], [34], [35]). In [14], we analyze the complex roots. In this paper, we continue the research along this line.

We assume that the matrix A has no pure imaginary eigenvalues, i.e. the eigenvalues of the form $q\mathrm{i}$, where $q \in \mathbb{R}$, $\mathrm{i}^2 = -1$. Then there exists the matrix $(U - \mathrm{e}^{At})^{-1}$ and putting $X(0) = X(T) = X_0$, we have

$$X_0 = (U - \mathrm{e}^{AT})^{-1} \int_0^T \mathrm{e}^{A(T-\tau)}(BF(\sigma(\tau)) + Kf(\tau))\,\mathrm{d}\tau,$$

where U is the identity matrix. For the analytical representation of the solution to (2.1), we use Cauchy's form. We may thus consider

$$\sigma(t) = \left\langle \Gamma, \mathrm{e}^{At}X_0 + \int_0^t \mathrm{e}^{A(t-\tau)}(BF(\sigma(\tau)) + Kf(\tau))\,\mathrm{d}\tau \right\rangle. \tag{2.2}$$

Equation (2.2) makes it possible to obtain the system of equations for the instants and points of intersection of the solutions with the switch surfaces $\langle \Gamma, X\rangle = l_i$ $(i = 1, 2)$, which determine splitting the function $F(\sigma)$ into linear parts. We show that in the next sections.

The outline of the paper is as follows. In Section 3, the sufficient condition for the steady-state motions of (2.1) to be bounded is established. In Section 4, the system of transcendental equations with respect to the switch instants and formulae for switch points are presented. The main result on the configuration robustness of periodic motions is given in Theorem 4.1. In Section 5, we apply Theorem 4.1 to the N-dimensional system of ODEs in case when the disturbance is absent (Theorem 5.1) and provide a supporting example with $N = 2$.

3. Bounded solutions

Relay feedback systems show a few various behaviours. Unbounded solutions correspond to the resonance in systems [28]. In this section we consider bounded (periodic as well as nonperiodic) solutions, which are also of interest in applications.

In the phase space, the trajectory of any solution to system (2.1) can be pieced out of the trajectories in virtue of the linear systems

$$\dot{X} = AX + Bm_1 + Kf(t), \quad \dot{X} = AX + Bm_2 + Kf(t). \tag{3.1}$$

Sewing together these trajectories happens by continuity at the points on the switch surfaces $\langle \Gamma, X \rangle = l_i$ $(i = 1, 2)$. We assume that the solutions to (2.1) are in the class of periodic functions with even number of switch points. These switch points coincide with the sewing points. A closed bounded phase trajectory corresponds to the periodic solution of (2.1). In the $(N + 1)$-dimensional space (X, t), the whole integrated curve consisting of the pieces of integrated curves in virtue of (3.1) conforms to the periodic solution of (2.1). These pieces are repeated with a period that we call the period of forced oscillations in system (2.1). We denote this period by T_f. The switch points of the periodic solution have the following properties:

$$X^i = X(t_0, m_j, t_0) = X(t_0, m_j, t_0 + T_f), \quad \langle \Gamma, X^i \rangle = l_k \quad \forall\, i, j, k = 1, 2,$$

where t_0 is the initial instant. Generally, along with the periodic solutions, system (2.1) can have nonperiodic solutions with the switch points that belong to the switch surfaces $\langle \Gamma, X \rangle = l_i$ $(i = 1, 2)$ as well.

Now we give Theorem 3.1.

Theorem 3.1. *Let the above assumptions with respect to the right-hand side of system* (2.1) *hold. In addition, let all the eigenvalues of the matrix A have negative real parts and let the conditions $-\langle \Gamma, A^{-1}Bm_2 \rangle < l_1$, $-\langle \Gamma, A^{-1}Bm_1 \rangle > l_2$ be fulfilled. Then all the steady-state motions of system* (2.1) *belong to some bounded domain of the phase space or, in other words, the representative point of any solution to* (2.1) *reaches some bounded domain of the phase space for a finite time and stays inside this domain.*

P r o o f. Let us consider the geometry of the phase space of system (2.1). Since A has the eigenvalues with negative real parts, one can allocate two convex compact sets on the switch surfaces, using the Lyapunov functions. By virtue of the solution to system (2.1), these sets are mapped into themselves. Zero solution of the system $\dot{X} = AX$ is asymptotically stable. As a consequence, there exist both the positive definite function $V(X)$ and the negative definite function $W(X)$ that are related by $\mathrm{d}V/\mathrm{d}t = W(X)$ in view of the system $\dot{X} = AX$. Here $V(X) = X^\top V_0 X$ and V_0 is a constant, where $\top$ means the transpose. Next we establish the conditions under which the total derivative of the function $V(X)$ is negative by virtue of (2.1), i.e.

