NON-LOCAL DAMAGE MODELLING OF QUASI-BRITTLE COMPOSITES

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Abstract. Most building materials can be characterized as quasi-brittle composites with a cementitious matrix, reinforced by some stiffening particles or elements. Their massive exploitation motivates the development of numerical modelling and simulation of behaviour of such material class under mechanical, thermal, etc. loads, including the evaluation of the risk of initiation and development of micro- and macro-fracture. This paper demonstrates the possibility of certain deterministic prediction, applying the dynamical approach using the Kelvin viscoelastic model and cohesive interface properties. The existence and convergence results rely on the semilinear computational scheme coming from the method of discretization in time, using several types of Rothe sequences, coupled with the extended finite element method (XFEM) for practical calculations. Numerical examples refer to cementitious samples reinforced by short steel fibres, with increasing number of applications as constructive parts in civil engineering.

Keywords: quasi-brittle composite; steel fibre concrete, micro- and macro-fracture, non-local viscoelasticity; cohesive interface; partial differential equations of evolution; method of discretization in time; extended finite element method

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1. Introduction

Study of behaviour of quasi-brittle composites under mechanical, thermal, etc., loads belongs to research priorities in civil engineering, utilizing such composites as constructive parts of buildings and engineering structures. These parts are often made from a material with a cementitious matrix, reinforced by some stiffening (e.g. various metal, glass or plastics as polyethylene or polypropylene fibres), with the aim to reduce the danger of cracking in tension: cf. [22] for the fibre reinforced

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concrete technology, [30] for the review of steel fibre reinforced composites (as the most frequently applied ones) and [25] for certain simplified methodology of computational design of such fibre composites, with the relevant software support [12]. However, a proper computational prediction of strain, stress, etc. development in such composites cannot be based on simple calculations well-known from linear elasticity and related fracture mechanics. Following [54], two stages of damage can be recognized:

i) formation of micro-fractured zones, reducing the stiffness of a structure,

ii) creation of macro-cracks, whose later opening and closing is conditioned by the cohesive characteristics of new interfaces.

At least the following scales should be distinguished: matrix particles (at 10^{-3} m), hardening fibres (at 10^{-2} m) and laboratory samples (at 10^{-1} m) or real structures in situ (even greater). Consequently, a reasonable setting of material parameters on the macroscopic scale supported by appropriate experiments, producing some (typically incomplete) data on material structure as random or intentionally oriented fibre directions, may be complicated in general. Selected problems of this kind preferring non-destructive or low-invasive testing approaches, namely direct photographic, roentgenographic and tomographic ones, and indirect electromagnetic ones relying on certain changes in stationary magnetic or harmonic electromagnetic fields, are discussed in [57] with numerous further references.

Various arguments on the non-negligible non-deterministic character of both input data and relevant physical processes motivate some authors to the attempts to handle the evolution of damage by stochastic considerations, genetic algorithms or other soft computing approaches like [53], [14] or [41], by statistical physics using [48], or by computational peridynamics, avoiding all gradient evaluations, following [38], [15] and [27]. Unlike such approaches, in this paper we shall try to develop a rather simple deterministic physical, mathematical and computational model, up to its software implementation, based on the principle of energy conservation from classical mechanics, incorporating the kinetic and deformation energy, similarly to [37], together with certain energy dissipation (structural and mass damping).

However, the detailed description and computational analysis of particular micro-cracks cannot be performed easily in most engineering applications. The thermodynamic approach of [51] and [40], especially in Part 4.3, introduces additional internal variables to displacements and temperature and combines a fully implicit discretization, based on both Galerkin and Rothe methods, with the analysis of non-linear Nemytskiĭ operators and enthalpy transformation to verify the existence of certain energetic solution of a needed initial and boundary value problem. In our paper, initiation and development of particular micro-cracks will be incorporated using the damage zone representation by [21] and [28], utilizing numerous ideas of [47], adopt-

ing the non-local model from [16]. This model was later reformulated by [17] and is frequently referred as the Eringen one in the last 2 decades. Fortunately, the recent result [18] on the ill-possedness of the non-local approach [17], referring to the incomplete existence analysis of [1] for boundary conditions significant in practical applications, is not addressed to our formulation as explained by [58]. Moreover, according to [55], such conception can be considered as a suitable procedure for a multi-scale approach, avoiding any interpolation between macro- and micro-scale variables. For the strain-stress relations we shall work with the viscoelastic Kelvin law, generating the so-called structural damping, accompanied by the mass damping due to the Rayleigh model in the sense of [43].

The above sketched considerations will be incorporated in our model problem. As its natural generalization, we shall consider also a finite set of matrix/fibre interfaces, as well as of interfaces inside the matrix or even inside any fibre, depending on the process of activation of macro-cracks. All such interfaces will be assumed to satisfy the cohesive model developed by [46], [32], [6] (for various types of fibre composites), [33] (for ceramics) and [36] (for a rather general class of damage propagation).

For most existence and convergence proofs we shall use the method of discretization in time, based on the convergence properties of Rothe sequences, following [49], devoted to linear problems. Moreover, we need to handle 2 types of non-linear terms, coming from i) and ii), as introduced above. For practical evaluations of fully discretized problems we shall prefer the extended finite element method (XFEM), working with the adaptive enrichment of the set of base functions near geometric singularities. This method, including numerous modifications with their special names and specific notations, as generalized finite element method (GFEM) or partition of unity method (PoUM), has its own rich history; the progress in several decades can be traced from the comparison of pioneering works [2], [3] and [19] with the later monograph [31] and the recent articles [37] and [56].

However, we shall pay attention namely to the convergence properties independent of the choice of XFEM adaptive strategies as discussed by [29].

2. Physical and mathematical preliminaries

For our first model problem, let us consider a domain Ω in the 3-dimensional Euclidean space $\mathbb{R}^3$, whose exterior Lipschitz boundary $\partial\Omega$ consists of 2 disjoint parts Θ (for homogeneous Dirichlet boundary conditions) and Γ (for non-homogeneous Neumann boundary conditions), Θ having a non-zero measure on $\partial\Omega$ (to avoid insufficient support). Let $\mathbb{R}^3$ be supplied by a Cartesian coordinate system $x = (x_1, x_2, x_3)$. Moreover, we shall work with the time $t \in I$ from an interval $I = [0, T]$ with some final time value T, assumed as finite here. For the brevity of notation we shall work

with the Hamilton operator $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$ and with upper dots instead of $\partial/\partial t$. Moreover, any comma followed by $k \in \{1,2,3\}$ will be seen as $\partial/\partial x_k$ applied to the preceding variable: e.g. $2\varepsilon_{ij}(v) = v_{i,j} + v_{j,i}$ with $i,j \in \{1,2,3\}$ can be understood as the definition of linearized strain, applicable to any differentiable virtual displacement $v(x) = (v_1(x), v_2(x), v_3(x))$, related to an initial configuration. The Einstein summation rule for indices $i,j,k,l \in \{1,2,3\}$ will be active, too.

The introduction of Lebesgue, Sobolev and Bochner spaces of functions on Ω and $\partial\Omega$ and abstract functions mapping I to them is compatible with [50]. To present our approach as simply as possible now, we shall work namely with the special Hilbert spaces $H = L^2(\Omega)^3$, $Z = L^2(\partial\Omega)^3$, $Z_\Gamma = L^2(\Gamma)^3$ and $V = \{v \in W^{1,2}(\Omega)^3 \colon\ v = O \text{ on } \Theta\}$, supplied with norms denoted by $|\cdot|$ both in H and $H \times H$, $|\cdot|_\Gamma$ in Z_Γ and $\|\cdot\|$ in V, as well as with scalar products $(\cdot,\cdot)$ both in H and $H \times H$, together with $\langle\cdot,\cdot\rangle_\Gamma$ in Z_Γ; O means the zero vector from $\mathbb{R}^3$ here. Slight natural generalizations (which may bring technical difficulties in proofs), motivated by much more detailed references from [50], are left to the curious reader. We shall also utilize upper star symbols for dual spaces, $\subset$ for continuous embeddings, $\Subset$ for compact embeddings, $\cong$ for the identification of a space with its dual in the sense of the Riesz representation theorem.

