ON THE STABILIZATION OF LAMINATED BEAMS WITH DELAY

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Abstract. Of concern in this paper is the laminated beam system with frictional damping and an internal constant delay term in the transverse displacement. Under suitable assumptions on the weight of the delay, we establish that the system's energy decays exponentially in the case of equal wave speeds of propagation, and polynomially in the case of non-equal wave speeds.

Keywords: laminated beam; interfacial slip; delay; exponential and polynomial decay

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1. Introduction

The laminated beam model describes a vibrating structure of an interfacial slip. It consists of two-layered beams of uniform thickness which are attached by an adhesive layer of small thickness in such a way that small amount of slip is possible while they are continuously in contact with each other. The model which consists of three coupled hyperbolic equations was derived by Hansen et al. [15] using the assumption of Timoshenko beam theory and is given as follows:

$$\begin{aligned} &\varrho w_{tt} + G(\psi - w_x)_x = 0, \\ &I_\varrho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0, \\ &3I_\varrho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t = 0, \end{aligned} \tag{1.1}$$

with $x \in (0,1)$ and $t \geqslant 0$. The subscripted t and x denote differentiation with respect to time and to the longitudinal spatial variable, respectively. The function $w = w(x,t)$ is the traverse displacement, $\psi = \psi(x,t)$ is the rotation angle, $s = s(x,t)$ is proportional to the amount of slip along the interface and $3s - \psi$ denotes the effective rotation angle. The positive parameters ϱ, I_ϱ, G, D, γ, and β are the density, mass

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moment of inertia, shear stiffness, flexural rigidity, adhesive stiffness, and adhesive damping parameter, respectively. Laminated beams are considered to be very important especially in the field of engineering because of their applicability in building and construction of different structures. In recent years, researchers have focused on the study of the well-posedness and asymptotic stability properties of these structures by adding some damping mechanisms to the system. Let us mention some of the results.

Wang et al. [33] considered (1.1) with some boundary feedback controls and established an exponential decay result provided $\sqrt{\varrho/G} \neq \sqrt{I_\varrho/D}$. Tatar [32] and Mustafa [23] improved the result in [33] when they established the exponential stability for the system under weak assumptions on ϱ, G, I_ϱ, and D. Similar results were also established by Cao et al. [10] with different boundary controls. Apart from boundary control feedback, researchers also considered some other damping mechanisms like introducing additional damping terms in order to achieve the desired stability results. For instance, Raposo [30] proved exponential stability by introducing additional frictional damping in the form of αw_t and $\beta(3s-\psi)_t$ on the transverse displacement and rotation angle, respectively. Similarly, classical heat effect and second sound were also considered in the literature by Apalara [7] and [6], respectively. For other damping mechanisms, we refer the reader to [24], [11], [20], [21].

Often, delay effects appear in various physical problems. Thus, to exhaustively analyse such problems, delay differential equations are developed. Over the years, the control of PDEs with time delay effects has become a center of attraction to researchers. Generally, time delay is observed to have a significant effect on the stability of most of the systems. This effect may take different directions. For example, it was established that delayed positive feedback can stabilize purely oscillatory systems, see [1]. On the other hand, Zhang et al. [35] numerically illustrated a direct proportionality between time delay and diffusion in their study of a semi-linear fractional partial differential equation with time delay. This implied a negative effect of time delay on stability. Furthermore, it was established that the presence of an arbitrarily small delay may destabilize a system which is uniformly or asymptotically stable in the absence of delay. For instance, consider the system

$$(1.2) \qquad \begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0, \\ \dfrac{\partial u}{\partial \nu}(x,t) = -\mu_1 u_t(x,t) - \mu_2 u_t(x,t-\tau), & x \in \Gamma_1,\ t > 0. \end{cases}$$

It is known that in the absence of delay ($\mu_2 = 0$, $\mu_1 > 0$), the system is exponentially stable, see [17], [18], [36]. Whereas, in the presence of delay ($\mu_2 > 0$), Nicaise and Pignotti [25] proved, under the assumption $\mu_2 < \mu_1$, that the energy is exponentially stable. However, for the opposite case ($\mu_2 \geqslant \mu_1$), they were able to construct a sequence of delays for which the corresponding solution is unstable. Similar conclusions were reached by [12], [34]. In some instances, the system's stability may not be affected by the delay, because the delay effect is insignificantly small to have any repercussions or the damping is strong enough to neutralize the delay effect. For instance, Mustafa [22] studied a thermoelastic system with boundary time-varying delay in one-dimensional space and showed that the damping effect through heat conduction is still strong enough to uniformly stabilize the system even in the presence of boundary time-varying delay. For more works regarding time delay, we refer the reader to [2], [16], [27], [29], [3], [5], [8], [14], [28], [26] and references therein.

Regarding the laminated beam with delay, very little has been done. We found the work of Feng [13] where he considered a laminated system with three internal constant time delays. Using boundary feedbacks coupled with some assumptions on internal parameters, he proved the well-posedness of the system and the exponential stability.

It is important to note that when $s(x,t)$ in (1.1) is identically zero, the standard Timoshenko system is obtained. Consequently, in this work, we extend the result in [4], which is on the Timoshenko system, to a laminated beam system and establish an exponential stability result. Precisely, we consider the following system of laminated beams with frictional damping and an internal constant delay term in the transverse displacement:

(1.3)
$$\begin{cases} \varrho w_{tt} + G(\psi - w_x)_x + \mu_1 w_t + \mu_2 w_t(x,t-\tau) = 0 & \text{in } (0,1)\times(0,\infty), \\ I_\varrho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0 & \text{in } (0,1)\times(0,\infty), \\ 3I_\varrho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t = 0 & \text{in } (0,1)\times(0,\infty), \\ w_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } (0,1)\times(0,\tau), \\ w(x,0) = w_0,\ w_t(x,0) = w_1,\ \psi(x,0) = \psi_0,\ \psi_t(x,0) = \psi_1 & \text{in } (0,1), \\ s(x,0) = s_0,\ s_t(x,0) = s_1 & \text{in } (0,1), \\ w(0,t) = w_x(1,t) = s_x(0,t) = s(1,t) = \psi_x(0,t) = \psi(1,t) = 0 & \text{in } (0,\infty). \end{cases}$$

Here, w_0, w_1, ψ_0, ψ_1, s_0, and s_1 are initial data with f_0 being the history function in an appropriate space, and τ is time delay, μ_1 is a positive constant and μ_2 is a real number. Under suitable assumptions on the delay term and coefficients of wave propagation speed, we establish the exponential decay as well as the polynomial decay results of the energy of system (1.3).