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.1)} = X^\top W_0 X + (B^\top F(\sigma) + K^\top f(t)) V_0 X + X^\top V_0 (BF(\sigma) + Kf(t)) < 0, \tag{3.2}$$

where $W_0 = A^\top V_0 + V_0 A$.

Inequality (3.2) is valid if the inequality

$$\|X\| > \frac{1}{\|A\|}\big(\max_{i=1,2}|m_i|\cdot\|B\| + M\cdot\|K\|\big) \tag{3.3}$$

holds, where M is a constant such that $|f(t)| \leqslant M$.

The equations of the form $V(X) = C$, where C is a constant, provide closed surfaces in the phase space. Obviously, there is the minimum value of C at which inequalities (3.2) and (3.3) are fulfilled.

In order for the surface $V(X) = \min C$ to cross the switch surfaces, it is necessary that the conditions $-\langle\Gamma, A^{-1}Bm_2\rangle < l_1$, $-\langle\Gamma, A^{-1}Bm_1\rangle > l_2$ are met.

The last conditions mean that system (2.1) has the form $\dot{X} = 0$ at the points $X_i = -A^{-1}Bm_i$ $(i = 1,2)$ when $f(t) \equiv 0$ and, in the phase space of the system, these points lie outside the ambiguity zone of the function $F(\sigma)$.

Now if the initial points X are taken from the domain bounded by the surface $V(X) = \min C$, then the trajectory of the representative point does not leave this domain of the phase space in virtue of (2.1). The intersection of the set given by the inequality $V(X) \leqslant \min C$ with the switch surfaces provides the convex compact sets S_i $(i = 1,2)$ defined by the system

$$\begin{gathered}\langle\Gamma, X\rangle = l_i, \quad i = 1,2,\\ \|X\| \leqslant \frac{1}{\|A\|}\big(\max_{i=1,2}|m_i|\cdot\|B\| + M\cdot\|K\|\big).\end{gathered}$$

The solution to (2.1) defines the continuous operator of the form

$$P(X_0, T(X_0)) = \mathrm{e}^{A(T(X_0)-t_0)}\left(X_0 + \int_{t_0}^{T(X_0)} \mathrm{e}^{-A(\tau-t_0)}(BF(\sigma) + Kf(\tau))\,\mathrm{d}\tau\right),$$

where $X_0 = X(t_0)$ and $T(X_0)$ is the time for the representative point to return into the set S_i along the trajectory of the solution to (2.1).

Thus, in the phase space, we have found the sets mapped into themselves in virtue of the solution to (2.1). The switch points belong to the sets S_i $(i = 1,2)$ in the phase space. Theorem 3.1 is proved. □

4. Periodic solutions

Let system (2.1) have at least one periodic solution with $2n$ switch points, where n is an integer. We denote the switch surfaces $\sigma = l_i$ of $\mathbb{R}^N$ by L_i $(i = 1, 2)$. Let the representative point of the required solution begin its motion at the point $X^1 \in L_1$ at $t_0 = 0$ and reach the point $X^2 \in L_2$ at t_1 in virtue of (3.1) provided that $m_i = m_1$. Next let it reach $X^3 \in L_1$ at t_2 in virtue of (3.1) provided that $m_i = m_2$, then $X^4 \in L_2$ at t_3 in virtue of (3.1) provided that $m_i = m_1$, and so on. At last, let the representative point come back at the initial point $X^{2n+1} = X^1$ at t_{2n} in virtue of (3.1) provided that $m_i = m_2$. Then the instant t_{2n} coincides with T_f, where T_f is the period of forced oscillations.

Considering the form of $F(\sigma)$ and the selected class of periodic solutions to (2.1), we construct the system of equations with respect to the parameters (switch instants and corresponding points) of the periodic solutions

$$
\begin{aligned}
(4.1)\qquad X^2 &= \mathrm{e}^{At_1} X^1 + \int_0^{t_1} \mathrm{e}^{A(t_1-\tau)}(Bm_1 + Kf(\tau))\,\mathrm{d}\tau,\\
X^3 &= \mathrm{e}^{A(t_2-t_1)} X^2 + \int_{t_1}^{t_2} \mathrm{e}^{A(t_2-\tau)}(Bm_2 + Kf(\tau))\,\mathrm{d}\tau,\\
X^4 &= \mathrm{e}^{A(t_3-t_2)} X^3 + \int_{t_2}^{t_3} \mathrm{e}^{A(t_3-\tau)}(Bm_1 + Kf(\tau))\,\mathrm{d}\tau,\\
&\vdots\\
X^{2n+1} = X^1 &= \mathrm{e}^{A(t_{2n}-t_{2n-1})} X^{2n} + \int_{t_{2n-1}}^{t_{2n}} \mathrm{e}^{A(t_{2n}-\tau)}(Bm_2 + Kf(\tau))\,\mathrm{d}\tau,
\end{aligned}
$$

where $X^1, \ldots, X^{2n}$ are the switch points and $t_1, \ldots, t_{2n}$ are the switch instants.