The following properties of the above introduced spaces (for all notations see [50] again) will be needed:

Lemma 2.1 (Sobolev embedding)**.** *We have $V \Subset H$. Consequently, from any weakly convergent sequence in V, a strongly convergent subsequence in H can be selected.*

P r o o f. See [50], p. 16, and [13], p. 40. □

Lemma 2.2 (trace operator)**.** *We have $V \Subset Z$; $|v|_\Gamma^2 \leqslant \mathfrak{T}\|v\|^2$ for any $v \in V$ with a positive $\mathfrak{T}$ independent of v. Consequently, from any weakly convergent sequence in V, a strongly convergent subsequence in Z can be selected.*

P r o o f. See [50], p. 17, and [13], p. 275. □

Lemma 2.3 (Korn)**.** *There holds $|\varepsilon(v)|^2 \geqslant \mathfrak{K}\|v\|^2$ for any $v \in V$ with a positive $\mathfrak{K}$ independent of v. Consequently, to the standard norm $\|v\|^2 = |v|^2 + |\nabla v|^2$ an alternative norm is generated by $|\varepsilon(v)|^2$ in V.*

P r o o f. For the inequality see [50], p. 22. The consequence follows from the obvious estimate, referring to linear elasticity, $\mathfrak{K}\|v\|^2 \leqslant |\varepsilon(v)|^2 = \frac{1}{4}(v_{i,j}+v_{j,i})(v_{i,j}+v_{j,i}) \leqslant v_{i,j}v_{i,j} = |\nabla v|^2 \leqslant \|v\|^2$. □

Lemma 2.4 (Eberlein-Shmul'yan). *All spaces H, V, $L^2(I,H)$ and $L^2(I,V)$ are reflexive. Consequently, from any bounded sequence in such space, a weakly convergent subsequence can be selected.*

P r o o f. For H and V (see [50], p. 15) $L^2(I,H)$ can be interpreted as $L^2(\Omega \times I)$ with a quite similar result. For the details on duality pairing $L^2(I,V)^* \cong L^2(I,V^*)$ see [50], p. 201. For the consequence cf. [50], pp. 5 and 210, with [13], p. 67. □

Lemma 2.5 (Gelfand triple). *In the triple $V \subset H \cong H^* \subset V^*$ both inclusions are dense; $W^{1,2,2}(I,V,V^*) \subset C(I,H)$.*

P r o o f. See [50], p. 190; $W^{1,2,2}(I,V,V^*)$ here denotes a Bochner-Sobolev space of abstract functions from $L^2(I,V)$ with time derivatives belonging to $L^2(I,V^*)$. □

Lemma 2.6 (Aubin-Lions lemma). *We have $W^{1,2,2}(I,V,V^*) \Subset L^2(I,X)$ with $X \in \{H,Z\}$.*

P r o o f. See [50], p. 194. □

3. A model problem with micro-cracks

Let us introduce a displacement in a deformable body (a priori uknown), occupying the domain Ω, $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$, related to the reference initial configuration (at $t=0$) such that the homogeneous Cauchy initial conditions $u_1(x,0) = u_2(x,0) = u_3(x,0) = 0$ and $\dot{u}_1(x,0) = \dot{u}_2(x,0) = \dot{u}_3(x,0) = 0$ are satisfied for almost every $x \in \Omega$. We shall assume that $u \in \mathcal{V} = W^{1,2,2,2}(I,V,V,V^*)$; thus $u(\cdot,t), \dot{u}(\cdot,t) \in V$ and $\ddot{u}(\cdot,t) \in V^*$ for any $t \in I$ — cf. (3.22) in the proof of Theorem 3.1, including the identification of particular limits below. Both initial conditions can be written as

$$u(\cdot,0) = O, \quad \dot{u}(\cdot,0) = O \quad \text{on } \Omega. \tag{3.1}$$

Analogous simplified notations will be used for further functions, too. Let us note that just the zero-valued $u(\cdot,0)$ and $\dot{u}(\cdot,0)$ in (3.1), referring to certain stationary initial status, are considered in numerous engineering applications; if needed, the following considerations can be repeated for a non-homogeneous form of (3.1) without substantial difficulties.

For an arbitrary $v \in V$ the energy conservation for our model problem can be presented as

$$(v, \varrho\ddot{u}) + \beta(v, \varrho\dot{u}) + \alpha(\varepsilon(v), \dot{\sigma}) + \ (\varepsilon(v), (1-\mathfrak{D})\sigma) = (v,f) + \langle v, g\rangle_\Gamma \quad \text{on } I, \tag{3.2}$$

where $\varrho \in L^\infty(\Omega)$ is the material density and $\sigma \in L^2(I, L^2(\Omega)^{3\times 3}_{\mathrm{sym}})$ refers to all stress components. Its symmetry comes from the assumptions on Boltzmann continuum; for much more general considerations of this type, including constitutive laws, cf. [4], p. 18. The energy dissipation in (3.2), driven by the prescribed body forces $f \in L^2(I, H)$ and surface forces $g \in L^2(I, Z^3_\Gamma)$, is taken into account using the positive damping factors α for structural damping due to the parallel Kelvin viscoelastic model, and the real non-negative factor β for mass damping, compatible with the Rayleigh damping model by [43]. We shall assume that $\varrho \geqslant \varrho_0$ on Ω for a positive constant ϱ_0. Finally, $\mathfrak{D}$ can be presented as a local damage factor with values between 0 and $1-\varsigma$, using an additional positive constant ς; $\mathfrak{D} = 0$ holds always for $t = 0$ (no micro-cracking is present). This factor should depend on σ or $\varepsilon(u)$ directly, non-increasing in time $t \in I$, which can be guaranteed by its evaluation in the form

$$\mathfrak{D}(u)(t) = \max_{\xi\in[0,t]} \mathfrak{D}_*(u(\xi)) \tag{3.3}$$

etc., for particular $t \in I$. Its practical design, namely the form of the continuous mapping $\mathfrak{D}_*$ from V to $L^\infty(\Omega)$, based on the non-local Eringen theory, will be discussed later.

Let us notice that the strong formulation corresponding to (3.2) can be derived, at least in the sense of distributions, from integration by parts. Following the approach of [58] (where the quasi-static case is discussed in all details), for each i (respecting the brief notation $\sigma_{ij,j}$ forcing divergence $\partial\sigma_{ij}/\partial x_j$, etc.) we receive

$$\begin{aligned} \varrho(\ddot{u}_i + \beta\dot{u}_i) - \sigma_{ij,j} &= f_i \quad \text{on } \Omega\times I,\\ \sigma_{ij}\nu_j &= g_i \quad \text{on } \Gamma\times I,\\ u_i &= O \quad \text{on } \Theta\times I, \end{aligned} \tag{3.4}$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ means the local unit normal vector associated with Γ. In addition to the 1st evolution equation of (3.4), referring to the classical Cauchy equilibrium condition, we can see both an explicit Neumann boundary condition in the 2nd equation and a Dirichlet one in the 3rd equation.

The local stress-strain relation can be taken in the simple form

$$\sigma = (1-\mathfrak{D})C\varepsilon(u) + \alpha C\varepsilon(\dot{u}) \tag{3.5}$$

with $C \in L^\infty(\Omega)^{(3\times 3)\times(3\times 3)}_{\mathrm{sym}}$, containing (in general) 21 material parameters, $C(x)$ being positive definite in the sense $C_{ijkl}(x)a_{ij}a_{kl} \geqslant C_0 a_{ij}a_{ij}$, involving a positive constant C_0. In particular, for an isotropic homogeneous medium, setting $\alpha = 0$ formally (the needed generalization with $\alpha > 0$ is straightforward) and zero-valued $\mathfrak{D}$

(no damage occurs yet), using the Kronecker symbol $\delta_{ij} = 1$ for $i = j$, 0 otherwise, we have $\sigma_{ij} = 2\lambda_1\varepsilon_{ij}(u) + \lambda_2\delta_{ij}\varepsilon_{kk}(u)$ with only 2 positive Lamé factors λ_1 and λ_2; frequently they are expressed as $\lambda_1 = \mu E/(1+\mu)/(1-2\mu)$, $2\lambda_2 = E/(1+\mu)$ utilizing the well-known Young modulus E and the Poisson ratio μ; such characteristics will be referenced e.g. from Section 5.

Inserting (3.5) into (3.2), taking (3.3) into account (without explicit highlighting), we obtain

$$(3.6)\quad (v, \varrho\ddot{u}) + \beta(v, \varrho\dot{u}) + \alpha(\varepsilon(v), C\varepsilon(\dot{u})) + (\varepsilon(v), (1-\mathfrak{D})C\varepsilon(u)) = (v, f) + \langle v, g\rangle_\Gamma \text{ on } I.$$

Let I be divided into a finite number m of subsets $I_s^m = \{t \in I\colon (s-1)\tau < t \leqslant s\tau\}$, where $s \in \{1, \dots, m\}$, with the final aim $m \to \infty$; $\tau(m) = T/m$ is considered (the argument m will be omitted formally). We are able to work with the Clément quasi-interpolation f^m of f in $L^2(I, H)$ and g^m of g in $L^2(I, Z_\Gamma)$, assuming $t \in I_s^m$, $s \in \{1, \dots, m\}$, i.e. $f^m(t) = f_s^m$ and $g^m(t) = g_s^m$, where

$$f_s^m = \frac{1}{\tau}\int_{(s-1)\tau}^{s\tau} f(\xi)\,\mathrm{d}\xi, \quad g_s^m = \frac{1}{\tau}\int_{(s-1)\tau}^{s\tau} g(\xi)\,\mathrm{d}\xi.$$

This yields

$$(3.7)\qquad \begin{aligned} \tau\sum_{s=1}^{m}|f_s^m|^2 &= \frac{1}{\tau}\sum_{s=1}^{m}\left|\int_{(s-1)\tau}^{s\tau} f(\xi)\,\mathrm{d}\xi\right|^2 \leqslant \int_I |f(\xi)|^2\,\mathrm{d}\xi \\ \tau\sum_{s=1}^{m}|g_s^m|_\Gamma^2 &= \frac{1}{\tau}\sum_{s=1}^{m}\left|\int_{(s-1)\tau}^{s\tau} g(\xi)\,\mathrm{d}\xi\right|_\Gamma^2 \leqslant \int_I |g(\xi)|_\Gamma^2\,\mathrm{d}\xi, \end{aligned}$$

which will be needed later, cf. (3.17).

