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ON INVERSIONS OF VAN DER GRINTEN PROJECTIONS

Tomáš Bayer, Milada Kočandrlová, Praha

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Abstract. Approximately 150 map projections are known, but the inverse forms have been published for only two-thirds of them. This paper focuses on finding the inverse forms of van der Grinten projections I–IV, both by non-linear partial differential equations and by the straightforward inverse of their projection equations. Taking into account the particular cases, new derivations of coordinate functions are also presented. Both the direct and inverse equations have the analytic form, are easy to implement and are applicable to the coordinate transformations.

Keywords: mathematical cartography; inverse form; map; projection; van der Grinten; GIS

1. Introduction

The wider aspects of the spherical representation of the Earth's curved surface on a flat map are studied in mathematical cartography. Currently, many map projections of different distortion characteristics exist. Since for large-scale, maps conformity is preferred, small-scale maps are usually equal-area (the true area of the map objects is preserved) or compromise (balanced angular, length and areal distortions).

Approximately 150 map projections are known. However, many of them represent more cartographic art. In cartographic practice, approximately 100 is used, but actively less than 30. Inverse forms were published for less than two-thirds of them (including Grinten projections I and III). In this paper, we will try to improve existing solutions and find a more straightforward analytical representation.

The van der Grinten projections (I–IV) called globular (evoking a spherical globe), belong to the important map projections widely used in cartography. Providing a conventional image of the Earth, in which the hemisphere or planisphere is depicted as a circle and the meridian or parallel images are circular arcs, they are mainly used in atlases or on wall maps. Grinten projections are

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popular especially in American cartography [10], they are frequently used for political maps; see Figure 1. While they are designed to allow an easy construction of meridians and parallels by ruler and compass, finding the analytic form of the projection equations is relatively complicated. The inverse forms have been published for projections I and III, otherwise, only the numerical solution exists (Newton-Raphson method). Unfortunately, the iterative methods do not provide good results around singular points, where a convergence may be lost.

Knowledge of the inverse form of projections is crucial in many applications, especially when reprojecting a map into different projections, processing heterogeneous data in GIS or cartometric analyses of early maps [1], [2], [4]. The numerical approach to inverse brought several unwanted and disturbing artifacts or missing pixels, especially around the singular points of digitized maps. Block-based non-linear transformations with raster partitioning represent a slight improvement, but they bring problems with C^1 continuity along the boundaries of the blocks.

Therefore, finding the inverse in the analytic form represents a problem which can be utilized in many cartographic applications.

Initially, this paper provides a unified approach of derivation based on the intersection of the meridian and parallel arcs, leading to a system of quadratic and, eventually, cubic, equations. Subsequently, two approaches for finding the inverse forms are compared: a) the straightforward inverse of the projection equations, b) analytic solution of the partial differential equations. However, for the Grinten I and IV projections, the analytic form for the φ coordinate cannot be found; only a numerical solution of the partial differential equations is available.

2. State of art

The globular-like projection, later called van der Grinten I, was invented by the American cartographer Alphons Johann van der Grinten in 1898 [16]. As well as the second, this apple-shaped projection, showing the Earth as the union of two circular segments, later called van der Grinten IV, was published in two papers [17], [18]. The alternative approach to the derivation of projection IV can be found in [12], its equations in [19]. Two different modifications of the original projection have been described in [20]. The first version, where the meridians and parallels intersect each other at right angles, was later denoted in Roman numerals as II and the variant with straight parallels as III. Altogether, they belong to the family of van der Grinten projections. While their geometric properties, allowing direct construction by ruler and compass, have been described in the above-mentioned papers, the projection equations were derived later; see [16], [7].

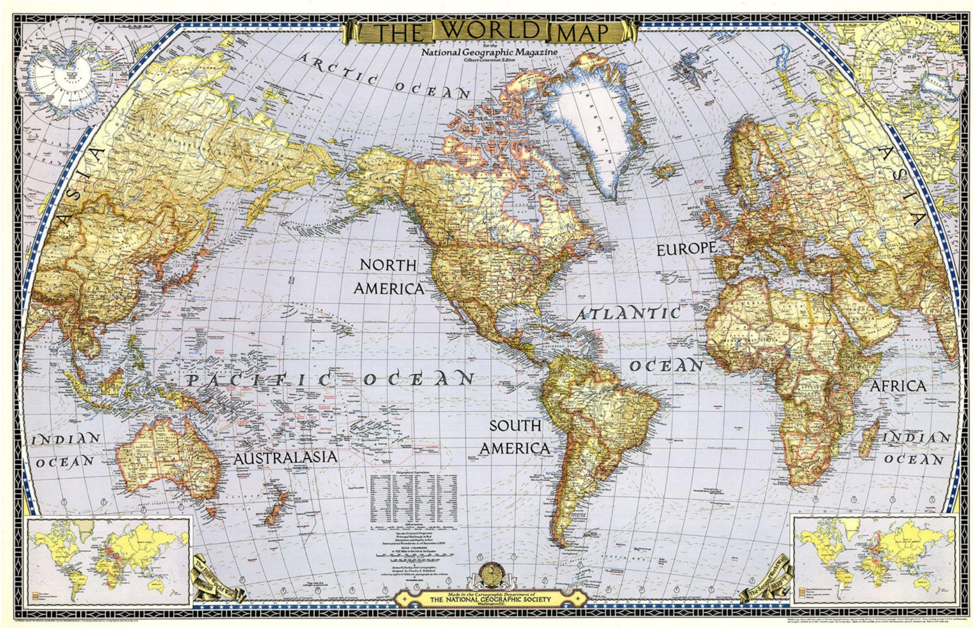

Figure 1. Reprint of the world map in the van der Grinten I. projection [9].

An analysis of the cartographic properties of projection III as well as the derivation of projection equations and attempts to find its inverse form can be found in [5]. More effort is concentrated on projection I. The analytic form of the projection I equations using the trigonometric functions was presented in [11], some corrections are in [14]. These equations were published in [15], [7] and implemented in the well-known open-source library Proj [6], which is the standard tool for coordinate transformations in geoinformatics. A modified derivation of projection I, leading to the cubic equations as well as its inverse form, can be found in [13].

Unlike Grinten projections I and III, the inverse forms for the remaining projections have not been published yet. For projections I and III, we will try to improve the existing solution and find a more suitable analytic form.

3. Map projection and its properties

Let the sphere $S^2 \subset \mathbb{R}^3$ with radius R be centered at the origin of the Cartesian coordinate system, let the local coordinate system $\{\varphi, \lambda\}$-spherical latitude and longitude be the reference surface, and let the plane σ with Cartesian coordinate system $\{x, y\}$ be the projected surface.

Definition 3.1. The projection of the set $M = \langle -\pi/2, \pi/2\rangle \times \langle -\pi, \pi\rangle$ into $\mathbb{R}^2$, given by the equations $x = f(\varphi, \lambda)$, $y = g(\varphi, \lambda)$, where f and g are real continuous functions defined on M, is called the *map projection* $\mathbb{P}$ of the sphere S^2 into the plane σ.

Definition 3.2. The *projection* $\mathbb{P}\colon S^2 \to \sigma$ is called to be *continuous at the point* $Q = [\varphi, \lambda]$ if the coordinate functions f, g are continuous at Q.

Definition 3.3. The *projection* $\mathbb{P}\colon S^2 \to \sigma$ is *injective on the set* M if for any different points $Q_1, Q_2 \in S^2$ we have $\mathbb{P}(Q_1) \neq \mathbb{P}(Q_2)$.

Theorem 3.4 (see [8])**.** *If the projection $\mathbb{P}$ on the boundary of Q is non-singular, then the inverse projection $\mathbb{P}^{-1}$ is non-singular on the boundary of $\mathbb{P}(Q)$ and the Jacobian determinants D, D' of both projections are reciprocal.*

Remark 3.5. The projection $\mathbb{P}$ will still be non-singular if, instead of the closed interval M, an open interval $M = (-\pi/2, \pi/2) \times (-\pi, \pi)$ will be considered. From the sphere S^2, the North Pole $A = [\pi/2, 0]$ and South Pole $B = [-\pi/2, 0]$ will be removed, since they are singular due to parameterization $\{\varphi, \lambda\}$.

The regular projection $\mathbb{P}$ between local coordinate systems $[\varphi, \lambda] \in (-\pi/2, \pi/2) \times (-\pi, \pi)$ of the sphere S^2 and $[x, y] \in \mathbb{R}^2$ of the plane is given by

$$x = f(\varphi, \lambda), \quad y = g(\varphi, \lambda). \tag{3.1}$$

In accordance with Theorem 3.4, the inverse projection $\mathbb{P}^{-1}$ exists such that

$$\varphi = f^{-1}(x, y), \quad \lambda = g^{-1}(x, y). \tag{3.2}$$

The differentials of the coordinate functions are

$$\mathbb{P}\colon \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \varphi} & \dfrac{\partial x}{\partial \lambda} \\ \dfrac{\partial y}{\partial \varphi} & \dfrac{\partial y}{\partial \lambda} \end{bmatrix} \begin{bmatrix} \mathrm{d}\varphi \\ \mathrm{d}\lambda \end{bmatrix}, \quad \mathbb{P}^{-1}\colon \begin{bmatrix} \mathrm{d}\varphi \\ \mathrm{d}\lambda \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \varphi}{\partial x} & \dfrac{\partial \varphi}{\partial y} \\ \dfrac{\partial \lambda}{\partial x} & \dfrac{\partial \lambda}{\partial y} \end{bmatrix} \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix}. \tag{3.3}$$

From (3.3) the vector of the differentials of the spherical coordinates can be found using the inverse matrix

$$\begin{bmatrix} \mathrm{d}\varphi \\ \mathrm{d}\lambda \end{bmatrix} = \frac{1}{D} \begin{bmatrix} \dfrac{\partial y}{\partial \lambda} & -\dfrac{\partial x}{\partial \lambda} \\ -\dfrac{\partial y}{\partial \varphi} & \dfrac{\partial x}{\partial \varphi} \end{bmatrix} \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix},$$

and, after comparison with $\mathbb{P}^{-1}$, we get

$$\frac{\partial y}{\partial \lambda} = D\frac{\partial \varphi}{\partial x}, \quad \frac{\partial x}{\partial \lambda} = -D\frac{\partial \varphi}{\partial y}, \quad \frac{\partial y}{\partial \varphi} = -D\frac{\partial \lambda}{\partial x}, \quad \frac{\partial x}{\partial \varphi} = D\frac{\partial \lambda}{\partial y},$$

the Jacobian determinants are

$$D = \frac{\partial x}{\partial \varphi}\frac{\partial y}{\partial \lambda} - \frac{\partial x}{\partial \lambda}\frac{\partial y}{\partial \varphi}, \quad D' = \frac{1}{D} = \frac{\partial \varphi}{\partial x}\frac{\partial \lambda}{\partial y} - \frac{\partial \varphi}{\partial y}\frac{\partial \lambda}{\partial x}. \tag{3.4}$$

Matrix $\mathbb{P}$ rows are the tangent vectors of curves in the plane with squares

$$\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 = \frac{1}{D^2}\left[\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2\right] = \frac{G}{D^2}, \tag{3.5}$$

$$\left(\frac{\partial \lambda}{\partial x}\right)^2 + \left(\frac{\partial \lambda}{\partial y}\right)^2 = \frac{1}{D^2}\left[\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2\right] = \frac{E}{D^2}, \tag{3.6}$$

their dot product is

$$\frac{\partial \varphi}{\partial x}\frac{\partial \lambda}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{\partial \lambda}{\partial y} = -\frac{1}{D^2}\left(\frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial \lambda} + \frac{\partial y}{\partial \varphi}\frac{\partial y}{\partial \lambda}\right) = -\frac{F}{D^2}. \tag{3.7}$$

Definition 3.6. The quadratic differential form

$$(\mathrm{d}s')^2 = E\,\mathrm{d}\varphi^2 + 2F\,\mathrm{d}\varphi\,\mathrm{d}\lambda + G\,\mathrm{d}\lambda^2 \tag{3.8}$$

is called the *first fundamental form of the plane* σ, where the factors E, F, G are the right-hand sides of (3.5)–(3.7).

Definition 3.7. The quadratic differential form

$$\mathrm{d}s^2 = \mathrm{d}\varphi^2 + \cos^2\varphi\,\mathrm{d}\lambda^2 \tag{3.9}$$

is called the *first fundamental form of the unit sphere* S^2, parametrized with $X(\varphi, \lambda) = [\cos\varphi\cos\lambda, \cos\varphi\sin\lambda, \sin\varphi]$.

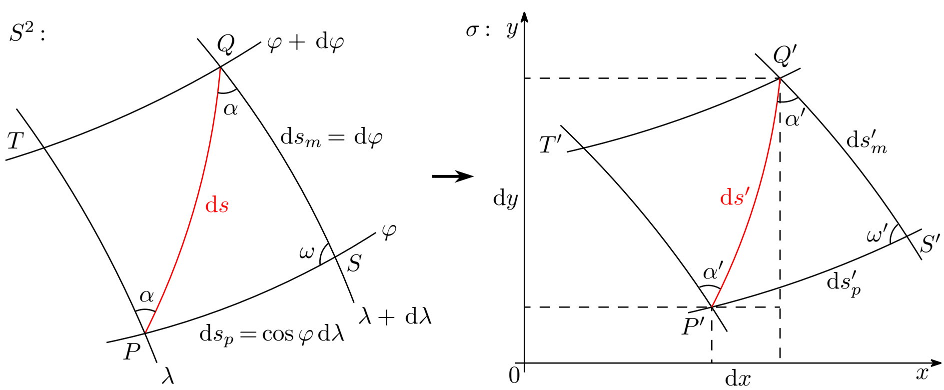

Figure 2. Spherical quadrangle $PTQS$ and its image $P'T'Q'S'$ in $\mathbb{P}$.