System (4.1) is written on account of the conditions necessary for the existence of at least one periodic solution to system (2.1) with $2n$ switch points. For later use, we introduce the notations

$$
\begin{aligned}
\int_0^{t_1} \mathrm{e}^{A(t_1-\tau)}(Bm_1 + Kf(\tau))\,\mathrm{d}\tau &= Q_1,\\
\int_{t_1}^{t_2} \mathrm{e}^{A(t_2-\tau)}(Bm_2 + Kf(\tau))\,\mathrm{d}\tau &= Q_2,\\
\int_{t_2}^{t_3} \mathrm{e}^{A(t_3-\tau)}(Bm_1 + Kf(\tau))\,\mathrm{d}\tau &= Q_3,\\
&\vdots\\
\int_{t_{2n-1}}^{t_{2n}} \mathrm{e}^{A(t_{2n}-\tau)}(Bm_2 + Kf(\tau))\,\mathrm{d}\tau &= Q_{2n}.
\end{aligned}
$$

It is clear that all the vectors X^j can be expressed in terms of X^1, i.e., we have the relation

$$X^j = E_j X^1 + I_j, \quad j = \overline{2, 2n+1}, \tag{4.2}$$

where E_j and I_j are given by the expressions

$$E_j = \mathrm{e}^{At_{j-1}},$$
$$I_j = \mathrm{e}^{A(t_{j-1}-t_1)}Q_1 + \delta_{j2}\mathrm{e}^{A(t_{j-1}-t_2)}Q_2 + \delta_{j3}\mathrm{e}^{A(t_{j-1}-t_3)}Q_3 + \ldots + \delta_{j2n}Q_{2n},$$

and

$$\delta_{jk} = \begin{cases} 0 & \text{for } j \leqslant k, \\ 1 & \text{for } j > k, \end{cases} \quad \text{where } k = \overline{2, 2n}.$$

After that we write $2n$ scalar equations according to the conditions for the points X^j to belong to L_i $(i = 1, 2)$, namely,

$$\langle \Gamma, X^{2l-1} \rangle = l_1, \quad \langle \Gamma, X^{2l} \rangle = l_2, \quad l = \overline{1, n}, \tag{4.3}$$

where X^j $(j = \overline{2, 2n+1})$ are defined by (4.2). We use (4.3) together with the last equation of (4.2),

$$X^{2n+1} = X^1 = E_{2n+1}X^1 + I_{2n+1}, \tag{4.4}$$

for finding the T_f-periodic solutions to (2.1) with $2n$ switch points. Equations (4.3) and (4.4) are equivalent to system (4.1).

In case when the matrix $U - E_{2n+1}$ is nonsingular, the point X^1 is eliminated from (4.3) via the equality

$$X^1 = (U - E_{2n+1})^{-1} I_{2n+1}.$$

Then we obtain $2n$ transcendental equations with respect to t_j $(j = \overline{1, 2n})$.

Assume that the solutions X^1, t_j $(j = \overline{1, 2n})$ exist under some conditions on the parameters of (4.3) and (4.4). Further, all the other points X^j $(j = \overline{2, 2n})$ are calculated by (4.2). If we connect the obtained points by arches of the integrated curves in virtue of (2.1) in the given sequence, then we may say that there exist the specified periodic solutions to (2.1).

Next, eliminating X^1 from (4.3), we come to the system

$$\begin{aligned} &\langle \Gamma, (U - E_{2n+1})^{-1} I_{2n+1} \rangle = l_1, \\ &\langle \Gamma, E_{2n}(U - E_{2n+1})^{-1} I_{2n+1} + I_{2n} \rangle = l_2, \\ &\langle \Gamma, E_{2n-1}(U - E_{2n+1})^{-1} I_{2n+1} + I_{2n-1} \rangle = l_1, \\ &\quad\vdots \\ &\langle \Gamma, E_2(U - E_{2n+1})^{-1} I_{2n+1} + I_2 \rangle = l_2. \end{aligned} \tag{4.5}$$

Note that system (4.5), as a rule, cannot be solved analytically. However, it is possible to obtain conditions under which the specified system has a solution.

The following theorem is valid.

