LOGARITHMIC STABILIZATION OF THE KIRCHHOFF PLATE TRANSMISSION SYSTEM WITH LOCALLY DISTRIBUTED KELVIN-VOIGT DAMPING

Gimyong Hong, Hakho Hong, Pyongyang

Received April 14, 2020. Published online May 28, 2021.

Abstract. We are concerned with a transmission problem for the Kirchhoff plate equation where one small part of the domain is made of a viscoelastic material with the Kelvin-Voigt constitutive relation. We obtain the logarithmic stabilization result (explicit energy decay rate), as well as the wellposedness, for the transmission system. The method is based on a new Carleman estimate to obtain information on the resolvent for high frequency. The main ingredient of the proof is some careful analysis for the Kirchhoff transmission plate equation.

Keywords: transmission problem; Kirchhoff plate; Kelvin-Voigt damping; energy decay; Carleman estimate

MSC 2020: 35Q74, 93D15, 35L57, 74M05

1. Introduction

Engineering applications give rise to fluid-structure interactions, composite laminates in smart materials and structures, structural-acoustic systems, and other interactive physical processes, which are modeled by coupled partial differential equations (transmission systems). Control design and stability analysis for such systems have become active over the past decades: we refer to [4], [18], [35], [39] for the stability analysis of the heat-wave transmission system, to [17], [9], [6], [5], [25], [7], [8] for the uniform stabilization, polynomial stability and backward uniqueness of the fluid-structure transmission system, and to [37] for the stabilization of heat-plate transmission systems, respectively. We are interested in the transmission system of wave and/or plate equations, which attracts much attention and has strong physical backgrounds; for example, it can describe the displacement of flexible structures

DOI: 10.21136/AM.2021.0104-20

consisting of two physically different types of materials. One approach to the suppression of vibration of elastic structures is to bond patches made of special materials to the underlying structures as passive or active controllers. Due to the presence of the patches, the material properties of the structure, such as the elasticity moduli, damping coefficient, and Poisson ratio, are changed. In particular, the jump discontinuity at the location of the edges of the patches is usually introduced to these properties. This passive method, on the one hand, makes the distributed control practically applicable, but on the other hand, brings some new mathematical challenges which attract increasing research interests.

In recent years, the study of the stabilization problem for wave and/or plate transmission systems has drawn a lot of attention. The stabilization for the wave transmission system was discussed in [33], [16], [15], [10], [14], [34], [38]. For the 1D transmission system, Liu-Williams [33] and Bastos-Raposo [10] proved exponential decay under some conditions on the difference between the speeds of propagation. Later, Chai-Liu [15] and Chai [14] studied the stabilization and uniform decay rate for the wave transmission system with variable coefficients, respectively. Also, Chai-Liu-Liu [16] showed the stability for elastic systems with the global or local Kelvin-Voigt damping. Recently, Ramos-Souza [34] considered the equivalence between observability at the boundary and stabilization for the 1D transmission system, while Zhang [38] proved that the energy for a multi-dimensional elastic-viscoelastic wave transmission system does not decay exponentially.

Also, the stabilization for the wave/plate or string/beam transmission system was discussed in [1], [2], [22], [21], [30]. Ammari-Jellouli-Mehrenberger [1] studied the feedback stabilization for the 1D string/beam transmission system and recently, Li-Han-Xu [30] showed that the energy decay rate of this system depends on the location of frictional damping. Ammari-Nicaise [2] established the exponential stability for a damped wave equation coupled with a damped Kirchhoff plate equation under some geometric condition. Recently, Hassine [22] studied a polynomial stabilization for the 1D wave/plate transmission system with the local Kelvin-Voigt damping and Hassine [21] proved an exponential stability result for the multidimensional wave/plate transmission system.

Next, we discuss some recent results for the stabilization of a plate or beam transmission system this paper is concerned with. Liu-Liu [31] first obtained the exponential stability for the Euler-Bernoulli beam equation with the local Kelvin-Voigt damping (see also [32]). Recently, the same result was proved for the Euler-Bernoulli beam transmission equation in Hassine [20]. Ammari-Vodev [3] obtained the exponential result by a boundary stabilization for the Euler-Bernoulli plate transmission system. Recently, Hassine [23] proved the logarithmic stability for the Euler-Bernoulli plate transmission system with the local Kelvin-Voigt damping.

However, to our knowledge, very little is known about the stabilization for the Kirchhoff plate transmission system (see [24] for the Kirchhoff plate model and its boundary stabilization). The main purpose of this paper is to study this problem and as the first step toward this goal, we consider the following initial boundary value problem for the Kirchhoff plate transmission equation with the local Kelvin-Voigt damping:

$$
\begin{cases}
\partial_t^2(u_1-\beta\Delta u_1)+\Delta(c_1^2\Delta u_1+a\Delta\partial_t u_1)=0, & x\in\Omega_1,\ t>0,\\
\partial_t^2(u_2-\beta\Delta u_2)+\Delta(c_2^2\Delta u_2)=0, & x\in\Omega_2,\ t>0,\\
u_1=u_2,\quad \partial_\nu u_1=\partial_\nu u_2, & x\in S,\ t>0,\\
c_1^2\Delta u_1=c_2^2\Delta u_2,\quad \partial_\nu(c_1^2\Delta u_1)=\partial_\nu(c_2^2\Delta u_2), & x\in S,\ t>0,\\
u_2=0,\quad \Delta u_2=0, & x\in\Gamma,\ t>0,\\
u_1|_{t=0}=u_1^0(x),\quad \partial_t u_1|_{t=0}=u_1^1(x), & x\in\Omega_1,\ t>0,\\
u_2|_{t=0}=u_2^0(x),\quad \partial_t u_2|_{t=0}=u_2^1(x), & x\in\Omega_2,\ t>0,
\end{cases}
\tag{1.1}
$$

where Ω and Ω_1 are two open, bounded and connected domains in $\mathbb{R}^n$ $(n\geqslant 2)$ with smooth boundary (of C^∞-class) Γ and S, respectively, such that $\Omega_1\subset\Omega$ and $\overline{S}\cap\overline{\Gamma}=\emptyset$, $\Omega_2=\Omega\setminus\overline{\Omega}_1$ which is an open connected domain with the boundary $\partial\Omega_2=\Gamma\cup S$. Also, u_i $(i=1,2)$ denote the displacements of the plates at time t and position x, ν denotes the unit outward normal vector to Ω_2 and Ω on S and Γ, respectively, $c_k>0$ $(k=1,2)$ are positive constants, $a:=a(x)$ are non-negative bounded functions in Ω_1 and $\beta>0$ is a parameter in front of the inertial term.

We assume that a vanishes near the boundary S and there exists a non-empty open domain $\omega\subset\Omega_1$ such that $a\geqslant a_0$ in $\overline{\omega}$ for some strictly positive constant a_0, which implies that in a viscoelastic material with the Kelvin-Voigt constitutive relation, a transmission effect has been established in such a way that the damping is locally effective on only one side of the interface.

When $\beta=0$, the system (1.1) reduces to the Euler-Bernoulli plate transmission problem.

The energy of the solution of the system (1.1) at time $t\geqslant 0$ is defined by

$$
\begin{aligned}
E(t)=\frac{1}{2}\int_{\Omega_1}&(|\partial_t u_1(x,t)|^2+\beta|\partial_t\nabla u_1(x,t)|^2+c_1^2|\Delta u_1(x,t)|^2)\,\mathrm{d}x\\
&+\frac{1}{2}\int_{\Omega_2}(|\partial_t u_2(x,t)|^2+\beta|\partial_t\nabla u_2(x,t)|^2+c_2^2|\Delta u_2(x,t)|^2)\,\mathrm{d}x.
\end{aligned}
$$

By Green's formula we can prove that for all t_1, $t_2>0$ we have

$$
E(t_1)-E(t_2)=-\int_{t_1}^{t_2}\int_{\Omega_1}a(x)|\Delta\partial_t u_1(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t
$$

and this means that the energy is decreasing over the time.

We introduce the function spaces

$$H = L^2(\Omega_1) \times L^2(\Omega_2) \text{ with the norm } \|(u_1, u_2)\|_H^2 = \sum_{j=1}^{2} \int_{\Omega_j} u_j^2(x)\,\mathrm{d}x, \tag{1.2}$$

$$V = \{(u_1, u_2) \in H\colon\ u_1 \in H^1(\Omega_1),\ u_2 \in H^1(\Omega_2),\ u_2|_\Gamma = 0,\ u_1|_S = u_2|_S\} \tag{1.3}$$

$$\text{with the norm } \|(u_1, u_2)\|_V^2 = \sum_{j=1}^{2} \int_{\Omega_j} (u_j^2(x) + \beta|\nabla u_j(x)|^2)\,\mathrm{d}x,$$

$$W = \{(u_1, u_2) \in V\colon\ u_1 \in H^2(\Omega_1),\ u_2 \in H^2(\Omega_2),\ \partial_\nu u_1|_S = \partial_\nu u_2|_S\} \tag{1.4}$$

$$\text{with the norm } \|(u_1, u_2)\|_W^2 = \sum_{j=1}^{2} \int_{\Omega_j} c_j^2|\Delta u_j(x)|^2\,\mathrm{d}x.$$

Then H, V, and W are Hilbert spaces satisfying $W \subset_{\to}^{d} V \subset_{\to}^{d} H$, where $\subset_{\to}^{d}$ denotes continuous dense embedding (see Lemma 2.1). So, if we identify the Hilbert space H with its dual space H^*, then we have

$$W \subset_{\to}^{d} V \subset_{\to}^{d} H \subset_{\to}^{d} V^* \subset_{\to}^{d} W^*. \tag{1.5}$$

Next, we define linear and bounded operators $\mathcal{A}$, $\mathcal{B} \in \mathcal{L}(W, W^*)$ and $\mathcal{C} \in \mathcal{L}(V, V^*)$ by

$$\begin{aligned} \langle \mathcal{A}u, v\rangle_{W^*,W} &= \sum_{j=1}^{2} \int_{\Omega_j} c_j^2 \Delta u_j(x)\overline{\Delta v_j(x)}\,\mathrm{d}x, \\ \langle \mathcal{B}u, v\rangle_{W^*,W} &= \int_{\Omega_1} a(x)\Delta u_1(x)\overline{\Delta v_1(x)}\,\mathrm{d}x \end{aligned} \tag{1.6}$$

for $u = (u_1, u_2)$, $v = (v_1, v_2) \in W$ and

$$\langle \mathcal{C}u, v\rangle_{V^*,V} = \sum_{k=1}^{2} \int_{\Omega_k} (u_k\overline{v_k} + \beta\nabla u_k \cdot \overline{\nabla v_k})\,\mathrm{d}x \tag{1.7}$$

for $u = (u_1, u_2)$, $v = (v_1, v_2) \in V$, respectively, where $\langle\cdot,\cdot\rangle_{X^*,X}$ denotes the duality pairing between X and X^*. It is easy to check that the operator $\mathcal{C}$ is the isomorphism from V onto V^* by the Lax-Milgram theorem.

Now we are able to state our main results. To this end, we define the operator $\mathbb{A}$ by

$$\mathbb{A}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\mathcal{C}^{-1}(\mathcal{A}u + \mathcal{B}v) \end{pmatrix}, \quad (u, v) \in D(\mathbb{A}), \tag{1.8}$$

in the Hilbert space $\mathcal{H} = W \times V$ with the domain

$$D(\mathbb{A}) = \{(u, v) \in W \times W\colon\ \mathcal{A}u + \mathcal{B}v \in V^*\}. \tag{1.9}$$

Then, we have the following resolvent estimate.

Theorem 1.1. *There exist positive constants $C > 0$ and $c > 0$ such that for every $\mu \in \mathbb{R}$ with $|\mu|$ large, we have*

$$\|(\mathbb{A} - \mathrm{i}\mu I_{\mathcal{H}})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant C\mathrm{e}^{c|\mu|}, \tag{1.10}$$

where $I_{\mathcal{H}}$ is the identity operator in the space $\mathcal{H}$.

As an immediate consequence of the previous theorem (for example [11], Theorem 1.5 or [13]), we get the following rate of decrease of energy.