The rest of the paper is organized as follows. We give some preliminaries which include the well-posedness result in Section 2. In Section 3, we state and prove some technical lemmas. In Sections 4 and 5, we establish exponential and polynomial decay results, respectively. Throughout this article $u_x = \partial u/\partial x$.

2. Preliminaries

In this section, we introduce some necessary transformations. Moreover, we give the proof of the well-posedness result.

As in [25], we introduce a new variable

$$z(x,\sigma,t) = w_t(x, t-\tau\sigma) \quad \text{in } (0,1)\times(0,1)\times(0,\infty). \tag{2.1}$$

It easy to show that z satisfies

$$\tau z_t(x,\sigma,t) + z_\sigma(x,\sigma,t) = 0 \quad \text{in } (0,1)\times(0,1)\times(0,\infty). \tag{2.2}$$

As a result, system (1.3) is equivalent to

$$\begin{cases} \varrho w_{tt} + G(\psi - w_x)_x + \mu_1 w_t + \mu_2 z(x,1,t) = 0 & \text{in } (0,1)\times(0,\infty),\\ I_\varrho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0 & \text{in } (0,1)\times(0,\infty),\\ 3I_\varrho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t = 0 & \text{in } (0,1)\times(0,\infty),\\ \tau z_t(x,\sigma,t) + z_\sigma(x,\sigma,t) = 0 & \text{in } (0,1)\times(0,1)\times(0,\infty),\\ z(x,1,t) = f_0(x,t-\tau) & \text{in } (0,1)\times(0,\tau),\\ z(x,0,t) = w_t(x,t) & \text{in } (0,1)\times(0,\infty),\\ z(x,\sigma,0) = f_0(x,-\sigma\tau) & \text{in } (0,1)\times(0,1),\\ w(x,0) = w_0,\ s(x,0) = s_0,\ \psi(x,0) = \psi_0 & \text{in } (0,1),\\ w_t(x,0) = w_1,\ s_t(x,0) = s_1,\ \psi_t(x,0) = \psi_1 & \text{in } (0,1),\\ w(0,t) = s_x(0,t) = \psi_x(0,t) & \\ \qquad = w_x(1,t) = s(1,t) = \psi(1,t) = 0 & \text{in } (0,\infty). \end{cases} \tag{2.3}$$

Thus, we consider (2.3) instead of (1.3). With respect to the weight of delay, we assume that

$$|\mu_2| < \mu_1 \tag{2.4}$$

and prove that this condition is sufficient to establish the well-posedness and the stability of the system (2.3) with the energy E, defined by

$$\begin{aligned} E(t) = {} & \frac{1}{2}\int_0^1 [\varrho w_t^2 + I_\varrho(3s_t - \psi_t)^2 + D(3s_x - \psi_x)^2 + 3I_\varrho s_t^2 + 3Ds_x^2 + 4\gamma s^2]\,\mathrm{d}x \\ & + \frac{1}{2}\int_0^1 \left[G(\psi - w_x)^2 + \tau|\mu_2|\int_0^1 z^2(x,\sigma,t)\,\mathrm{d}\sigma\right]\mathrm{d}x. \end{aligned} \tag{2.5}$$

Regarding the existence, uniqueness, and smoothness of the solution of problem (2.3), let $\xi = 3s - \psi$ and then write (2.3) in the form of a system of first-order equations:

$$
\text{(2.6)} \qquad \begin{cases} \dfrac{\partial w}{\partial t} = u, \\ \dfrac{\partial u}{\partial t} = -\dfrac{1}{\varrho}(G(3s - \xi - w_x)_x + \mu_1 u + \mu_2 z(x,1,t)), \\ \dfrac{\partial \xi}{\partial t} = v, \\ \dfrac{\partial v}{\partial t} = \dfrac{1}{I_\varrho}(D\xi_{xx} + G(3s - \xi - w_x)), \\ \dfrac{\partial s}{\partial t} = y, \\ \dfrac{\partial y}{\partial t} = \dfrac{1}{I_\varrho}\Big(Ds_{xx} - G(3s - \xi - w_x) - \dfrac{4\gamma}{3}s - \dfrac{4\beta}{3}y\Big), \\ \dfrac{\partial z}{\partial t} = -\dfrac{1}{\tau} z_\sigma(x,\sigma,t). \end{cases}
$$

We set $\Phi = (w, u, \xi, v, s, y, z)^\top$, so that (2.6) becomes

$$
\text{(2.7)} \qquad \begin{cases} \dfrac{\mathrm{d}\Phi}{\mathrm{d}t} = \mathcal{A}\Phi, \quad t > 0, \\ \Phi(0) = \Phi_0 = (w_0, w_1, 3s_0 - \psi_0, 3s_1 - \psi_1, s_0, s_1, f_0)^\top, \end{cases}
$$

where

$$
\mathcal{A}\Phi = \begin{pmatrix} u \\ -\dfrac{1}{\varrho}(G(3s - \xi - w_x)_x + \mu_1 u + \mu_2 z(x,1,t)) \\ v \\ \dfrac{1}{I_\varrho}(D\xi_{xx} + G(3s - \xi - w_x)) \\ y \\ \dfrac{1}{I_\varrho}\Big(Ds_{xx} - G(3s - \xi - w_x) - \dfrac{4\gamma}{3}s - \dfrac{4\beta}{3}y\Big) \\ -\dfrac{1}{\tau} z_\sigma(x,\sigma,t) \end{pmatrix}.
$$

R e m a r k 2.1. System (1.3) is the main problem. The transformation to system (2.3) is well known in the literature; it is necessary (not compulsory) because of the delay term. System (2.6) is the semigroup setting necessary for the proof of the well-posedness result. See [6], [4], [31] for a similar approach.