Theorem 4.1. *Let system* (2.1) *satisfy the conditions of Theorem* 3.1 *and also have at least one periodic solution with* $2n$ $(n \in \mathbb{N})$ *isolated switch points located on the switch surfaces* $\langle \Gamma, X \rangle = l_i$ $(i = 1, 2)$*. Then these* $2n$ *switch points depend locally continuously on the specified parameters if*

$$\Delta = \begin{vmatrix} \langle \Gamma, (X^1)'_{t_1} \rangle \ldots \langle \Gamma, (X^1)'_{t_{2n}} \rangle \\ \vdots \\ \langle \Gamma, (X^{2n})'_{t_1} \rangle \ldots \langle \Gamma, (X^{2n})'_{t_{2n}} \rangle \end{vmatrix} \neq 0,$$

where $(X^i)'_{t_j}$ *are the partial derivatives of the elements of the vectors* X^i *with respect to* t_j $(i, j = \overline{1, 2n})$.

P r o o f. In the phase space, the switch surface-to-switch surface move times t_j are the functions of the system parameters, i.e.

$$t_j = t_j(a_{ik}, b_k, k_k, \gamma_k, m_1, m_2, l_1, l_2, \beta_1, \ldots, \beta_v), \quad j = \overline{1, 2n}.$$

Here a_{ik} are the elements of A; b_k, k_k, and γ_k are the elements of the vectors B, K, and Γ $(i, k = \overline{1, N})$, respectively; m_1, m_2, l_1, l_2 are the parameters of $F(\sigma)$; the constants $\beta_1, \ldots, \beta_v$ $(v \in \mathbb{N})$ are the parameters of $f(t)$. We take these parameters for independent variables. Next we apply the implicit function theorem. System (4.5) has the real solution t_j $(j = \overline{1, 2n})$ if $\Delta \neq 0$. More precisely, the functions defined by (4.3) meet the conditions of the implicit function theorem and the Jacobian Δ is not equal to zero at some point $\widetilde{R} = \widetilde{R}(\tilde{a}_{ik}, \tilde{b}_k, \tilde{k}_k, \tilde{\gamma}_k, \widetilde{m}_1, \widetilde{m}_2, \tilde{l}_1, \tilde{l}_2, \tilde{\beta}_1, \ldots, \tilde{\beta}_v, \tilde{t}_1, \ldots, \tilde{t}_{2n})$, where the point $\widetilde{R}$ satisfies (4.3). Then there exists a neighbourhood O of $\widetilde{R}$ and the functions t_j $(j = \overline{1, 2n})$ which are uniquely determined and continuous in O so that

$$t_j = t_j(a_{ik}, b_k, k_k, \gamma_k, m_1, m_2, l_1, l_2, \beta_1, \ldots, \beta_v),$$
$$\tilde{t}_j = t_j(\tilde{a}_{ik}, \tilde{b}_k, \tilde{k}_k, \tilde{\gamma}_k, \widetilde{m}_1, \widetilde{m}_2, \tilde{l}_1, \tilde{l}_2, \tilde{\beta}_1, \ldots, \tilde{\beta}_v),$$

where $j = \overline{1, 2n}$ and $i, k = \overline{1, N}$. Theorem 4.1 is proved. □

R e m a r k 4.1. On practical grounds, some parameters of system (2.1) can be fixed. Then you should not consider them as the parameters on which the functions t_j $(j = \overline{1, 2n})$ depend.

R e m a r k 4.2. The geometrical meaning of Theorem 4.1 is that the space configuration of the periodic mode of the system remains unchanged while the parameter values vary only slightly. In other words, the switch points move continuously along the switch surfaces, and also the number of these points does not change. Thus there exists the configuration robustness for the periodic mode.

5. Autonomous system

We consider system (2.1) when $f(t) \equiv 0$, that is, the autonomous system

$$\dot{X} = AX + BF(\sigma), \quad \sigma = \langle \Gamma, X \rangle. \tag{5.1}$$

System (5.1) has been studied for many years (see, for example, [29]), but by now there are no comprehensive results on the dynamics of this system. Obviously, it is convenient for applied scientists to have at least sufficient conditions on the parameters of the matrix A, the vectors B, Γ, and the function $F(\sigma)$ under which there exist stationary oscillations of system (5.1) with specific properties. In this section, we use Theorem 4.1 to study the dynamics of (5.1). Let the representative point of the solution to (5.1) begin its motion at the point $X^1 \in L_2$ at $t = 0$. The next theorem is valid.

Theorem 5.1. *Let system* (5.1) *meet the conditions of both Theorem* 3.1 *and Theorem* 4.1 *concerning the parameters of the matrix* A*, the vectors* B, Γ*, and the function* $F(\sigma)$*. In addition, let* $\langle \Gamma, B \rangle \neq 0$*. Then the unimodal periodic solution to system* (5.1) *is locally continuously dependent on the parameters if the condition*