Theorem 1.2. *Assume that $\mathrm{i}\mathbb{R} \cap \sigma(\mathbb{A}) = \emptyset$, where $\sigma(\mathbb{A})$ denotes the spectrum set of $\mathbb{A}$, and (1.10) holds. Then for any $l \in \mathbb{N}$, there exists a constant $C > 0$ such that for any initial data $(u^0, u^1) \in D(\mathbb{A}^l)$, the energy $E(t)$ of the system (1.1) whose solution $u(x,t)$ starts from (u^0, u^1) satisfies*

$$E(t) \leqslant \frac{C}{(\ln(2+t))^{2l}} \|(u^0, u^1)\|^2_{D(\mathbb{A}^l)},$$

where $u^0 = (u_1^0, u_2^0)$ and $u^1 = (u_1^1, u_2^1)$.

Now we make some comments on the analysis in this paper. We aim to discuss the stabilization of a transmission system of coupling plate equations with the Kelvin-Voigt damping, which is one type of the viscoelastic damping. Here, we would like to emphasize that the operator corresponding to the Kelvin-Voigt damping is unbounded on the underlying space and is not a lower-order perturbation of the elastic operator. Compared with the case [23] of the Euler-Bernoulli plate transmission system, the main difficulties are due to the appearance of $-\beta\Delta\partial_t^2 u_j$ $(j = 1, 2)$ in $(1.1)_{1,2}$, which consists of terms including the fourth order derivative with respect to spatial and time variable. In order to show the existence of mild solutions for the C_0-semigroup of contractions, we need to consider the operator equation of the type $\mathcal{C}u_{tt} + \mathcal{B}u_t + \mathcal{A}u = 0$, where $\mathcal{C}$ is not the identity operator, which is the first difficulty. To circumvent the difficulty, we construct function spaces V and W with new equivalent norms and it is essential to prove that the operator $\mathbb{A}$ is a generator of a C_0-semigroup of contractions in $W \times V$ (see Theorem 2.1). Next, to prove the resolvent estimation (1.10), we should obtain Carleman type estimates for a new system (3.18), which is different from the case of $\beta = 0$. To this end, we transform the fourth order system (3.18) into the second order system (3.22), add the terms for $\beta > 0$, and obtain the estimate of the solution for the system (3.22) (see Lemma 3.1).

The outline of this paper is as follows. In Section 2 we prove the existence and uniqueness of solution to the problem (1.1). Section 3 is mainly devoted to the resolvent estimate given by Theorem 1.1.

2. Existence and uniqueness

This section is devoted to the existence, uniqueness and regularity of solutions to the system (1.1).

2.1. Reduction to an operator differential equation. First, we have

Lemma 2.1. *Let H, V and W be the function spaces defined in* (1.2)–(1.4), *respectively. Then they are Hilbert spaces satisfying $W \subset^d_{\to} V \subset^d_{\to} H$.*

Proof. It is obvious that H is Hilbert space.

We prove that V is a Hilbert space and $V \subset^d_{\to} H$. It is sufficient to prove that $u \in H^1_0(\Omega)$ and $\|(u_1, u_2)\|_V$ is equivalent to $\|u\|_{H^1(\Omega)}$ when $(u_1, u_2) \in V$, where

$$u(x) = \begin{cases} u_1(x), & x \in \Omega_1, \\ u_2(x), & x \in \Omega_2. \end{cases}$$

In fact, using that $u_1 \in H^1(\Omega_1)$, $u_2 \in H^1(\Omega_2)$ and $u_1|_S = u_2|_S$, we obtain $u \in H^1(\Omega)$, which implies $u \in H^1_0(\Omega)$ together with $u|_\Gamma = 0$. The equivalence of norms is obtained from

$$\min\{1, \beta\}\|u\|^2_{H^1(\Omega)} \leqslant \sum_{j=1}^{2} \int_{\Omega_j} (u_j^2 + \beta|\nabla u_j|^2)\,\mathrm{d}x \leqslant \max\{1, \beta\}\|u\|^2_{H^1(\Omega)}.$$

Next, we prove that W is Hilbert space and $W \subset^d_{\to} V$.

It also is sufficient to prove that $u \in H^2(\Omega) \cap H^1_0(\Omega)$ and $\|(u_1, u_2)\|_W$ is equivalent to $\|u\|_{H^2(\Omega)}$ when $(u_1, u_2) \in W$, where

$$u(x) = \begin{cases} u_1(x), & x \in \Omega_1, \\ u_2(x), & x \in \Omega_2. \end{cases}$$

Obviously, $W \subset V$ and $u \in H^1_0(\Omega)$.

As $u_1|_S = u_2|_S$ and $\partial_\nu u_1|_S = \partial_\nu u_2|_S$, the direct calculation shows

$$\begin{aligned} \int_\Omega u \frac{\partial^2 \phi}{\partial x_i \partial x_j}\,\mathrm{d}x &= \int_{\Omega_1} u_1 \frac{\partial^2 \phi}{\partial x_i \partial x_j}\,\mathrm{d}x + \int_{\Omega_2} u_2 \frac{\partial^2 \phi}{\partial x_i \partial x_j}\,\mathrm{d}x \\ &= -\int_{\Omega_1} \frac{\partial u_1}{\partial x_i}\frac{\partial \phi}{\partial x_j}\,\mathrm{d}x - \int_{\Omega_2} \frac{\partial u_2}{\partial x_i}\frac{\partial \phi}{\partial x_j}\,\mathrm{d}x \\ &= \int_{\Omega_1} \frac{\partial^2 u_1}{\partial x_i \partial x_j}\phi\,\mathrm{d}x + \int_{\Omega_2} \frac{\partial^2 u_2}{\partial x_i \partial x_j}\phi\,\mathrm{d}x \end{aligned}$$

for $\phi \in C_0^\infty(\Omega)$ and $i, j = 1, \dots, n$, which means

$$\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(\Omega)$$

and

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \begin{cases} \dfrac{\partial^2 u_1(x)}{\partial x_i \partial x_j}, & x \in \Omega_1, \\ \dfrac{\partial^2 u_2(x)}{\partial x_i \partial x_j}, & x \in \Omega_2. \end{cases} \tag{2.1}$$

Therefore, $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

To prove the equivalence of $\|(u_1, u_2)\|_W$ and $\|u\|_{H^2(\Omega)}$, it is sufficient to check that

$$\exists\, m > 0\ \forall\, (u_1, u_2) \in W, \quad \sum_{j=1}^{2} \int_{\Omega_j} c_j^2 |\Delta u_j(x)|^2 \,\mathrm{d}x \geqslant m \|u\|_{H^2(\Omega)}^2. \tag{2.2}$$

Using (2.1) and the estimate $\|y\|_{H^2(\Omega)} \leqslant M \|f\|_{L^2(\Omega)}$ for the solution y to the boundary value problem

$$\begin{cases} -\Delta y = f \in L^2(\Omega) & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma, \end{cases}$$

we have

$$\sum_{j=1}^{2} \int_{\Omega_j} c_j^2 |\Delta u_j(x)|^2 \,\mathrm{d}x \geqslant \min\{c_1^2, c_1^2\} \|\Delta u\|_{L^2(\Omega)}^2 \geqslant \frac{\min\{c_1^2, c_1^2\}}{M^2} \|u\|_{H^2(\Omega)}^2,$$

which implies (2.2). The proof of the lemma is completed. □

Next, we derive the second order operator differential equation from the system (1.1).

Let (u_1, u_2) be the classical solution to the system (1.1). Then, multiplying $(1.1)_1$ and $(1.1)_2$ by $\overline{\phi_1}$ and $\overline{\phi_2}$ with $(\phi_1, \phi_2) \in W$, respectively, and integrating them over Ω_j $(j = 1, 2)$ to add the resulting equalities yields

$$\sum_{j=1}^{2} \Big(\int_{\Omega_j} \partial_t^2 (u_j \overline{\phi_j} - \beta \Delta u_j \overline{\phi_j}) \,\mathrm{d}x + \int_{\Omega_j} \Delta(c_j^2 \Delta u_j) \overline{\phi_j} \,\mathrm{d}x \Big) + \int_{\Omega_1} \Delta(a \partial_t \Delta u_1) \overline{\phi_1} \,\mathrm{d}x = 0. \tag{2.3}$$

By $(1.1)_3$–$(1.1)_5$ and $\mathrm{supp}(a) \subset \Omega_1$, we have

$$
\begin{aligned}
-\sum_{j=1}^{2} \int_{\Omega_j} \partial_t^2 \Delta u_j \overline{\phi_j}\,\mathrm{d}x &= \sum_{j=1}^{2} \int_{\Omega_j} \nabla \partial_t^2 u_j \cdot \overline{\nabla \phi_j}\,\mathrm{d}x, \\
\sum_{j=1}^{2} \int_{\Omega_j} \Delta(c_j^2 \Delta u_j) \overline{\phi_j}\,\mathrm{d}x &= \sum_{j=1}^{2} \int_{\Omega_j} c_j^2 \Delta u_j \overline{\Delta \phi_j}\,\mathrm{d}x, \\
\int_{\Omega_1} \Delta(a \partial_t \Delta u_1) \overline{\phi_1}\,\mathrm{d}x &= \int_{\Omega_1} a \Delta \partial_t u_1 \overline{\Delta \phi_1}\,\mathrm{d}x.
\end{aligned} \tag{2.4}
$$

Substituting (2.4) into (2.3) yields

$$
\begin{aligned}
\sum_{j=1}^{2} \int_{\Omega_j} (\partial_t^2 u_j \overline{\phi_j} + \beta \nabla \partial_t^2 u_j \cdot \overline{\nabla \phi_j})\,\mathrm{d}x \\
+ \sum_{j=1}^{2} \int_{\Omega_j} c_j^2 \Delta u_j \overline{\Delta \phi_j}\,\mathrm{d}x + \int_{\Omega_1} a \Delta \partial_t u_1 \overline{\Delta \phi_1}\,\mathrm{d}x = 0.
\end{aligned} \tag{2.5}
$$

By (2.5), (1.6) and (1.7), we have

$$
\langle \mathcal{C} u_{tt}, \phi \rangle_{V^*,V} + \langle \mathcal{A} u, \phi \rangle_{W^*,W} + \langle \mathcal{B} u_t, \phi \rangle_{W^*,W} = 0
$$

for $\phi = (\phi_1, \phi_2) \in W$, which means the operator differential equation of the second order

$$
\mathcal{C} u_{tt} + \mathcal{A} u + \mathcal{B} u_t = 0 \quad \text{in } W^*. \tag{2.6}
$$

Noticing that the operator $\mathcal{C}$ is an isomorphism from V onto V^*, we rewrite (2.6) as

$$
u_{tt} + \mathcal{C}^{-1}(\mathcal{A} u + \mathcal{B} u_t) = 0 \quad \text{in } V \tag{2.7}
$$

for $(u, u_t) \in D(\mathbb{A})$ (see (1.9)).

Setting $v = u_t$ and using (1.8), the equation (2.7) is reduced to the system of first order operator differential equations

$$
\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \mathbb{A} \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{in } W \times V.
$$

2.2. Existence and uniqueness of a mild solution. The main result of this subsection is

Theorem 2.1. *The operator $\mathbb{A}$ defined in* (1.8) *generates a C_0-semigroup of contractions in $\mathcal{H} = W \times V$.*

P r o o f. According to the Lumer-Phillips theorem (see for example [19]) we only have to show that $\mathbb{A}$ is m-dissipative.

Let $(u,v) \in D(\mathbb{A})$. Then by (1.8), $\mathcal{C} \in \mathrm{Isom}(V, V^*)$ and (1.3), we have

$$\left\langle \mathbb{A}\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_{\mathcal{H}} = \langle v, u\rangle_W - \left\langle \mathcal{C}^{-1}(\mathcal{A}u + \mathcal{B}v), v\right\rangle_V = \langle v, u\rangle_W - \langle \mathcal{A}u, v\rangle_{W^*,W} - \langle \mathcal{B}v, v\rangle_{W^*,W}, \tag{2.8}$$

where $\langle \cdot, \cdot \rangle_X$ is the scalar product in the Hilbert space X.

By (1.4) and (1.6), we obtain

$$\begin{aligned} \mathrm{Re}\,\langle v, u\rangle_W &= \mathrm{Re} \sum_{j=1}^{2} \int_{\Omega_j} c_j^2 \Delta v_j(x) \overline{\Delta u_j(x)}\,\mathrm{d}x \\ &= \mathrm{Re} \sum_{j=1}^{2} \int_{\Omega_j} c_j^2 \Delta u_j(x) \overline{\Delta v_j(x)}\,\mathrm{d}x = \mathrm{Re}\,\langle \mathcal{A}u, v\rangle_{W^*,W}, \\ \langle \mathcal{B}v, v\rangle_{W^*,W} &= \int_{\Omega_1} a(x) |\Delta v_1(x)|^2\,\mathrm{d}x \geqslant 0. \end{aligned} \tag{2.9}$$

By (2.8) and (2.9), we get

$$\mathrm{Re}\left\langle \mathbb{A}\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_{\mathcal{H}} \leqslant 0,$$

which implies that $\mathbb{A}$ is dissipative.