Let $P[\varphi,\lambda]$ and $Q[\varphi+\mathrm{d}\varphi,\lambda+\mathrm{d}\lambda]$ be two infinitesimally close points on the great circle of the sphere S^2, $P'[x,y]$ and $Q'[x+\mathrm{d}x,y+\mathrm{d}y]$ their images in σ: $\mathbb{P}(P)=P'$, $\mathbb{P}(Q)=Q'$, see Figure 2. The meridians $m(\varphi)$, $m(\varphi+\mathrm{d}\varphi)$ and parallels $p(\lambda)$, $p(\lambda+\mathrm{d}\lambda)$ form the infinitesimal spherical quadrangle $PTQS$. The lengths $\mathrm{d}s_m$ of the meridian arc between points P,S, and $\mathrm{d}s_p$ of the parallel arc between points S, Q are

$$\mathrm{d}s_m=\mathrm{d}\varphi,\quad \mathrm{d}s_p=\cos\varphi\,\mathrm{d}\lambda.$$

The great circle passing through the points P, Q intersects the meridian $m(\varphi)$ at angle α representing the azimuth, where

$$\sin\alpha=\frac{\mathrm{d}s_p}{\mathrm{d}s},\quad \cos\alpha=\frac{\mathrm{d}s_m}{\mathrm{d}s},$$

$\mathrm{d}s$ is the arc of PQ. Then

$$\mathrm{d}\varphi=\cos\alpha\,\mathrm{d}s,\quad \mathrm{d}\lambda=\frac{\sin\alpha}{\cos\varphi}\,\mathrm{d}s. \tag{3.10}$$

Definition 3.8. The ratio

$$\mu=\frac{\mathrm{d}s'}{\mathrm{d}s}$$

of an infinitesimal element $\mathrm{d}s'$ in the plane σ to that of the corresponding infinitesimal element $\mathrm{d}s$ on the sphere S^2 is called a *local linear scale at P'*.

Substituting for $\mathrm{d}s'$, $\mathrm{d}\varphi$, $\mathrm{d}\lambda$ from (3.8) and (3.10), we get

$$\mu^2=\Big(\frac{\mathrm{d}s'}{\mathrm{d}s}\Big)^2=E\cos^2\alpha+\frac{G}{\cos^2\varphi}\sin^2\alpha+\frac{F}{\cos\varphi}\sin 2\alpha.$$

It is obvious that the local linear scale μ depends both on the position (φ,λ) of P and the azimuth α of PQ.

R e m a r k 3.9. For the azimuth $\alpha=0$,

$$h^2=\mu^2=E \tag{3.11}$$

is the local linear scale μ^2 along a meridian. For the azimuth $\alpha=\pi/2$,

$$k^2=\mu^2=\frac{G}{\cos^2\varphi} \tag{3.12}$$

is the local linear scale μ^2 along a parallel.

Substituting for $\mathrm{d}x,\ \mathrm{d}y$ from (3.3), the direction γ of the line tangent to the curve $P'Q'$ at P' is

$$\tan\gamma = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\partial y}{\partial\varphi}\,\mathrm{d}\varphi + \frac{\partial y}{\partial\lambda}\,\mathrm{d}\lambda}{\frac{\partial x}{\partial\varphi}\,\mathrm{d}\varphi + \frac{\partial x}{\partial\lambda}\,\mathrm{d}\lambda} = \frac{\frac{\partial y}{\partial\varphi}\cos\varphi\cos\alpha + \frac{\partial y}{\partial\lambda}\sin\alpha}{\frac{\partial x}{\partial\varphi}\cos\varphi\cos\alpha + \frac{\partial x}{\partial\lambda}\sin\alpha}.$$

Remark 3.10. The direction $\gamma_m = \gamma$ $(\alpha = 0)$ of a meridian and $\gamma_p = \gamma$ $(\alpha = \pi/2)$ of a parallel at P' are

$$\tan\gamma_m = \frac{\partial y}{\partial\varphi}\Big/\frac{\partial x}{\partial\varphi}, \quad \tan\gamma_p = \frac{\partial y}{\partial\lambda}\Big/\frac{\partial x}{\partial\lambda}.$$

Angle ω' between the projected meridian $m(\varphi)$ and parallel $p(\lambda)$ at P' is their difference

$$\omega' = \gamma_m - \gamma_p,$$

where

$$\tan\omega' = \tan(\gamma_m - \gamma_p) = \frac{\tan\gamma_m - \tan\gamma_m}{1 + \tan\gamma_m\tan\gamma_p} = \frac{H}{F}$$

and

$$H = \sqrt{EG - F^2} = \frac{\partial x}{\partial\lambda}\frac{\partial y}{\partial\varphi} - \frac{\partial x}{\partial\varphi}\frac{\partial y}{\partial\lambda} = -D.$$

Remark 3.11. Since both the meridians and parallels form the regular parametrization of the sphere, the angle between $m(\varphi)$ and parallel $p(\lambda)$ at P is $\omega = \pi/2$.

Definition 3.12. The *projection* $\mathbb{P}$ is called *orthogonal* if the angle between the projected meridian $m(\varphi)$ and parallel $p(\lambda)$ at P' is $\omega' = \pi/2$, and

$$F = \frac{\partial x}{\partial\varphi}\frac{\partial x}{\partial\lambda} + \frac{\partial y}{\partial\varphi}\frac{\partial y}{\partial\lambda} = 0. \tag{3.13}$$

The area $\mathrm{d}\Sigma$ of the corresponding infinitesimal triangle PSQ on the sphere S^2 and $\mathrm{d}\Sigma'$ of its image $P'S'Q'$ in the plane σ are

$$\mathrm{d}\Sigma = \frac{1}{2}\,\mathrm{d}s_m\,\mathrm{d}s_p\sin\omega, \quad \mathrm{d}\Sigma' = \frac{1}{2}\,\mathrm{d}s'_m\,\mathrm{d}s'_p\sin\omega',$$

where

$$\sin\omega' = \frac{\tan\omega'}{\sqrt{1 + \tan^2\omega'}} = \frac{H}{\sqrt{F^2 + H^2}} = \frac{H}{\sqrt{EG}}.$$

Definition 3.13. The ratio of areas

$$\wp = \frac{\mathrm{d}\Sigma'}{\mathrm{d}\Sigma} = hk\sin\omega' = \frac{H}{\cos\varphi} \tag{3.14}$$

of the infinitesimal triangle $P'S'Q'$ in the plane σ to that of the corresponding infinitesimal triangle PSQ on the sphere S^2 is called a *local area scale at* P'.

3.1. Partial differential equations of the inverse transformation. A substitution for $h, k, \wp$ into (3.5)–(3.6) leads to the following differential equations.

Definition 3.14. The partial differential equations

$$\left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2 = \frac{h^2}{\wp^2\cos^2\varphi}, \tag{3.15}$$

$$\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 = \frac{k^2}{\wp^2} \tag{3.16}$$

are the *equations of the projection* $\mathbb{P}^{-1}$ *inverse to* $\mathbb{P}$, where h, k are the local linear scales (3.11)–(3.12), and $\wp$ is the local area scale (3.14) at P'.

Remark 3.15. For most Grinten projections, the partial differential equations (3.15)–(3.16) have the general form of

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = h(x, y)g(w), \tag{3.17}$$

the right-hand side is the function of x, y, w. Then (3.17) is the non-linear partial differential equation of the first order

$$f(x, y, w, p, q) = 0 \tag{3.18}$$

with the solution $w(x, y)$, where $p = \partial w/\partial x, q = \partial w/\partial y$.

Definition 3.16. The partial differential equation (3.18), which is equivalent to the system of ordinary differential equations

$$\frac{\mathrm{d}p}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{\mathrm{d}q}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{\mathrm{d}z}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{\mathrm{d}x}{-\frac{\partial f}{\partial p}} = \frac{\mathrm{d}y}{-\frac{\partial f}{\partial q}}, \tag{3.19}$$

has the solution $(x(t), y(t), z(t), p(t), q(t))$ satisfying the condition $f(x(t), y(t), z(t), p(t), q(t)) = 0$, which is called a *Monge strip.*

Remark 3.17. A set of ordinary differential equations (3.19) is called *Lagrange-Charpit equations* of (3.18).

Definition 3.18. Let $f(x(t), y(t), z(t), p(t), q(t))$ be the Monge strip satisfying (3.19). The curve in (x, y)-plane given by $(x(t), y(t))$ is called a *characteristic curve* of (3.18).

Remark 3.19. For most map projections except simple cases (including elementary projections), where $x = f(\lambda)$, $y = g(\varphi)$, finding a closed-form solution for the problem is difficult.

Using the substitution

$$\mathrm{d}z = \frac{\mathrm{d}w}{\sqrt{h(x,y)}}, \quad z = \int \frac{1}{\sqrt{h(x,y)}}\,\mathrm{d}w, \tag{3.20}$$

(3.17) is transformed to a simpler form

$$\Big(\frac{\partial z}{\partial x}\Big)^2 + \Big(\frac{\partial z}{\partial y}\Big)^2 = h(x,y),$$

which represents the non-linear partial differential equation of the first order

$$f(x,y,z,P,Q) = 0, \quad P = \frac{\partial z}{\partial x}, \quad Q = \frac{\partial z}{\partial y}. \tag{3.21}$$

The closed-form solution can be found only for some specific types of the right-hand side $h(x,y)$. For the Grinten III projection, (3.21) has a simple form of

$$P^2 + Q^2 - h(x) = 0,$$

see Section 4.1.1. For some kinds of map projections, their inverse forms are more straightforward.

Remark 3.20. In orthogonal projections, where $\omega' = \pi/2$ and $\wp = hk$, the partial differential equations (3.15)–(3.16) simplify to

$$\Big(\frac{\partial\lambda}{\partial x}\Big)^2 + \Big(\frac{\partial\lambda}{\partial y}\Big)^2 = \frac{1}{k^2\cos^2\varphi} = \frac{1}{(\frac{\partial x}{\partial\lambda})^2 + (\frac{\partial y}{\partial\lambda})^2}, \tag{3.22}$$

$$\Big(\frac{\partial\varphi}{\partial x}\Big)^2 + \Big(\frac{\partial\varphi}{\partial y}\Big)^2 = \frac{1}{h^2} = \frac{1}{(\frac{\partial x}{\partial\varphi})^2 + (\frac{\partial y}{\partial\varphi})^2}. \tag{3.23}$$

Remark 3.21. Since $y = g(\varphi)$, parallels are represented by equally spaced straight lines, and (3.16) transforms into

$$\mathrm{d}\varphi = \frac{k}{\wp}\,\mathrm{d}y. \tag{3.24}$$

Since $x = f(\lambda)$, meridians are represented by straight lines, and (3.15) transforms into

$$\mathrm{d}\lambda = \frac{h}{\wp\cos\varphi}\,\mathrm{d}x. \tag{3.25}$$

For the Grinten III. projection, the solution of (3.24) leads to the exact differential equation, see Section 4.1.1.

4. Inverse forms of van der Grinten projections

Like other globular projections, van der Grinten projections can easily be constructed by ruler and compass. While the meridians are formed by circular arcs equally spaced along the equator and concave toward the central meridian, the parallels are circular arcs concave to the nearest pole or straight lines. The planisphere is represented by a circle (I–III) or by the union of two circular segments (IV); see Figure 3. Van der Grinten projections are neither conformal, nor equal-area, nor equidistant.

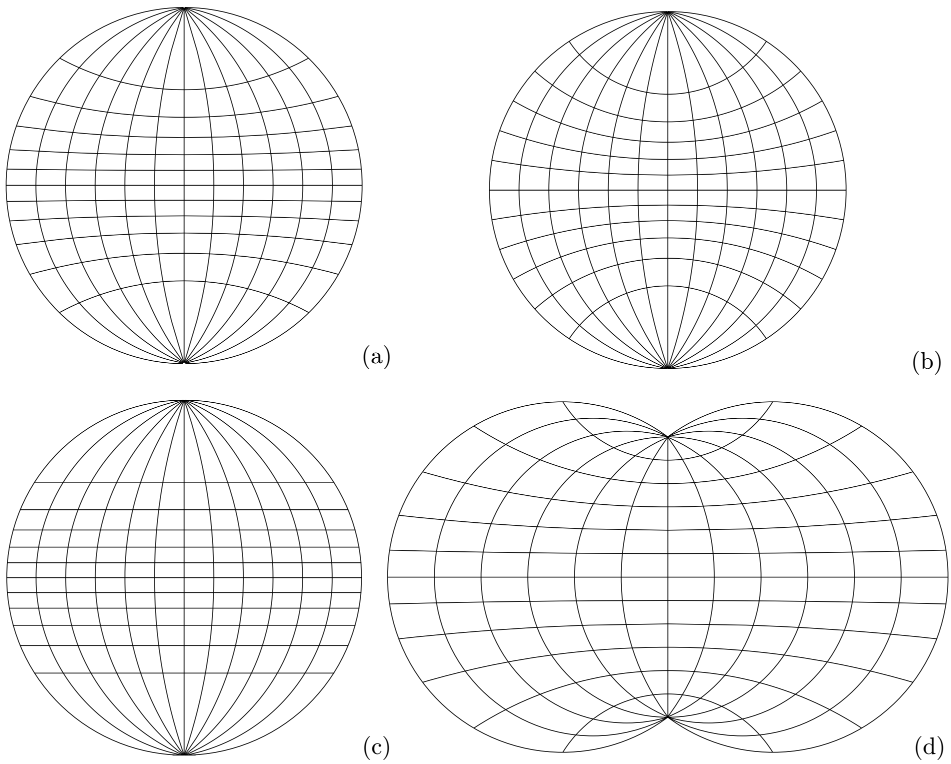

Figure 3. Graticules of van der Grinten projections: (a) projection I, (b) projection II, (c) projection III, (d) projection IV.