$$\begin{aligned}
&\langle \Gamma, \Theta_1(\mathrm{e}^{AT}Bm_1 + \mathrm{e}^{A\tau_1}B(m_2 - m_1) - Bm_2)\rangle\langle \Gamma, \Theta_2\mathrm{e}^{A\tau_2}B(m_1 - m_2)\rangle \\
&\quad + \langle \Gamma, \Theta_2\mathrm{e}^{A\tau_1}B(m_2 - m_1)\rangle\langle \Gamma, \Theta_1(\mathrm{e}^{AT}Bm_2 + \mathrm{e}^{A\tau_2}B(m_1 - m_2) - Bm_1)\rangle \\
&\quad + \langle \Gamma, \Theta_2(\mathrm{e}^{AT}Bm_1 + \mathrm{e}^{A\tau_1}B(m_2 - m_1))\rangle\langle \Gamma, \Theta_2(\mathrm{e}^{AT}Bm_2 + \mathrm{e}^{A\tau_2}B(m_1 - m_2))\rangle \\
&\quad - \langle \Gamma, \Theta_2\mathrm{e}^{AT}Bm_1\rangle\langle \Gamma, \Theta_2\mathrm{e}^{AT}Bm_2\rangle \neq 0
\end{aligned}$$

is satisfied. Here $\Theta_1 = (U - \mathrm{e}^{AT})^{-2}\mathrm{e}^{AT}$, $\Theta_2 = (U - \mathrm{e}^{AT})^{-1}$*, and* $T = \tau_1 + \tau_2$*, where* τ_1, τ_2 *are the switch surface-to-switch surface move times.*

P r o o f. The switch points of the unimodal periodic solution [30] to an autonomous system belong to the switch surfaces and are defined by the expressions

$$X^1 = (U - \mathrm{e}^{AT})^{-1}\Big(\mathrm{e}^{A\tau_1}\int_{\tau_1}^{T} \mathrm{e}^{A(T-\tau)}Bm_1\,\mathrm{d}\tau + \int_0^{\tau_1} \mathrm{e}^{A(\tau_1-\tau)}Bm_2\,\mathrm{d}\tau\Big), \tag{5.2}$$

$$X^2 = (U - \mathrm{e}^{AT})^{-1}\Big(\mathrm{e}^{A\tau_2}\int_0^{\tau_1} \mathrm{e}^{A(\tau_1-\tau)}Bm_2\,\mathrm{d}\tau + \int_{\tau_1}^{T} \mathrm{e}^{A(T-\tau)}Bm_1\,\mathrm{d}\tau\Big). \tag{5.3}$$

Here, by analogy to (4.1), formulae (5.2) and (5.3) are written for the particular case when the number of switch points is two, the period is T such that $T = \tau_1 + \tau_2$, and $\langle \Gamma, X^1 \rangle = l_2$, $\langle \Gamma, X^2 \rangle = l_1$.

We consider the functions

$$(5.4) \qquad F_1(\tau_1, \tau_2) = \langle \Gamma, X^1 \rangle - l_2 = 0, \quad F_2(\tau_1, \tau_2) = \langle \Gamma, X^2 \rangle - l_1 = 0.$$

By virtue of (5.2), (5.3), the functions F_1 and F_2 depend on the system parameters, namely, on the parameters of A, B, Γ, and $F(\sigma)$. On the other hand, in a general case, system (5.4) cannot be solved with respect to τ_1 and τ_2. Then (5.4) can be considered as the equations defining the implicit functions $\tau_i(a_{jk}, b_k, \gamma_k, m_1, m_2, l_1, l_2)$, where $i = 1, 2$ and $j, k = \overline{1, N}$.

In the multidimensional space of the parameters of A, B, Γ, and $F(\sigma)$, we fix the point G that goes with the values τ_1^*, τ_2^*. In some neighborhood of the point (τ_1^*, τ_2^*, G), the functions F_1, F_2 are continuously differentiable with respect to τ_1, τ_2 and all their parameters. Now we construct the determinant

$$d(\tau_1, \tau_2) = \begin{vmatrix} \dfrac{\partial F_1}{\partial \tau_1} & \dfrac{\partial F_1}{\partial \tau_2} \\ \dfrac{\partial F_2}{\partial \tau_1} & \dfrac{\partial F_2}{\partial \tau_2} \end{vmatrix}$$

and calculate it at the point G. If $d(\tau_1, \tau_2) \neq 0$, then there exist open sets V and W ($G \in V$, $(\tau_1^*, \tau_2^*) \in W$) such that for any point of V there exists a unique point (τ_1, τ_2) of W for which (5.4) holds. At the same time the functions $\tau_1(a_{jk}, b_k, \gamma_k, m_1, m_2, l_1, l_2)$, $\tau_2(a_{jk}, b_k, \gamma_k, m_1, m_2, l_1, l_2)$ ($j, k = \overline{1, N}$) are differentiable with respect to their arguments.