We proceed by introducing a one-dimensional Sobolev space [9], pp. 202–203,

$$W^{1,p}(0,1) = \Big\{u \in L^p(0,1);\ \exists\, g \in L^p(0,1) \text{ such that} \\ \int_0^1 u\varphi_x \,\mathrm{d}x = -\int_0^1 g\varphi \,\mathrm{d}x\ \forall\, \varphi \in C_0^1(0,1)\Big\}$$

with φ being a test function. The space $W^{1,p}$ is equipped with the norm

$$\|u\|_{W^{1,p}(0,1)} = (\|u\|^p_{L^p(0,1)} + \|u_x\|^p_{L^p(0,1)})^{1/p}.$$

If $p = 2$, we have

$$W^{1,2}(0,1) = H^1(0,1).$$

The space $H^1(0,1)$ is equipped with the inner product

$$(u,v)_{H^1(0,1)} = (u,v)_{L^2(0,1)} + (u_x,v_x)_{L^2(0,1)} = \int_0^1 (uv + u_xv_x)\,\mathrm{d}x$$

and with the associated norm

$$\|u\|_{H^1(0,1)} = (\|u\|^2_{L^2(0,1)} + \|u_x\|^2_{L^2(0,1)})^{1/2}.$$

Furthermore, the space H^2 (see [9], p. 216) is defined as

$$H^2(0,1) = \{u \in W^{1,2}(0,1);\ u_x \in W^{1,2}(0,1)\}$$

and is equipped with an inner product

$$(u,v)_{H^2(0,1)} = (u,v)_{L^2(0,1)} + (u_x,v_x)_{L^2(0,1)} + (u_{xx},v_{xx})_{L^2(0,1)}.$$

Concerning our problem, we consider the spaces

$$H_a^1 = \{v\colon\ v \in H^1(0,1),\ v(0) = 0\}, \quad H_b^1 = \{v\colon\ v \in H^1(0,1),\ v(1) = 0\}$$

and let

$$\mathcal{H} := H_a^1(0,1) \times L^2(0,1) \times H_b^1(0,1) \times L^2(0,1) \times H_b^1(0,1) \times L^2(0,1) \times L^2((0,1)\times(0,1))$$

be the Hilbert space equipped with the inner product

$$(\Phi,\widetilde{\Phi})_{\mathcal{H}} = \varrho\int_0^1 u\tilde{u}\,\mathrm{d}x + G\int_0^1 (3s-\xi-w_x)(3\tilde{s}-\tilde{\xi}-\widetilde{w}_x)\,\mathrm{d}x + I_\varrho\int_0^1 v\tilde{v}\,\mathrm{d}x \\ + 3I_\varrho\int_0^1 y\tilde{y}\,\mathrm{d}x + D\int_0^1 \xi_x\tilde{\xi}_x\,\mathrm{d}x + 4\gamma\int_0^1 s\tilde{s}\,\mathrm{d}x + 3D\int_0^1 s_x\tilde{s}_x\,\mathrm{d}x \\ + \tau|\mu_2|\int_0^1\int_0^1 z(x,\sigma,t)\tilde{z}(x,\sigma,t)\,\mathrm{d}\sigma\,\mathrm{d}x. \tag{2.8}$$

Instead of dealing with (2.3), we will consider (2.7) with the domain of the operator $\mathcal{A}$ given by

$$\begin{aligned} D(\mathcal{A}) = \{\Phi = (w,u,\xi,v,s,y,z)^\top \in \mathcal{H};\ & w \in H^2(0,1)\cap H_a^1(0,1),\\ & \xi, s \in H^2(0,1)\cap H_b^1(0,1),\ u\in H_a^1(0,1),\ v,y\in H_b^1(0,1),\\ & z, z_\sigma \in L^2((0,1)\times(0,1)),\ w_x(1) = \xi_x(0) = s_x(0) = 0\}. \end{aligned}$$

Note that $D(\mathcal{A})$ is independent of time $t>0$. Furthermore, it is obvious that $D(\mathcal{A})$ is dense in $\mathcal{H}$.

Proposition 2.1. *$\mathcal{A}$ is a dissipative operator.*

Proof. Using the inner product defined by (2.8) and integration by parts, we have

$$\begin{aligned} (2.9)\qquad (\mathcal{A}\Phi,\Phi)_{\mathcal{H}} = &-\Big(\mu_1 - \frac{|\mu_2|}{2}\Big)\int_0^1 u^2\,\mathrm{d}x - \frac{|\mu_2|}{2}\int_0^1 z^2(x,1,t)\,\mathrm{d}x\\ &- 4\beta\int_0^1 y^2\,\mathrm{d}x - \mu_2\int_0^1 uz(x,1,t)\,\mathrm{d}x. \end{aligned}$$

Using Young's inequality, the last term in (2.9) gives

$$(2.10)\qquad -\mu_2\int_0^1 uz(x,1,t)\,\mathrm{d}x \leqslant \frac{|\mu_2|}{2}\int_0^1 u^2\,\mathrm{d}x + \frac{|\mu_2|}{2}\int_0^1 z^2(x,1,t)\,\mathrm{d}x.$$

Combining (2.9) and (2.10), we end up with

$$(\mathcal{A}\Phi,\Phi)_{\mathcal{H}} \leqslant -(\mu_1 - |\mu_2|)\int_0^1 u^2\,\mathrm{d}x - 4\beta\int_0^1 y^2\,\mathrm{d}x.$$