Two approaches for finding the inverse forms are presented. Since the straightforward inverse leads to the system of quadratic or cubic equations, the partial differential equations of the inverse transformation need to be transformed to the set of ordinary differential equations. However, this method has some limitations if the right-hand side has an overcomplicated form; this is typical for Grinten I and IV projections when the analytic solution cannot be found.

4.1. Van der Grinten projection III. The sphere projects into a unit circle, the meridians are circular arcs equally spaced along the Equator, the parallels are straight lines. The projection preserves true scales along the Equator, but the areal distortion increases rapidly towards the boundaries.

Let us place the center of the unit circle at the origin O of the Cartesian coordinate system. The projected Equator CD is aligned with the x-axis, the projected central meridian AB with the y-axis, where A, B are the images of the North and South Poles, see Figure 4.

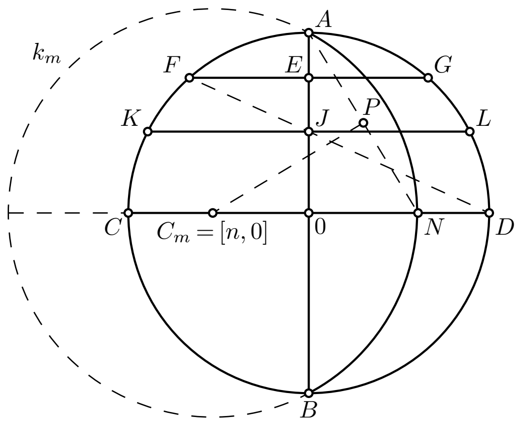

Figure 4. Van der Grinten projection III, a geometric construction of meridians and parallels.

For the parallel of latitude φ construction, the line AO is divided by equally spaced points $E[0, t]$, where $t : 1 = \varphi : \pi/2$, $t \in (0, 1\rangle$. Point E defines the auxiliary line FG, $F[-\sqrt{1-t^2}, t]$, parallel to CD. The point F is projected from D to AO as the point $J[0, y]$. From two similar triangles ODJ, EFJ we have

$$\frac{|FE|}{t-y} = \frac{1}{y},$$

the y coordinate satisfies

$$y = \frac{1-\sqrt{1-t^2}}{t}. \tag{4.1}$$

The line KL parallel to CD, passing through the point J, represents the required parallel image. The meridian arc passing through the point $N[s, 0]$ has the center at the point of intersection $C_m[n, 0]$ of the bisector AN and the x-axis. Since $s \in (0, 1\rangle$, we have $\lambda : \pi = s : 1$, the arc center is

$$n = \frac{s^2-1}{2s} = \frac{\lambda^2-\pi^2}{2\pi\lambda}, \tag{4.2}$$

its radius r_m is

$$r_m = \sqrt{1+n^2} = s + n = \frac{1+s^2}{2s}.$$

The point of intersection of the meridian and parallel arcs can be found by solving the system of equations

$$(x-n)^2 + y^2 = 1 + n^2, \quad y = \frac{1-\sqrt{1-t^2}}{t}.$$

The results are summarized in Theorem 4.1.

Theorem 4.1. *The Projection* III *is given by the equations*

$$x = n \pm \sqrt{1 - y^2 + n^2}, \quad y = \frac{1-\sqrt{1-t^2}}{t}, \tag{4.3}$$

where n is the coordinate of the center of the meridian arc and $t = 2\varphi/\pi$ is the parameter.

Remark 4.2. Substituting for y and n into (4.1) leads to the quadratic equation for the coordinate x

$$ax^2 + bx + c = 0$$

with the factors

$$a = st^2, \quad b = t^2(1-s^2), \quad c = 2s(1 - t^2 - \sqrt{1-t^2}),$$

where $s = \lambda/\pi$ is the parameter.

Remark 4.3. The particular cases are solved as follows:
If $\lambda = 0$, then $x = 0$ and the y coordinate holds (4.1).
If $\varphi = 0$, then $x = \lambda/\pi$ and $y = 0$.
If $\varphi = \pm\pi/2$, then $x = 0$ and $y = \pm 1$.

4.1.1. Inverse form using partial differential equations. Initially, the inverse form of the Grinten III projection will be found by solving (3.15)–(3.24). For the x-coordinate, which is a function of φ, λ, the solution will be more difficult.

Recall that the coordinate function for y depends only on φ:

$$y(\varphi) = \frac{1-\sqrt{1-t^2(\varphi)}}{t(\varphi)}, \tag{4.4}$$

where $t(\varphi) = 2\varphi/\pi$. The local linear scale along a parallel is

$$k = \frac{1}{\cos\varphi}\frac{\partial x}{\partial n}\frac{\partial n}{\partial \lambda},$$

the local area scale is

$$\wp = \frac{1}{\cos\varphi}\frac{\partial x}{\partial n}\frac{\partial n}{\partial \lambda}\frac{\partial y}{\partial t}\frac{\partial t}{\partial \varphi}, \tag{4.5}$$

their fraction is

$$\frac{k}{P} = \frac{1}{\frac{\partial y}{\partial t}\frac{\partial t}{\partial \varphi}} = \frac{\pi}{2\frac{\partial y}{\partial t}}.$$

Rewriting (4.4) as an implicit function

$$t^2 - 2yt + y^2t^2 = 0,$$

after the differentiation we have

$$\frac{\partial y}{\partial t} = \frac{1}{1-yt} - \frac{y}{t} = \pi\frac{2\varphi(1+y^2)-\pi y}{2\varphi(\pi - 2\varphi y)}.$$

Because $\partial\varphi/\partial x = 0$, (3.24) transforms to

$$\frac{\partial \varphi}{\partial y} = \frac{\pi}{2\frac{\partial y}{\partial t}},$$

which is the exact differential equation

$$(2\varphi(1+y^2) - \pi y)\,\mathrm{d}\varphi + \varphi(2\varphi y - \pi)\,\mathrm{d}y = 0,$$

since

$$\frac{\partial}{\partial y}(2\varphi(1+y^2) - \pi y) = \frac{\partial}{\partial \varphi}(\varphi(2\varphi y - \pi)).$$

The general solution is

$$\varphi(2\varphi(1+y^2) - \pi y) + \varphi y(2\varphi y - \pi) = c,$$

where $c \in \mathbb{R}$ is an arbitrary constant of integration. For $c = 0$, the inverse equation of the Grinten III projection has the form of

$$\varphi = \frac{\pi y}{1+2y^2}. \tag{4.6}$$

Recall that the coordinate function for x depends both on φ, λ:

$$x(\varphi, \lambda) = n(\lambda) \pm \sqrt{1 - y^2(\varphi) + n^2(\lambda)}, \quad n(\lambda) = \frac{s^2(\lambda) - 1}{2s(\lambda)},$$

where $s(\lambda) = \lambda/\pi$. The local linear scale along a meridian is

$$h = \frac{\partial t}{\partial \varphi}\sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2}. \tag{4.7}$$

Using the substitution $c = \sqrt{1 - y^2 + n^2}$, the partial derivatives have the form of

$$\frac{\partial n}{\partial \lambda} = \frac{\pi^2 + \lambda^2}{2\pi\lambda^2}, \quad \frac{\partial x}{\partial n} = \frac{c \pm n}{c}, \quad \frac{\partial x}{\partial t} = \frac{\partial x}{\partial y}\frac{\partial y}{\partial t} = \pm\frac{y}{c}\frac{\partial y}{\partial t}, \quad \frac{\partial x}{\partial \lambda} = \frac{\partial x}{\partial n}\frac{\partial n}{\partial \lambda} = \frac{c \pm n}{c}\frac{\pi^2 + \lambda^2}{2\pi\lambda^2}.$$

After back substitution into (4.7)

$$h = \frac{2\sqrt{c^2 + y^2}}{c\pi}\frac{\partial y}{\partial t}, \quad \wp = \frac{2}{\pi}\frac{(c \pm n)}{c\cos\varphi}\frac{\partial n}{\partial \lambda}\frac{\partial y}{\partial t},$$

the fraction transforms into

$$\frac{h}{\wp\cos\varphi} = \frac{\sqrt{1 + n^2}}{(c \pm n)\frac{\partial n}{\partial \lambda}}.$$

Then the partial differential equation (3.15) has the form of

$$\left(\frac{\partial \lambda}{\partial x}\right)^2 + \left(\frac{\partial \lambda}{\partial y}\right)^2 = \frac{1 + n^2}{(c \pm n)^2(\frac{\partial n}{\partial \lambda})^2}. \tag{4.8}$$

Taking into account that

$$(c \pm n)^2 = x^2, \quad 1 + n^2 = \frac{(\lambda^2 + \pi^2)^2}{4\pi^2\lambda^2},$$

(4.8) simplifies into

$$\left(\frac{\partial \lambda}{\partial x}\right)^2 + \left(\frac{\partial \lambda}{\partial y}\right)^2 = \frac{\lambda^2}{x^2}. \tag{4.9}$$

Considering (3.20), after substitution

$$\mathrm{d}z = \frac{\mathrm{d}\lambda}{\lambda}, \quad z = \ln\lambda, \quad \lambda = \mathrm{e}^z,$$

equation (4.9) transforms into

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \frac{1}{x^2},$$

which is the non-linear partial differential equation of the first order. Using the Lagrange-Charpit method, it can be rewritten to

$$x^2P^2 + x^2Q^2 - 1 = 0, \quad P = \frac{\partial z}{\partial x}, \quad Q = \frac{\partial z}{\partial y}. \tag{4.10}$$

Then the auxiliary Lagrange-Charpit equations, transforming the partial differential equation into a set of ordinary differential equations, are

$$\frac{\mathrm{d}P}{-x(P^2+Q^2)} = \frac{\mathrm{d}Q}{0} = \frac{\mathrm{d}z}{x^2(P^2+Q^2)} = \frac{\mathrm{d}x}{Px^2} = \frac{\mathrm{d}y}{Qx^2}.$$

The second member has to be understood as the asymptotic notation. Since $\mathrm{d}Q = 0$, we get $Q = a$, where $a \in \mathbb{R}$ is the arbitrary constant of integration. Putting it into (4.10), we get

$$x^2P^2 + x^2a^2 = 1 \Rightarrow P^2 = \frac{1-a^2x^2}{x^2}.$$

Since we know that

$$\mathrm{d}z = \frac{\partial z}{\partial x}\,\mathrm{d}x + \frac{\partial z}{\partial y}\,\mathrm{d}z = P\,\mathrm{d}x + Q\,\mathrm{d}y,$$

integrating on both sides, we get

$$\mathrm{d}z = \frac{\sqrt{1-a^2x^2}}{x}\,\mathrm{d}x + a\,\mathrm{d}y,$$

where

$$z = \int \frac{\sqrt{1-a^2x^2}}{x}\,\mathrm{d}x + ay + b, \tag{4.11}$$

$b \in \mathbb{R}$ is the arbitrary constant of integration. Using the substitution $t^2 = 1 - a^2x^2$, the integral can be rewritten to the form of

$$\int \frac{\sqrt{1-a^2x^2}}{x}\,\mathrm{d}x = \int \frac{t^2}{t^2-1}\,\mathrm{d}t = t + \frac{1}{2}\ln\frac{t-1}{t+1} = \sqrt{1-a^2x^2} + \ln\frac{1-\sqrt{1-a^2x^2}}{ax}.$$

After back substitution into (4.11)

$$z = \sqrt{1-a^2x^2} + \ln\frac{1-\sqrt{1-a^2x^2}}{ax} + ay + b = \ln\lambda,$$

the general solution of the partial differential equation is

$$\lambda = \mathrm{e}^{ay+b+\sqrt{1-a^2x^2}}\,\frac{1-\sqrt{1-a^2x^2}}{ax}. \tag{4.12}$$

The arbitrary constants of integration a, b are chosen so that the curve (4.12) passes through the unit circle $x^2 + y^2 = 1$ and $z = 0$. Since

$$y + \sqrt{1 - x^2} = 0,$$

we choose $a = 1$, $b = 0$, and after comparison with (4.12), the associated particular solution is

$$\lambda = \frac{1 - \sqrt{1 - x^2}}{x} = \frac{1 + y}{x}. \tag{4.13}$$

4.1.2. Straightforward inverse of projection equations. Due to the simple form of the projection equations, the straightforward inverse is efficient.