Next we find the determinant $d(\tau_1, \tau_2)$. From (5.2), (5.3), it follows that

$$X^1 = (U - \mathrm{e}^{AT})^{-1}(\mathrm{e}^{A\tau_1} A^{-1}(\mathrm{e}^{A\tau_2} - U)Bm_1 + A^{-1}(\mathrm{e}^{A\tau_1} - U)Bm_2),$$
$$X^2 = (U - \mathrm{e}^{AT})^{-1}(\mathrm{e}^{A\tau_2} A^{-1}(\mathrm{e}^{A\tau_1} - U)Bm_2 + A^{-1}(\mathrm{e}^{A\tau_2} - U)Bm_1).$$

From (5.4), we obtain that

$$(5.5) \qquad \begin{aligned} \frac{\partial F_1}{\partial \tau_1} &= \langle \Gamma, (X^1)'_{\tau_1} \rangle, & \frac{\partial F_1}{\partial \tau_2} &= \langle \Gamma, (X^1)'_{\tau_2} \rangle, \\ \frac{\partial F_2}{\partial \tau_1} &= \langle \Gamma, (X^2)'_{\tau_1} \rangle, & \frac{\partial F_2}{\partial \tau_2} &= \langle \Gamma, (X^2)'_{\tau_2} \rangle. \end{aligned}$$

We calculate these derivatives, considering that the matrices $(U-\mathrm{e}^{At})^{-1}$, e^{At}, A are commutative and $T=T(\tau_1,\tau_2)=\tau_1+\tau_2$. Then we have

$$
\begin{aligned}
(5.6)\qquad (X^1)'_{\tau_1} &= (U-\mathrm{e}^{AT})^{-2}\mathrm{e}^{AT}(\mathrm{e}^{A\tau_1}(\mathrm{e}^{A\tau_2}-U)Bm_1+(\mathrm{e}^{A\tau_1}-U)Bm_2)\\
&\quad + (U-\mathrm{e}^{AT})^{-1}(\mathrm{e}^{A\tau_1}(\mathrm{e}^{A\tau_2}-U)Bm_1+\mathrm{e}^{A\tau_1}Bm_2),\\
(X^1)'_{\tau_2} &= (U-\mathrm{e}^{AT})^{-2}\mathrm{e}^{AT}(\mathrm{e}^{A\tau_1}(\mathrm{e}^{A\tau_2}-U)Bm_1+(\mathrm{e}^{A\tau_1}-U)Bm_2)\\
&\quad + (U-\mathrm{e}^{AT})^{-1}\mathrm{e}^{A\tau_1}\mathrm{e}^{A\tau_2}Bm_1,\\
(X^2)'_{\tau_1} &= (U-\mathrm{e}^{AT})^{-2}\mathrm{e}^{AT}(\mathrm{e}^{A\tau_2}(\mathrm{e}^{A\tau_1}-U)Bm_2+(\mathrm{e}^{A\tau_2}-U)Bm_1)\\
&\quad + (U-\mathrm{e}^{AT})^{-1}\mathrm{e}^{A\tau_1}\mathrm{e}^{A\tau_2}Bm_2,\\
(X^2)'_{\tau_2} &= (U-\mathrm{e}^{AT})^{-2}\mathrm{e}^{AT}(\mathrm{e}^{A\tau_2}(\mathrm{e}^{A\tau_1}-U)Bm_2+(\mathrm{e}^{A\tau_2}-U)Bm_1)\\
&\quad + (U-\mathrm{e}^{AT})^{-1}(\mathrm{e}^{A\tau_2}(\mathrm{e}^{A\tau_1}-U)Bm_2+\mathrm{e}^{A\tau_2}Bm_1).
\end{aligned}
$$

We substitute (5.6) into (5.5). We put $\Theta_1=(U-\mathrm{e}^{AT})^{-2}\mathrm{e}^{AT}$, $\Theta_2=(U-\mathrm{e}^{AT})^{-1}$, $\Psi_1=\mathrm{e}^{AT}Bm_1+\mathrm{e}^{A\tau_1}B(m_2-m_1)$, and $\Psi_2=\mathrm{e}^{AT}Bm_2+\mathrm{e}^{A\tau_2}B(m_1-m_2)$. Then