So, to show that $\mathbb{A}$ is m-dissipative we find $(u,v) \in D(\mathbb{A})$ such that

$$(I_{\mathcal{H}} - \mathbb{A})\begin{pmatrix} u \\ v \end{pmatrix} \equiv \begin{pmatrix} u - v \\ v + \mathcal{C}^{-1}(\mathcal{A}u + \mathcal{B}v) \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \quad \text{in } W \times V$$

for any $(f,g) \in W \times V$. It is sufficient to prove that there exists $v \in W$ such that

$$\mathcal{C}v + \mathcal{A}v + \mathcal{B}v = g \quad \text{in } V^* \tag{2.10}$$

for any $g \in V^*$. By (1.5)–(1.7) and $(2.9)_2$, we have

$$\begin{aligned} \langle \mathcal{C}w + \mathcal{A}w + \mathcal{B}w, w\rangle_{W^*,W} &= \langle \mathcal{C}w, w\rangle_{V^*,V} + \langle \mathcal{A}w, w\rangle_{W^*,W} + \langle \mathcal{B}w, w\rangle_{W^*,W} \\ &\geqslant \|w\|_V^2 + \|w\|_W^2 \geqslant m_1 \|w\|_W^2 \end{aligned} \tag{2.11}$$

for any $w \in W$, where $m_1 > 0$ is a positive constant independent of w. Using (2.11) and the Lax-Milgram theorem, we have (2.10). The proof of the theorem is completed. $\square$

A consequence of Theorem 2.1 is that, if we assume that $(u^0, v^1) \in D(\mathbb{A})$, there exists the unique solution to the system (1.1) which can be expressed by means of a semigroup on $\mathcal{H}$ as

$$\begin{pmatrix} u \\ u_t \end{pmatrix} = \mathrm{e}^{t\mathbb{A}} \begin{pmatrix} u^0 \\ u^1 \end{pmatrix}, \tag{2.12}$$

where $\mathrm{e}^{-t\mathbb{A}}$ is the C_0-semigroup of contractions generated by the operator $-\mathbb{A}$, $u = (u_1, u_2)$, $u^0 = (u_1^0, u_2^0)$ and $u^1 = (u_1^1, u_2^1)$. Moreover, we have the regularity of the solution

$$\begin{pmatrix} u \\ u_t \end{pmatrix} \in C([0,\infty); D(\mathbb{A})) \cap C^1((0,\infty); \mathcal{H}).$$

Besides, if $(u^0, u^1) \in \mathcal{H}$, then the function (u, u_t) given by (2.12) is the mild solution of the system (1.1), $(u, u_t)(t) \in D(\mathbb{A})$ for all $t > 0$ and $\begin{pmatrix} u \\ u_t \end{pmatrix} \in C([0,\infty); \mathcal{H})$.

3. Resolvent estimate

3.1. Carleman estimate and construction of weight functions. In this subsection, we give the Carleman estimate and construction of the weight functions, which plays an important role in the proof of Theorem 1.1.

We first recall the local Carleman estimate at the interface described in [29] by Le Rousseau-Robbiano.

In the neighborhood of a point (y_0, y) of $(0,1) \times S$, we denote by x_d the variable that is normal to the interface S and by x' the remaining spatial variables, that is, $x = (x', x_d)$. In particular, $y = (y', 0)$. The interface is now given by $S = \{x; x_d = 0\}$.

In a sufficiently small neighborhood $V \subset \mathbb{R}^{d+1}$ of (y_0, y), we employ normal geodesic coordinates (with respect to the spatial variables x). For convenience, we take the neighborhood V of the form $(y_0 - \varepsilon, y_0 + \varepsilon) \times V_{y'} \times (-\varepsilon, \varepsilon)$, where $V_{y'}$ is a sufficiently small neighborhood of y'. We use the notations

$$\mathbb{R}^{d+1}_+ = \{(x_0, x);\ x_d > 0\},\ \mathbb{R}^{d+1}_- = \{(x_0, x);\ x_d < 0\},\ V^+ = V \cap \mathbb{R}^{d+1}_+,\ V^- = V \cap \mathbb{R}^{d+1}_-.$$

We introduce the operator

$$A(x_0, x, D) := \begin{cases} A_1(x_0, x, D) = -\partial_{x_0}^2 - \partial_{x_d}^2 + R_1\Big(x, \dfrac{\partial_{x'}}{\mathrm{i}}\Big), & x_d > 0, \\ A_2(x_0, x, D) = -\partial_{x_0}^2 - \partial_{x_d}^2 + R_2\Big(x, \dfrac{\partial_{x'}}{\mathrm{i}}\Big), & x_d < 0, \end{cases}$$

where

$$R(x, \xi') = \begin{cases} R_1(x, \xi'), & x_d > 0, \\ R_2(x, \xi'), & x_d < 0, \end{cases}$$

is a second order polynomial in ξ' with coefficients in $\mathbb{R}$ with the principal symbol

$$r(x,\xi') = \begin{cases} r_1(x,\xi'), & x_d > 0, \\ r_2(x,\xi'), & x_d < 0, \end{cases} \text{ satisfying } \begin{cases} r_1(x,\xi') \geqslant C|\xi'|^2 & \forall\, x_d > 0\ \forall\, \xi' \in \mathbb{R}^{d-1}, \\ r_2(x,\xi') \geqslant C|\xi'|^2 & \forall\, x_d < 0\ \forall\, \xi' \in \mathbb{R}^{d-1}. \end{cases}$$

Then we consider the transmission problem

$$\begin{cases} A(x_0,x,D)w = f \quad \forall\, x_d \neq 0, \\ w(x_0,x',0^+) = w(x_0,x',0^-) + \theta, \\ \partial_\nu w(x_0,x',0^+) = \partial_\nu w(x_0,x',0^-) + \Theta, \end{cases} \tag{3.1}$$

where θ, Θ are error terms and ν is the unit outward normal vector of $\mathbb{R}^{d+1}_+$ and $\mathbb{R}^{d+1}_-$ on the interface S.

Let φ be a weight function and we define on both sides of S the conjugate operator

$$A_\varphi = h^2 \mathrm{e}^{\varphi/h} A \mathrm{e}^{-\varphi/h}$$

with a small semi-classical parameter and the principal symbol

$$a_\varphi(x_0,x,\xi) = \begin{cases} (\xi_0 + \mathrm{i}\partial_{x_0}\varphi)^2 + (\xi_n + \mathrm{i}\partial_{x_n}\varphi)^2 + r_1(x,\xi' + \mathrm{i}\partial_{x'}\varphi), & x_d > 0, \\ (\xi_0 + \mathrm{i}\partial_{x_0}\varphi)^2 + (\xi_n + \mathrm{i}\partial_{x_n}\varphi)^2 + r_2(x,\xi' + \mathrm{i}\partial_{x'}\varphi), & x_d < 0. \end{cases}$$

Assumptions: we suppose that the weight function φ is in $C^\infty(\overline{V})$, $\phi|_{\overline{\mathbb{R}^{d+1}_+}} \in C^\infty(\overline{V}^+)$, $\phi|_{\overline{\mathbb{R}^{d+1}_-}} \in C^\infty(\overline{V}^-)$ and such that

1. $|\nabla_{(x_0,x)}\varphi|(x_0,x) > 0$ in $\overline{V}$.
2. For all x_0 and x,

$$\partial_\nu\varphi(x_0,x',0^+) > 0 \ \text{ and } \ \partial_\nu\varphi(x_0,x',0^-) > 0, \ \ \partial_\nu\varphi(x_0,x',0^+) - \partial_\nu\varphi(x_0,x',0^-) > 1.$$

3. The sub-ellipticity condition

$$\forall\,(x_0,x,\xi_0,\xi) \in \overline{V}^\pm \times \mathbb{R}^{n+1};\ a_{\varphi_1}(x_0,x,\xi_0,\xi) = 0 \Rightarrow \{\mathrm{Re}(a_{\varphi_1}), \mathrm{Im}(a_{\varphi_1})\}(x,\xi) > 0.$$

Then, we have

Proposition 3.1 ([29], Theorem 2.1)**.** *Let K be a compact subset of V and φ a weight function satisfying the above assumptions. Then there exist positive constants $h_0 > 0$ and $C > 0$ such that*

$$\begin{aligned} h\|\mathrm{e}^{\varphi/h}w\|^2_{L^2(K)} &+ h^3\|\mathrm{e}^{\varphi/h}\nabla_{(x_0,x)}w\|^2_{L^2(K)} + h\|\mathrm{e}^{\varphi/h}w\|^2_{L^2(K_S)} \\ &+ h^3\|\mathrm{e}^{\varphi/h}\nabla_{(x_0,x)}w\|^2_{L^2(K_S)} + h^3\|\mathrm{e}^{\varphi/h}\partial_\nu w\|^2_{L^2(K_S)} \\ &\leqslant C(h^4\|\mathrm{e}^{\varphi/h}f\|^2_{L^2(K)} + h\|\mathrm{e}^{\varphi_1/h}\theta\|^2_{L^2(K_S)} \\ &\qquad + h^3\|\mathrm{e}^{\varphi/h}\partial_{x_0,x'}\theta\|^2_{L^2(K_S)} + h^3\|\mathrm{e}^{\varphi/h}\Theta\|^2_{L^2(K_S)}) \end{aligned}$$

for all $h \in (0, h_0]$ *and* w *satisfying the system* (3.1)*, where* $w|_{\overline{\mathbb{R}^{d+1}_+}} \in C^\infty(K^+)$, $w|_{\overline{\mathbb{R}^{d+1}_-}} \in C^\infty(K^-)$ *and* $K_S = K \cap S$.

Next, we give the Carleman estimate needed for the resolvent estimate. To this end, we consider two open and disjoint domains $\mathcal{O}_1$ and $\mathcal{O}_2$ in which we define the second order elliptic semi-classical operators $P_1 = -h^2\Delta - \alpha_1 h$ and $P_2 = -h^2\Delta - \alpha_2 h$, respectively, with the principal symbol $p(x,\xi) = |\xi|^2$, where $h > 0$ is a very small semi-classical parameter and α_1, α_2 are two positive constants, and we suppose that

$$\partial\mathcal{O}_1 = \gamma \cup \gamma_1, \quad \partial\mathcal{O}_2 = \gamma \cup \gamma_2 \quad \text{and} \quad \overline{\gamma} \cap \overline{\gamma}_1 = \overline{\gamma}_2 \cap \overline{\gamma} = \emptyset.$$

Let $\varphi_1 \in C^\infty(\overline{\mathcal{O}}_1)$ and $\varphi_2 \in C^\infty(\overline{\mathcal{O}}_2)$ be two real valued functions. We define two adjoint operators $P_{\varphi_1} = \mathrm{e}^{\varphi_1/h} P_1 \mathrm{e}^{\varphi_1/h}$ and $P_{\varphi_2} = \mathrm{e}^{\varphi_2/h} P_2 \mathrm{e}^{\varphi_2/h}$ of the principal symbols $p_{\varphi_1}(x,\xi) = p(x, \xi + \mathrm{i}\nabla\varphi_1)$ and $p_{\varphi_2}(x,\xi) = p(x, \xi + \mathrm{i}\nabla\varphi_2)$, respectively.