Theorem 4.4. *The inverse formulas of van der Grinten projection* III *have the form of*

$$\varphi = \frac{\pi}{2} t, \quad \lambda = \pi s, \tag{4.14}$$

where s is the solution of a system of quadratic equations

$$1 + 2sn - s^2 = 0, \quad x^2 + y^2 - 2nx = 1. \tag{4.15}$$

It leads to the quadratic equation for s

$$xs^2 + (1 - x^2 - y^2)s - x = 0, \tag{4.16}$$

where longitude λ takes the sign of x. Subsequently, the parameter t is determined from (4.1)

$$t = \frac{2y}{1 + y^2}. \tag{4.17}$$

Remark 4.5. The particular cases are solved as follows:
If $x = 0$, then $\varphi = \pi/2t$ and $\lambda = 0$.
If $y = 0$, then $\varphi = 0$ and the quadratic equation transforms to

$$xs^2 + (1 - x^2)s - x = 0$$

with the solution of

$$s = \frac{x^2 - 1 \pm (x^2 + 1)}{2x}.$$

Because $s \in (0, 1\rangle$, then $\lambda = \pi x$.

4.2. Van der Grinten projection II. For the derivation of projection II, the above-mentioned facts regarding projection III will be used. While the world is enclosed in a unit circle, the parallels are circular arcs unequally spaced along the central meridian. All meridians intersect parallels at right angles, but the projection is not conformal. For the construction, see Figure 5.

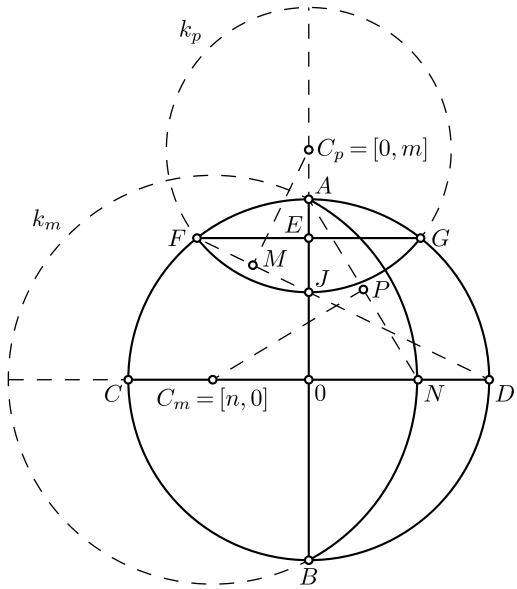

Figure 5. Van der Grinten projection II, a geometric construction of meridians and parallels.

Analogously, the line AO is divided by equally spaced points $E[0,t]$, $t \in (0,1\rangle$. The auxiliary line FG, $F[-\sqrt{1-t^2}, t]$, parallel to CD, passes through the point E. The point F is projected from D to AO as the point $J[0, y]$. Likewise, the meridian arc is defined by the three points F, J, G, where J lies on the Equator.

The radius of the parallel arc is determined from the right triangle C_pFO using the geometric mean theorem $|FE|^2 = |C_pE| \cdot t$, which leads to

$$|C_pE| = \frac{1-t^2}{t}.$$

The parallel center is

$$m = |C_pE| + t = \frac{1}{t}, \tag{4.18}$$

its radius is determined from the right triangle C_pGO or C_pFE, as

$$r_p^2 = m^2 - 1 = \frac{1-t^2}{t^2}.$$

The meridian arc is constructed analogously to projection III. The point of intersection of the meridian and parallel arcs can be found by solving the system of quadratic equations

$$(x-n)^2+y^2=1+n^2, \tag{4.19}$$

$$x^2+(y-m)^2=m^2-1. \tag{4.20}$$

Subtracting one from the other leads to the linear equation

$$nx-my+1=0. \tag{4.21}$$

Substituting for y into (4.19) leads to the quadratic equation for x

$$(m^2+n^2)x^2+2xn(1-m^2)+1-m^2=0. \tag{4.22}$$

Its factors a, b, c are functions of s, t:

$$a=s^2+\frac{1}{s^2}+\frac{4}{t^2}-2, \quad b=\frac{4(s^2-1)(t^2-1)}{st^2}, \quad c=4\Big(1-\frac{1}{t^2}\Big).$$

Substituting for x into (4.20) leads to the quadratic equation for y

$$(m^2+n^2)y^2-2ym(1+n^2)+1+n^2=0, \tag{4.23}$$

the factors a, b, c can be expressed as

$$a=s^2+\frac{1}{s^2}+\frac{4}{t^2}-2, \quad b=-\frac{2(s^2+1)^2}{s^2t}, \quad c=\frac{(s^2+1)^2}{s^2}.$$

Let us summarize the results into a theorem.

Theorem 4.6. *The image of the point $P[\varphi,\lambda]$ in van der Grinten projection* II *is given by* (4.19)–(4.20), *where from* (4.18) *it is $m=1/t$ and $n=(s^2-1)/(2s)$.*

Remark 4.7. Alternatively, instead of solving (4.23), from (4.19), the y coordinate holds

$$y=\sqrt{1-x^2+2xn}.$$

Remark 4.8. Particular situations when $\varphi=0$ or $\lambda=0$ are solved analogously to projection III.

4.2.1. Inverse form using partial differential equations. The projection equations can be rewritten to the form of

$$x^2 - 2xn + y^2 - 1 = 0, \tag{4.24}$$

$$x^2 + y^2 - 2ym + 1 = 0. \tag{4.25}$$

Initially, it needs to be verified whether the projection is orthogonal. Since the projection equations are composite functions of m, n, s, t, we have

$$\frac{\partial x}{\partial \varphi} = \frac{\partial x}{\partial m}\frac{\partial m}{\partial t}\frac{\partial t}{\partial \varphi}, \quad \frac{\partial y}{\partial \varphi} = \frac{\partial y}{\partial m}\frac{\partial m}{\partial t}\frac{\partial t}{\partial \varphi}, \quad \frac{\partial x}{\partial \lambda} = \frac{\partial x}{\partial n}\frac{\partial n}{\partial s}\frac{\partial s}{\partial \lambda}, \quad \frac{\partial y}{\partial \lambda} = \frac{\partial y}{\partial n}\frac{\partial n}{\partial s}\frac{\partial s}{\partial \lambda}.$$

The orthogonality condition (3.13) transforms into

$$\frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial \lambda} + \frac{\partial y}{\partial \varphi}\frac{\partial y}{\partial \lambda} = \frac{\partial m}{\partial t}\frac{\partial t}{\partial \varphi}\frac{\partial n}{\partial s}\frac{\partial s}{\partial \lambda}\left(\frac{\partial x}{\partial m}\frac{\partial x}{\partial n} + \frac{\partial y}{\partial m}\frac{\partial y}{\partial n}\right) = 0.$$

Hence, the projection is orthogonal if

$$\frac{\partial x}{\partial m}\frac{\partial x}{\partial n} + \frac{\partial y}{\partial m}\frac{\partial y}{\partial n} = 0. \tag{4.26}$$

Partial derivatives of m, n with respect to φ, λ are:

$$\frac{\partial m}{\partial t} = -\frac{1}{t^2} = -\frac{\pi^2}{4\varphi^2}, \quad \frac{\partial t}{\partial \varphi} = \frac{2}{\pi}, \quad \frac{\partial m}{\partial \varphi} = \frac{\partial m}{\partial t}\frac{\partial t}{\partial \varphi} = -\frac{\pi}{2\varphi^2},$$

and

$$\frac{\partial n}{\partial s} = \frac{s^2+1}{2s^2} = \frac{\pi^2+\lambda^2}{2\lambda^2}, \quad \frac{\partial s}{\partial \lambda} = \frac{1}{\pi}, \quad \frac{\partial n}{\partial \lambda} = \frac{\partial n}{\partial s}\frac{\partial s}{\partial \lambda} = \frac{\pi^2+\lambda^2}{2\pi\lambda^2},$$

partial derivatives of (4.21) with respect to m, n are:

$$\frac{\mathrm{d}x}{\mathrm{d}m} = \frac{y}{n}, \quad \frac{\mathrm{d}x}{\mathrm{d}n} = -\frac{x}{n}, \quad \frac{\mathrm{d}y}{\mathrm{d}m} = -\frac{y}{m}, \quad \frac{\mathrm{d}y}{\mathrm{d}n} = \frac{x}{m}, \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{n}{m}, \quad \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{m}{n}.$$

For the projection equations (4.24)–(4.25), the implicit differentiation with respect to x, y, m, n is used. The partial derivatives with respect to m are

$$2x\,\mathrm{d}x - 2n\,\mathrm{d}x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x + 2y\frac{\mathrm{d}y}{\mathrm{d}m}\mathrm{d}m = 0 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}m} = \frac{y^2}{xm - nm + yn},$$

$$2x\frac{\mathrm{d}x}{\mathrm{d}y}\,\mathrm{d}y - 2m\,\mathrm{d}y + 2y\,\mathrm{d}y + 2x\frac{\mathrm{d}x}{\mathrm{d}m}\,\mathrm{d}m - 2y\,\mathrm{d}m = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}m} = \frac{yn - xy}{xm - nm + yn}.$$

Analogously, the partial derivatives with respect to n are

$$2x\,\mathrm{d}x - 2n\,\mathrm{d}x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x - 2x\,\mathrm{d}n + 2y\frac{\mathrm{d}y}{\mathrm{d}n}\,\mathrm{d}n = 0 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}n} = \frac{xm - yx}{xm - nm + yn},$$
$$2x\frac{\mathrm{d}x}{\mathrm{d}y}\,\mathrm{d}y - 2m\,\mathrm{d}y + 2y\,\mathrm{d}y + 2x\frac{\mathrm{d}x}{\mathrm{d}n}\,\mathrm{d}n = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}n} = \frac{x^2}{xm - nm + yn}.$$

Proceed with verification of the projection orthogonality. After substituting into partial derivatives they become

$$\frac{\partial x}{\partial m}\frac{\partial x}{\partial n} = \frac{y^2x(m-y)}{(xm - nm + yn)^2}, \quad \frac{\partial y}{\partial m}\frac{\partial y}{\partial n} = \frac{x^2y(n-x)}{(xm - nm + yn)^2}.$$

Then, using condition (4.26), substituting from (4.21), and with respect to (4.25), we have

$$\frac{\partial x}{\partial m}\frac{\partial x}{\partial n} + \frac{\partial y}{\partial m}\frac{\partial y}{\partial n} = \frac{yx(ym - y^2 + xn - x^2)}{(xm - nm + yn)^2} = -\frac{yx(x^2 - 2my + y^2 + 1)}{(xm - nm + yn)^2} = 0.$$

Since the projection is orthogonal, the area scale holds $\wp = hk$, and the partial differential equations transform into (3.22)

$$\Big(\frac{\partial\lambda}{\partial x}\Big)^2 + \Big(\frac{\partial\lambda}{\partial y}\Big)^2 = \frac{1}{(\frac{\partial x}{\partial\lambda})^2 + (\frac{\partial y}{\partial\lambda})^2}, \quad \Big(\frac{\partial\varphi}{\partial x}\Big)^2 + \Big(\frac{\partial\varphi}{\partial y}\Big)^2 = \frac{1}{(\frac{\partial x}{\partial\varphi})^2 + (\frac{\partial y}{\partial\varphi})^2}.$$

The squares of the first derivatives are

$$\Big(\frac{\partial x}{\partial\lambda}\Big)^2 + \Big(\frac{\partial y}{\partial\lambda}\Big)^2 = \Big(\frac{\partial n}{\partial s}\frac{\partial s}{\partial\lambda}\Big)^2\Big[\Big(\frac{\partial x}{\partial n}\Big)^2 + \Big(\frac{\partial y}{\partial n}\Big)^2\Big],$$
$$\Big(\frac{\partial x}{\partial\varphi}\Big)^2 + \Big(\frac{\partial y}{\partial\varphi}\Big)^2 = \Big(\frac{\partial m}{\partial t}\frac{\partial t}{\partial\varphi}\Big)^2\Big[\Big(\frac{\partial x}{\partial m}\Big)^2 + \Big(\frac{\partial y}{\partial m}\Big)^2\Big],$$

where

$$\Big(\frac{\partial x}{\partial m}\Big)^2 + \Big(\frac{\partial y}{\partial m}\Big)^2 = \frac{y^4 + y^2(n-x)^2}{(xm - nm + yn)^2}, \tag{4.27}$$
$$\Big(\frac{\partial x}{\partial n}\Big)^2 + \Big(\frac{\partial y}{\partial n}\Big)^2 = \frac{x^2(m-y)^2 + x^4}{(xm - nm + yn)^2}. \tag{4.28}$$

With the use of (4.24)–(4.25), the numerators of both fractions can be rewritten to the form of

$$y^4 + y^2(n-x)^2 = y^2(1+n^2), \quad x^2(m-y)^2 + x^4 = x^2(m^2-1),$$

as well as with (4.24)–(4.25) and (4.21), the denominator is

$$(xm - nm + yn)^2 = m^2 - n^2 + m^2n^2 - (my - nx)^2 = (1 + n^2)(m^2 - 1).$$

After back substitution into (4.27)–(4.28), the squares of the first derivatives simplify to

$$\Big(\frac{\partial x}{\partial m}\Big)^2 + \Big(\frac{\partial y}{\partial m}\Big)^2 = \frac{y^2}{m^2 - 1}, \quad \Big(\frac{\partial x}{\partial n}\Big)^2 + \Big(\frac{\partial y}{\partial n}\Big)^2 = \frac{x^2}{n^2 + 1},$$

and their reciprocals become

$$\frac{1}{(\frac{\partial x}{\partial m})^2 + (\frac{\partial y}{\partial m})^2} = \frac{m^2 - 1}{y^2}, \quad \frac{1}{(\frac{\partial x}{\partial n})^2 + (\frac{\partial y}{\partial n})^2} = \frac{n^2 + 1}{x^2}.$$