$$
\begin{aligned}
d(\tau_1,\tau_2) &= \langle\Gamma,(X^1)'_{\tau_1}\rangle\langle\Gamma,(X^2)'_{\tau_2}\rangle-\langle\Gamma,(X^1)'_{\tau_2}\rangle\langle\Gamma,(X^2)'_{\tau_1}\rangle\\
&= \langle\Gamma,\Theta_1\Psi_1\rangle\langle\Gamma,\Theta_2\Psi_2\rangle+\langle\Gamma,\Theta_1(-Bm_2)\rangle\langle\Gamma,\Theta_2\Psi_2\rangle+\langle\Gamma,\Theta_2\Psi_1\rangle\langle\Gamma,\Theta_2\Psi_2\rangle\\
&\quad - \langle\Gamma,\Theta_1\Psi_1\rangle\langle\Gamma,\Theta_2\mathrm{e}^{AT}Bm_2\rangle-\langle\Gamma,\Theta_1(-Bm_2)\rangle\langle\Gamma,\Theta_2\mathrm{e}^{AT}Bm_2\rangle\\
&\quad - \langle\Gamma,\Theta_2\mathrm{e}^{AT}Bm_1\rangle\langle\Gamma,\Theta_1\Psi_2\rangle-\langle\Gamma,\Theta_2\mathrm{e}^{AT}Bm_1\rangle\langle\Gamma,\Theta_1(-Bm_1)\rangle\\
&\quad - \langle\Gamma,\Theta_2\mathrm{e}^{AT}Bm_1\rangle\langle\Gamma,\Theta_2\mathrm{e}^{AT}Bm_2\rangle+\langle\Gamma,\Theta_2\Psi_1\rangle\langle\Gamma,\Theta_1\Psi_2\rangle\\
&\quad - \langle\Gamma,\Theta_2\Psi_1\rangle\langle\Gamma,\Theta_1 Bm_1\rangle.
\end{aligned}
$$

Whence, it follows that

$$
\begin{aligned}
(5.7)\qquad d(\tau_1,\tau_2) &= \langle\Gamma,\Theta_1(\Psi_1-Bm_2)\rangle\langle\Gamma,\Theta_2\mathrm{e}^{A\tau_2}B(m_1-m_2)\rangle\\
&\quad + \langle\Gamma,\Theta_2\mathrm{e}^{A\tau_1}B(m_2-m_1)\rangle\langle\Gamma,\Theta_1(\Psi_2-Bm_1)\rangle\\
&\quad + \langle\Gamma,\Theta_2\Psi_1\rangle\langle\Gamma,\Theta_2\Psi_2\rangle-\langle\Gamma,\Theta_2\mathrm{e}^{AT}Bm_1\rangle\langle\Gamma,\Theta_2\mathrm{e}^{AT}Bm_2\rangle\\
&= \langle\Gamma,\Theta_1(\mathrm{e}^{AT}Bm_1+\mathrm{e}^{A\tau_1}B(m_2-m_1)-Bm_2)\rangle\langle\Gamma,\Theta_2\mathrm{e}^{A\tau_2}B(m_1-m_2)\rangle\\
&\quad + \langle\Gamma,\Theta_2\mathrm{e}^{A\tau_1}B(m_2-m_1)\rangle\langle\Gamma,\Theta_1(\mathrm{e}^{AT}Bm_2+\mathrm{e}^{A\tau_2}B(m_1-m_2)-Bm_1)\rangle\\
&\quad + \langle\Gamma,\Theta_2(\mathrm{e}^{AT}Bm_1+\mathrm{e}^{A\tau_1}B(m_2-m_1))\rangle\langle\Gamma,\Theta_2(\mathrm{e}^{AT}Bm_2+\mathrm{e}^{A\tau_2}B(m_1-m_2))\rangle\\
&\quad - \langle\Gamma,\Theta_2\mathrm{e}^{AT}Bm_1\rangle\langle\Gamma,\Theta_2\mathrm{e}^{AT}Bm_2\rangle.
\end{aligned}
$$

If $d(\tau_1,\tau_2)\neq 0$, then the assumptions of the implicit function theorem are satisfied for

$$\tau_i(a_{jk},b_k,\gamma_k,m_1,m_2,l_1,l_2)$$

$(i=1,2,\ j,k=\overline{1,N})$. Theorem 5.1 is proved. □

E x a m p l e 5.1. Now we apply Theorem 5.1 to system (5.1) when $N = 2$. Let

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad \text{and} \quad F(\sigma) = \begin{cases} m_1 = -1 & \text{if } \sigma < l_2 = 1, \\ m_2 = 1 & \text{if } \sigma > l_1 = -1. \end{cases}$$

In (5.1), we replace the function $F(\sigma)$ by its values either m_1 or m_2 in accordance with the description of this function in Section 2. In the phase space, the switch surfaces as well as the virtual stability points of the system $\dot{X} = AX + Bm_i$ $(i = 1, 2)$ are symmetric with respect to the point $X = 0$. We have $A^{-1} = A$. According to Theorem 3.1, the feedback vector Γ has to meet the conditions $-\langle \Gamma, A^{-1}Bm_2 \rangle < l_1$, $-\langle \Gamma, A^{-1}Bm_1 \rangle > l_2$. From here it follows that the vector $\Gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$ has to satisfy the condition $\gamma_1 + \gamma_2 > 1$. We put $\gamma_1 = \gamma_2 = 1$. Since $\sigma = \langle \Gamma, X \rangle$, we have $\dot{\sigma} = \langle \Gamma, \dot{X} \rangle$. Therefore, in virtue of (5.1), we obtain $\dot{\sigma} = \langle \Gamma, AX \rangle + \langle \Gamma, B \rangle m_i$ $(i = 1, 2)$. Now we have the equation $\dot{\sigma} = \alpha\sigma + \langle \Gamma, B \rangle m_i$, where $\alpha = -1$, and its solution