For any unknown u^m, introducing the differences $Du_s^m = u_s^m - u_{s-1}^m$ with $s \in \{1, \dots, m\}$, taking $u_0^m = O$ and $Du_0^m = O$ formally, due to (3.1), we can set some linear Lagrange splines

$$(3.8)\qquad u^m(t) = u_{s-1}^m + \frac{t-(s-1)\tau}{\tau}Du_s^m$$

and standard and retarded simple functions

$$(3.9)\qquad \bar{u}^m(t) = u_s^m, \quad \breve{u}^m(t) = u_{s-1}^m$$

for $t \in I_s^m$. Let us recall that $u^m(t)$ and $\bar{u}^m(t)$ for $m \in \{1, 2, \dots\}$ by (3.8) and (3.9) are just 2 classical sequences of Rothe, as introduced in [49]. To handle the 1st additive term of (3.6) properly, we need also certain quadratic interpolation

$$(3.10)\qquad U^m(t) = u_{s-1}^m + \frac{t-(s-1)\tau}{2\tau}(Du_s^m + Du_{s-1}^m) + \frac{(t-(s-1)\tau)^2}{2\tau^2}D^2u_s^m,$$

where $D^2u_s^m = Du_s^m - Du_{s-1}^m$. Thus, we are able to rewrite (3.6) in its time-discretized form

$$(3.11)\qquad \begin{aligned}(v,\varrho\ddot{U}^m) + \beta(v,\varrho\dot{u}^m) + \alpha(\varepsilon(v), C\varepsilon(\dot{u}^m))\\ + (\varepsilon(v),(1-\breve{\mathfrak{D}}^m)C\varepsilon(\bar{u}^m)) = (v,f^m) + \langle v, g^m\rangle_\Gamma,\end{aligned}$$

where $\breve{\mathfrak{D}}^m$ refers to the evaluation of $\mathfrak{D}$ with $\breve{u}^m$, instead of u by (3.6); this is a simple function with certain values u_{s-1}^m, denoted by $\mathfrak{D}_{s-1}^m$ for brevity, cf. (3.12). For any step-by-step evaluation with $s \in \{1,\dots,m\}$, taking $t \in I_s^m$ only, (3.11) gets the form

$$(3.12)\qquad \begin{aligned}\frac{1}{\tau^2}(v,\varrho D^2u_s^m) + \frac{\beta}{\tau}(v,\varrho Du_s^m) + \frac{\alpha}{\tau}(\varepsilon(v), C\varepsilon(Du_s^m))\\ + (\varepsilon(v),(1-\mathfrak{D}_{s-1}^m)C\varepsilon(u_s^m)) = (v,f_s^m) + \langle v, g_s^m\rangle_\Gamma.\end{aligned}$$

The following theorem guarantees the solvability of (3.6) assuming (3.1), utilizing the computational construction of sequences by (3.12).

Theorem 3.1. *Let us consider a damage factor by* (3.3). *There exist a solution* $u \in \mathcal{V}$ *satisfying* (3.6) *for any* $v \in V$ *together with the Cauchy initial condition* (3.1). *Moreover,* $u, \dot{u} \in C(I,H)$ *and up to subsequences,*

$$(3.13)\qquad \begin{aligned}
\{\ddot{U}^m\}_{m=1}^\infty &\ \textit{converges weakly to } \ddot{u} && \textit{in } L^2(I,V^*),\\
\{\dot{u}^m\}_{m=1}^\infty &\ \textit{converges weakly to } \dot{u} && \textit{in } L^2(I,V),\\
\{\dot{u}^m(t)\}_{m=1}^\infty &\ \textit{converges weakly to } \dot{u} && \textit{in } H \textit{ for any } t\in I,\\
\{\bar{u}^m(t)\}_{m=1}^\infty &\ \textit{converges weakly to } u && \textit{in } V \textit{ for any } t\in I,\\
\{\breve{u}^m(t)\}_{m=1}^\infty &\ \textit{converges weakly to } u && \textit{in } V \textit{ for any } t\in I,\\
\{\breve{u}^m(t)\}_{m=1}^\infty &\ \textit{converges strongly to } u && \textit{in } H \textit{ for any } t\in I,\\
\{\dot{U}^m\}_{m=1}^\infty &\ \textit{converges strongly to } \dot{u} && \textit{in } L^2(I,H),
\end{aligned}$$

using the sequences $\{u^m\}_{m=1}^\infty$, $\{\bar{u}^m\}_{m=1}^\infty$, $\{\breve{u}^m\}_{m=1}^\infty$ *and* $\{U^m\}_{m=1}^\infty$, *induced by* (3.8), (3.9) *and* (3.10), *for the time-discretization scheme* (3.11).

Proof. Let us choose $v = Du_s^m$ in (3.12), with the aim to derive some a priori bounds for the above introduced sequences generated by U^m, u^m, $\bar{u}^m$ and $\breve{u}^m$ with integer m. We receive

$$(3.14)\qquad \begin{aligned}\frac{1}{\tau^2}(Du_s^m,\varrho D^2u_s^m) + \frac{\beta}{\tau}(Du_s^m,\varrho Du_s^m) + \frac{\alpha}{\tau}(\varepsilon(Du_s^m), C\varepsilon(Du_s^m))\\ + (\varepsilon(Du_s^m),(1-\mathfrak{D}_{s-1}^m)C\varepsilon(u_s^m)) = (Du_s^m,f_s^m) + \langle Du_s^m, g_s^m\rangle_\Gamma.\end{aligned}$$

The same results remain true with arbitrary $r \in \{1, \dots, s\}$ instead of s. Using the obvious relation $2a(a-b) = a^2 - b^2 + (a-b)^2$ valid for any real a and b, the sum of all equations (3.14), understanding r as well as $p \in \{1, \dots, s-1\}$ (needed in the following 6th left-hand-side additive term exclusively) as Einstein summation indices, is then

$$
\begin{aligned}
(3.15)\qquad & \frac{1}{2\tau^2}(Du_s^m, \varrho Du_s^m) + \frac{1}{2\tau^2}(D^2u_r^m, \varrho D^2u_r^m) \\
& \quad + \frac{\beta}{\tau}(Du_r^m, \varrho Du_r^m) + \frac{\alpha}{\tau}(\varepsilon(Du_r^m), C\varepsilon(Du_r^m)) \\
& \quad + \frac{1}{2}(\varepsilon(u_s^m), (1-\mathfrak{D}_{s-1}^m)C\varepsilon(u_s^m)) + \frac{1}{2}(\varepsilon(u_p^m), (\mathfrak{D}_p^m - \mathfrak{D}_{p-1}^m)\varepsilon(u_p^m)) \\
& \quad + \frac{1}{2}(\varepsilon(Du_r^m), (1-\mathfrak{D}_{r-1}^m)C\varepsilon(Du_r^m)) \\
& = (Du_r^m, f_r^m) + \langle Du_r^m, g_r^m\rangle_\Gamma.
\end{aligned}
$$

All left-hand-side additive terms are non-negative, namely the 6th one thanks to (3.3), thus the 2nd, 3rd and 6th ones can be bounded by zero from below. The more precise estimates for the 1st, 4th and 5th terms, applying Lemma 2.3 to the 4th and 5th ones, are

$$
\begin{aligned}
(3.16)\qquad \frac{1}{2\tau^2}(Du_s^m, \varrho Du_s^m) &\geqslant \frac{\varrho_0}{2\tau^2}|Du_s^m|^2, \\
\frac{\alpha}{\tau}(\varepsilon(Du_r^m), C\varepsilon(Du_r^m)) &\geqslant \frac{\alpha C_0 \mathfrak{K}}{\tau}\delta_{rr}\|Du_r^m\|^2, \\
\frac{1}{2}(\varepsilon(u_s^m), (1-\mathfrak{D}_s^m)C\varepsilon(u_s^m)) &\geqslant \frac{\varsigma C_0 \mathfrak{K}}{2}\|u_s^m\|^2.
\end{aligned}
$$