The right-hand side of (3.23) is

$$\frac{1}{(\frac{\partial x}{\partial \varphi})^2 + (\frac{\partial y}{\partial \varphi})^2} = \frac{4\varphi^4}{\pi^2}\frac{m^2 - 1}{y^2} = \frac{\varphi^2(\pi^2 - 4\varphi^2)}{\pi^2 y^2},$$

the right-hand side of (3.22) is

$$\frac{1}{(\frac{\partial x}{\partial \lambda})^2 + (\frac{\partial y}{\partial \lambda})^2} = \frac{4\pi^2\lambda^4}{(\pi^2 + \lambda^2)^2}\frac{n^2 + 1}{x^2} = \frac{4\pi^2\lambda^4}{(\pi^2 + \lambda^2)^2}\frac{(\lambda^2 - \pi^2)^2 + 4\pi^2\lambda^2}{4\pi^2\lambda^2 x^2} = \frac{\lambda^2}{x^2}.$$

Then the partial differential equations of the inverse transformation have the form of

(4.29)
$$\Big(\frac{\partial \varphi}{\partial x}\Big)^2 + \Big(\frac{\partial \varphi}{\partial y}\Big)^2 = \frac{\varphi^2(\pi^2 - 4\varphi^2)}{\pi^2 y^2},$$
$$\Big(\frac{\partial \lambda}{\partial x}\Big)^2 + \Big(\frac{\partial \lambda}{\partial y}\Big)^2 = \frac{\lambda^2}{x^2}.$$

Since the second equation is analogous to Grinten III, our effort concentrates on the first one. Using the substitution

$$\mathrm{d}z = \frac{\pi}{\varphi\sqrt{(\pi^2 - 4\varphi^2)}}\,\mathrm{d}\varphi,$$

we have

(4.30)
$$z = \pi\int \frac{1}{\varphi\sqrt{(\pi^2 - 4\varphi^2)}}\,\mathrm{d}\varphi.$$

The next substitution

$$u = \sqrt{\pi^2 - 4\varphi^2}, \quad \mathrm{d}u = -\frac{4\varphi}{\sqrt{\pi^2 - 4\varphi^2}}\,\mathrm{d}\varphi,$$

transforms the integral into the form of

$$\int \frac{1}{\varphi\sqrt{(\pi^2-4\varphi^2)}}\,\mathrm{d}\varphi = -\int \frac{1}{\pi^2-u^2}\,\mathrm{d}u$$
$$= \frac{1}{\pi}\Big(\ln|\pi-u| - \frac{1}{2}\ln|\pi^2-u^2|\Big) = \frac{1}{\pi}\ln\frac{|\pi-\sqrt{\pi^2-4\varphi^2}|}{|2\varphi|}.$$

Putting it into (4.30) leads to

$$z = \ln\frac{|\pi-\sqrt{\pi^2-4\varphi^2}|}{|2\varphi|},$$

and the partial differential equation (4.29) simplifies to

$$\Big(\frac{\partial\varphi}{\partial x}\Big)^2 + \Big(\frac{\partial\varphi}{\partial y}\Big)^2 = \frac{1}{y^2}.$$

Using the Lagrange-Charpit method, it can be rewritten to

$$y^2P^2 + y^2Q^2 - 1 = 0, \quad P = \frac{\partial z}{\partial x}, \quad Q = \frac{\partial z}{\partial y}. \tag{4.31}$$

Then the auxiliary Lagrange-Charpit equations are

$$\frac{\mathrm{d}P}{0} = \frac{\mathrm{d}Q}{-y(P^2+Q^2)} = \frac{\mathrm{d}z}{y^2(P^2+Q^2)} = \frac{\mathrm{d}x}{Py^2} = \frac{\mathrm{d}y}{Qy^2}.$$

Their solution is analogous to Grinten III. From the first member, we get $P = a$. Putting it into (4.31), we get $Q^2 = (1-a^2y^2)/y^2$, z is the solution of

$$z = ax + b + \int \frac{\sqrt{1-a^2y^2}}{y}\,\mathrm{d}y, \tag{4.32}$$

where

$$\int \frac{\sqrt{1-a^2y^2}}{y}\,\mathrm{d}y = \sqrt{1-a^2y^2} + \ln\frac{1-\sqrt{1-a^2y^2}}{ay},$$

$a, b \in \mathbb{R}$ are the arbitrary constants of integration. Putting it into (4.32), we get

$$z = ax + b + \sqrt{1-a^2y^2} + \ln\frac{1-\sqrt{1-a^2y^2}}{ay} = \pi\ln\frac{|\pi-\sqrt{\pi^2-4\varphi^2}|}{|2\varphi|}. \tag{4.33}$$

Using the substitution

$$K = \mathrm{e}^{ax+b+\sqrt{1-a^2y^2}}\frac{1-\sqrt{1-a^2y^2}}{ay}, \tag{4.34}$$

equation (4.33) transforms into the quadratic equation

$$|2\varphi|K = |\pi - \sqrt{\pi^2 - 4\varphi^2}|, \quad \varphi(\varphi + \varphi K^2 - \pi K) = 0.$$

For $\varphi \neq 0$ the general solution of the partial differential equation is

$$\varphi = \frac{\pi K}{1 + K^2}. \tag{4.35}$$

The arbitrary constants of integration a, b are chosen analogously to the Grinten III projection, when the curve (4.34) passes through the unit circle. Since

$$x + \sqrt{1 - y^2} = 0,$$

we choose $a = 1$, $b = 0$, and after comparison with (4.34), the associated particular solution is

$$K = \frac{1 - \sqrt{1 - y^2}}{y} = \frac{1 + x}{y}. \tag{4.36}$$

4.2.2. Straightforward inverse of projection equations. The inverse formulas are analogous to projection III, see (4.14). Using (4.18), t is determined from the linear equation

$$t = \frac{1}{m} = \frac{2y}{1 + x^2 + y^2}.$$

Analogously, s can be obtained using (4.15).

Remark 4.9. Alternatively, n is determined from (4.22)–(4.23) as

$$n = \frac{x^2 + y^2 - 1}{2x}.$$

Substitution into (4.2) leads to the quadratic equation for λ

$$\lambda = \frac{\pi}{2x}(x^2 + y^2 - 1 \pm \sqrt{(x^2 + y^2 - 1)^2 + 4x^2}),$$

where λ takes the sign of x.

Remark 4.10. Particular situations when $x = 0$ or $y = 0$ are solved analogously to projection III.

4.3. Van der Grinten projection I. The world is enclosed in a circle; the meridians and parallels are circular arcs. Unlike projection II, they do not intersect each other at right angles. The graticule is similar to the Lagrange projection, but the projection is not conformal. In general, projection I is the most popular of the Grinten family.

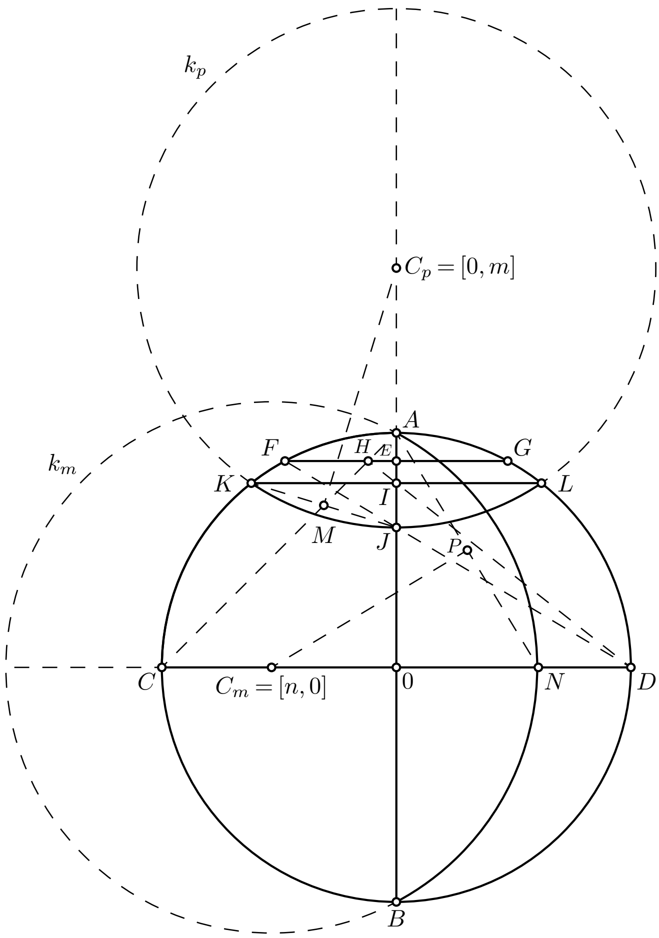

Figure 6. Van der Grinten projection I, a geometric construction of meridians and parallels.

Since the meridian arc construction is analogous to its predecessors, the parallel arc requires more effort, see Figure 6. Initially, the line OA is divided by equally spaced points $E[0, t]$, $t \in (0, 1)$. The end point $F[-\sqrt{(1 - t^2}, t)]$ of auxiliary line FE is projected from $D[1, 0]$ to AO as point $J[0, j]$; it will be the lowest point of the parallel arc. From the equation of FD

$$y(1 + \sqrt{1 - t^2}) + t(x - 1) = 0,$$

the second coordinate of the point J holds

$$j = \frac{t}{1 + \sqrt{1 - t^2}}. \tag{4.37}$$

Unlike previous projections, two additional points of the parallel arc are required. Point H, $H[t - 1, t]$, lies on the intersection of lines CA and FG of two similar triangles, COA and HEA. The intersection point of two lines, HD and AO is $I[0, v]$, its coordinate $v = t/(2 - t)$ is determined from the similar triangles, HEI, and IOD. The line KL parallel to CD passing through the point J represents the lowest point of the parallel arc. For further calculations, let us denote its coordinate as $K[k_1, k_2]$, where

$$k_1 = -\sqrt{1 - v^2} = -\frac{2\sqrt{1 - t}}{2 - t}$$

and

$$k_2 = v = \frac{t}{2 - t}. \tag{4.38}$$

The center of the parallel arc represents an intersection of the bisector MC_p of the line KJ and the line AB. From the equation of MC_p, $M[\frac{1}{2}k_1, \frac{1}{2}(k_2 + j)]$,

$$k_1 x + (k_2 - j)y = \frac{1 - j^2}{2}, \tag{4.39}$$

the parallel arc center coordinate is

$$m = \frac{1 - j^2}{2(k_2 - j)} = \frac{(2 - t)(t + 1 + \sqrt{1 - t^2})}{2t^2}, \tag{4.40}$$

and its radius

$$r_p = \frac{1 + j(j - 2k_2)}{2(k_2 - j)} = \frac{2 - t - t^2}{t(t - 1 + \sqrt{1 - t^2})}.$$

The point of intersection of the meridian and parallel arcs can be found by solving the system of quadratic equations

$$(x - n)^2 + y^2 = 1 + n^2, \tag{4.41}$$

$$x^2 + (y - m)^2 = (m - j)^2. \tag{4.42}$$

Subtracting one from the other leads to the linear equation

$$-2nx + 2my = 1 - j^2 + 2mj. \tag{4.43}$$

Substituting for y into (4.41) leads to the quadratic equation for x

$$4(m^2+n^2)x^2+4nx(1-j^2+2mj-2m^2)+(1-j^2+2mj)^2-4m^2=0, \tag{4.44}$$

the factors are functions of s, t

$$\begin{aligned} a &= s^2+\frac{1}{s^2}+\frac{(t-2)^2(t+t'+1)^2}{t^4}-2, \\ b &= \frac{2(1-s^2)[t^6+16(t'+1)+t^4(t'+7)-8t^2(2t'+3)]}{st^4(t'+1)^2}, \\ c &= \frac{8(t^2-1)(t'+1)}{t^4}, \end{aligned}$$

where $t'=\sqrt{1-t^2}$, and x takes the sign of λ. Substituting for x into (4.42) leads to the quadratic equation for y

$$4(m^2+n^2)y^2+4my(j^2-2mj-2n^2-1)+(j^2-2mj-1)^2-4jn^2(j-2m)=0, \tag{4.45}$$

where the factors are functions of s, t

$$\begin{aligned} a &= s^2+\frac{1}{s^2}+\frac{(t-2)^2(t+t'+1)^2}{t^4}-2, \\ b &= \frac{(t-2)(t+t'+1)[t+s^4t+2s^2(t'+1)]}{s^2t^3}, \\ c &= \frac{(t+s^4t+2s^2)(t'+1)}{s^2t^2}. \end{aligned}$$

Let us summarize the results in a new theorem.

Theorem 4.11. *The image of the point $P[\varphi,\lambda]$ in van der Grinten projection* I *is given by* (4.44)–(4.45).

Remark 4.12. Alternatively, instead of solving (4.45), the second coordinate may be determined from (4.41)

$$y=\sqrt{1-x^2+2xn}. \tag{4.46}$$

Remark 4.13. Particular situations when $\varphi=0$ or $\lambda=0$ are solved analogously to projection III.