Moreover, by (2.4), it follows that $(\mathcal{A}\Phi,\Phi)_{\mathcal{H}} \leqslant 0$. Thus, $\mathcal{A}$ is dissipative. □

Proposition 2.2. *$I-\mathcal{A}$ is surjective.*

Proof. Since $\mathcal{A}$ dissipative and $D(\mathcal{A})$ is dense in $\mathcal{H}$, it is sufficient to show that $\mathcal{A}$ is maximal. Given $H = (f_1,\dots,f_7)\in\mathcal{H}$, we must show that there exists $\Phi = (w,u,\xi,v,s,z)\in D(\mathcal{A})$ satisfying

$$(2.11)\qquad (I-A)\Phi = H,$$

which implies

$$w - u = f_1 \in H_a^1(0,1), \tag{2.12}$$

$$\varrho u + G(3s - \xi - w_x)_x + \mu_1 u + \mu_2 z(x,1) = \varrho f_2 \in L^2(0,1), \tag{2.13}$$

$$\xi - v = f_3 \in H_b^1(0,1), \tag{2.14}$$

$$I_\varrho v - D\xi_{xx} - G(3s - \xi - w_x) = I_\varrho f_4 \in L^2(0,1), \tag{2.15}$$

$$s - y = f_5 \in H_b^1(0,1), \tag{2.16}$$

$$3I_\varrho y - 3Ds_{xx} + 3G(3s - \xi - w_x) + 4\gamma s + 4\beta y = 3I_\varrho f_6 \in L^2(0,1), \tag{2.17}$$

$$\tau z(x,\sigma,t) + z_\sigma(x,\sigma,t) = \tau f_7 \in L^2((0,1)\times(0,1)). \tag{2.18}$$

We observe that (2.18) with $z(x,0) = u$ has a unique solution given by

$$z(x,\sigma,t) = \mathrm{e}^{-\tau\sigma} u + \tau \mathrm{e}^{-\tau\sigma} \int_0^\sigma \mathrm{e}^{\tau q} f_7(x,q)\,\mathrm{d}q \in L^2((0,1)\times(0,1)). \tag{2.19}$$

Using (2.12), (2.14), and (2.16), we end up with

$$\begin{aligned} G(3s - \xi - w_x)_x + \bar\varrho w &= h_1 \ \in L^2(0,1), \\ -D\xi_{xx} - G(3s - \xi - w_x) + I_\varrho \xi &= h_2 \ \in L^2(0,1), \\ -3Ds_{xx} + 3G(3s - \xi - w_x) + \lambda s &= h_3 \ \in L^2(0,1), \end{aligned} \tag{2.20}$$

where

$$\begin{gathered} h_1 = \bar\varrho f_1 + \varrho f_2 - \tau \mathrm{e}^{-\tau} \int_0^1 \mathrm{e}^{\tau q} f_7(x,q)\,\mathrm{d}q, \quad h_2 = I_\varrho(f_3 + f_4), \\ h_3 = (4\beta + 3I_\varrho) f_5 + 3I_\varrho f_6, \quad \bar\varrho = \varrho + \mu_1 + \mu_2 \mathrm{e}^{-\tau}, \quad \lambda = 4\gamma + 4\beta + 3I_\varrho. \end{gathered} \tag{2.21}$$

To solve (2.20) we consider

$$B((w,\xi,s),(\widetilde w,\tilde\xi,\tilde s)) = F(\widetilde w,\tilde\xi,\tilde s), \tag{2.22}$$

where $B\colon\ [H_a^1(0,1) \times H_b^1(0,1) \times H_b^1(0,1)]^2 \to \mathbb{R}$ is the bilinear form given by

$$\begin{aligned} B((w,\xi,s),(\widetilde w,\tilde\xi,\tilde s)) = G \int_0^1 (3s - \xi - w_x)(3\tilde s - \tilde\xi - \widetilde w_x)\,\mathrm{d}x + D \int_0^1 \xi_x \tilde\xi_x\,\mathrm{d}x \\ + 3D \int_0^1 s_x \tilde s_x\,\mathrm{d}x + \bar\varrho \int_0^1 w\widetilde w\,\mathrm{d}x + \lambda \int_0^1 s\tilde s\,\mathrm{d}x + I_\varrho \int_0^1 \xi\tilde\xi\,\mathrm{d}x \end{aligned}$$

and $F\colon\ [H_a^1(0,1) \times H_b^1(0,1) \times H_b^1(0,1)] \to \mathbb{R}$ is the linear form defined by

$$F(\widetilde w,\tilde\xi,\tilde s) = \int_0^1 h_1 \widetilde w\,\mathrm{d}x + \int_0^1 h_2 \tilde\xi\,\mathrm{d}x + \int_0^1 h_3 \tilde s\,\mathrm{d}x.$$

Now, for $V = H_a^1(0,1) \times H_b^1(0,1) \times H_b^1(0,1)$ equipped with the norm

$$\|(w, \xi, s, q)\|_V^2 = \|3s - \xi - w_x\|_2^2 + \|w\|_2^2 + \|\xi_x\|_2^2 + \|s_x\|_2^2,$$

one can easily see that B and F are bounded. Furthermore, using integration by parts, we obtain

$$\begin{aligned} B((w,\xi,s),(w,\xi,s)) &= G\int_0^1 (3s - \xi - w_x)^2 \,\mathrm{d}x + D\int_0^1 \xi_x^2 \,\mathrm{d}x + 3D\int_0^1 s_x^2 \,\mathrm{d}x \\ &\quad + \bar{\varrho}\int_0^1 w^2 \,\mathrm{d}x + \lambda\int_0^1 s^2 \,\mathrm{d}x + I_\varrho\int_0^1 \xi^2 \,\mathrm{d}x \\ &\geqslant c\|(w,\xi,s)\|_V^2. \end{aligned}$$