By denoting ν the unit outward normal vector to $\mathcal{O}_1$ and $\mathcal{O}_2$ on $\gamma \cup \gamma_1$ and γ_2, respectively, we assume that the weight functions φ_1 and φ_2 satisfy Hörmander's condition

(1) $|\nabla\varphi_1|(x) > 0$ for all $x \in \mathcal{O}_1$ and $|\nabla\varphi_2|(x) > 0$ for all $x \in \mathcal{O}_2$.
(2) $\partial_\nu\varphi_1|_{\gamma_1} \neq 0$ and $\partial_\nu\varphi_2|_{\gamma_2} < 0$.
(3) $\varphi_1|_\gamma = \varphi_2|_\gamma$.
(4) $\partial_\nu\varphi_1|_\gamma < 0$, $\partial_\nu\varphi_2|_\gamma < 0$ and $(\partial_\nu\varphi_1)^2|_\gamma - (\partial_\nu\varphi_2)^2|_\gamma > 0$.
(5) the sub-ellipticity conditions in $\mathcal{O}_1$ and $\mathcal{O}_2$,

$$\forall\,(x,\xi) \in \overline{\mathcal{O}}_1 \times \mathbb{R}^n;\ p_{\varphi_1}(x,\xi) = 0 \Rightarrow \{\mathrm{Re}(p_{\varphi_1}), \mathrm{Im}(p_{\varphi_1})\}(x,\xi) > 0,$$
$$\forall\,(x,\xi) \in \overline{\mathcal{O}}_2 \times \mathbb{R}^n;\ p_{\varphi_2}(x,\xi) = 0 \Rightarrow \{\mathrm{Re}(p_{\varphi_2}), \mathrm{Im}(p_{\varphi_2})\}(x,\xi) > 0,$$

respectively. Then, we consider the transmission boundary value problem

$$\begin{cases} -\Delta w_1 - \dfrac{\alpha_1^2}{h^2} w_1 = f_1 & \text{in } \mathcal{O}_1, \\ -\Delta w_2 - \dfrac{\alpha_2^2}{h^2} w_2 = f_2 & \text{in } \mathcal{O}_2, \\ w_1 = w_2 & \text{on } \gamma, \\ \partial_\nu w_1 = \partial_\nu w_2 & \text{on } \gamma, \\ w_2 = 0 & \text{on } \gamma_2, \end{cases} \tag{3.2}$$

to which the Carleman estimate given by the following proposition belongs.

Proposition 3.2. *Under the above assumptions on the weight functions φ_1 and φ_2, there exist positive constants $h_0 > 0$ and $C > 0$ such that*

$$
\begin{aligned}
(3.3)\quad & \sum_{j=1}^{2}(h\|\mathrm{e}^{\varphi_j/h}w_j\|^2_{L^2(\mathcal{O}_j)} + h^3\|\mathrm{e}^{\varphi_j/h}\nabla w_j\|^2_{L^2(\mathcal{O}_j)}) \\
& \quad + \sum_{j=1}^{2}(h\|\mathrm{e}^{\varphi_j/h}w_j\|^2_{L^2(\gamma)} + h^3\|\mathrm{e}^{\varphi_j/h}\nabla w_j\|^2_{L^2(\gamma)} + h^3\|\mathrm{e}^{\varphi_j/h}\partial_\nu w_j\|^2_{L^2(\gamma)}) \\
& \leqslant C\Big(\sum_{j=1}^{2} h^4\|\mathrm{e}^{\varphi_j/h}f_j\|^2_{L^2(\mathcal{O}_j)} + h\|\mathrm{e}^{\varphi_1/h}w_1\|^2_{L^2(\gamma_1)} + h^3\|\mathrm{e}^{\varphi_1/h}\partial_\nu w_1\|^2_{L^2(\gamma_1)}\Big)
\end{aligned}
$$

for all $w_j \in C^\infty(\overline{\mathcal{O}}_j)$ satisfying the system (3.2) *and $h \in (0, h_0]$ $(j = 1, 2)$.*

P r o o f. Setting

$$
v_i(x_0, x) = \mathrm{e}^{\alpha_i x_0/h}w_i(x), \quad i = 1, 2,
$$

where $x_0 \in (0, 1)$ is an additional variable, the system (3.2) is changed into the system

$$
(3.4)\qquad \begin{cases}
-\Delta v_1 - \partial_{x_0} v_1 = f_1^{x_0} & \text{in } \mathcal{O}_1^{x_0} = \mathcal{O}_1 \times (0,1), \\
-\Delta v_2 - \partial_{x_0} v_2 = f_2^{x_0} & \text{in } \mathcal{O}_2^{x_0} = \mathcal{O}_2 \times (0,1), \\
v_1 = v_2 & \text{on } \gamma^{x_0} = \gamma \times (0,1), \\
\partial_\nu v_1 = \partial_\nu v_2 & \text{on } \gamma^{x_0} = \gamma \times (0,1), \\
v_2 = 0 & \text{on } \gamma_2^{x_0} = \gamma_2 \times (0,1),
\end{cases}
$$

where $f_i^{x_0} = \mathrm{e}^{\alpha_i x_0/h} f_i$ $(i = 1, 2)$.

We apply Proposition 3.1 to the system (3.4) by taking into account [27], Proposition 1 and [26], Proposition 2. Then we get

$$
\begin{aligned}
(3.5)\qquad & \sum_{j=1}^{2}(h\|\mathrm{e}^{\varphi_j^{x_0}/h}v_j\|^2_{L^2(\mathcal{O}_j^{x_0})} + h^3\|\mathrm{e}^{\varphi_j^{x_0}/h}\nabla_{(x,x_0)}v_j\|^2_{L^2(\mathcal{O}_j^{x_0})}) \\
& \quad + \sum_{j=1}^{2}(h\|\mathrm{e}^{\varphi_j^{x_0}/h}v_j\|^2_{L^2(\gamma^{x_0})} + h^3\|\mathrm{e}^{\varphi_j^{x_0}/h}\nabla_{(x',x_0)}v_j\|^2_{L^2(\gamma^{x_0})} \\
& \qquad\qquad + h^3\|\mathrm{e}^{\varphi_j^{x_0}/h}\partial_\nu v_j\|^2_{L^2(\gamma^{x_0})}) \\
& \leqslant C\Big(\sum_{j=1}^{2} h^4\|\mathrm{e}^{\varphi_j^{x_0}/h}f_j^{x_0}\|^2_{L^2(\mathcal{O}_j^{x_0})} + h\|\mathrm{e}^{\varphi_1^{x_0}/h}v_1\|^2_{L^2(\gamma_1^{x_0})} \\
& \qquad\qquad + h^3\|\mathrm{e}^{\varphi_1^{x_0}/h}\partial_\nu v_1\|^2_{L^2(\gamma_1^{x_0})}\Big)
\end{aligned}
$$

with the weight functions $\varphi_j^{x_0} = \varphi_j - \alpha_j x_0$, where $\gamma_1^{x_0} = \gamma_1 \times (0,1)$, from which (3.3) follows immediately. Indeed, one first chooses the partition of unity (ζ_i) on some neighborhood of $\partial\mathcal{O}_1^{x_0}$ and $\partial\mathcal{O}_2^{x_0}$ such that any element of this partition ζ belongs to one of the following cases:

(i) $\operatorname{supp}(\zeta) \cap \gamma_1^{x_0} \neq \emptyset$, $\operatorname{supp}(\zeta) \cap \gamma_2^{x_0} = \emptyset$ and $\operatorname{supp}(\zeta) \cap \gamma^{x_0} = \emptyset$.
(ii) $\operatorname{supp}(\zeta) \cap \gamma_2^{x_0} \neq \emptyset$, $\operatorname{supp}(\zeta) \cap \gamma_1^{x_0} = \emptyset$ and $\operatorname{supp}(\zeta) \cap \gamma^{x_0} = \emptyset$.
(iii) $\operatorname{supp}(\zeta) \cap \gamma^{x_0} \neq \emptyset$, $\operatorname{supp}(\zeta) \cap \gamma_1^{x_0} = \emptyset$ and $\operatorname{supp}(\zeta) \cap \gamma_2^{x_0} = \emptyset$.

Next, if $\operatorname{supp}(\zeta)$ is chosen sufficiently small, one defines $\zeta \cdot v$. Working in local coordinates, we may apply to function $\zeta \cdot v$

- ▷ in case (i) [27], Proposition 1 where especially we need the assumption $\partial_\nu \varphi_1|_{\gamma_1} \neq 0$,
- ▷ in case (ii) [26], Proposition 2 since $\partial_\nu \varphi_2|_{\gamma_2} < 0$,
- ▷ in case (iii), Proposition 3.1 where assumptions (3) and (4) are needed here, and, summing up these inequalities, we directly get the estimate (3.5).

The proof of the proposition is completed. □

Last, we recall a way to find two phases that satisfy Hörmander's condition except for a finite number of balls where one of them does not satisfy this condition while the other does and is strictly greater.

Proposition 3.3 ([23], Proposition 4.1)**.** *Let $\mathcal{O}$ be a bounded open subset with the boundary $\gamma_1 \cup \gamma_2$, where $\overline{\gamma_1} \cap \overline{\gamma_2} = \emptyset$. Then there exist two real functions $\psi_1, \psi_2 \in C^\infty(\mathcal{O})$ continuous on $\overline{\mathcal{O}}$ satisfying for $k = 1,2$ that $\partial_\nu \psi_1|_\gamma < 0$ and $\partial_\nu \psi_2|_\gamma < 0$, and having only a finite number degenerate critical points such that when $\nabla\psi_k = 0$ then $\nabla\psi_{k+1} \neq 0$ and $\psi_{k+1} > \psi_k$ where we assume that $k+1 = 2$ if $k = 1$ and $k+1 = 1$ if $k = 2$.*

Remark 3.1 ([23], Remark 4.2). (1) A consequence of Proposition 3.3 is that for $k = 1,2$ we can find a finite number of points x_{kj} where $j = 1, \ldots, N_k$ and $\varepsilon > 0$ such that $\overline{B(x_{kj}, 2\varepsilon)} \subset \mathcal{O}$ and $B(x_{1j_1}, 2\varepsilon) \cap B(x_{2j_2}, 2\varepsilon) = \emptyset$, for all $k = 1,2$ and $j_k = 1, \ldots, N_k$, and in $B(x_{kj}, 2\varepsilon)$ we have $\psi_{k+1} > \psi_k$ for all $j = 1, \ldots, N_k$.

(2) For all $\lambda > 0$ large enough the weight functions $\varphi_k = \mathrm{e}^{\lambda\psi_k}$ satisfy Hörmander's condition in $U_k = \mathcal{O} \cap \Big(\bigcup_{j=1}^{N_k} B(x_{kj}, \varepsilon)\Big)^c$.

Now we construct the weight functions needed in the proof of our lemma (see Lemma 3.1). Setting $\widetilde{\Omega}_1 = \Omega_1 \setminus \overline{B}_r$, where B_r is an open ball in Ω_1 with radius $r > 0$ such that $\overline{B}_r \subset \Omega_1$, and applying Proposition 3.2 and Remark 3.1, by the same arguments as in [23], we can find four weight functions $\varphi_{1,1}$, $\varphi_{1,2}$, $\varphi_{2,1}$, and $\varphi_{2,2}$ satisfying Hörmander's condition in $U_{1,1} = \widetilde{\Omega}_1 \cap \Big(\bigcup_{j=1}^{N_{11}} B(x_{11}^j, \varepsilon)\Big)^c$, $U_{1,2} = \widetilde{\Omega}_1 \cap \Big(\bigcup_{j=1}^{N_{12}} B^j(x_{12}, \varepsilon)\Big)^c$,

$U_{2,1} = \Omega_2 \cap \Big(\bigcup_{j=1}^{N_{21}} B(x_{21}^j, \varepsilon)\Big)^c$ and $U_{2,2} = \Omega_2 \cap \Big(\bigcup_{j=1}^{N_{22}} B(x_{22}^j, \varepsilon)\Big)^c$, respectively, moreover $\varphi_{1,k} < \varphi_{1,k+1}$ in $B(x_{1k}^j, 2\varepsilon)$ for all $j = 1, \dots, N_{1k}$ and $\varphi_{2,k} < \varphi_{2,k+1}$ in $B(x_{2k}^j, 2\varepsilon)$ for all $j = 1, \dots, N_{2k}$. Furthermore, for all $k = 1, 2$ we have

$$(\partial_\nu \varphi_{1,k})|_S < 0, \quad (\partial_\nu \varphi_{2,k})|_S < 0, \quad \text{and} \quad (\partial_\nu \varphi_{2,k})|_\Gamma < 0.$$

Also, we can suppose that $\varphi_{1,k}|_S = \varphi_{2,k}|_S$ and $(\partial_\nu \varphi_{1,k})^2|_S - (\partial_\nu \varphi_{2,k})^2|_S > 0$. For more details of that construction of the weight functions we refer the reader to [12] and [21].