4.3.1. Inverse form using partial differential equations. The projection equations can be rewritten as

$$x^2 - 2xn + y^2 - 1 = 0, \tag{4.47}$$

$$x^2 + y^2 - 2ym + 2mj - j^2 = 0. \tag{4.48}$$

Taking into account (4.37), t can be expressed as a function of j

$$t = \frac{2j}{1+j^2}, \tag{4.49}$$

we obtain

$$\sqrt{1-t^2} = \frac{1-j^2}{1+j^2}. \tag{4.50}$$

Analogously, considering (4.38), for k_2 we have

$$k_2 = \frac{j}{j^2 - j + 1},$$

after substitution into (4.40), we get

$$m = \frac{1+j^3}{2j^2}.$$

Since the projection equations are composite functions of j, m, n, s, t, the partial derivatives $\partial \cdot /\partial\varphi$ have the general form of

$$\frac{\partial x}{\partial\varphi} = \frac{\partial j}{\partial t}\frac{\partial t}{\partial\varphi}\Big(\frac{\partial x}{\partial m}\frac{\partial m}{\partial j} + \frac{\partial x}{\partial j}\Big), \quad \frac{\partial y}{\partial\varphi} = \frac{\partial j}{\partial t}\frac{\partial t}{\partial\varphi}\Big(\frac{\partial y}{\partial m}\frac{\partial m}{\partial j} + \frac{\partial y}{\partial j}\Big),$$

the partial derivatives $\partial \cdot /\partial\lambda$ are

$$\frac{\partial x}{\partial\lambda} = \frac{\partial x}{\partial n}\frac{\partial n}{\partial s}\frac{\partial s}{\partial\lambda}, \quad \frac{\partial y}{\partial\lambda} = \frac{\partial y}{\partial n}\frac{\partial n}{\partial s}\frac{\partial s}{\partial\lambda}.$$

From (4.43), the coordinates can be expressed as

$$x = \frac{j^2 + 2my - 2mj - 1}{2n}, \quad y = \frac{1 - j^2 + 2mj + 2nx}{2m},$$

their partial derivatives are

$$\frac{\mathrm{d}x}{\mathrm{d}m} = \frac{y-j}{n}, \quad \frac{\mathrm{d}x}{\mathrm{d}n} = -\frac{x}{n}, \quad \frac{\mathrm{d}y}{\mathrm{d}m} = \frac{j-y}{m}, \quad \frac{\mathrm{d}y}{\mathrm{d}n} = \frac{x}{m},$$
$$\frac{\mathrm{d}x}{\mathrm{d}j} = \frac{j-m}{n}, \quad \frac{\mathrm{d}y}{\mathrm{d}j} = \frac{m-j}{m}.$$

For (4.47)–(4.48), the implicit differentiation with respect to x, y, m, n, j is used. The partial derivatives with respect to m are

$$2x\,\mathrm{d}x - 2n\,\mathrm{d}x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x + 2y\frac{\mathrm{d}y}{\mathrm{d}m}\,\mathrm{d}m = 0 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}m} = \frac{y(y-j)}{xm - nm + yn},$$

$$2x\frac{\mathrm{d}x}{\mathrm{d}y}\,\mathrm{d}y - 2m\,\mathrm{d}y + 2y\,\mathrm{d}y + 2x\frac{\mathrm{d}x}{\mathrm{d}m}\,\mathrm{d}m - 2y\,\mathrm{d}m + 2j\,\mathrm{d}m = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}m} = \frac{(j-y)(x-n)}{xm - nm + yn},$$

the partial derivatives with respect to n are

$$2x\,\mathrm{d}x - 2n\,\mathrm{d}x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x - 2x\,\mathrm{d}n + 2y\frac{\mathrm{d}y}{\mathrm{d}n}\,\mathrm{d}n = 0 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}n} = \frac{x(m-y)}{xm - nm + yn},$$

$$2x\frac{\mathrm{d}x}{\mathrm{d}y}\,\mathrm{d}y - 2m\,\mathrm{d}y + 2y\,\mathrm{d}y + 2x\frac{\mathrm{d}x}{\mathrm{d}n}\,\mathrm{d}n = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}n} = \frac{x^2}{xm - nm + yn}.$$

Analogously, the partial derivatives with respect to j are

$$2x\,\mathrm{d}x - 2n\,\mathrm{d}x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x + 2y\frac{\mathrm{d}y}{\mathrm{d}j}\,\mathrm{d}j = 0 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}j} = \frac{y(j-m)}{xm - nm + yn},$$

$$2x\frac{\mathrm{d}x}{\mathrm{d}y}\,\mathrm{d}y - 2m\,\mathrm{d}y + 2y\,\mathrm{d}y + 2x\frac{\mathrm{d}x}{\mathrm{d}j}\,\mathrm{d}j + 2m\,\mathrm{d}j - 2j\,\mathrm{d}j = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}j} = \frac{(j-m)(n-x)}{xm - nm + yn}.$$

Using the chain rule, the local linear scales hold

$$h = \frac{\partial j}{\partial t}\frac{\partial t}{\partial \varphi}\sqrt{\left(\frac{\partial x}{\partial m}\frac{\partial m}{\partial j} + \frac{\partial x}{\partial j}\right)^2 + \left(\frac{\partial y}{\partial m}\frac{\partial m}{\partial j} + \frac{\partial y}{\partial j}\right)^2},$$

$$k = \frac{\partial n}{\partial s}\frac{\partial s}{\partial \lambda}\sqrt{\left(\frac{\partial x}{\partial n}\right)^2 + \left(\frac{\partial y}{\partial n}\right)^2}\,\frac{1}{\cos\varphi},$$

and the local area scale is

$$\wp = \frac{\partial n}{\partial s}\frac{\partial s}{\partial \lambda}\frac{\partial j}{\partial t}\frac{\partial t}{\partial \varphi}\left[\frac{\partial x}{\partial n}\left(\frac{\partial y}{\partial m}\frac{\partial m}{\partial j} + \frac{\partial y}{\partial j}\right) - \frac{\partial y}{\partial n}\left(\frac{\partial x}{\partial m}\frac{\partial m}{\partial j} + \frac{\partial x}{\partial j}\right)\right]\frac{1}{\cos\varphi}.$$

Since

$$\frac{\partial m}{\partial j} = \frac{1}{2} - \frac{1}{j^3}, \quad \frac{\partial m}{\partial \varphi} = \frac{\partial m}{\partial t}\frac{\partial t}{\partial \varphi} = -\frac{\pi}{2\varphi^2}, \quad \frac{\partial j}{\partial t} = \frac{1}{1 - t^2 + \sqrt{1-t^2}} = \frac{(1+j^2)^2}{2(1-j^2)},$$

the products of partial derivatives are

$$\frac{\partial x}{\partial m}\frac{\partial m}{\partial j} + \frac{\partial x}{\partial j} = \frac{y[(y-j)(j^3-2) + 2j^3(j-m)]}{2j^3(xm - nm + yn)},$$

$$\frac{\partial y}{\partial m}\frac{\partial m}{\partial j} + \frac{\partial y}{\partial j} = \frac{(n-x)[(y-j)(j^3-2) + 2j^3(j-m)]}{2j^3(xm - nm + yn)}.$$

Taking into account that

$$j - m = \frac{j^3 - 1}{2j^2}$$

and using the substitution

$$c = (y - j)(j^3 - 2) + 2j^3(j - m) = y(j^3 - 2) + j,$$

they can be rewritten to the form of

$$\frac{\partial x}{\partial m}\frac{\partial m}{\partial j} + \frac{\partial x}{\partial j} = \frac{cy}{2j^3(xm - nm + yn)}, \quad \frac{\partial y}{\partial m}\frac{\partial m}{\partial j} + \frac{\partial y}{\partial j} = \frac{c(n - x)}{2j^3(xm - nm + yn)}.$$

The products are

$$\frac{\partial y}{\partial n}\Big(\frac{\partial x}{\partial m}\frac{\partial m}{\partial j} + \frac{\partial x}{\partial j}\Big) = \frac{cx^2 y}{2j^3(xm - nm + yn)^2},$$
$$\frac{\partial x}{\partial n}\Big(\frac{\partial y}{\partial m}\frac{\partial m}{\partial j} + \frac{\partial y}{\partial j}\Big) = \frac{cx(m - y)(n - x)}{2j^3(xm - nm + yn)^2}.$$

Taking into account (4.41), the sum of the squares of the derivatives is

$$\Big(\frac{\partial x}{\partial m}\frac{\partial m}{\partial j} + \frac{\partial x}{\partial j}\Big)^2 + \Big(\frac{\partial y}{\partial m}\frac{\partial m}{\partial j} + \frac{\partial y}{\partial j}\Big)^2 = \frac{c^2(y^2 + (n - x)^2)}{4j^6(xm - nm + yn)^2} = \frac{c^2(1 + n^2)}{4j^6(xm - nm + yn)^2}.$$

Since the difference of the products of the derivatives is

$$\frac{\partial x}{\partial n}\Big(\frac{\partial y}{\partial m}\frac{\partial m}{\partial j} + \frac{\partial y}{\partial j}\Big) - \frac{\partial y}{\partial n}\Big(\frac{\partial x}{\partial m}\frac{\partial m}{\partial j} + \frac{\partial x}{\partial j}\Big) = \frac{c[x(m - y)(n - x) - x^2 y]}{2j^3(xm - nm + yn)^2} = -\frac{cx}{2j^3(xm - nm + yn)},$$

its square is

$$\Big[\frac{\partial x}{\partial n}\Big(\frac{\partial y}{\partial m}\frac{\partial m}{\partial j} + \frac{\partial y}{\partial j}\Big) - \frac{\partial y}{\partial n}\Big(\frac{\partial x}{\partial m}\frac{\partial m}{\partial j} + \frac{\partial x}{\partial j}\Big)\Big]^2 = \frac{c^2 x^2}{4j^6(xm - nm + yn)^2}.$$

Then the right-hand side of the first differential equation has the general form of

$$\frac{h^2}{\wp^2 \cos^2 \varphi} = \Big(\Big(\frac{\partial x}{\partial m}\frac{\partial m}{\partial j} + \frac{\partial x}{\partial j}\Big)^2 + \Big(\frac{\partial y}{\partial m}\frac{\partial m}{\partial j} + \frac{\partial y}{\partial j}\Big)^2\Big) \times \Big(\frac{\partial n}{\partial s}\frac{\partial s}{\partial \lambda}\Big)^{-2}\Big[\frac{\partial x}{\partial n}\Big(\frac{\partial y}{\partial m}\frac{\partial m}{\partial j} + \frac{\partial y}{\partial j}\Big) - \frac{\partial y}{\partial n}\Big(\frac{\partial x}{\partial m}\frac{\partial m}{\partial j} + \frac{\partial x}{\partial j}\Big)\Big]^{-2}.$$

After the substitution, the fraction simplifies to

$$\frac{h^2}{\wp^2\cos^2\varphi} = \pi^2\frac{c^2(1+n^2)}{4j^6(xm-nm+yn)^2}\frac{4j^6(xm-nm+yn)^2}{c^2x^2}\frac{4s^2}{(s^2+1)^2}$$
$$= 4\pi^2\frac{(1+n^2)}{x^2}\frac{s^4}{(s^2+1)^2} = \frac{\lambda^2}{x^2}.$$

The first partial differential equation of the inverse transformation

$$\left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2 = \frac{\lambda^2}{x^2} \tag{4.51}$$

is similar to Grinten III and II equations. For the second partial differential equation we use an analogous approach. Taking into account (4.42), the squares of the derivatives are

$$\left(\frac{\partial x}{\partial n}\right)^2 + \left(\frac{\partial y}{\partial n}\right)^2 = \frac{x^2(m-y)^2+x^4}{(xm-nm+yn)^2} = \frac{x^2(m-j)^2}{(xm-nm+yn)^2}.$$

The right-hand side of the second partial differential equation has the general form of

$$\frac{k^2}{\wp^2} = \left(\left(\frac{\partial x}{\partial n}\right)^2 + \left(\frac{\partial y}{\partial n}\right)^2\right)\left(\frac{\partial j}{\partial t}\frac{\partial t}{\partial\varphi}\right)^{-2}\left[\frac{\partial x}{\partial n}\left(\frac{\partial y}{\partial m}\frac{\partial m}{\partial j}+\frac{\partial y}{\partial j}\right) - \frac{\partial y}{\partial n}\left(\frac{\partial x}{\partial m}\frac{\partial m}{\partial j}+\frac{\partial x}{\partial j}\right)\right]^{-2}.$$

After the substitution, the fraction simplifies to

$$\frac{k^2}{\wp^2} = \frac{x^2(m-j)^2}{(xm-nm+yn)^2}\frac{4j^6(xm-nm+yn)^2}{c^2x^2}\frac{4\pi^2(1-j^2)^2}{4(1+j^2)^4}$$
$$= 4\pi^2\frac{j^6(m-j)^2}{c^2}\frac{(1-j^2)^2}{(1+j^2)^4} = \pi^2\frac{j^2(1-j^3)^2}{(y(j^3-2)+j)^2}\frac{(1-j^2)^2}{(1+j^2)^4}.$$

Taking into account (4.49)–(4.50), we have

$$\frac{k^2}{\wp^2} = \frac{[t(1-t)(2+t)(1-\sqrt{1-t^2})(1+t-\sqrt{1-t^2})]^2}{4[(1-\sqrt{1-t^2})((4-t^2)y+t^2)-2t^2(1+t)y]^2}$$
$$= \frac{[t(1-t)(2+t)(1+t-\sqrt{1-t^2})]^2}{4[y(4-t^2-2(1+\sqrt{1-t^2})(1+t))+t^2]^2}.$$

After the substitution for φ, the second partial differential equation has the form of

$$\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 = \frac{[\varphi(\pi-2\varphi)(\pi+\varphi)(\pi+2\varphi-\sqrt{1-4\varphi^2})]^2}{\pi^4[2\varphi^2+y(2\pi^2-2\varphi^2-(\pi+2\varphi)(\pi+\sqrt{1-4\varphi^2}))]^2}. \tag{4.52}$$

Due to the fact that the right-hand side is the function of φ, y and cannot be expressed as a product of $h(x,y)g(\varphi)$, the partial differential equation has no known analytical solution. Therefore, the numerical approach is recommended.