Thus B is coercive. Consequently, by Lax-Milgram lemma, system (2.20) has a unique solution

$$w \in H_a^1(0,1), \quad \xi, s \in H_b^1(0,1).$$

Substituting w, ξ, and s into (2.12), (2.14), and (2.16), respectively, we obtain

$$u \in H_a^1(0,1), \quad v, y \in H_b^1(0,1).$$

Similarly, inserting u into (2.19), bearing in mind (2.18), we end up with

$$z, z_\sigma \in L^2((0,1)\times(0,1)).$$

Now, if $(\tilde{\xi}, \tilde{s}) \equiv (0,0) \in H_a^1(0,1) \times H_a^1(0,1)$, then (2.22) reduces to

$$(2.23) \qquad -G\int_0^1 (3s - \xi - w_x)\widetilde{w}_x \,\mathrm{d}x + \bar{\varrho}\int_0^1 w\widetilde{w} \,\mathrm{d}x = \int_0^1 h_1\widetilde{w} \,\mathrm{d}x \quad \forall\, \widetilde{w} \in H_a^1(0,1),$$

which implies

$$(2.24) \qquad -Gw_{xx} = -3Gs_x + G\xi_x - \varrho w + h_1 \in L^2(0,1).$$

Consequently, by the regularity theory for the linear elliptic equations, it follows that

$$w \in H^2(0,1) \cap H_a^1(0,1).$$

Moreover, (2.23) is also true for any $\phi \in C^1([0,1])$, $\phi(0) = 0$, which is in $H_a^1(0,1)$. Hence, we have

$$G\int_0^1 w_x\phi_x \,\mathrm{d}x + \int_0^1 (3Gs_x - G\xi_x + \varrho w - h_1)\phi \,\mathrm{d}x = 0 \quad \forall\, \phi \in C^1([0,1]),\ \phi(0) = 0.$$

Thus, using integration by parts and bearing in mind (2.24), we obtain

$$w_x(1)\phi(1) = 0 \quad \forall\, \phi \in C^1([0,1]),\ \phi(0) = 0.$$

Therefore, $w_x(1) = 0$. Similarly, we obtain

$$\begin{aligned} -D\xi_{xx} &= G(3s - \xi - w_x) - I_\varrho \xi + h_2 \in L^2(0,1), \\ -3Ds_{xx} &= -3G(3s - \xi - w_x) - \lambda s + h_3 \in L^2(0,1). \end{aligned}$$

Consequently, we have

$$\xi, s \in H^2(0,1) \cap H^1_b(0,1), \quad \xi_x(0) = s_x(0) = 0.$$

Finally, the application of the regularity theory for the linear elliptic equations guarantees the existence of a unique $\Phi \in D(\mathcal{A})$ such that (2.11) is satisfied. Thus the operator $I - \mathcal{A}$ is surjective. □

As a consequence of the Hille-Yosida theorem [19], Theorem 1.2.2, p. 3, we have that $\mathcal{A}$ generates a C_0-*semigroup* of contractions $S(t) = \mathrm{e}^{t\mathcal{A}}$ on $\mathcal{H}$. From the semigroup theory, $\Phi(t) = \mathrm{e}^{t\mathcal{A}}\Phi_0$ is the unique solution of (2.7) satisfying the conditions of the following theorem.

Theorem 2.1 (well-posedness result)**.** *Let $\Phi_0 \in \mathcal{H}$. Then there exists a unique weak solution $\Phi \in C(\mathbb{R}^+, \mathcal{H})$ of problem* (2.7)*. Moreover, if $\Phi_0 \in D(\mathcal{A})$, then $\Phi \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

Remark 2.2. Theorem 2.1 guarantees the existence of system (2.3) in a weak sense.

3. Technical lemmas

In this section, we state and prove the necessary lemmas required to construct a suitable Lyapunov functional, which is used to establish some stability results for the energy of the solution of system (2.3). Throughout this section, c is a generic positive constant, precisely,

$$c \geqslant \max\Big\{\frac{\mu_1^2}{G}, \frac{\mu_2^2}{G}, \frac{\varrho^2}{4}, \frac{9\varrho^2}{4}, \frac{G^2}{2D}, \frac{9G^2}{4\gamma}, \frac{3I_\varrho\mu_1^2}{2D\varrho^2}, \frac{3I_\varrho\mu_2^2}{2D\varrho^2}, \frac{3G^2}{2DI_\varrho}\Big\}.$$

Lemma 3.1. *Let (w, ψ, s, z) be a solution of* (2.3)*. Then the energy functional E, defined by* (2.5)*, satisfies*

$$E'(t) \leqslant -m_0 \int_0^1 w_t^2 \,\mathrm{d}x - 4\beta \int_0^1 s_t^2 \,\mathrm{d}x, \quad t \geqslant 0, \tag{3.1}$$

for some positive constant m_0.

Proof. We begin with multiplying the first three equations in system (2.3) by $w_t, (3s_t - \psi_t)$ and s_t, respectively, then integrate by parts over $(0,1)$ using the boundary conditions. This leads to

$$\begin{aligned}(3.2)\qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 [\varrho w_t^2 + I_\varrho(3s_t - \psi_t)^2 + D(3s_x - \psi_x)^2 \\ + 3I_\varrho s_t^2 + 3Ds_x^2 + 4\gamma s^2 + G(\psi - w_x)^2]\,\mathrm{d}x \\ = -\mu_1\int_0^1 w_t^2\,\mathrm{d}x - \mu_2\int_0^1 z(x,1,t)w_t\,\mathrm{d}x - 4\beta\int_0^1 s_t^2\,\mathrm{d}x.\end{aligned}$$

Next, we multiply $(2.3)_4$ by $|\mu_2|z$, and then integrate the product over $(0,1)\times(0,1)$. Using the fact that $z(x,0,t) = w_t$, we obtain

$$(3.3)\qquad \frac{\tau|\mu_2|}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^1 z^2(x,\sigma,t)\,\mathrm{d}\sigma\,\mathrm{d}x = -\frac{|\mu_2|}{2}\int_0^1 z^2(x,1,t)\,\mathrm{d}x + \frac{|\mu_2|}{2}\int_0^1 w_t^2\,\mathrm{d}x.$$