Using the Cauchy-Schwarz and the Young inequalities, the 1st and 2nd rigth-hand-side terms of (3.14) then admit the estimates

$$
\begin{aligned}
(3.17)\qquad (Du_r^m, f_r^m) \leqslant |Du_r^m||f_r^m| &\leqslant \frac{\varepsilon}{2\tau}\delta_{rr}|Du_r^m|^2 + \frac{\tau}{2\varepsilon}\delta_{rr}|f_r^m|^2 \\
&\leqslant \frac{\varepsilon}{2\tau}\delta_{rr}\|Du_r^m\|^2 + \frac{\tau}{2\varepsilon}\delta_{rr}|f_r^m|^2, \\
\langle Du_r^m, g_r^m\rangle_\Gamma \leqslant |Du_r^m|_\Gamma|g_r^m|_\Gamma &\leqslant \frac{\varepsilon}{2\tau}\delta_{rr}|Du_r^m|_\Gamma^2 + \frac{\tau}{2\varepsilon}\delta_{rr}|g_r^m|_\Gamma^2 \\
&\leqslant \frac{\varepsilon\mathfrak{T}}{2\tau}\delta_{rr}\|Du_r^m\|^2 + \frac{\tau}{2\varepsilon}\delta_{rr}|g_r^m|_\Gamma^2,
\end{aligned}
$$

where ε is an arbitrary positive constant; the constant $\mathfrak{T}$ in the last inequality comes from Lemma 2.2. Comparing (3.16) and (3.17) with respect to (3.7), we obtain

$$
(3.18)\qquad \frac{1}{\tau^2}|Du_s^m|^2 + \frac{1}{\tau}\delta_{rr}\|Du_r^m\|^2 + \|u_s\|^2 \leqslant c,
$$

where c is a positive constant independent of τ (as well as of m, s, etc.). The evident consequences of (3.18) are:

$$
\begin{aligned}
(3.19) \qquad \{\dot{u}^m\}_{m=1}^{\infty} &\quad \text{is bounded in } L^2(I,V),\\
\{\dot{u}^m(t)\}_{m=1}^{\infty} &\quad \text{is bounded in } H \text{ for any } t \in I,\\
\{\bar{u}^m(t)\}_{m=1}^{\infty} &\quad \text{is bounded in } V \text{ for any } t \in I,\\
\{\breve{u}^m(t)\}_{m=1}^{\infty} &\quad \text{is bounded in } V \text{ for any } t \in I.
\end{aligned}
$$

Moreover, from (3.11), converted to the form

$$
\begin{aligned}
(3.20) \qquad (v, \varrho\ddot{U}^m) = &-\beta(v, \varrho\dot{u}^m) - \alpha(\varepsilon(v), C\varepsilon(\dot{u}^m))\\
&- (\varepsilon(v), (1-\breve{\mathfrak{D}}^m)C\varepsilon(\bar{u}^m)) + (v, f^m) + \langle v, g^m\rangle_\Gamma,
\end{aligned}
$$

we are able to derive an additional estimate for $\ddot{U}^m$, using the dual space to $L^2(I,V)$ as suggested by [50], p. 205; cf. also the comment to the proof of Lemma 2.4. Taking $\|v\| \leqslant 1$, $(v, \varrho\ddot{U}^m)$ in (3.20) can be bounded from its right-hand side again, using the Cauchy-Schwarz inequalities and further arguments similarly to (3.17), together with the knowledge of all results (3.19); thus, we have $(v, \varrho\ddot{U}^m) \leqslant \hat{c}$ for a positive constant $\hat{c}$ and consequently,

$$
(3.21) \qquad \{\varrho\ddot{U}^m\}_{m=1}^{\infty} \text{ is bounded in } L^2(I,V^*).
$$

Let us notice that (3.21) remains true without the positive multiplier $\varrho \in L^\infty(\Omega)$.

From (3.21) and (3.19) we can now conclude, following Lemma 2.5, up to subsequences that

$$
\begin{aligned}
(3.22) \qquad \{\ddot{U}^m\}_{m=1}^{\infty} &\quad \text{converges weakly to } u'' &&\text{in } L^2(I,V^*),\\
\{\dot{u}^m\}_{m=1}^{\infty} &\quad \text{converges weakly to } \hat{u} &&\text{in } L^2(I,V),\\
\{\dot{u}^m(t)\}_{m=1}^{\infty} &\quad \text{converges weakly to } u' &&\text{in } H \text{ for any } t \in I,\\
\{\bar{u}^m(t)\}_{m=1}^{\infty} &\quad \text{converges weakly to } \bar{u} &&\text{in } V \text{ for any } t \in I,\\
\{\breve{u}^m(t)\}_{m=1}^{\infty} &\quad \text{converges weakly to } \breve{u} &&\text{in } V \text{ for any } t \in I,\\
\{\breve{u}^m(t)\}_{m=1}^{\infty} &\quad \text{converges strongly to } \breve{u} &&\text{in } H \text{ for any } t \in I,\\
\{\dot{U}^m\}_{m=1}^{\infty} &\quad \text{converges strongly to } u^\diamond &&\text{in } L^2(I,H),
\end{aligned}
$$

etc., where u'', $\hat{u}$, u', $\bar{u}$ and $\breve{u}$ are some elements of corresponding spaces, see Lemma 2.6 for the last proposition, too. The strong convergence of $\{\breve{\mathfrak{D}}^m\}_{m=1}^{\infty}$, seemingly as in that for $\{\breve{u}^m(t)\}_{m=1}^{\infty}$ in the 6th proposition of (3.22), is inherited from the formal introduction of $\mathfrak{D}$ here and will need more detailed analysis.

In particular, considering $t \in I$, by Lemma 2.6 $u^\diamond(t)$ coincides with $\int_0^t u''(\xi)\,\mathrm{d}\xi$; also further limits can be unified. The 2nd and 3rd propositions (3.22) manifest the weak convergence of the same sequence both to $\hat{u}$ and to u', e.g. in $L^2(I,H)$, thus $\hat{u} = u'$. The obvious estimate $\max\limits_{t\in I}(|u^m(t)-\bar{u}^m(t)|, |u^m(t)-\breve{u}^m(t)|) \leqslant \max\limits_{s\in\{1,\dots,m\}} |Du_s^m| \leqslant \sqrt{c}\tau$, referring to (3.18), implies $u = \bar{u} = \breve{u}$ and $\dot{u} = u'$, where $u(t) = \int_0^t u'(\xi)\,\mathrm{d}\xi$. Thus, it remains to identify u'' with $\ddot{u}$ only, as the most delicate task. Let us work with symbols $[\cdot]$ for integration over I for brevity. The following integration by parts, inspired by [50], p. 210, can be helpful:

$$
\begin{aligned}
[(w,u'')] &= \lim_{m\to\infty}[(w,\ddot{U}^m)] = -\lim_{m\to\infty}[(\dot{w},\dot{U}^m)] \\
&= -\lim_{m\to\infty}[(\dot{w},\dot{U}^m-\dot{u}^m)] - \lim_{m\to\infty}[(\dot{w},\dot{u}^m)] \\
&= -\lim_{m\to\infty}[(\dot{w},\dot{U}^m-\dot{u}^m)] - [(\dot{w},\dot{u})]
\end{aligned}
\tag{3.23}
$$

is valid for each w from the space of distributions $C_0^\infty(I)$. Moreover, for arbitrary $t \in I_s^m$, $s \in \{1,\dots,m\}$, we can write

$$
\begin{aligned}
\dot{U}^m - \dot{u}^m &= \frac{t-(s-1)\tau}{\tau^2}D^2u_s^m + \frac{1}{2\tau}(Du_s^m + Du_{s-1}^m) - \frac{1}{\tau}Du_s^m \\
&= \frac{t-(s-1)\tau}{\tau^2}D^2u_s^m + \frac{1}{2\tau}(Du_{s-1}^m - Du_s^m) \\
&= \frac{t-(s-1/2)\tau}{\tau^2}D^2u_s^m = (t-(s-1/2)\tau)\ddot{U}^m.
\end{aligned}
\tag{3.24}
$$

Let us remind that $0 \leqslant t-(s-\frac{1}{2})\tau \leqslant \frac{1}{2}\tau$ here. Thus, inserting (3.24) into the result of (3.23), the limit in its 1st additive term vanishes, whereas the 2nd additive term is sufficient to identify u'' with $\ddot{u}$, etc., as explained by Buncure [7], p. 49. Thus, the modified form of (3.22) is just (3.13); also the convergence of $\{\breve{\mathfrak{D}}^m\}_{m=1}^\infty$ can work with u. This enables the limit passage from (3.11) to (3.6) finally; $u, \dot{u} \in C(I,H)$ follows from Lemma 2.5. $\square$

Let us recall that the crucial step for the design of a model with micro-cracks is the reasonable choice of the damage factor $\mathfrak{D}$. Here we shall demonstrate how to express it as an appropriate function of σ, with certain regularizing properties. This can be done using some kernel (typically radial basis or similar) operator $K \in L^2(\Omega\times\Omega)$, introduced as

$$
A(w(x)) = \int_\Omega K(x,\tilde{x})w(\tilde{x})\,\mathrm{d}\tilde{x}
\tag{3.25}
$$

for $x \in \Omega$ and $w \in H$ by [17]. From the mathematical point of view, such nonlocal approach to engineering mechanics relies on the properties of compact linear

operators, discussed in [13], Part 2.2, in details; for its computational implementation cf. [5] and [39]. The following regularization (compactness) property of the kernel K, taken from $L^2(\Omega\times\Omega)$, is useful: if $\{w^k\}_{k=1}^{\infty}$ is a sequence converging weakly to w in H, then taking $\widetilde{w} = A(w)$ and $\widetilde{w}^k = A(w^k)$ up to a subsequence, $\{\widetilde{w}^k\}_{k=1}^{\infty}$ converges strongly to $\widetilde{w}$ in H. Two different ways of verification of this result can be found in [13], pp. 80 and 81.