3.2. Resolvent estimate. This subsection is devoted to the proof of the resolvent estimate (1.10). We suppose that the resolvent estimate (1.10) is false. Then by the continuity of the resolvent and the resonance theorem there exist $K_m > 0$, $\mu_m \in \mathbb{R}$, and two sequences $(u^m, v^m) \in D(\mathbb{A})$ and $(f^m, g^m) \in \mathcal{H}$, $m = 1, 2, \dots$ such that

$$|\mu_m| \to \infty, \quad K_m \to \infty, \quad \|(u^m, v^m)\|_{\mathcal{H}} = 1 \tag{3.6}$$

and

$$\mathrm{e}^{K_m|\mu_m|}(\mathbb{A} - \mathrm{i}\mu_m I_{\mathcal{H}}) \begin{pmatrix} u^m \\ v^m \end{pmatrix} = \begin{pmatrix} f^m \\ g^m \end{pmatrix} \to 0 \quad \text{in } \mathcal{H}. \tag{3.7}$$

Using (1.8), $\mathcal{H} = W \times V$ and $\mathcal{C} \in \mathrm{Isom}(V, V^*)$, we obtain from (3.7)

$$\begin{aligned} \mathrm{e}^{K_m|\mu_m|}(v^m - \mathrm{i}\mu_m u^m) &= f^m \to 0 \quad \text{in } W, \\ -\mathrm{e}^{K_m|\mu_m|}(\mathcal{A}u^m + \mathcal{B}v^m + \mathrm{i}\mu_m \mathcal{C}v^m) &= G^m \to 0 \quad \text{in } V^*, \end{aligned} \tag{3.8}$$

where $G^m = \mathcal{C}g^m \in V^*$.

Noticing that

$$\left\langle (\mathbb{A} - \mathrm{i}\mu_m I_{\mathcal{H}}) \begin{pmatrix} u^m \\ v^m \end{pmatrix}, \begin{pmatrix} u^m \\ v^m \end{pmatrix} \right\rangle_{\mathcal{H}} = -\int_{\Omega_1} a|\Delta v_1^m|^2 \,\mathrm{d}x - \mathrm{i}\mu_m(\|u^m\|_W^2 + \|v^m\|_V^2),$$

and by (3.6) and (3.7), we have

$$\mathrm{Re}\left\langle \mathrm{e}^{K_m|\mu_m|}(\mathbb{A} - \mathrm{i}\mu_m I_{\mathcal{H}}) \begin{pmatrix} u^m \\ v^m \end{pmatrix}, \begin{pmatrix} u^m \\ v^m \end{pmatrix} \right\rangle_{\mathcal{H}} = -\mathrm{e}^{K_m|\mu_m|} \int_{\Omega_1} a|\Delta v_1^m|^2 \,\mathrm{d}x \to 0. \tag{3.9}$$

Using $(3.8)_1$, we obtain from (3.9)

$$|\mu_m|^2 \mathrm{e}^{K_m|\mu_m|/2} \int_{\Omega_1} a|\Delta u_1^m|^2 \,\mathrm{d}x \to 0. \tag{3.10}$$

By (3.9), (3.10) and $\operatorname{supp}(a) = \overline{\omega}$, we find that

$$\mathrm{e}^{K_m|\mu_m|/2}\left(\int_\omega |\Delta u_1^m|^2\,\mathrm{d}x + \int_\omega |\Delta v_1^m|^2\,\mathrm{d}x\right) \to 0. \tag{3.11}$$

Noticing that if we set

$$u(x) = \begin{cases} u_1(x), & x \in \Omega_1, \\ u_2(x), & x \in \Omega_2, \end{cases}$$

then $\|(u_1, u_2)\|_W$ is equivalent to $\|u\|_{H^2(\Omega)}$, we get from $(3.8)_1$

$$\begin{aligned} \mathrm{e}^{K_m|\mu_m|}(v_1^m - \mathrm{i}\mu_m u_1^m) &= f_1^m \to 0 \quad \text{in } H^2(\Omega_1), \\ \mathrm{e}^{K_m|\mu_m|}(v_2^m - \mathrm{i}\mu_m u_2^m) &= f_2^m \to 0 \quad \text{in } H^2(\Omega_2). \end{aligned} \tag{3.12}$$

Also, by using (1.6) and (1.7), we obtain from $(3.8)_2$

$$\begin{aligned} \mathrm{e}^{K_m|\mu_m|}\left(\sum_{j=1}^2 \int_{\Omega_j} c_j^2 \Delta u_j^m \overline{\Delta\phi_j}\,\mathrm{d}x + \int_{\Omega_1} a\Delta v_1^m \overline{\Delta\phi_1}\,\mathrm{d}x\right) \\ + \mathrm{i}\mu_m \mathrm{e}^{K_m|\mu_m|} \sum_{j=1}^2 \int_{\Omega_j} (v_j^m \overline{\phi_j} + \beta\nabla v_j^m \cdot \overline{\nabla\phi_j})\,\mathrm{d}x \\ = -\langle G^m, \phi\rangle_{V^*,V} \to 0 \quad \text{for any } \phi \in W. \end{aligned} \tag{3.13}$$

By $(3.12)_1$ and (3.6), we have

$$\frac{1}{|\mu_m|^2}\|(\psi v_1^m)\|_{H^2(\Omega_1)}^2 = O(1) \text{ for any real function } \psi \in C^\infty(\overline{\Omega}_1). \tag{3.14}$$

Taking $\phi = \mu_m^{-1}\psi v^m$ in (3.13) yields

$$\begin{aligned} \frac{\mathrm{e}^{K_m|\mu_m|}}{\mu_m}\left(\int_\omega c_1^2 \Delta u_1^m \Delta(\psi\overline{v_1^m})\,\mathrm{d}x + \int_\omega a\Delta v_1^m \Delta(\psi\overline{v_1^m})\,\mathrm{d}x\right) \\ + \mathrm{i}\mathrm{e}^{K_m|\mu_m|}\int_\omega (|v_1^m|^2\psi + \beta\nabla v_1^m \cdot \nabla(\psi\overline{v_1^m}))\,\mathrm{d}x \to 0, \end{aligned} \tag{3.15}$$

where $\psi \in C^\infty(\overline{\Omega})$ is any real function satisfying $\operatorname{supp}(\psi) \subset \omega$. By (3.11), (3.14), and (3.15), we have

$$\mathrm{e}^{K_m|\mu_m|/4}\int_\omega |v_1^m|^2\psi\,\mathrm{d}x \to 0. \tag{3.16}$$

If we set B_{4r} to be a ball with radius $r > 0$ such that $B_{4r} \subset \omega$, then it follows that

$$\mathrm{e}^{K_m|\mu_m|/4}\int_{B_{4r}} |v_1^m|^2\,\mathrm{d}x \to 0,$$

which implies together with $(3.12)_1$

$$\mathrm{e}^{K_m|\mu_m|/4}\int_{B_{4r}}|u_1^m|^2\,\mathrm{d}x\to 0. \tag{3.17}$$

Let us consider now the transmission problem

$$\begin{cases} v_1-\mathrm{i}\mu u_1=f_1, & x\in\Omega_1,\\ v_2-\mathrm{i}\mu u_2=f_2, & x\in\Omega_2,\\ -\Delta(c_1^2\Delta u_1+a\Delta v_1)-\mathrm{i}\mu(v_1-\beta\Delta v_1)=g_1-\beta\Delta g_1, & x\in\Omega_1,\\ -\Delta(c_2^2\Delta u_2)-\mathrm{i}\mu(v_2-\beta\Delta v_2)=g_2-\beta\Delta g_2, & x\in\Omega_2,\\ u_1=u_2,\quad \partial_\nu u_1=\partial_\nu u_2, & x\in S,\\ c_1^2\Delta u_1=c_2^2\Delta u_2,\quad \partial_\nu(c_1^2\Delta u_1)=\partial_\nu(c_2^2\Delta u_2), & x\in S,\\ u_2=0,\quad \Delta u_2=0, & x\in\Gamma, \end{cases} \tag{3.18}$$

where $(f_1,f_2)\in W$ and $(g_1,g_2)\in W$. Then the solution (u_1,u_2,v_1,v_2) of (3.18) satisfies

$$\begin{cases} v_1=\mathrm{i}\mu u_1+f_1, & x\in\Omega_1,\\ v_2=\mathrm{i}\mu u_2+f_2, & x\in\Omega_2,\\ (\mu^2-\mu^2\beta\Delta-c_1^2\Delta^2)u_1-\Delta(a\Delta v_1-\beta g_1)=g_1+\mathrm{i}\mu f_1^c, & x\in\Omega_1,\\ (\mu^2-\mu^2\beta\Delta-c_2^2\Delta^2)u_2+\beta\Delta g_2=g_2+\mathrm{i}\mu f_2^c, & x\in\Omega_2,\\ u_1=u_2,\quad \partial_\nu u_1=\partial_\nu u_2, & x\in S,\\ c_1^2\Delta u_1=c_2^2\Delta u_2,\quad \partial_\nu(c_1^2\Delta u_1)=\partial_\nu(c_2^2\Delta u_2), & x\in S,\\ u_2=0,\quad \Delta u_2=0, & x\in\Gamma, \end{cases} \tag{3.19}$$

where $f_k^c=f_k-\beta\Delta f_k$, $k=1,2$.

Noticing that

$$\mu^2-\mu^2\beta\Delta-c_k^2\Delta^2=\Big(-\Delta-\frac{b_k^+(\mu)}{2c_k^2}\Big)\Big(c_k^2\Delta-\frac{b_k^-(\mu)}{2}\Big)$$

for $k=1,2$, where

$$b_k^+(\mu)=|\mu|\Big(\sqrt{|\mu|^2\beta^2+4c_k^4}+|\mu|\beta\Big),$$
$$b_k^-(\mu)=|\mu|\Big(\sqrt{|\mu|^2\beta^2+4c_k^4}-|\mu|\beta\Big),$$

we can rewrite (3.19) as

$$
(3.20)\quad \begin{cases}
v_1 = \mathrm{i}\mu u_1 + f_1, & x \in \Omega_1,\\
v_2 = \mathrm{i}\mu u_2 + f_2, & x \in \Omega_2,\\
\Big(-\Delta - \dfrac{b_1^+(\mu)}{2c_1^2}\Big)\Big(c_1^2\Delta u_1 - \dfrac{b_1^-(\mu)}{2}u_1 + a(x)\Delta v_1 - \beta g_1\Big) & \\
\qquad\qquad = \Phi_1 := g_1 + \mathrm{i}\mu f_1^c - \dfrac{b_1^+(\mu)}{2c_1^2}(a(x)\Delta v_1 - \beta g_1), & x \in \Omega_1,\\
\Big(-\Delta - \dfrac{b_2^+(\mu)}{2c_2^2}\Big)\Big(c_2^2\Delta u_2 - \dfrac{b_2^-(\mu)}{2}u_2 - \beta g_2\Big) & \\
\qquad\qquad = \Phi_2 := g_2 + \mathrm{i}\mu f_2^c + \dfrac{b_2^+(\mu)\beta}{2c_2^2}g_2, & x \in \Omega_2,\\
u_1 = u_2, \quad \partial_\nu u_1 = \partial_\nu u_2, & x \in S,\\
c_1^2\Delta u_1 = c_2^2\Delta u_2, \quad \partial_\nu(c_1^2\Delta u_1) = \partial_\nu(c_2^2\Delta u_2), & x \in S,\\
u_2 = 0, \quad \Delta u_2 = 0, & x \in \Gamma.
\end{cases}
$$

Setting

$$
(3.21)\qquad \begin{aligned}
w_1 &= c_1^2\Delta u_1 - \frac{b_1^-(\mu)}{2}u_1 + a(x)\Delta v_1 - \beta g_1,\\
w_2 &= c_2^2\Delta u_2 - \frac{b_2^-(\mu)}{2}u_2 - \beta g_2,
\end{aligned}
$$

and using (3.20), $g_1|_S = g_2|_S$ and $(\partial_\nu g_1)|_S = (\partial_\nu g_2)|_S$, it is easy to show that w_1 and w_2 satisfy the simple transmission problem

$$
(3.22)\qquad \begin{cases}
-\Delta w_1 - \dfrac{b_1^+(\mu)}{2c_1^2}w_1 = \Phi_1, & x \in \Omega_1,\\
-\Delta w_2 - \dfrac{b_1^+(\mu)}{2c_2^2}w_2 = \widetilde{\Phi}_2, & x \in \Omega_2,\\
w_1 = w_2, \quad \partial_\nu w_1 = \partial_\nu w_2, & x \in S,\\
w_2 = 0, & x \in \Gamma,
\end{cases}
$$

where

$$
(3.23)\qquad \widetilde{\Phi}_2 = \Phi_2 + (b_2^+(\mu) - b_1^+(\mu))\frac{w_2}{2c_2^2}.
$$

The main ingredient of the resolvent estimate is the following lemma which is essentially a consequence of the Carleman estimate.