Putting together (3.2) and (3.3) leads to

$$\begin{aligned}(3.4)\qquad E'(t) = &-\Big(\mu_1 - \frac{|\mu_2|}{2}\Big)\int_0^1 w_t^2\,\mathrm{d}x - \mu_2\int_0^1 z(x,1,t)w_t\,\mathrm{d}x \\ &-\frac{|\mu_2|}{2}\int_0^1 z^2(x,1,t)\,\mathrm{d}x - 4\beta\int_0^1 s_t^2\,\mathrm{d}x.\end{aligned}$$

Applying Young's inequality on the second term of (3.4) gives

$$(3.5)\qquad -\mu_2\int_0^1 w_t z(x,1,t)\,\mathrm{d}x \leqslant \frac{|\mu_2|}{2}\int_0^1 z^2(x,1,t)\,\mathrm{d}x + \frac{|\mu_2|}{2}\int_0^1 w_t^2\,\mathrm{d}x$$

and lastly, substituting (3.5) in (3.4), and using (2.5) completes the proof of (3.1). □

Lemma 3.2. *If (w,ψ,s,z) is a solution of (2.3), then the functional F_1, defined by*

$$F_1(t) := \varrho\int_0^1 ww_t\,\mathrm{d}x - \varrho\int_0^1 w_t\int_0^x \psi(y)\,\mathrm{d}y\,\mathrm{d}x, \quad t \geqslant 0,$$

satisfies, for any $\varepsilon_1, \varepsilon_2 > 0$, the estimate

$$\begin{aligned}(3.6)\qquad \frac{\mathrm{d}}{\mathrm{d}t}F_1(t) \leqslant &-\frac{G}{2}\int_0^1 (\psi - w_x)^2\,\mathrm{d}x + \varepsilon_1\int_0^1 (3s_t - \psi_t)^2\,\mathrm{d}x \\ &+ c\Big(1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\Big)\int_0^1 w_t^2\,\mathrm{d}x + \varepsilon_2\int_0^1 s_t^2\,\mathrm{d}x + c\int_0^1 z^2(x,1,t)\,\mathrm{d}x.\end{aligned}$$

P r o o f. Differentiating F_1 and using $(2.3)_1$, i.e., $-\varrho w_{tt} = G(\psi - w_x)_x + \mu_1 w_t + \mu_2 z(x,1,t)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F_1(t) = &-G\int_0^1 (\psi - w_x)_x w\,\mathrm{d}x - \mu_1\int_0^1 w_t w\,\mathrm{d}x + \varrho\int_0^1 w_t^2\,\mathrm{d}x\\ &+\mu_1\int_0^1 w_t\int_0^x \psi(y)\,\mathrm{d}y\,\mathrm{d}x - \mu_2\int_0^1 z(x,1,t)w\,\mathrm{d}x\\ &+G\int_0^1 (\psi - w_x)_x\int_0^x \psi(y)\,\mathrm{d}y\,\mathrm{d}x - \varrho\int_0^1 w_t\int_0^x \psi_t(y)\,\mathrm{d}y\,\mathrm{d}x\\ &+\mu_2\int_0^1 z(x,1,t)\int_0^x \psi(y)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Integrating by parts the terms involving G and using $w_x = -(\psi - w_x) + \psi$ leads to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F_1(t) = &-G\int_0^1 (\psi - w_x)^2\,\mathrm{d}x + \varrho\int_0^1 w_t^2\,\mathrm{d}x - \mu_1\int_0^1 w_t w\,\mathrm{d}x\\ &+\mu_1\int_0^1 w_t\int_0^x \psi(y)\,\mathrm{d}y\,\mathrm{d}x - \mu_2\int_0^1 z(x,1,t)w\,\mathrm{d}x\\ &+\mu_2\int_0^1 z(x,1,t)\int_0^x \psi(y)\,\mathrm{d}y\,\mathrm{d}x - \varrho\int_0^1 w_t\int_0^x \psi_t(y)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Using $\psi_t = -(3s_t - \psi_t) + 3s_t$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F_1(t) = &-G\int_0^1 (\psi - w_x)^2\,\mathrm{d}x + \varrho\int_0^1 w_t^2\,\mathrm{d}x\\ &+\mu_1\int_0^1 w_t\Big(\int_0^x \psi(y)\,\mathrm{d}y - w\Big)\,\mathrm{d}x\\ &+\mu_2\int_0^1 \Big(\int_0^x \psi(y)\,\mathrm{d}y - w\Big)z(x,1,t)\,\mathrm{d}x\\ &+\varrho\int_0^1 w_t\int_0^x (3s_t - \psi_t)(y)\,\mathrm{d}y\,\mathrm{d}x\\ &-3\varrho\int_0^1 w_t\int_0^x s_t(y)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}\tag{3.7}$$

By Young's, Poincaré's and Cauchy-Schwarz inequalities, we have the right-hand side of (3.7) as follows:

$$\begin{aligned}&\mu_1\int_0^1 w_t\Big(\int_0^x \psi(y)\,\mathrm{d}y - w\Big)\,\mathrm{d}x\\ &\quad\leqslant \frac{G}{4}\int_0^1\Big(\int_0^x \psi(y)\,\mathrm{d}y - w\Big)^2\,\mathrm{d}x + \frac{\mu_1^2}{G}\int_0^1 w_t^2\,\mathrm{d}x\\ &\quad\leqslant \frac{G}{4}\int_0^1 (\psi - w_x)^2\,\mathrm{d}x + c\int_0^1 w_t^2\,\mathrm{d}x,\end{aligned}\tag{3.8}$$