$$\sigma(t) = \mathrm{e}^{\alpha(t - t_0)} \left(\sigma_0 + \langle \Gamma, B \rangle \int_{t_0}^{t} \mathrm{e}^{-\alpha(\tau - t_0)} m_i \, \mathrm{d}\tau \right), \tag{5.8}$$

where $\sigma_0 = \sigma(t_0)$. Let $t_0 = 0$ and $\sigma(0) = l_2 = 1$, $\sigma(\tau_1) = l_1 = -1$. Next we find the move time τ_1 from $\langle \Gamma, X \rangle = l_2 = 1$ to $\langle \Gamma, X \rangle = l_1 = -1$. Substituting the initial conditions into (5.8), we obtain $\tau_1 = \ln 3$. If $\sigma(0) = l_1$ and $\sigma(\tau_2) = l_2$, then $\tau_2 = \ln 3$ due to symmetry of the phase portrait. The period of the solution with two switch points is $T = \tau_1 + \tau_2 = 2 \ln 3$. Since $A^\top \Gamma = \alpha \Gamma$, such solution is unique. Uniqueness of the solution is true for any vector Γ satisfying the condition $\gamma_1 + \gamma_2 > 1$. Further, we have

$$\Theta_1 = \begin{pmatrix} (\mathrm{e}^{\ln 3} - \mathrm{e}^{-\ln 3})^{-2} & 0 \\ 0 & (\mathrm{e}^{\ln 3} - \mathrm{e}^{-\ln 3})^{-2} \end{pmatrix}, \quad \Psi_1 = \begin{pmatrix} (\mathrm{e}^{-\ln 3} - 2)\mathrm{e}^{-\ln 3} \\ (\mathrm{e}^{-\ln 3} - 2)\mathrm{e}^{-\ln 3} \end{pmatrix},$$

$$\Theta_2 = \begin{pmatrix} (1 - \mathrm{e}^{-2\ln 3})^{-1} & 0 \\ 0 & (1 - \mathrm{e}^{-2\ln 3})^{-1} \end{pmatrix}, \quad \Psi_2 = \begin{pmatrix} 2\mathrm{e}^{-\ln 3} - \mathrm{e}^{-2\ln 3} - 1 \\ 2\mathrm{e}^{-\ln 3} - \mathrm{e}^{-2\ln 3} - 1 \end{pmatrix},$$

and

$$\mathrm{e}^{AT} = \begin{pmatrix} \mathrm{e}^{-2\ln 3} & 0 \\ 0 & \mathrm{e}^{-2\ln 3} \end{pmatrix}.$$

We substitute the last expressions into (5.7). After some transformations, we obtain

$$d(\tau_1, \tau_2) = \frac{2\mathrm{e}^{-2\ln 3}}{(\mathrm{e}^{-\ln 3} - 1)(\mathrm{e}^{-\ln 3} + 1)^3} (\gamma_1 + \gamma_2)^2.$$

Then $d(\tau_1, \tau_2) = 0$ if $\gamma_1 + \gamma_2 = 0$. Except that, the vector Γ has to meet the condition $\gamma_1 + \gamma_2 > 1$. Therefore, for the vector Γ and corresponding move times τ_1, τ_2, we have $d(\tau_1, \tau_2) \neq 0$. The solutions with τ_1 and τ_2 describe the orbitally asymptotically stable closed trajectories with two switch points. Consequently, there exists a neighborhood of the point $(a_{jk}, b_k, \gamma_k, m_1, m_2, l_1, l_2)$ in which τ_1 and τ_2 are continuously dependent on the system parameters, where $j, k = 1, 2$. This means the robust asymptotic stability of unimodal periodic solutions.

6. Conclusion

In applied problems of nonlinear oscillation theory and control theory, much attention is paid to qualitative behaviour of the studied systems. Sufficiently small changes of the values of the system parameters should not lead to the qualitatively different splitting of the phase spaces. The results obtained give us a reliable sufficient condition for the boundedness of the solutions to the essentially nonlinear nonautonomous systems (Theorem 3.1). In case when such systems have periodic solutions, we have established the sufficient conditions under which the configurations of these solutions are independent of sufficiently small changes of the system parameters (Theorem 4.1, Theorem 5.1). Moreover, we have written out the formulae allowing applied researchers to limit the choice of the system parameters. In the supporting example we have shown the way how to apply these results to the real automatic control system with the nonlinearity as a control and how to adjust the settings to obtain the oscillations with desired properties.

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