The needed generalization introduced by [20] for $w \in L^2(\Omega)^{3\times 3}_{\mathrm{sym}}$ or that for $w \in H$, referring to principal stresses by [23], Part 1.5, is straightforward. Namely, [34] works with $K(x,\tilde{x}) = \mathcal{K}(|x-\tilde{x}|_3)$, where $|\cdot|_3$ means the norm in $\mathbb{R}^3$ and $\mathcal{K}(|x-\tilde{x}|_3)$ is obtained using Green functions of a special bi-Helmholtz equation; for certain class of brittle fracture this can be traced up to atomistic considerations, working with dislocation and disclination defects. However, for more complicated material structures such transparent theory is not available; e.g. for practical computational simulations of behaviour of fibre-reinforced concrete structures under mechanical loads [12] recommends the "generalized Mazars model" with several heuristic parameters, respecting anisotropy together with different behaviour under tension and pressure like [28] and [24], inspired by [44], [45] and [21].

Thus, we are ready, using σ from (3.5), for any fixed time $\xi \in I$, to derive (at least theoretically) all non-local stress values

$$\widetilde{\sigma}(\xi) = A(\sigma(\xi)) \tag{3.26}$$

belonging to $L^2(\Omega)^{3\times 3}_{\mathrm{sym}}$. Thanks to (3.26), it only remains to set

$$\mathfrak{D}_*(u(\xi)) = \omega(|\widetilde{\sigma}(\xi)|_{3\times 3}), \tag{3.27}$$

taking ω as a real continuous non-decreasing function (containing some additional experimentally validated parameters typically) for the right-hand side of (3.3), $|\cdot|_{3\times 3}$ here means the norm in $\mathbb{R}^{3\times 3}$. Clearly, the resulting damage factor $\mathfrak{D}$, obtained from (3.3) with (3.27), depends on u from (3.2) in a rather complicated way. Nevertheless, such formulation of (3.3) together with (3.25) enables us to exploit the results on Nemytyskiĭ mappings by [13], p. 134: if a sequence converges weakly to $u \in V$ for a fixed $t \in I$ together with the sequence or corresponding time derivatives converging to $\dot{u}$, then thanks to (3.5), the operator A generates a weakly convergent sequence to $\widetilde{\sigma} \in H$; after the regularization (3.26), the same, up to a subsequence, converges strongly to $\widetilde{\sigma} \in H$, etc. Consequently, thanks to the continuity of ω by (3.27), we are allowed to come to the strong limit of the corresponding sequence induced by (3.3), which may be helpful to overcome the non-linearity of our model problem. However, the design of a sufficiently general class of functions ω admitting all above sketched mathematical considerations and applicable in engineering

practice (regardless of both physical and geometrical linearizations, connected with the existence of a positive ς) cannot be seen as a closed problem; for a particular examples cf. [28] and [58].

4. Implementation of macro-cracks

Instead of one domain Ω, as introduced in Section 3, let us consider a union of a finite number of adjacent domains, denoted again by Ω, whose boundary $\partial\Omega$ consists of 3 parts: of 2 exterior ones, analogical to Γ and Θ, and of a set of internal interfaces Λ. Such notion of interfaces can cover both potential locations of macro-cracks as well as existing interfaces between particular components of a composite, e.g. between a cementitious matrix and stiffening fibres in building applications.

For simplicity, let us consider a material specimen occupying an open set Ξ with its boundary $\partial\Xi$ in the 3-dimensional Euclidean space $\mathbb{R}^3$, compounded from a finite number of domains $\Omega_\times$ with their boundaries $\partial\Omega_\times$ in the following sense:

a) The union of closures of all domains $\Omega_\times$ is identical to the closure of the domain Ω in $\mathbb{R}^3$.
b) Every boundary $\partial\Omega_\times$ consists of a part belonging to $\partial\Xi$ (external boundary) and from that non-belonging to $\partial\Xi$ (internal boundary); the 1st one will be denoted by $\Psi_\times$, the 2nd one by $\Lambda_\times$. (Some of them can be empty.) Cohesive interface conditions will be applied later on $\Lambda_\times$.
c) Every boundary part $\Psi_\times$ is the union of its disjoint subsets $\Theta_\times$ and $\Gamma_\times$. (Some of them can be empty.) Homogeneous Dirichlet boundary conditions will be then prescribed on $\Theta_\times$ (supported boundary part), unlike Neumann boundary conditions (inhomogeneous in general) on $\Gamma_\times$ (unsupported boundary part).
d) The unions of above introduced sets $\Theta_\times$, $\Gamma_\times$ and $\Lambda_\times$ are certain sets Ω, Γ and Λ. Similarly, the union of all $\Omega_\times$ generates an open set Ω (i.e. Ξ without interior boundaries) with its boundary $\partial\Omega$.

We shall also work with the notation $\delta v(x)$ for the differences of triples of values $v(x)$ from the neighbour domains $\Omega_\times$; the same notation is applicable to arbitrary $\tilde{v}(x,t)$, dependent also on $t \in I$, replacing $v(x)$ here.

One can notice that such rather extensive list of assumptions tries to save the validity of Lemmas 2.1, ..., 2.6, to be able to adopt the proof of Theorem 3.1 without serious difficulties. The potential modification of this approach for a finite dimension other than 3 (namely as 2 in illustrative examples) is left to the patient reader. Evidently a model problem with macro-cracks could be studied separately; however, we shall now try to implement macro-cracking to the results of Section 3 directly.

All notations need certain extensions. Namely, we shall utilize also the Hilbert space $Z_\Lambda = L^2(\Lambda)^3$, its norm $|\cdot|_\Lambda$ and its scalar product $\langle\cdot,\cdot\rangle_\Lambda$. For the analysis of potential opening and further behaviour of cracks let us consider such surface tractions $\mathcal{T} \in L^2(I, Z_\Lambda)$ that

$$\mathcal{T} = \lambda(\delta u) \tag{4.1}$$

on $\Lambda \times I$; possible forms of just introduced function λ can be found in [32] and [33]. Especially $\lambda(\delta u) = \lambda_0 \delta u$ with a real constant $\lambda_0 \to \infty$ forces $\delta u \to O$ on Λ, i.e. the continuity of u without no active macro-cracking. In general, we shall assume the Lipschitz continuity of λ in the sense

$$|\lambda(\delta v)|_\Lambda \leqslant \lambda_\star |\delta v|_\Lambda \tag{4.2}$$

for any $v \in V$ and a positive $\lambda_\star$. For certain finite N independent of v, from (4.2) we have:

$$|\delta v|_\Lambda \leqslant N\sqrt{\mathfrak{T}}\|v\|, \quad |\lambda(\delta v)|_\Lambda \leqslant \lambda_\star N\sqrt{\mathfrak{T}}\|v\|. \tag{4.3}$$

The existence of N comes from b), d) above: traces by the analogy of Lemma 2.2 are related to any $\Omega_\times$ at most N-times, from corresponding cohesive boundary parts $\Lambda_\times$. In the proof of Theorem 4.1 below we are allowed to take only $\mathfrak{T}$ instead of $N^2\mathfrak{T}$ formally, without any loss of generality.

Thus, we obtain the slight modification of (3.2):

$$\begin{aligned}(v, \varrho\ddot{u}) + \beta(v, \varrho\dot{u}) + \alpha(\varepsilon(v), \dot{\sigma}) + (\varepsilon(v), (1-\mathfrak{D})\sigma)\\ = (v, f) + \langle v, g\rangle_\Gamma + \langle \delta v, \mathcal{T}\rangle_\Lambda \quad \text{on } I.\end{aligned} \tag{4.4}$$

Inserting (4.1) into (4.4), we receive the analogy of (3.2):

$$\begin{aligned}(v, \varrho\ddot{u}) + \beta(v, \varrho\dot{u}) + \alpha(\varepsilon(v), \dot{\sigma}) + (\varepsilon(v), (1-\mathfrak{D})\sigma)\\ = (v, f) + \langle v, g\rangle_\Gamma + \langle \delta v, \lambda(\delta u)\rangle_\Lambda \quad \text{on } I.\end{aligned} \tag{4.5}$$

In most of the following equations, unlike (4.4) extending (3.2) and (4.5) extending (3.6), we shall discuss only additional rigth-hand-side terms for brevity. In particular, this means $\langle \delta v, \lambda(\delta \check{u}^m)\rangle_\Lambda$ in (3.11), $\langle \delta v, \lambda(\delta u^m_{s-1})\rangle_\Lambda$ in (3.12); no improved linearization will be considered, although the evaluation of $\lambda(\cdot)$ is typically less complicated as that of $\mathfrak{D}(\cdot)$. Such approach will be useful in the proof of the following theorem, too.