$$\begin{aligned}(3.9)\qquad \mu_2\int_0^1\Big(\int_0^x\psi(y)\,\mathrm{d}y-w\Big)z(x,1,t)\,\mathrm{d}x &\leqslant \frac{G}{4}\int_0^1\Big(\int_0^x\psi(y)\,\mathrm{d}y-w\Big)^2\,\mathrm{d}x+\frac{\mu_2^2}{G}\int_0^1 z^2(x,1,t)\,\mathrm{d}x\\ &\leqslant \frac{G}{4}\int_0^1(\psi-w_x)^2\,\mathrm{d}x+c\int_0^1 z^2(x,1,t)\,\mathrm{d}x,\end{aligned}$$

and for $\varepsilon_1>0$ and $\varepsilon_2>0$,

$$\begin{aligned}(3.10)\qquad \varrho\int_0^1 w_t\int_0^x(3s_t-\psi_t)(y)\,\mathrm{d}y\,\mathrm{d}x &\leqslant \varepsilon_1\int_0^1\Big(\int_0^x(3s_t-\psi_t)(y)\,\mathrm{d}y\Big)^2\,\mathrm{d}x+\frac{\varrho^2}{4\varepsilon_1}\int_0^1 w_t^2\,\mathrm{d}x\\ &\leqslant \varepsilon_1\int_0^1(3s_t-\psi_t)^2\,\mathrm{d}x+\frac{c}{\varepsilon_1}\int_0^1 w_t^2\,\mathrm{d}x,\end{aligned}$$

and

$$\begin{aligned}(3.11)\qquad -3\varrho\int_0^1 w_t\Big(\int_0^x s_t(y)\,\mathrm{d}y\Big)\,\mathrm{d}x &\leqslant \varepsilon_2\int_0^1\Big(\int_0^x s_t(y)\,\mathrm{d}y\Big)^2\,\mathrm{d}x+\frac{9\varrho^2}{4\varepsilon_2}\int_0^1 w_t^2\,\mathrm{d}x\\ &\leqslant \varepsilon_2\int_0^1 s_t^2\,\mathrm{d}x+\frac{c}{\varepsilon_2}\int_0^1 w_t^2\,\mathrm{d}x,\end{aligned}$$

respectively. Consequently, the estimate (3.6) follows by substituting (3.8)–(3.10) into (3.7). □

Lemma 3.3. *If (w,ψ,s,z) is a solution of (2.3), then the functional F_2, defined by*

$$F_2(t):=-I_\varrho\int_0^1(3s_t-\psi_t)(3s-\psi)\,\mathrm{d}x,\quad t\geqslant 0,$$

satisfies

$$\begin{aligned}(3.12)\qquad \frac{\mathrm{d}}{\mathrm{d}t}F_2(t)\leqslant &-I_\varrho\int_0^1(3s_t-\psi_t)^2\,\mathrm{d}x\\ &+\frac{3D}{2}\int_0^1(3s_x-\psi_x)^2\,\mathrm{d}x+c\int_0^1(\psi-w_x)^2\,\mathrm{d}x.\end{aligned}$$

P r o o f. First we differentiate F_2 and use $(2.3)_2$ to obtain

$$\begin{aligned}(3.13)\qquad \frac{\mathrm{d}}{\mathrm{d}t}F_2(t)= &-I_\varrho\int_0^1(3s_t-\psi_t)^2\,\mathrm{d}x+D\int_0^1(3s_x-\psi_x)^2\,\mathrm{d}x\\ &-G\int_0^1(\psi-w_x)(3s-\psi)\,\mathrm{d}x.\end{aligned}$$

Exploiting Young's and Poincaré's inequality, we have

$$
\begin{aligned}
(3.14)\qquad -G\int_0^1(\psi-w_x)(3s-\psi)\,\mathrm{d}x &\leqslant \frac{G^2}{2D}\int_0^1(\psi-w_x)^2\,\mathrm{d}x+\frac{D}{2}\int_0^1(3s-\psi)^2\,\mathrm{d}x\\
&\leqslant c\int_0^1(\psi-w_x)^2\,\mathrm{d}x+\frac{D}{2}\int_0^1(3s_x-\psi_x)^2\,\mathrm{d}x.
\end{aligned}
$$

Finally, substituting (3.14) into (3.13) completes the proof. □

Lemma 3.4. *Let* (w,ψ,s,z) *be a solution of* (2.3). *Then the functional* F_3, *defined by*

$$
F_3(t):=3I_\varrho\int_0^1 s_t s\,\mathrm{d}x+2\beta\int_0^1 s^2\,\mathrm{d}x,\quad t\geqslant 0,
$$

satisfies

$$
\begin{aligned}
(3.15)\qquad \frac{\mathrm{d}}{\mathrm{d}t}F_3(t)\leqslant &-3D\int_0^1 s_x^2\,\mathrm{d}x+\frac{9G^2}{4\gamma}\int_0^1(\psi-w_x)^2\,\mathrm{d}x\\
&-3\gamma\int_0^1 s^2\,\mathrm{d}x+3I_\varrho\int_0^1 s_t^2\,\mathrm{d}x.
\end{aligned}
$$

Proof. Direct computations involving simple differentiation of F_3, followed by substitution for the integral of s_{tt} using $(2.3)_3$, then integrating by parts the term containing $s_{xx}s$, lead to

$$
\begin{aligned}
(3.16)\qquad \frac{\mathrm{d}}{\mathrm{d}t}F_3(t)= &-3D\int_0^1 s_x^2\,\mathrm{d}x-4\gamma\int_0^1 s^2\,\mathrm{d}x\\
&+3I_\varrho\int_0^1 s_t^2\,\mathrm{d}x-3G\int_0^1(\psi-w_x)s\,\mathrm{d}x.
\end{aligned}
$$

Using Young's inequality, we estimate the last terms of (3.16) as follows:

$$
\begin{aligned}
-3G\int_0^1(\psi-w_x)s\,\mathrm{d}x &\leqslant \frac{9G^2}{4\gamma}\int_0^1(\psi-w_x)^2\,\mathrm{d}x+\gamma\int_0^1 s^2\,\mathrm{d}x\\
&\leqslant c\int_0^1(\psi-w_x)^2\,\mathrm{d}x+\gamma\int_0^1 s^2\,\mathrm{d}x.
\end{aligned}
$$