Lemma 3.1. *There exist constants $C > 0$ and $r_0 > 0$ such that for any solution $(u, v) \in D(\mathbb{A})$ of the system* (3.18) *the estimate*

$$\begin{aligned}(3.24)\quad &\|\Delta u_1\|^2_{L^2(\Omega_1)} + \|\Delta u_2\|^2_{L^2(\Omega_2)} + \|v_1\|^2_{L^2(\Omega_1)} + \|v_2\|^2_{L^2(\Omega_2)} \\ &\leqslant C\mathrm{e}^{C|\mu|}\|\Delta f_1\|^2_{L^2(\Omega_1)} + C\mathrm{e}^{C|\mu|}\Big(\|\Delta f_2\|^2_{L^2(\Omega_2)} + \|g_1\|^2_{L^2(\Omega_1)} + \|g_2\|^2_{L^2(\Omega_2)} \\ &\qquad + \int_{\Omega_1} a|\Delta v_1|^2\,\mathrm{d}x + \int_{B_{4r}} |u_1|^2\,\mathrm{d}x\Big)\end{aligned}$$

holds for all $0 < r < r_0$ and $\mu \in \mathbb{R}$ large enough, where B_r is an open ball with radius $r > 0$ such that $B_{4r} \subset \omega$.

P r o o f. We introduce the cut-off function $\chi \in C^\infty(\Omega_1)$ by setting

$$\chi(x) = \begin{cases} 1 & \text{in } B^c_{3r}, \\ 0 & \text{in } B_{2r}. \end{cases}$$

Next, put $\widetilde{w}_1 = \chi w_1$. Then by $(3.22)_1$, one sees that

$$(3.25)\qquad -\Delta\widetilde{w}_1 - \frac{b_1^+(\mu)}{2c_1^2}\widetilde{w}_1 = \widetilde{\Phi}_1 \equiv \chi\Phi_1 - [\Delta, \chi]w_1,$$

where $[\Delta, \chi]f = \Delta(\chi f) - \chi\Delta f$.

Now keeping the same notations as in the previous subsection, let $\varphi_{1,1}$, $\varphi_{1,2}$, $\varphi_{2,1}$, and $\varphi_{2,2}$ be four weight functions that satisfy the conclusion of Subsection 3.1.

Let $\chi_{1,1}$, $\chi_{1,2}$, $\chi_{2,1}$ and $\chi_{2,2}$ be four cut-off functions equal to 1 in $\Big(\bigcup\limits_{j=1}^{N_{11}} B(x^j_{11}, 2\varepsilon)\Big)^c$, $\Big(\bigcup\limits_{j=1}^{N_{12}} B^j(x_{12}, 2\varepsilon)\Big)^c$, $\Big(\bigcup\limits_{j=1}^{N_{21}} B(x^j_{21}, 2\varepsilon)\Big)^c$ and $\Big(\bigcup\limits_{j=1}^{N_{22}} B(x^j_{22}, 2\varepsilon)\Big)^c$, respectively, and supported in $\Big(\bigcup\limits_{j=1}^{N_{11}} B(x^j_{11}, \varepsilon)\Big)^c$, $\Big(\bigcup\limits_{j=1}^{N_{12}} B^j(x_{12}, \varepsilon)\Big)^c$, $\Big(\bigcup\limits_{j=1}^{N_{21}} B(x^j_{21}, \varepsilon)\Big)^c$ and $\Big(\bigcup\limits_{j=1}^{N_{22}} B(x^j_{22}, \varepsilon)\Big)^c$, respectively (in order to eliminate the critical point of the weight functions $\varphi_{1,1}$, $\varphi_{1,2}$, $\varphi_{2,1}$, and $\varphi_{2,2}$).

Setting

$$w_{1,1} = \chi_{1,1}\widetilde{w}_1, \quad w_{1,2} = \chi_{1,2}\widetilde{w}_1, \quad w_{2,1} = \chi_{2,1}w_2, \quad \text{and} \quad w_{2,2} = \chi_{2,2}w_2,$$

and using (3.25), we obtain from (3.22)

$$(3.26)\qquad \begin{cases} -\Delta w_{1,k} - \dfrac{b_1^+(\mu)}{2c_1^2}w_{1,k} = \Psi_{1,k}, & x \in \Omega_1, \\ -\Delta w_{2,k} - \dfrac{b_1^+(\mu)}{2c_2^2}w_{2,k} = \Psi_{2,k}, & x \in \Omega_2, \\ w_{1,k} = w_{2,k}, \quad \partial_\nu w_{1,k} = \partial_\nu w_{2,k}, & x \in S, \\ w_{2,k} = 0, & x \in \Gamma \end{cases}$$

for $k=1,2$, where

$$\begin{cases} \Psi_{1,k} = \chi_{1,k}\widetilde{\Phi}_1 - [\Delta, \chi_{1,k}]\widetilde{w}_1, \\ \Psi_{2,k} = \chi_{2,k}\widetilde{\Phi}_2 - [\Delta, \chi_{2,k}]w_2. \end{cases} \tag{3.27}$$

Taking

$$h = \frac{1}{\sqrt{b_1^+(\mu)}}, \quad \mathcal{O}_j = U_{j,k} \quad (j=1,2), \quad \gamma = S,$$
$$\gamma_2 = \Gamma \quad \text{and} \quad \gamma_1 = (\partial U_{1,k} \setminus S) \cap (\partial U_{2,k} \setminus (S \cup \Gamma)),$$

and applying Proposition 3.2 to the system (3.26), we have

$$\begin{aligned} h\|\mathrm{e}^{\varphi_{1,k}/h} w_{1,k}\|^2_{L^2(U_{1,k})} &+ h^3\|\mathrm{e}^{\varphi_{1,k}/h}\nabla w_{1,k}\|^2_{L^2(U_{1,k})} \\ &+ h\|\mathrm{e}^{\varphi_{2,k}/h} w_{2,k}\|^2_{L^2(U_{2,k})} + h^3\|\mathrm{e}^{\varphi_{2,k}/h}\nabla w_{2,k}\|^2_{L^2(U_{2,k})} \\ &\leqslant Ch^4(\|\mathrm{e}^{\varphi_{1,k}/h}\Psi_{1,k}\|^2_{L^2(U_{1,k})} + \|\mathrm{e}^{\varphi_{2,k}/h}\Psi_{2,k}\|^2_{L^2(U_{2,k})}) \end{aligned} \tag{3.28}$$

for $k=1,2$, where we used that

$$w_{1,k}|_{\gamma_1} = w_{2,k}|_{\gamma_1} = \partial_\nu w_{1,k}|_{\gamma_1} = \partial_\nu w_{2,k}|_{\gamma_1} = 0.$$

We estimate the right-hand side in (3.28) by using (3.27), (3.23), and (3.25). Then we get

$$\begin{aligned} h\|\mathrm{e}^{\varphi_{1,k}/h} w_{1,k}\|^2_{L^2(U_{1,k})} &+ h^3\|\mathrm{e}^{\varphi_{1,k}/h}\nabla w_{1,k}\|^2_{L^2(U_{1,k})} \\ &+ h\|\mathrm{e}^{\varphi_{2,k}/h} w_{2,k}\|^2_{L^2(U_{2,k})} + h^3\|\mathrm{e}^{\varphi_{2,k}/h}\nabla w_{2,k}\|^2_{L^2(U_{2,k})} \\ &\leqslant Ch^4(\|\mathrm{e}^{\varphi_{1,k}/h}\Phi_1\|^2_{L^2(U_{1,k})} + \|\mathrm{e}^{\varphi_{2,k}/h}\Phi_2\|^2_{L^2(U_{2,k})}) \\ &\quad + Ch^4(\|\mathrm{e}^{\varphi_{1,k}/h}[\Delta, \chi_{1,k}]\widetilde{w}_1\|^2_{L^2(U_{1,k})} \\ &\quad + \|\mathrm{e}^{\varphi_{2,k}/h}[\Delta, \chi_{2,k}]w_2\|^2_{L^2(U_{2,k})}) \\ &\quad + Ch^4\|\mathrm{e}^{\varphi_{1,k}/h}[\Delta, \chi]w_1\|^2_{L^2(U_{1,k})} + Ch^4(b_2^+(\mu) \\ &\quad - b_1^+(\mu))^2\|\mathrm{e}^{\varphi_{2,k}/h}w_{2,k}\|^2_{L^2(U_{2,k})}. \end{aligned} \tag{3.29}$$

By the definition $b_j^+(\mu)$ $(j=1,2)$, we find that

$$b_2^+(\mu) - b_1^+(\mu) = \frac{4|\mu|(c_2^4 - c_1^4)}{\sqrt{|\mu|^2\beta^2 + 4c_2^4} + \sqrt{|\mu|^2\beta^2 + 4c_2^4}} \leqslant C \tag{3.30}$$

for any $\mu \in \mathbb{R}$ large enough. Also, in the same lines as in [23], we obtain the following fact: using the properties $\varphi_{1,k} < \varphi_{1,\sigma(k)}$ in $\left(\bigcup_{j=1}^{N_{1k}} B(x_{1k}^j, 2\varepsilon)\right)$ and $\varphi_{2,k} < \varphi_{2,\sigma(k)}$ in

$\Big(\bigcup_{j=1}^{N_{2k}} B(x_{2k}^j, 2\varepsilon)\Big)$, where $\sigma(k) = 2$ if $k = 1$ and $\sigma(k) = 1$ if $k = 2$, we can absorb the terms $[\Delta, \chi_{1,k}]\widetilde{w}_1$ and $[\Delta, \chi_{2,k}]w_2$ at the right-hand side in (3.29) into the left-hand side for small $h > 0$. By using the above fact and (3.30), we obtain from (3.29) that

$$\begin{aligned}
(3.31)\quad & \int_{\widetilde{\Omega}_1} h\Big(\mathrm{e}^{2\varphi_{1,1}/h} + \mathrm{e}^{2\varphi_{1,2}/h}\Big)|\widetilde{w}_1|^2\,\mathrm{d}x + \int_{\Omega_2} h(\mathrm{e}^{2\varphi_{2,1}/h} + \mathrm{e}^{2\varphi_{2,2}/h})|w_2|^2\,\mathrm{d}x \\
&\leqslant Ch^4\bigg(\int_{\Omega_1}(\mathrm{e}^{2\varphi_{1,1}/h} + \mathrm{e}^{2\varphi_{1,2}/h})|\Phi_1|^2\,\mathrm{d}x + \int_{\Omega_2}(\mathrm{e}^{2\varphi_{2,1}/h} + \mathrm{e}^{2\varphi_{2,2}/h})|\Phi_2|^2\,\mathrm{d}x\bigg) \\
&\quad + Ch^4\int_{\widetilde{\Omega}_1}(\mathrm{e}^{2\varphi_{1,1}/h} + \mathrm{e}^{2\varphi_{1,2}/h})|[\Delta, \chi]w_1|^2\,\mathrm{d}x.
\end{aligned}$$