Theorem 4.1. *Let us consider a damage factor by* (3.3) *and a cohesive interface by* (4.1), (4.2). *There exists a solution* $u \in \mathcal{V}$ *satisfying* (4.5) *for any* $v \in V$ *together with the Cauchy initial condition* (3.1). *Moreover,* $u, \dot{u} \in C(I, H)$ *and up to subsequences,* (3.13) *remains valid.*

P r o o f. All common ideas are the same as in the proof of Theorem 3.1. The above announced additional terms are $\langle \delta D u_s^m, \lambda(\delta u_{s-1}^m)\rangle_\Lambda$ in (3.14) and $\langle \delta D u_r^m, \lambda(\delta u_{r-1}^m)\rangle_\Lambda$ in (3.15) (involving the sum over $r \in \{1,\dots,s\}$). The extension of (3.17), making use of (4.3), reads

$$\begin{aligned}(4.6)\qquad \langle \delta D u_r^m, \lambda(\delta u_r^m)\rangle_\Lambda &\leqslant |\delta D u_r^m|_\Gamma |\delta u_{r-1}^m|_\Gamma \leqslant \frac{\varepsilon}{2\tau}\delta_{rr}|\delta D u_r^m|_\Gamma^2 + \frac{\tau}{2\varepsilon}\delta_{rr}|\delta u_{r-1}^m|_\Gamma^2 \\ &\leqslant \frac{\varepsilon\mathfrak{T}}{2\tau}\delta_{rr}\|D u_r^m\|^2 + \frac{\tau\mathfrak{T}}{2\varepsilon}\delta_{rr}\|u_{r-1}^m\|^2.\end{aligned}$$

The last rigth-hand-side additive term of (4.6) causes the modification of (3.18)

$$(4.7)\qquad \frac{1}{\tau^2}|D u_s^m|^2 + \frac{1}{\tau}\delta_{rr}\|D u_r^m\|^2 + \|u_s\|^2 \leqslant c + c\tau\delta_r r\|u_r\|^2.$$

However, the 2nd right-hand-side additive term can be removed from (4.7), using the Gronwall lemma by [13], p. 99; its simple discrete version from [9] is sufficient here. Thus, we come back to (3.18) with some larger value of c. All remaining steps can be then performed following the proof of Theorem 4.1 with obvious modifications. □

5. Computational strategy with illustrative examples

The computational scheme (3.12) by Section 3, including its extension by Section 4, for the evaluation of u_s^m, $s \in \{1,\dots,m\}$ refers to the numerical analysis of m elliptic problems of infinite dimension. In practical calculations, instead of v in (3.12) from an infinite-dimensional space V, we consider a finite number n of test functions φ_i^n, where $i \in \{1,\dots,n\}$ refers to a new Einstein summation index; the approximation of u_s^m from (3.12) with n unknown parameters can be then constructed as

$$(5.1)\qquad u_s^m(x) = u_{is}^{nm}\varphi_i^n(x)$$

for any x from Ω or its suitable approximation. Consequently, step-by-step with $s \in \{1,\dots,m\}$, by (3.12) and (5.1) we choose φ_i^n as particular elements from a basis of certain finite-dimensional space V^n, approximating V (which can be a subspace of V in a special case).

Typically φ_i^n are functions with small compact support, applicable in Ω, as well as on Θ, Γ and Λ, or their approximations, to create a sparse system of linear algebraic equations, and u_{is}^{nm} refer to nodal displacement values. The guarantee of solvability of such system, together with the convergence properties for $n \to \infty$, depend on certain (semi-)regularity of such decomposition due to the XFEM-based adaptive

enrichment functions, namely near geometric singularities. Here we shall apply the approach of [29] to demonstrate the possibility of effective numerical simulations.

As an illustrative 2-dimensional example, the test task is a relatively simple body with an a priori crack of a circular shape (fulfilling the plane strain condition). A uniform load was applied to the surface of this a priori crack, and thus the formation of the following cracks emanating from this stress concentrator is assumed using XFEM. The basic calculation system was the commercial software Abaqus 2018, into which a user subroutine in the Fortran 90 language was implemented, realizing the modelling of matrix damage using exponential law, based on the planar element CPE4. The following basic input data corresponding to reinforced cement paste were used for this task: the Young modulus $E = 3.2\,\mathrm{GPa}$, the Poisson constant $\mu = 0.3$ and the tensile strength 10 MPa—cf. the discussion under (3.5) on some special structures of C. For approximately 20 mm long and 3 mm thick circular steel fibres, the Young modulus $E = 190\,\mathrm{GPa}$ and the same Poisson constant $\mu = 0.3$ were used.

All figures show a typical distribution of principal stresses under plain strain conditions. Figure 1 shows the distribution for quasi-static loads in certain representative time for the pure cement paste. In the initial period, four germs of initial cracks are formed evenly distributed along the circular initiator. The germs closer to the plane of symmetry are running first as expected. Directions that are not blocked by the fibres will run. Figure 2 presents the comparable result near 1 or 2 stiffening fibres, where reinforcing metal fibres are introduced into the structure. The influence of fibre blocking and their orientation is clear. Directions that are not blocked by the fibres will run. Figure 3 demonstrates the non-local handling of stress near the crack tip. Its left part attempts to illustrate the algorithm used to calculate the stress concentrators in front of the crack tip. This picture has a schematic character, indicates how the stresses are calculated with the help of a non-local approach. The question is from which distance from the crack front it is appropriate to calculate the stress distribution ahead the crack tip. Figure 4 reflects the Mazars model, evaluating the stress by certain exponential formula (cf. [44] and [21]). The former Mazars model has gaps in the modelling of the behaviour of concrete during loading; the new formulation was proposed to improve behaviour in bi-compression and shearing. It was achieved by introduction of one new internal variable into the classical Mazars model. It corresponds to the maximum of equivalent deformation reached during loading. The damage factor is dependent on the stress concentration in front of the local crack, therefore the real stress value is essential for the prediction of the crack growth, especially the non-local approach can give credible direction of damage. Figure 5 tries to implement some homogenized material structure with "smeared cracks", inspired by [28], where reinforcement has the influence to the whole structure. XFEM approach reflects the increasing strength of structure.

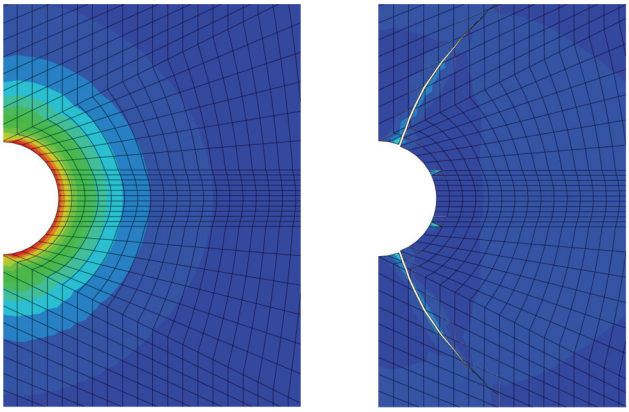

Figure 1. Principle stresses in the pure cement paste.

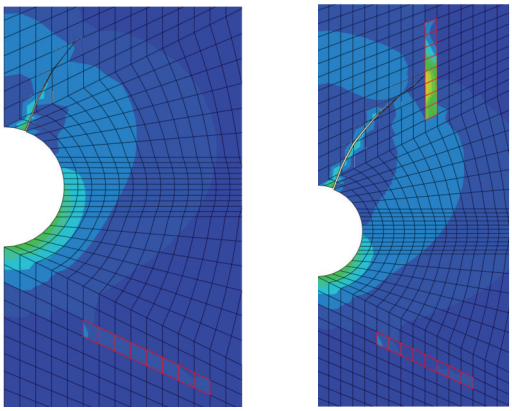

Figure 2. Principle stresses near 1 or 2 stiffening fibres.