4.3.2. Straightforward inverse of projection equations. The inverse formulas of van der Grinten projection I are analogous to projection III, see (4.14). The parameter s is obtained from (4.16). Substituting for m, r from (4.37), and (4.39) into (4.45) leads to the cubic equation for t

$$at^3 + bt^2 + ct + d = 0 \tag{4.53}$$

with the factors

$$\begin{aligned} a &= x^4 + 2x^2y(y+1) + (y+1)^2(y^2+1), \\ b &= -2[x^2(y-1) + y(y+1)^2], \\ c &= -4y(x^2+y^2+1), \\ d &= 8y^2, \end{aligned}$$

where $\varphi = \pi t/2$ takes the sign of y.

Remark 4.14. Particular situations when $x = 0$ or $y = 0$ are solved analogously to projection III.

4.4. Van der Grinten projection IV. Projection IV, called *apple-shaped*, depicts the planisphere as a union of two circular fragments with a common boundary at the central meridian $AB = \frac{1}{2}|CD|$, where CD is the equator image. Both the meridians and the parallels are circular arcs. Unlike previous projections, the parallels are equally spaced along the central meridian, see Figure 7.

The planisphere boundary, the meridians of the longitude of $\lambda = \pm\pi$, has the implicit equation

$$\left(x \pm \frac{3}{4}\right)^2 + y^2 = \frac{25}{16}. \tag{4.54}$$

For the parallel construction we use an analogous approach when the line OA is divided by equally spaced points $E[0,t]$, $t \in (0,1)$. The meridian of longitude $\lambda = -\pi$ intersects the Equator CD at the point $I[\frac{1}{2}, 0]$. The point of intersection $J[j_1, t]$ of the lines AC and FE, where

$$j_1 = -\frac{|CO|}{|OA|}|EA| = -2(1-t),$$

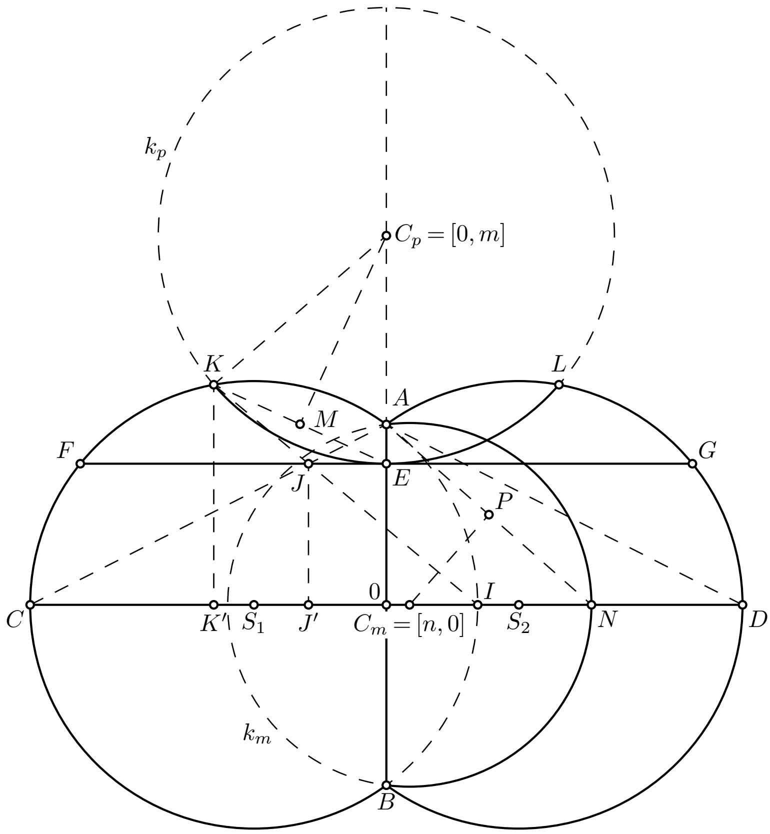

Figure 7. Van der Grinten projection IV, a geometric construction of meridians and parallels.

is projected from I into $\lambda = -\pi$ as $K[k_1, k_2]$. The parallel arc passes through the points K, E, L. Their coordinates are determined from the similar triangles $KK'I$ and $JJ'I$,

$$\frac{k_2}{-k_1 + \frac{1}{2}} = \frac{t}{2(1-t) + \frac{1}{2}}.$$

Substituting for K into (4.54) leads to

$$k_1 = \frac{2(1-t)(3t-5)}{5 + 4t(t-2)}, \quad k_2 = \frac{t(5-4t)}{5+4t(t-2)}.$$

Analogously, the radius r_p of the parallel arc is evaluated using the similarity

$$\frac{r_p}{|ME|} = \frac{|KE|}{k_2 - t},$$

where $|ME| = 0.5|KE|$, and

$$r_p = \frac{k_1^2 + (k_2 - t)^2}{2(k_2 - t)} = \frac{(1-t)[5 + t(2+t)]}{2t^2}.$$

Its center is $C_p = [0, m]$, where

$$m = r_p + t = \frac{5 + t(t^2 - t - 3)}{2t^2}. \tag{4.55}$$

The circular arc passing through the point $N[s, 0]$ has the center at the point of intersection $C_m[n, 0]$ of the bisector AN and the $x-$ axis. Since $s \in (0, 2\rangle$, we have $s = 2\lambda/\pi$. The point of intersection of the meridian and parallel arcs can be found by solving the system of quadratic equations

$$(x - n)^2 + y^2 = 1 + n^2, \tag{4.56}$$

$$x^2 + (y - m)^2 = (m - t)^2. \tag{4.57}$$

Subtracting one from the other leads to the linear equation

$$-2nx + 2my = 1 - t^2 + 2mt. \tag{4.58}$$

Substituting for y into (4.56) leads to the quadratic equation for x

$$4(m^2 + n^2)x^2 + 4nx(1 - t^2 + 2mt - 2m^2) + (1 - t^2 + 2mt)^2 - 4m^2 = 0, \tag{4.59}$$

where the factors are functions of s, t

$$a = s^2 + \frac{1}{s^2} + \frac{[5 + t(t^2 - t - 3)]^2}{t^4} - 2,$$
$$b = \frac{(1 - s^2)(t - 1)[t(t^4 + t^3 + 6t + 5) - 25]}{st^4},$$
$$c = \frac{(t - 1)^2(t + 1)(2t + 5)(3t - 5)}{t^4}.$$

Substituting for x into (4.57) leads to the quadratic equation for y

$$4(m^2 + n^2)y^2 + 4my(t^2 - 2mt - 2n^2 - 1) + (t^2 - 2mt - 1)^2 - 4tn^2(t - 2m) = 0 \tag{4.60}$$

with the factors

$$a = s^2 + \frac{1}{s^2} + \frac{[5 + t(t^2 - t - 3)]^2}{t^4} - 2,$$
$$b = \frac{(-t^3 + t^2 + 3t - 5)[t + s^4 t - 2s^2(t^2 + 3t - 5)]}{s^2 t^3},$$
$$c = \frac{(t^2 + 3t - s^2 t - 5)[s^2(t^2 + 3t - 5) - t]}{s^2 t^2}.$$

The results are summarized in the theorem.

Theorem 4.15. *The image of the point* $P[\varphi, \lambda]$ *in van der Grinten projection* IV *is given by* (4.59)–(4.60).

R e m a r k 4.16. Alternatively, the second coordinate may be determined from equation (4.46).

R e m a r k 4.17. The particular cases are solved as follows:
If $\lambda = 0$, then $x = 0$ and $y = 2\varphi/\pi$.
If $\varphi = 0$, then $x = 2\lambda/\pi$ and $y = 0$.
If $\varphi = \pm\pi/2$, then $x = 0$ and $y = \pm 1$.

4.4.1. Inverse form using partial differential equations. The projection equations can be rewritten to

$$x^2 - 2xn + y^2 - 1 = 0, \tag{4.61}$$

$$x^2 + y^2 - 2ym + 2mt - t^2 = 0. \tag{4.62}$$

Since the projection equations are composite functions of m, n, s, t, the partial derivatives $\partial \cdot /\partial\varphi$ have the general form of

$$\frac{\partial x}{\partial \varphi} = \frac{\partial t}{\partial \varphi}\Big(\frac{\partial x}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial x}{\partial t}\Big), \quad \frac{\partial y}{\partial \varphi} = \frac{\partial t}{\partial \varphi}\Big(\frac{\partial y}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial y}{\partial t}\Big),$$

the partial derivatives $\partial \cdot /\partial\lambda$ are analogous to the Grinten I projection. Therefore,

$$\frac{\partial m}{\partial t} = \frac{t^3 + 3t - 10}{2t^3}, \quad \frac{\partial t}{\partial \varphi} = \frac{\partial s}{\partial \lambda} = \frac{2}{\pi}, \quad \frac{\partial n}{\partial s} = \frac{s^2+1}{2s^2} = \frac{4\pi^2 + \lambda^2}{8\lambda^2},$$

and

$$\frac{\partial m}{\partial \varphi} = \frac{\partial m}{\partial t}\frac{\partial t}{\partial \varphi} = \frac{t^3 + 3t - 10}{\pi t^3}, \quad \frac{\partial n}{\partial \lambda} = \frac{\partial n}{\partial s}\frac{\partial s}{\partial \lambda} = \frac{4\pi^2 + \lambda^2}{4\pi\lambda^2}.$$

From (4.58), the coordinates can be expressed as

$$x = \frac{t^2 + 2my - 2mt - 1}{2n}, \quad y = \frac{1 - t^2 + 2mt + 2nx}{2m},$$

their partial derivatives are

$$\frac{\mathrm{d}x}{\mathrm{d}m} = \frac{y-t}{n}, \quad \frac{\mathrm{d}x}{\mathrm{d}n} = -\frac{x}{n}, \quad \frac{\mathrm{d}y}{\mathrm{d}m} = \frac{t-y}{m}, \quad \frac{\mathrm{d}y}{\mathrm{d}n} = \frac{x}{m},$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{t-m}{n}, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{m-t}{m}.$$

For (4.61)–(4.62), the implicit differentiation with respect to x, y, m, n, t is used. The partial derivatives $\partial \cdot / \partial m$ are

$$2x\,\mathrm{d}x - 2n\,\mathrm{d}x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x + 2y\frac{\mathrm{d}y}{\mathrm{d}m}\,\mathrm{d}m = 0 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}m} = \frac{y(y-t)}{xm - nm + yn},$$
$$2x\frac{\mathrm{d}x}{\mathrm{d}y}\,\mathrm{d}y - 2m\,\mathrm{d}y + 2y\,\mathrm{d}y + 2x\frac{\mathrm{d}x}{\mathrm{d}m}\,\mathrm{d}m - 2y\,\mathrm{d}m + 2t\,\mathrm{d}m = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}m} = \frac{(t-y)(x-n)}{xm - nm + yn}.$$

The partial derivatives with respect to n are analogous to the Grinten I projection. The partial derivatives $\partial \cdot / \partial t$ are

$$2x\,\mathrm{d}x - 2n\,\mathrm{d}x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x + 2y\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t = 0 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{y(t-m)}{xm - nm + yn},$$
$$2x\frac{\mathrm{d}x}{\mathrm{d}y}\,\mathrm{d}y - 2m\,\mathrm{d}y + 2y\,\mathrm{d}y + 2x\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t + 2m\,\mathrm{d}t - 2t\,\mathrm{d}t = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{(t-m)(n-x)}{xm - nm + yn}.$$

Using the chain rule, the local linear scales hold

$$h = \frac{\partial t}{\partial \varphi}\sqrt{\left(\frac{\partial x}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial y}{\partial t}\right)^2},$$
$$k = \frac{\partial n}{\partial s}\frac{\partial s}{\partial \lambda}\sqrt{\left(\frac{\partial x}{\partial n}\right)^2 + \left(\frac{\partial y}{\partial n}\right)^2}\,\frac{1}{\cos\varphi},$$

and the local area scale is

$$\wp = \frac{\partial n}{\partial s}\frac{\partial s}{\partial \lambda}\frac{\partial t}{\partial \varphi}\left[\frac{\partial x}{\partial n}\left(\frac{\partial y}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial y}{\partial t}\right) - \frac{\partial y}{\partial n}\left(\frac{\partial x}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial x}{\partial t}\right)\right]\frac{1}{\cos\varphi}.$$