Noticing that

$$\begin{aligned}
&|\Phi_1| \leqslant |g_1| + |\mu|(|f_1| + \beta|\Delta f_1|) + \frac{b_1^+(\mu)}{2c_1^2}(|a\Delta v_1| + \beta|g_1|), \\
&|\Phi_2| \leqslant |g_2| + |\mu|(|f_2| + \beta|\Delta f_2|) + \frac{b_2^+(\mu)\beta}{2c_2^2}|g_2|, \\
&\frac{|\mu|}{b_1^+(\mu)} \to 0 \quad \text{and} \quad \frac{b_2^+(\mu)}{b_1^+(\mu)} \to 1 \quad \text{as } |\mu| \to \infty,
\end{aligned}$$

and using

$$h = \frac{1}{\sqrt{b_1^+(\mu)}}, \quad \Omega_1 = \widetilde{\Omega}_1 \cup B_{2r}, \quad \widetilde{w}_i = \chi w_i \quad (i = 1, 2)$$

and

$$\chi(x) = \begin{cases} 1 & \text{in } B_{3r}^c, \\ 0 & \text{in } B_{2r}, \end{cases}$$

we obtain from (3.31)

$$\begin{aligned}
(3.32)\qquad & \int_{\Omega_1} h(\mathrm{e}^{2\varphi_{1,1}/h} + \mathrm{e}^{2\varphi_{1,2}/h})|w_1|^2\,\mathrm{d}x + \int_{\Omega_2} h(\mathrm{e}^{2\varphi_{1,1}/h} + \mathrm{e}^{2\varphi_{1,2}/h})|w_2|^2\,\mathrm{d}x \\
&\leqslant C\sum_{j=1}^{2}\int_{\Omega_j}(\mathrm{e}^{2\varphi_{1,1}/h} + \mathrm{e}^{2\varphi_{1,2}/h})(|g_j|^2 + |f_j|^2 + |\Delta f_j|^2)\,\mathrm{d}x \\
&\quad + Ch\int_{B_{2r}}(\mathrm{e}^{2\varphi_{1,1}/h} + \mathrm{e}^{2\varphi_{1,2}/h})|w_1|^2 \\
&\quad + C\int_{\Omega_1}(\mathrm{e}^{2\varphi_{2,1}/h} + \mathrm{e}^{2\varphi_{2,2}/h})|a\Delta v_1|^2\,\mathrm{d}x \\
&\quad + Ch^4\int_{\widetilde{\Omega}_1}(\mathrm{e}^{2\varphi_{1,1}/h} + \mathrm{e}^{2\varphi_{1,2}/h})|[\Delta, \chi]w_1|^2\,\mathrm{d}x.
\end{aligned}$$

Inserting the maximum of $\varphi_{1,1}, \varphi_{2,1}, \varphi_{1,2}$, and $\varphi_{2,2}$ at the right-hand side of (3.32) and their minimum at the left-hand side, we obtain

$$
\begin{aligned}
(3.33)\qquad & \int_{\Omega_1} |w_1|^2 \,\mathrm{d}x + \int_{\Omega_2} |w_2|^2 \,\mathrm{d}x \\
&\leqslant C\mathrm{e}^{C/h} \sum_{j=1}^{2} \int_{\Omega_j} (|g_j|^2 + |f_j|^2 + |\Delta f_j|^2) \,\mathrm{d}x \\
&\quad + C\mathrm{e}^{C/h} \bigg(\int_{\Omega_1} |a\Delta v_1|^2 \,\mathrm{d}x + \int_{B_{2r}} |w_1|^2 \,\mathrm{d}x + \int_{\widetilde{\Omega}_1} |[\Delta, \chi] w_1|^2 \,\mathrm{d}x \bigg).
\end{aligned}
$$

We estimate now the two last terms on the right-hand side of (3.33). Let $\overline{\chi}$ be a cut-off function equal to 1 in a neighborhood of B_{3r} and supported in B_{4r}. Using $(3.22)_1$, we have

$$
(-1+\Delta)(\overline{\chi} w_1) = [\Delta, \overline{\chi}] w_1 - \overline{\chi} w_1 - \frac{b_1^+(\mu)}{2c_1^2} \overline{\chi} w_1 - \overline{\chi} \Phi_1
$$

and due to elliptic estimates (see [36]), we get

$$
\begin{aligned}
(3.34)\qquad \|w_1\|^2_{H^1(B_{3r})} &\leqslant C(\|(-1+\Delta)(\overline{\chi} w_1)\|^2_{H^{-1}(B_{4r})} + \|w_1\|^2_{L^2(B_{4r})}) \\
&\leqslant C(\|\Phi_1\|^2_{L^2(\Omega_1)} + (1 + |b_1^+(\mu)|^2) \|w_1\|^2_{L^2(B_{4r})}) \\
&\leqslant C[(1 + |b_1^+(\mu)|^2) \|g_1\|^2_{\Omega_1} + |\mu|^2 (\|f_1\|^2_{\Omega_1} + \|\Delta f_1\|^2_{\Omega_1})] \\
&\quad + C\bigg((1 + |b_1^+(\mu)|^2) \|w_1\|^2_{L^2(B_{4r})} + |b_1^+(\mu)|^2 \int_{\Omega_1} a |\Delta v_1|^2 \,\mathrm{d}x \bigg).
\end{aligned}
$$

Using that $\mathrm{supp}([\Delta, \chi]) \subset B_{3r}$, we deduce

$$
(3.35)\qquad \int_{B_{2r}} |w_1|^2 \,\mathrm{d}x + \int_{\widetilde{\Omega}_1} |[\Delta, \chi] w_1|^2 \,\mathrm{d}x \leqslant C \|w_1\|^2_{H^1(B_{3r})}.
$$

Also, using (3.21), $(3.19)_1$, $|b_1^-(\mu)| \leqslant C$ and $a > 0$ in B_{4r}, we have

(3.36)

$$
\begin{aligned}
\|w_1\|^2_{L^2(B_{4r})} &\leqslant C\bigg(\|\Delta u_1\|^2_{L^2(B_{4r})} + \int_{\Omega_1} |a\Delta v_1|^2 \,\mathrm{d}x + \|g_1\|^2_{L^2(B_{4r})} + \|u_1\|^2_{L^2(B_{4r})} \bigg) \\
&\leqslant C\bigg(|\mu|^{-1} \|\Delta f_1\|^2_{L^2(B_{4r})} + (|\mu|^{-1} + 1) \int_{\Omega_1} |a\Delta v_1|^2 \,\mathrm{d}x \\
&\qquad\qquad + \|g_1\|^2_{L^2(B_{4r})} + \|u_1\|^2_{L^2(B_{4r})} \bigg).
\end{aligned}
$$

By using (3.33)–(3.36) and $|b_1^+(\mu)| \leqslant C|\mu|^2$, we obtain

$$
\begin{aligned}
\|w_1\|^2_{L^2(\Omega_1)} + \|w_2\|^2_{L^2(\Omega_2)} &\leqslant C\mathrm{e}^{C|\mu|}(\|\Delta f_1\|^2_{L^2(\Omega_1)} + \|\Delta f_2\|^2_{L^2(\Omega_2)}) \\
&\quad + C\mathrm{e}^{C|\mu|}\bigg(\|g_1\|^2_{L^2(\Omega_1)} + \|g_2\|^2_{L^2(\Omega_2)} \\
&\qquad + \int_{\Omega_1} a|\Delta v_1|^2\,\mathrm{d}x + \int_{B_{4r}} |u_1|^2\,\mathrm{d}x\bigg).
\end{aligned} \tag{3.37}
$$

On the other hand, we get by (3.21)

$$
\begin{aligned}
&\Big\|c_1^2\Delta u_1 - \frac{b_1^-(\mu)}{2}u_1\Big\|^2_{L^2(\Omega_1)} + \Big\|c_2^2\Delta u_2 - \frac{b_2^-(\mu)}{2}u_2\Big\|^2_{L^2(\Omega_2)} \\
&\quad \leqslant C\bigg(\|w_1\|^2_{L^2(\Omega_1)} + \|w_2\|^2_{L^2(\Omega_2)} + \|g_1\|^2_{L^2(\Omega_1)} + \|g_2\|^2_{L^2(\Omega_2)} + \int_{\Omega_1} a|\Delta v_1|^2\,\mathrm{d}x\bigg).
\end{aligned} \tag{3.38}
$$

Noticing that due to the transmission conditions $(3.20)_{5,7}$,

$$
\begin{aligned}
&\int_S \partial_\nu u_1 u_1\,\mathrm{d}S = -\int_S \partial_\nu u_2 u_2\,\mathrm{d}S, \\
&\int_{\Omega_j} \Delta u_j u_j\,\mathrm{d}x = -\int_{\Omega_j} |\nabla u_j|^2\,\mathrm{d}x + \int_S \partial_\nu u_j u_j\,\mathrm{d}S, \quad j = 1, 2,
\end{aligned}
$$

we have

$$
\begin{aligned}
&-c_1^2 b_1^-(\mu)\int_{\Omega_1} \Delta u_1 u_1\,\mathrm{d}x - c_2^2 b_2^-(\mu)\int_{\Omega_2} \Delta u_2 u_2\,\mathrm{d}x \\
&\qquad = \sum_{j=1}^{2} c_j^2 b_j^-(\mu)\int_{\Omega_j} |\nabla u_j|^2\,\mathrm{d}x + (c_1^2 b_1^-(\mu) - c_2^2 b_2^-(\mu))\int_S \partial_\nu u_1 u_1\,\mathrm{d}S \\
&\qquad \geqslant \frac{c_1^2 b_1^-(\mu) - c_2^2 b_2^-(\mu)}{2}\bigg(\int_{\Omega_1} \Delta u_1 u_1\,\mathrm{d}x - \int_{\Omega_2} \Delta u_2 u_2\,\mathrm{d}x\bigg),
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\Big\|c_1^2\Delta u_1 - \frac{b_1^-(\mu)}{2}u_1\Big\|^2_{L^2(\Omega_1)} + \Big\|c_2^2\Delta u_2 - \frac{b_2^-(\mu)}{2}u_2\Big\|^2_{L^2(\Omega_2)} \\
&\quad \geqslant c_1^4\|\Delta u_1\|^2_{L^2(\Omega_1)} + c_2^4\|\Delta u_2\|^2_{L^2(\Omega_2)} \\
&\qquad + \frac{|b_1^-(\mu)|^2}{4}\|u_1\|^2_{L^2(\Omega_1)} + \frac{|b_2^-(\mu)|^2}{4}\|u_2\|^2_{L^2(\Omega_2)} \\
&\qquad + \frac{c_1^2 b_1^-(\mu) - c_2^2 b_2^-(\mu)}{2}\bigg(\int_{\Omega_1} \Delta u_1 u_1\,\mathrm{d}x - \int_{\Omega_2} \Delta u_2 u_2\,\mathrm{d}x\bigg) \\
&\quad \geqslant \frac{c_1^4}{2}\|\Delta u_1\|^2_{L^2(\Omega_1)} + \frac{c_2^4}{2}\|\Delta u_2\|^2_{L^2(\Omega_2)}.
\end{aligned} \tag{3.39}
$$

Also, using $(3.20)_1$ and $(3.20)_2$, we get

$$\begin{aligned} \|v_1\|^2_{L^2(\Omega_1)} &\leqslant \|f_1\|^2_{L^2(\Omega_1)} + |\mu|^2\|u_1\|^2_{L^2(\Omega_1)}, \\ \|v_2\|^2_{L^2(\Omega_2)} &\leqslant \|f_2\|^2_{L^2(\Omega_2)} + |\mu|^2\|u_2\|^2_{L^2(\Omega_2)}. \end{aligned} \tag{3.40}$$

By (3.37)–(3.40), we obtain (3.24). The proof of Lemma 3.1 is completed. □

We continue the proof of the resolvent estimate (1.10). Applying the inequality (3.24) to the system (3.12)–(3.13), we arrive at

$$\begin{aligned} &\|\Delta u_1^m\|^2_{L^2(\Omega_1)} + \|\Delta u_2^m\|^2_{L^2(\Omega_2)} + \|v_1^m\|^2_{L^2(\Omega_1)} + \|v_2^m\|^2_{L^2(\Omega_2)} \\ &\quad \leqslant C\mathrm{e}^{C|\mu_m|}\mathrm{e}^{-2K_m|\mu_m|}(\|f^m\|^2_W + \|g^m\|^2_H) \\ &\qquad + C\mathrm{e}^{C|\mu_m|}\mathrm{e}^{-K_m|\mu_m|/4}\biggl(\int_{\Omega_1} a|\Delta v_1|^2\,\mathrm{d}x + \int_{B_{4r}} |u_1|^2\,\mathrm{d}x\biggr)\mathrm{e}^{K_m|\mu_m|/4}, \end{aligned} \tag{3.41}$$

where $f^m = (f_1^m, f_2^m) \in W$ and $g^m = (g_1^m, g_2^m) \in V$.

By (3.8), (3.10), and (3.17), the right-hand side of (3.41) tends to zero as $m \to \infty$ which contradicts (3.6). The proof of the resolvent estimate (1.10) is completed.

P r o o f of Theorem 1.2. It just remains to show that $\mathbb{A}$ has no purely imaginary eigenvalue. It is easy to check that $0 \in \varrho(\mathbb{A})$, where $\varrho(\mathbb{A})$ stands for the resolvent set of $\mathbb{A}$.

Let $\mu \neq 0$ be a real number and assume that for some $(u, v) \in D(\mathbb{A})$,

$$\mathbb{A}\begin{pmatrix} u \\ v \end{pmatrix} = \mathrm{i}\mu\begin{pmatrix} u \\ v \end{pmatrix}. \tag{3.42}$$

Then we show that $u = v = 0$.