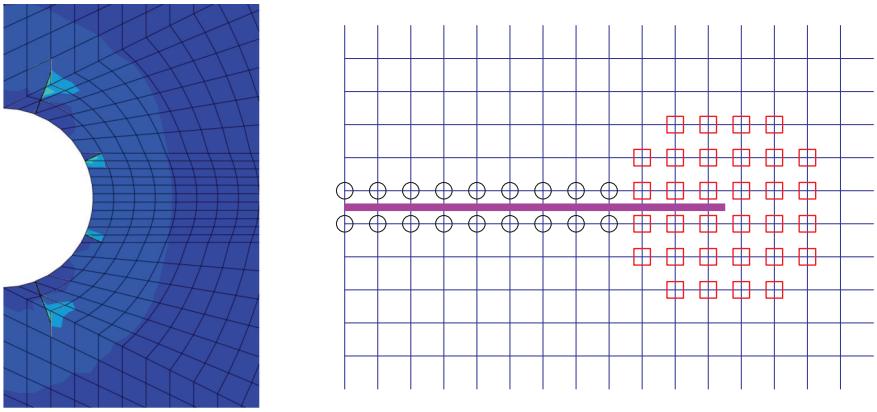

Figure 3. Non-local handling of stress near the crack tip.

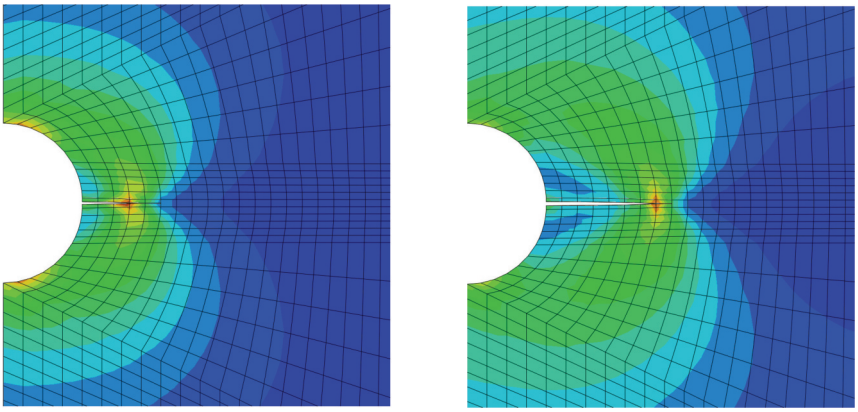

Figure 4. Application of Mazars model.

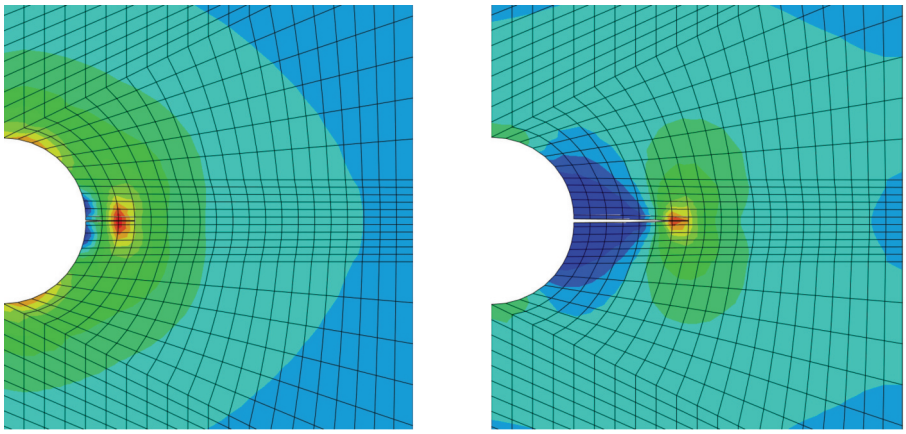

Figure 5. Application of crack homogenization.

6. Some modifications and generalizations

Numerous computational tools in fracture mechanics ignore the first couple of additional terms in (3.2), which switches to a quasi-static problem where the time evolution of damage relies on the 3rd additive term; even in our illustrative examples the 1st and 2nd terms are not dominant. Clearly, the 2nd condition (3.1) is not applicable. Some estimates from Sections 3 and 4 degenerate, namely the 1st inequality (3.16) to $0 = 0$. Thus, less regular results in comparison with the above discussed dynamic case can be expected and their derivation cannot be easily repeated. The remedy is to seek for $u \in W^{1,2,2}(I, V, V^\star)$ instead of $u \in W^{1,2,2,2}(I, V, V, V^\star)$; all details of such approach can be found in [58].

Other seemingly useful generalization ideas can rely on the removal of too strict assumptions, e.g. that on the Lipschitz boundary for Lemmas 2.1,..., 2.6, thus also for Theorems 3.1 and 4.1. Indeed, results like [26] (on Sobolev embeddinng on arbitrary domains), [7] (on improved trace operators), [8] (on weaker Bochner-Sobolev evolution triples), etc., offer this way. However, we have ignored it here, at least from three reasons: i) the original version of this paper, prepared for the (rarely physical) discussion at the seminar PANM (Programs and Algorithms of Numerical Mathematics) in Hejnice (Czech Republic) in June 2020 was intended to be as reader-friendly as possible, ii) numerous technical difficulties in proofs have to be overcome, often those occurring in practical modelling and simulation software tools exceptionally, iii) such generalizations do not handle more significant limitations of our approach: iii–1) the non-local damage factor implemented into certain linearized model, working with small strains and linearized empirical constitutive (strain-stress) relations in the case of micro-cracking, iii–2) the careful description of geometrical properties of Ω, Λ, etc., admitting the macroscopic cracks only at a finite number of prescribed interfaces. Because of iii–1) we are not able to detect a total loss of stiffness in some part of Ω properly, whereas iii–2) may not cover some practical XFEM techniques correctly. Certain inspiration for a future (much more complicated) proper finite-strain formulation can be found in [40], Part 4.2.4, in confrontation with [10] and [35], together with the scale-bridging using structured deformation, following [11], [42] and [52].

7. Conclusion

We have demonstrated the possibility of simultaneous deterministic study of dynamics of micro- and macro-cracking in quasi-brittle composites, using the standard linearized viscoelastic model with two non-linear terms, covering i) the non-local evaluation of damage factor $\mathfrak{D}$ for micro-cracking and ii) the cohesive behaviour of macroscopis cracks. Some unclosed problems occur even in such linearized theory, namely in the physically and mathematically proper interpretation of $\mathfrak{D}$, ad hoc implemented in available software packages.

The limitations of the presented approach, sketched in the preceding section, can be seen as motivations for continuing research in the near future. Its possible aim of high practical importance can be the development, verification and validation of the computational tool for prediction of quasi-brittle behaviour of structural components from fibre reinforced composites under mechanical loads, with methodology based on the physical model incorporating most significant physical processes, namely elastic and plastic deformation, crack initiation and propagation in a matrix and alternative debonding and rupture of fibres.

References

[1] *S. B. Altan*: Existence in nonlocal elasticity. Arch. Mech. *41* (1989), 25–36. zbl MR

[2] *I. Babuška, J. M. Melenk*: The partition of unity method. Int. J. Numer. Methods Eng. *40* (1997), 727–758. zbl MR doi

[3] *T. Belytschko, T. Black*: Elastic crack growth in finite elements with minimal remeshing. Int. J. Numer. Methods Eng. *45* (1999), 601–620. zbl doi

[4] *A. Bermúdez de Castro*: Continuum Thermomechanics. Progress in Mathematical Physics 43. Birkhäuser, Basel, 2005. zbl MR doi

[5] *G. S. Bhatia, G. Arora*: Radial basis function methods for solving partial differential equations: A review. Indian J. Sci. Technol. *9* (2016), Article ID 45, 18 pages. doi

[6] *L. Bouhala, A. Makradi, S. Belouettar, H. Kiefer-Kamal, P. Fréres*: Modelling of failure in long fibres reinforced composites by X-FEM and cohesive zone model. Composites B, Eng. *55* (2013), 352–361. doi

[7] *J. K. Bunkure*: Lebesgue-Bochner spaces and evolution triples. Int. J. Math. Appl. *7* (2019), 41–52.

[8] *A. Cianchi, V. Maz'ya*: Sobolev inequalities in arbitrary domains. Adv. Math. *293* (2016), 644–696. zbl MR doi

[9] *D. S. Clark*: Short proof of a discrete Gronwall inequality. Discrete Appl. Math. *16* (1987), 279–281. zbl MR doi

[10] *G. Dal Maso, G. Lazzaroni*: Crack growth with non-interpenetration: A simplified proof for the pure Neumann problem. Discrete Contin. Dyn. Syst. *31* (2011), 1219–1231. zbl MR doi

[11] *G. Del Piero, D. R. Owen*: Structured deformations of continua. Arch. Ration. Mech. Anal. *124* (1993), 99–155. zbl MR doi

[12] *L. Dlouhý, S. Pouillon*: Application of the design code for steel-fibre-reinforced concrete into finite element software. Beton *116* (2020), 8–13.