The products of the partial derivatives are

$$\begin{aligned}
\frac{\partial x}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial x}{\partial t} &= \frac{y(y-t)}{xm - nm + yn}\,\frac{t^3 + 3t - 10}{2t^3} + \frac{y(t-m)}{xm - nm + yn}\\
&= \frac{y(y-t)(t^3 + 3t - 10) + 2t^3(t-m)}{2t^3(xm - nm + yn)},\\
\frac{\partial y}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial y}{\partial t} &= \frac{(t-y)(x-n)}{xm - nm + yn}\,\frac{t^3 + 3t - 10}{2t^3} + \frac{(t-m)(n-x)}{xm - nm + yn}\\
&= \frac{(n-x)[(y-t)(t^3 + 3t - 10) + 2t^3(t-m)]}{2t^3(xm - nm + yn)}.
\end{aligned}$$

Taking into account that

$$t - m = \frac{(t-1)(t^2 + 2t + 5)}{2t^2} \tag{4.63}$$

and using the substitution

$$c = (y-t)(t^3 + 3t - 10) + 2t^3(t-m) = y(t^3 + 3t - 10) + t(t^2 + 5), \tag{4.64}$$

they can be rewritten to the form of

$$\frac{\partial x}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial x}{\partial t} = \frac{cy}{2t^3(xm - nm + yn)}, \qquad \frac{\partial y}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial y}{\partial t} = \frac{c(n-x)}{2t^3(xm - nm + yn)}.$$

The products are

$$\frac{\partial y}{\partial n}\Big(\frac{\partial x}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial x}{\partial t}\Big) = \frac{cx^2 y}{2t^3(xm - nm + yn)^2},$$
$$\frac{\partial x}{\partial n}\Big(\frac{\partial y}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial y}{\partial t}\Big) = \frac{cx(m-y)(n-x)}{2t^3(xm - nm + yn)^2}.$$

Taking into account (4.56), the sum of the squares of the derivatives is

$$\Big(\frac{\partial x}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial x}{\partial t}\Big)^2 + \Big(\frac{\partial y}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial y}{\partial t}\Big)^2 = \frac{c^2(1+n^2)}{4t^6(xm - nm + yn)^2}.$$

Since the difference between the products of the derivatives is

$$\frac{\partial x}{\partial n}\Big(\frac{\partial y}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial y}{\partial t}\Big) - \frac{\partial y}{\partial n}\Big(\frac{\partial x}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial x}{\partial t}\Big) = -\frac{cx}{2t^3(xm - nm + yn)},$$

its square is

$$\Big[\frac{\partial x}{\partial n}\Big(\frac{\partial y}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial y}{\partial t}\Big) - \frac{\partial y}{\partial n}\Big(\frac{\partial x}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial x}{\partial t}\Big)\Big]^2 = \frac{c^2x^2}{4t^6(xm - nm + yn)^2}.$$

Then, the right-hand side of the first differential equation has the general form of

$$\begin{aligned}\frac{h^2}{\wp^2\cos^2\varphi} = \Big(&\Big(\frac{\partial x}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial x}{\partial t}\Big)^2 + \Big(\frac{\partial y}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial y}{\partial t}\Big)^2\Big) \\ &\times \Big(\frac{\partial n}{\partial s}\frac{\partial s}{\partial \lambda}\Big)^{-2}\Big[\frac{\partial x}{\partial n}\Big(\frac{\partial y}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial y}{\partial t}\Big) - \frac{\partial y}{\partial n}\Big(\frac{\partial x}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial x}{\partial t}\Big)\Big]^{-2}.\end{aligned}$$

Then after the substitution, analogously to the Grinten I projection we have

$$\frac{h^2}{\wp^2\cos^2\varphi} = \frac{\lambda^2}{x^2}.$$

The first partial differential equation of the inverse transformation

$$\Big(\frac{\partial \lambda}{\partial x}\Big)^2 + \Big(\frac{\partial \lambda}{\partial y}\Big)^2 = \frac{\lambda^2}{x^2} \tag{4.65}$$

is similar to the Grinten I–III equations.

For the second partial differential equation we use an analogous approach. Taking into account (4.57), the squares of the derivatives are

$$\Big(\frac{\partial x}{\partial n}\Big)^2 + \Big(\frac{\partial y}{\partial n}\Big)^2 = \frac{x^2(m-y)^2 + x^4}{(xm - nm + yn)^2} = \frac{x^2(m-t)^2}{(xm - nm + yn)^2}.$$

The right-hand side of the second partial differential equation has the general form of

$$\frac{k^2}{\wp^2} = \Big(\Big(\frac{\partial x}{\partial n}\Big)^2 + \Big(\frac{\partial y}{\partial n}\Big)^2\Big)\Big(\frac{\partial t}{\partial \varphi}\Big)^{-2}\Big[\frac{\partial x}{\partial n}\Big(\frac{\partial y}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial y}{\partial t}\Big) - \frac{\partial y}{\partial n}\Big(\frac{\partial x}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial x}{\partial t}\Big)\Big]^{-2}.$$

Taking into account (4.55) and (4.63)–(4.64),

$$\frac{k^2}{\wp^2} = \pi^2\frac{t^6(m-t)^2}{c^2} = \frac{\pi^2}{4}\frac{t^2(t-1)^2(t^2+2t+5)^2}{(y(t^3+3t-10)+t(t^2+5))^2},$$

after the back substitution for φ we have

$$\frac{k^2}{\wp^2} = \frac{\varphi^2(2\varphi-\pi)^2(4\varphi^2+4\pi\varphi+5\pi^2)^2}{[y(8\varphi^3+6\pi^2\varphi-10\pi^3)+2\varphi(4\varphi^2+5\pi^2)]^2}.$$

The second partial differential equation has the form of

$$\Big(\frac{\partial \varphi}{\partial x}\Big)^2 + \Big(\frac{\partial \varphi}{\partial y}\Big)^2 = \frac{\varphi^2(2\varphi-\pi)^2(4\varphi^2+4\pi\varphi+5\pi^2)^2}{[y(8\varphi^3+6\pi^2\varphi-10\pi^3)+2\varphi(4\varphi^2+5\pi^2)]^2}. \tag{4.66}$$

Due to the complex form of its right-hand side, the partial differential equation has no known analytical solution. Therefore, the numerical approach is recommended.

4.5. Straightforward inverse of projection equations.

Theorem 4.18. *The inverse formulas of the van der Grinten projection* IV *have the form of*

$$\varphi = \frac{\pi}{2}t, \quad \lambda = \frac{\pi}{2}s, \tag{4.67}$$

where s is the solution of a system of quadratic equations

$$x^2 - 2xn + y^2 - 1 = 0, \quad x^2 + y^2 - 2ym + 2mt - t^2 = 0, \tag{4.68}$$

the parameters m, n are obtained from (4.55) and (4.2), the longitude λ takes the sign of x.

Substituting for m, r_p into (4.59) leads to the cubic equation for t

$$at^3 + bt^2 + ct + d = 0, \tag{4.69}$$

where

$$a = y + 1, \quad b = -(x^2 + y^2 + y - 3), \quad c = -(3y + 5), \quad d = 5y,$$

and φ takes the sign of y.

Remark 4.19. The particular cases are solved as follows:
If $x = 0$, then $\varphi = \pi/2y$ and $\lambda = 0$.
If $y = 0$, then $\varphi = 0$ and $\lambda = \pi/2x$.

4.6. Practical computation of projection equations. In cartography, the coordinate functions are evaluated for the sphere of the radius R

$$X = cRx, \quad Y = cRy,$$

where $c \in \mathbb{R}^+$ is the user-defined multiplication constant. For projections I–III, we usually choose $c = \pi$, for projection IV it is $c = \pi/2$, see [15]. To avoid problems with a quadrant adjustment for x, y as well as for φ, λ in the inverse form, we take their absolute values

$$t = 2\frac{|\varphi|}{\pi}, \quad s = \frac{|\lambda|}{\pi},$$

the calculations are performed for the first quadrant. Taking into account the central meridian and Equator symmetries, the correct quadrant of X, Y depends on the sign of φ, λ:

$$X = cR\,\mathrm{sign}(\lambda)x, \quad Y = cR\,\mathrm{sign}(\varphi)y.$$

Finding the point of intersection of the meridian and parallel arcs led to quadratic equations, where $D > 0$, so there are two distinct roots. For the x-coordinate, in (4.3), (4.22), (4.44) and (4.59), the solution is $x = (-b + \sqrt{D})/(2a)$. For the coordinate y, in (4.1), (4.23) and (4.45), or in (4.60) the solution is $y = (-b - \sqrt{D})/(2a)$ or $y = (-b + \sqrt{D})/(2a)$, respectively.

In the inverse form of projections, the coordinates X, Y are reduced on the unit sphere

$$x = \frac{|X|}{cR}, \quad y = \frac{|Y|}{cR}. \tag{4.70}$$

The cubic equation: $at^3 + bt^2 + ct + d = 0$, used in projections I, IV, can be solved using the Cardano formulas

$$Q = \frac{3B - A^2}{9}, \quad R = \frac{9AB - 27C - 2A^3}{54},$$

where t in (4.49) and (4.69) is the solution of

$$t = 2\sqrt{-Q}\cos\frac{(\theta + 4\pi)}{3} - \frac{A}{3}$$

and

$$\theta = \arccos\frac{R}{\sqrt{-Q^3}}, \quad A = \frac{b}{a}, \quad B = \frac{c}{a}, \quad C = \frac{d}{a}.$$

Analogously, in (4.16), the solution is $s = (-b + \sqrt{D})/(2a)$. The correct quadrant of φ, λ takes the sign of coordinates X, Y

$$\varphi = \operatorname{sign}(Y)\frac{\pi}{2}t, \quad \lambda = \operatorname{sign}(X)\pi s.$$

Finally, the inverse reprojection of the map [9] created in van der Grinten projection I into the Mercator projection (EPSG code 3857) popular in many web map servers, using the above-mentioned formulas, can be found in Figure 8.

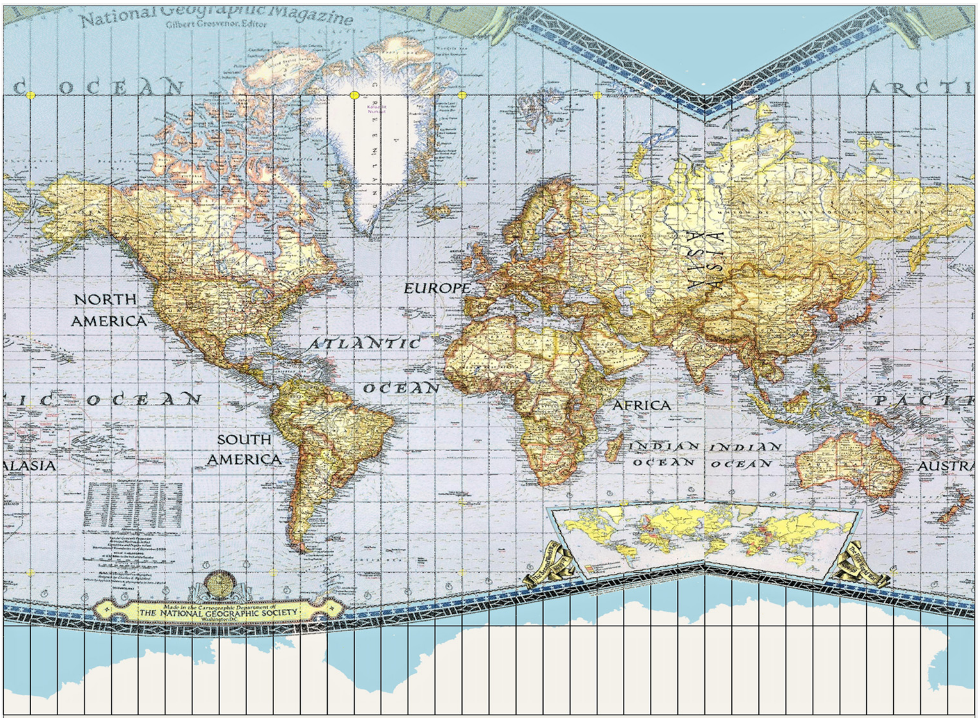

Figure 8. Reprojection of the map [9] from Figure 2.1 in van der Grinten projection I into the Mercator projection; a superimposition with Open-Street Maps.

5. Conclusion

In this paper, a new derivation of van der Grinten projection I–IV equations and their inverse have been presented. Two approaches—the straightforward inverse of the projection equations, as well as the solution of the partial differential equations—were compared. The straightforward inverse is easier to evaluate and works well for all Grinten projections. Particular cases, when the point coincides with the poles, lies on the Equator, or on meridians of the longitude $\lambda = \pm\pi$, are involved.

However, the second approach is computationally more challenging, the partial differential equations of the Grinten I and IV projections have no known analytical solution; the numerical approach is recommended.

Knowledge of the inverse form of the equations is important for the re-projection of maps into different projections, which is frequently used in digital cartography, GIS, environmental science, or in cartometric analyses of early maps.

The proposed methods are non-iterative, easy to implement, and applicable to the coordinate transformation software (e.g. the conversion library Proj. 4). Currently, they are supported by the software tool `detectproj` for the analysis of the map projection [3].

The source code in Matlab involving both the direct and inverse transformations for projections I–IV is available free of charge at `https://github.com/bayertom/vangrinten`

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Authors' addresses: *Tomáš Bayer* (corresponding author), Faculty of Science, Charles University, Albertov 6, 120 78 Praha 2, Czech Republic, e-mail: `bayertom@natur.cuni.cz`; *Milada Kočandrlová*, Faculty of Finance and Accounting, University of Economics, nám. W. Churchilla 1938/4, 130 67 Praha 3, Czech Republic, e-mail: `kocandrlova@hotmail.cz`.