Noticing that

$$\operatorname{Re}\biggl\langle \mathbb{A}\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\biggr\rangle_{W\times V} = -\int_{\Omega_1} a(x)|\Delta v_1(x)|^2\,\mathrm{d}x$$

and using (3.42), we obtain $\int_{\Omega_1} a(x)|\Delta v_1(x)|^2\,\mathrm{d}x = 0$, which means that $v_1 = 0$ on $\operatorname{supp}(a)$ by the similar arguments as in (3.9)–(3.16).

Using the definition of the operator $\mathbb{A}$, (3.42) can be recast as

$$\begin{cases} v_1 = \mathrm{i}\mu u_1, & x \in \Omega_1, \\ v_2 = \mathrm{i}\mu u_2, & x \in \Omega_2, \\ -\Delta(c_1^2\Delta u_1 + a\Delta v_1) - \mathrm{i}\mu(v_1 - \beta\Delta v_1) = 0, & x \in \Omega_1, \\ -\Delta(c_2^2\Delta u_2) - \mathrm{i}\mu(v_2 - \beta\Delta v_2) = 0, & x \in \Omega_2, \\ u_1 = u_2, \quad \partial_\nu u_1 = \partial_\nu u_2, & x \in S, \\ c_1^2\Delta u_1 = c_2^2\Delta u_2, \quad \partial_\nu(c_1^2\Delta u_1) = \partial_\nu(c_2^2\Delta u_2), & x \in S, \\ u_2 = 0, \quad \Delta u_2 = 0, & x \in \Gamma. \end{cases} \tag{3.43}$$

Since $v_1 = 0$ on $\mathrm{supp}(a)$, $(3.43)_1$ yields $u_1 = 0$ on $\mathrm{supp}(a)$.

In the same lines as in (3.20), we can rewrite $(3.43)_3$ combined with $(3.43)_1$ as

$$\Delta Z + \frac{b_1^+(\mu)}{2c_1^2} Z = 0 \quad \text{in } \Omega_1 \quad \text{and} \quad Z = 0 \quad \text{on } \mathrm{supp}(a),$$

where

$$Z = c_1^2 \Delta u_1 - \frac{b_1^-(\mu)}{2} u_1.$$

Then by Calderón's theorem for elliptic operators (see [28], Theorem 4.2), we find that $Z = 0$. This means that

$$c_1^2 \Delta u_1 - \frac{b_1^-(\mu)}{2} u_1 = 0,$$

which implies for the same argument as previously that $u_1 = 0$ in Ω_1.

Equations $(3.43)_2$ and $(3.43)_4$ lead to

$$c_2^2 \Delta^2 u_2 - \mu^2 (u_2 - \beta \Delta u_2) = 0, \quad x \in \Omega_2, \tag{3.44}$$

where the transmission conditions lead to

$$\begin{cases} u_2 = \partial_\nu u_2 = \Delta u_2 = \partial_\nu(\Delta u_1) = 0, & x \in S, \\ u_2 = 0, \quad \Delta u_2 = 0, & x \in \partial\Omega_2. \end{cases}$$

Following these boundary conditions, we can extend u_2 by zero into the whole Ω and (3.44) remains valid on all Ω. Then by using the same arguments as for u_1 above one can also show that $u_2 = 0$ in Ω_2. Using $(3.43)_2$, we get $v_2 = 0$ in Ω_2. Therefore, $\mathbb{A}$ has no purely imaginary eigenvalue.

Conflict of interests. The authors declare that there is no conflict of interests regarding the publication of this paper.

Data availability statement. The authors will permit all the data underlying the findings of their manuscripts to be shared by any researchers or groups who are interested in the article.

Acknowledgments. The authors would like to thank the anonymous referee for his/her deep insight, useful comments and suggestions, which led to the improvement of this paper.

References

[1] *K. Ammari, M. Jellouli, M. Mehrenberger*: Feedback stabilization of a coupled string-beam system. Netw. Heterog. Media *4* (2009), 19–34. zbl MR doi

[2] *K. Ammari, S. Nicaise*: Stabilization of a transmission wave/plate equation. J. Differ. Equations *249* (2010), 707–727. zbl MR doi

[3] *K. Ammari, G. Vodev*: Boundary stabilization of the transmission problem for the Bernoulli-Euler plate equation. Cubo *11* (2009), 39–49. zbl MR

[4] *G. Avalos, I. Lasiecka, R. Triggiani*: Heat-wave interaction in 2-3 dimensions: Optimal rational decay rate. J. Math. Anal. Appl. *437* (2016), 782–815. zbl MR doi

[5] *G. Avalos, R. Triggiani*: Backward uniqueness of the s.c. semigroup arising in parabolic-hyperbolic fluid-structure interaction. J. Differ. Equations *245* (2008), 737–761. zbl MR doi

[6] *G. Avalos, R. Triggiani*: Uniform stabilization of a coupled PDE system arising in fluid-structure interaction with boundary dissipation at the interface. Discrete Contin. Dyn. Syst. *22* (2008), 817–833. zbl MR doi

[7] *G. Avalos, R. Triggiani*: Fluid-structure interaction with and without internal dissipation of the structure: A contrast study in stability. Evol. Equ. Control Theory *2* (2013), 563–598. zbl MR doi

[8] *G. Avalos, R. Triggiani*: Rational decay rates for a PDE heat-structure interaction: A frequency domain approach. Evol. Equ. Control Theory *2* (2013), 233–253. zbl MR doi

[9] *V. Barbu, Z. Grujić, I. Lasiecka, A. Tuffaha*: Existence of the energy-level weak solutions for a nonlinear fluid-structure interaction model. Fluids and Waves: Recent Trends in Applied Analysis. Contemporary Mathematics 440. American Mathematical Society, Providence, 2007, pp. 55–82. zbl MR doi

[10] *W. D. Bastos, C. A. Raposo*: Transmission problem for waves with frictional damping. Electron. J. Differ. Equ. *2007* (2007), Article ID 60, 10 pages. zbl MR

[11] *C. J. K. Batty, T. Duyckaerts*: Non-uniform stability for bounded semi-groups on Banach spaces. J. Evol. Equ. *8* (2008), 765–780. zbl MR doi

[12] *M. Bellassoued*: Carleman estimates and distribution of resonances for the transparent obstacle and application to the stabilization. Asymptotic. Anal. *35* (2003), 257–279. zbl MR

[13] *N. Burq*: Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel. Acta Math. *180* (1998), 1–29. (In French.) zbl MR doi

[14] *S. Chai*: Uniform decay rate for the transmission wave equations with variable coefficients. J. Syst. Sci. Complex *24* (2011), 253–260. zbl MR doi

[15] *S. Chai, K. Liu*: Boundary stabilization of the transmission of wave equations with variable coefficients. Chin. Ann. Math., Ser. A *26* (2005), 605–612. (In Chinese.) zbl MR

[16] *S. Chen, K. Liu, Z. Liu*: Spectrum and stability for elastic systems with global or local Kelvin-Voigt damping. SIAM J. Appl. Math. *59* (1999), 651–668. zbl MR doi

[17] *Q. Du, M. D. Gunzburger, L. S. Hou, J. Lee*: Analysis of a linear fluid-structure interaction problem. Discrete Contin. Dyn. Syst. *9* (2003), 633–650. zbl MR doi

[18] *T. Duyckaerts*: Optimal decay rates of the energy of an hyperbolic-parabolic system coupled by an interface. Asymptotic. Anal. *51* (2007), 17–45. zbl MR

[19] *K.-J. Engel, R. Nagel*: One-Parameter Semigroups for Linear Evolution Equations. Graduate Texts in Mathematics 194. Springer, Berlin, 2000. zbl MR doi

[20] *F. Hassine*: Stability of elastic transmission systems with a local Kelvin-Voigt damping. Eur. J. Control *23* (2015), 84–93. zbl MR doi

[21] *F. Hassine*: Asymptotic behavior of the transmission Euler-Bernoulli plate and wave equation with a localized Kelvin-Voigt damping. Discrete Contin. Dyn. Syst., Ser. B *21* (2016), 1757–1774. zbl MR doi

[22] *F. Hassine*: Energy decay estimates of elastic transmission wave/beam systems with a local Kelvin-Voigt damping. Int. J. Control *89* (2016), 1933–1950. zbl MR doi
[23] *F. Hassine*: Logarithmic stabilization of the Euler-Bernoulli transmission plate equation with locally distributed Kelvin-Voigt damping. J. Math. Anal. Appl. *455* (2017), 1765–1782. zbl MR doi
[24] *J. E. Lagnese*: Boundary Stabilization of Thin Plates. SIAM Studies in Applied Mathematics 10. SIAM, Philadelphia, 1989. zbl MR doi
[25] *I. Lasiecka, R. Triggiani, J. Zhang*: Min-max game theory for elastic and visco-elastic fluid structure interactions. Open Appl. Math. J. *7* (2013), 1–17. zbl MR doi
[26] *G. Lebeau, L. Robbiano*: Contrôle exacte de l'équation de la chaleur. Commun. Partial Differ. Equations *20* (1995), 335–356. (In French.) zbl MR doi
[27] *G. Lebeau, L. Robbiano*: Stabilisation de l'équation des ondes par le bord. Duke Math. J. *86* (1997), 465–491. (In French.) zbl MR doi
[28] *J. Le Rousseau, G. Lebeau*: Introduction aux inégalités de Carleman pour les opérateurs elliptiques et paraboliques: Applications au prolongement unique et au contrôle des équations paraboliques. Available at `https://hal.archives-ouvertes.fr/hal-00351736v2` (2009), 27 pages. (In French.)
[29] *J. Le Rousseau, L. Robbiano*: Carleman estimate for elliptic operators with coefficients with jumps at an interface in arbitrary dimension and application to the null controllability of linear parabolic equations. Arch. Ration. Mech. Anal. *195* (2010), 953–990. zbl MR doi
[30] *Y.-F. Li, Z.-J. Han, G.-Q. Xu*: Explicit decay rate for coupled string-beam system with localized frictional damping. Appl. Math. Lett. *78* (2018), 51–58. zbl MR doi
[31] *K. Liu, Z. Liu*: Exponential decay of the energy of the Euler-Bernoulli beam with locally distributed Kelvin-Voigt damping. SIAM J. Control Optim. *36* (1998), 1086–1098. zbl MR doi
[32] *K. Liu, Z. Liu*: Exponential decay of energy of vibrating strings with local viscoelasticity. Z. Angew. Math. Phys. *53* (2002), 265–280. zbl MR doi
[33] *W. Liu, G. Williams*: The exponential stability of the problem of transmission of the wave equation. Bull. Aust. Math. Soc. *57* (1998), 305–327. zbl MR doi
[34] *A. J. A. Ramos, M. W. P. Souza*: Equivalence between observability at the boundary and stabilization for transmission problem of the wave equation. Z. Angew. Math. Phys. *68* (2017), Article ID 48, 11 pages. zbl MR doi
[35] *J. Rauch, X. Zhang, E. Zuazua*: Polynomial decay for a hyperbolic-parabolic coupled system. J. Math. Pures Appl., IX. Sér. *84* (2005), 407–470. zbl MR doi
[36] *J. T. Wloka, B. Rowley, B. Lawruk*: Boundary Value Problems for Elliptic System. Cambridge University Press, Cambridge, 1995. zbl MR doi
[37] *Q. Zhang*: Exponential stability of an elastic string with local Kelvin-Voigt damping. Z. Angew. Math. Phys. *61* (2010), 1009–1015. zbl MR doi
[38] *Q. Zhang*: On the lack of exponential stability for an elastic-viscoelastic waves interaction system. Nonlinear Anal., Real World Appl. *37* (2017), 387–411. zbl MR doi
[39] *X. Zhang, E. Zuazua*: Long-time behavior of a coupled heat-wave system arising in fluid-structure interaction. Arch. Ration. Mech. Anal. *184* (2007), 49–120. zbl MR doi

Authors' addresses: *Gimyong Hong*, Faculty of Mathematics, University of Sciences, Unjong District, 355 Pyongyang, Democratic People's Republic of Korea, e-mail: `saslib201902@star-co.net.kp`; *Hakho Hong* (corresponding author), Institute of Mathematics, State Academy of Sciences, Unjong District, 355 Pyongyang, Democratic People's Republic of Korea, e-mail: `hhhong@star.co-net.kp`.