[13] *P. Drábek, J. Milota*: Methods of Nonlinear Analysis: Applications to Differential Equations. Birkhäuser Advanced Texts. Basler Lehrbücher. Birkhäuser, Basel, 2013. zbl MR doi

[14] *J. Eliáš, M. Vořechovský, J. Skoček, Z. P. Bažant*: Stochastic discrete meso-scale simulations of concrete fracture: Comparison to experimental data. Eng. Fract. Mech. *135* (2015), 1–16. doi

[15] *E. Emmrich, D. Puhst*: Measure-valued and weak solutions to the nonlinear peridynamic model in nonlocal elastodynamics. Nonlinearity *28* (2015), 285–307. zbl MR doi

[16] *A. C. Eringen*: Theory of Nonlocal Elasticity and Some Applications. Technical Report 62. Princeton University Press, Princeton, 1984.

[17] *A. C. Eringen*: Nonlocal Continuum Field Theories. Springer, New York, 2002. zbl MR doi

[18] *A. Evgrafov, J. C. Bellido*: From non-local Eringen's model to fractional elasticity. Math. Mech. Solids *24* (2019), 1935–1953. zbl MR doi

[19] *T.-P. Fries, T. Belytschko*: The intrinsic XFEM: A method for arbitrary discontinuities without additional unknowns. Int. J. Numer. Methods Eng. *68* (2006), 1358–1385. zbl doi

[20] *Z. Gao, L. Zhang, W. Yu*: A nonlocal continuum damage model for brittle fracture. Eng. Fract. Mech. *189* (2018), 481–500. doi

[21] *C. Giry, F. Dufour, J. Mazars*: Stress-based nonlocal damage model. Int. J. Solids Struct. *48* (2011), 3431–3443. doi

[22] *S. Grija, D. Shanthini, S. Abinaya*: A review on fiber reinforced concrete. Int. J. Civil Eng. Technol. *7* (2016), 386–392.

[23] *K. Hashiguchi*: Elastoplasticity Theory. Lecture Notes in Applied and Computational Mechanics 69. Springer, Berlin, 2014. zbl MR doi

[24] *P. Havlásek, P. Grassl, M. Jirásek*: Analysis of size effect on strength of quasi-brittle materials using integral-type nonlocal models. Eng. Fract. Mech. *157* (2016), 72–85. doi

[25] *A. Hoekstra*: Design methodologies for steel-fibre-reinforced concrete and a new methodology for a real time quality control. Beton *116* (2020), 44–49.

[26] *C. O. Horgan*: Eigenvalue estimates and the trace theorem. J. Math. Anal. Appl. *69* (1979), 231–242. zbl MR doi

[27] *A. Javili, R. Morasata, E. Oterkus, S. Oterkus*: Peridynamics review. Math. Mech. Solids *24* (2019), 3714–3739. zbl MR doi

[28] *M. Jirásek*: Damage and smeared crack models. Numerical Modeling of Concrete Cracking. CISM Courses and Lectures 532. Springer, Wien, 2011, pp. 1–49. zbl doi

[29] *M. Kaliske, H. Dal, R. Fleischhauer, C. Jenkel, C. Netzker*: Characterization of fracture processes by continuum and discrete modelling. Comput. Mech. *50* (2012), 303–320. zbl MR doi

[30] *P. Kawde, A. Warudkar*: Steel fibre reinforced concrete: A review. Int. J. Eng. Sci. Res. Technol. *6* (2017), 130–133. doi

[31] *A. R. Khoei*: Extended Finite Element Method: Theory and Applications. Wiley Series in Computational Mechanics. John Wiley & Sons, New York, 2015. zbl doi

[32] *V. Kozák, Z. Chlup*: Modelling of fibre-matrix interface of brittle matrix long fibre composite by application of cohesive zone method. Key Eng. Materials *465* (2011), 231–234. doi

[33] *V. Kozák, Z. Chlup, P. Padělek, I. Dlouhý*: Prediction of the traction separation law of ceramics using iterative finite element modelling. Solid State Phenomena *258* (2017), 186–189. doi

[34] *M. Lazar, G. A. Maugin, E. C. Aifantis*: On a theory of nonlocal elasticity of bi-Helmholtz type and some applications. Int. J. Solids Struct. *43* (2006), 1404–1421. zbl MR doi

[35] *G. Lazzaroni*: Quasistatic crack growth in finite elasticity with Lipschitz data. Ann. Mat. Pura Appl. (4) *190* (2011), 165–194. zbl MR doi

[36] *X. Li, J. Chen*: An extended cohesive damage model for simulating arbitrary damage propagation in engineering materials. Comput. Methods Appl. Mech. Eng. *315* (2017), 744–759. zbl MR doi

[37] *X. Li, W. Gao, W. Liu*: A mesh objective continuum damage model for quasi-brittle crack modelling and finite element implementation. Int. J. Damage Mech. *28* (2019), 1299–1322. doi

[38] *R. W. Macek, S. A. Silling*: Peridynamics via finite element analysis. Finite Elem. Anal. Des. *43* (2007), 1169–1178. MR doi

[39] *Z. Majdisova, V. Skala*: Radial basis function approximations: Comparison and applications. Appl. Math. Modelling *51* (2017), 728–743. zbl MR doi

[40] *A. Mielke, T. Roubíček*: Rate-Independent Systems: Theory and Applications. Applied Mathematical Sciences 193. Springer, New York, 2015. zbl MR doi

[41] *M. Moradi, A. R. Bagherieh, M. R. Esfahani*: Constitutive modeling of steel fiber-reinforced concrete. Int. J. Damage Mech. *29* (2020), 388–412. doi

[42] *M. Morandotti*: Structured deformation of continua: Theory and applications. Mathematical Analysis of Continuum Mechanics and Industrial Applications II. Springer, Singapore, 2018, pp. 125–136. doi

[43] *N. Nakamura*: Extended Rayleigh damping model. Front. Built Environ. *2* (2016), Article ID 14, 13 pages. doi

[44] *R. H. J. Peerlings, R. de Borst, W, A. M. Brekelmans, M. Geers*: Gradient enhanced damage modelling of concrete fracture. Mech. Cohesive-frictional Mater. *3* (1998), 323–342. doi

[45] *G. Pijaudier-Cabot, J. Mazars*: Damage models for concrete. Section 6.13. Handbook of Materials Behavior Models. Volume II (J. Lemaitre, ed.). Academic Press, London, 2001, pp. 500–512. doi

[46] *M. G. Pike, C. Oskay*: XFEM modeling of short microfiber reinforced composites with cohesive interfaces. Finite Elem. Anal. Des. *106* (2005), 16–31. doi

[47] *Yu. Z. Povstenko*: The nonlocal theory of elasticity and its application to the description of defects in solid bodies. J. Math. Sci. *97* (1999), 3840–3845. doi
[48] *P. Ray*: Statistical physics perspective of fracture in brittle and quasi-brittle materials. Philos. Trans. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci. *377* (2019), Article ID 20170396, 13 pages. doi
[49] *K. Rektorys*: The Method of Discretization in Time and Partial Differential Equations. Mathematics and Its Applications 4. D. Reidel, Dordrecht, 1982. zbl MR
[50] *T. Roubíček*: Nonlinear Partial Differential Equations with Applications. ISNM. International Series of Numerical Mathematics 153. Birkhäuser, Basel, 2005. zbl MR
[51] *T. Roubíček*: Thermodynamics of rate-independent processes in viscous solids at small strains. SIAM J. Math. Anal. *42* (2010), 256–297. zbl MR doi
[52] *M. Šilhavý*: The general form of the relaxation of a purely interfacial energy for structured deformations. Math. Mech. Complex Syst. *5* (2017), 191–215. zbl MR doi
[53] *X. T. Su, Z. J. Yang, G. H. Liu*: Monte Carlo simulation of complex cohesive fracture in random heterogeneous quasi-brittle materials: A 3D study. Int. J. Solids Struct. *47* (2010), 2336–2345. zbl doi
[54] *Y. Sumi*: Mathematical and Computational Analyses of Cracking Formation. Mathematics for Industry (Tokyo) 2. Springer, Tokyo, 2014. zbl MR doi
[55] *E. Svenning, F. Larsson, M. Fagerström*: A two-scale modeling framework for strain localization in solids: XFEM procedures and computational aspects. Comput. Struct. *211* (2019), 43–54. doi
[56] *R. F. Swati, L. H. Wen, H. Elahi, A. A. Khan, S. Shad*: Extended finite element method (XFEM) analysis of fiber reinforced composites for prediction of micro-crack propagation and delaminations in progressive damage: A review. Microsyst. Technol. *25* (2019), 747–763. doi
[57] *J. Vala*: Structure identification of metal fibre reinforced cementitious composites. Algoritmy: 20th Conference on Scientific Computing. STU Bratislava, Bratislava, 2016, pp. 244–253.
[58] *J. Vala, V. Kozák*: Computational analysis of quasi-brittle fracture in fibre reinforced cementitious composites. Theor. Appl. Fract. Mech. *107* (2020), Article ID 102486, 8 pages. doi

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