Consequently, we obtain (3.15), which completes the proof. □

Lemma 3.5. *If* (w,ψ,s,z) *is a solution of* (2.3), *then the functional* F_4, *defined by*

$$
F_4(t):=-\int_0^1(3s_t-\psi_t)w_x\,\mathrm{d}x-\int_0^1(3s_x-\psi_x)w_t\,\mathrm{d}x+3\int_0^1(3s_t-\psi_t)s\,\mathrm{d}x,\quad t\geqslant 0,
$$

satisfies for any $\varepsilon_3 > 0$ *the estimate*

$$\frac{\mathrm{d}}{\mathrm{d}t}F_4(t) \leqslant -\frac{D}{2I_\varrho}\int_0^1 (3s_x-\psi_x)^2\,\mathrm{d}x + \varepsilon_3\int_0^1 (3s_t-\psi_t)^2\,\mathrm{d}x + \frac{9}{\varepsilon_3}\int_0^1 s_t^2\,\mathrm{d}x + c\int_0^1 w_t^2\,\mathrm{d}x + c\int_0^1 (\psi-w_x)^2\,\mathrm{d}x + c\int_0^1 z^2(x,1,t)\,\mathrm{d}x - \Big(\frac{D}{I_\varrho}-\frac{G}{\varrho}\Big)\int_0^1 (3s_x-\psi_x)(\psi-w_x)_x\,\mathrm{d}x. \tag{3.17}$$

P r o o f. Differentiating F_4, making use of the first two equations in system (2.3) coupled with the fact that $w_x = -(\psi - w_x) - (3s-\psi) + 3s$, and then integrating by parts the terms involving $(3s_{xx} - \psi_{xx})$, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}F_4(t) = \frac{\mu_1}{\varrho}\int_0^1 (3s_x-\psi_x)w_t\,\mathrm{d}x + \frac{\mu_2}{\varrho}\int_0^1 (3s_x-\psi_x)z(x,1,t)\,\mathrm{d}x + \frac{G}{I_\varrho}\int_0^1 (\psi-w_x)^2\,\mathrm{d}x + \frac{G}{I_\varrho}\int_0^1 (\psi-w_x)(3s-\psi)\,\mathrm{d}x + 3\int_0^1 (3s_t-\psi_t)s_t\,\mathrm{d}x - \frac{D}{I_\varrho}\int_0^1 (3s_x-\psi_x)^2\,\mathrm{d}x - \Big(\frac{D}{I_\varrho}-\frac{G}{\varrho}\Big)\int_0^1 (3s_x-\psi_x)(\psi-w_x)_x\,\mathrm{d}x. \tag{3.18}$$

Next, we exploit Young's and Poincaré's inequalities to estimate the non-square terms of (3.18),

$$\frac{\mu_1}{\varrho}\int_0^1 (3s_x-\psi_x)w_t\,\mathrm{d}x \leqslant \frac{D}{6I_\varrho}\int_0^1 (3s_x-\psi_x)^2\,\mathrm{d}x + \frac{3I_\varrho\mu_1^2}{2D\varrho^2}\int_0^1 w_t^2\,\mathrm{d}x \leqslant \frac{D}{6I_\varrho}\int_0^1 (3s_x-\psi_x)^2\,\mathrm{d}x + c\int_0^1 w_t^2\,\mathrm{d}x,$$

$$\frac{\mu_2}{\varrho}\int_0^1 (3s_x-\psi_x)z(x,1,t)\,\mathrm{d}x \leqslant \frac{D}{6I_\varrho}\int_0^1 (3s_x-\psi_x)^2\,\mathrm{d}x + \frac{3I_\varrho\mu_2^2}{2D\varrho^2}\int_0^1 z^2(x,1,t)\,\mathrm{d}x \leqslant \frac{D}{6I_\varrho}\int_0^1 (3s_x-\psi_x)^2\,\mathrm{d}x + c\int_0^1 z^2(x,1,t)\,\mathrm{d}x,$$

$$\frac{G}{I_\varrho}\int_0^1 (\psi-w_x)(3s-\psi)\,\mathrm{d}x \leqslant \frac{3G^2}{2DI_\varrho}\int_0^1 (\psi-w_x)^2\,\mathrm{d}x + \frac{D}{6I_\varrho}\int_0^1 (3s-\psi)^2\,\mathrm{d}x \leqslant c\int_0^1 (\psi-w_x)^2\,\mathrm{d}x + \frac{D}{6I_\varrho}\int_0^1 (3s_x-\psi_x)^2\,\mathrm{d}x,$$

and for $\varepsilon_3 > 0$

$$3\int_0^1 (3s_t-\psi_t)s_t\,\mathrm{d}x \leqslant \varepsilon_3\int_0^1 (3s_t-\psi_t)^2\,\mathrm{d}x + \frac{9}{\varepsilon_3}\int_0^1 s_t^2\,\mathrm{d}x.$$

Combining the above four estimates with (3.18) concludes our proof. □

Lemma 3.6. *Let (w,ψ,s,z) be a solution of (2.3). Then the functional F_5, defined by*

$$F_5(t) := \tau\int_0^1\int_0^1 \mathrm{e}^{-\sigma\tau} z^2(x,\sigma,t)\,\mathrm{d}\sigma\,\mathrm{d}x,\quad t\geqslant 0,$$

satisfies, for some $m_1>0$, the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}F_5(t)\leqslant -m_1\int_0^1 z^2(x,1,t)\,\mathrm{d}x - m_1\tau\int_0^1\int_0^1 z^2(x,\sigma,t)\sigma\,\mathrm{d}x + \int_0^1 w_t^2\,\mathrm{d}x. \tag{3.19}$$

Proof. Differentiating F_5 and using $(2.3)_4$ and $z(x,0,t)=w_t$, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F_5(t) &= -2\int_0^1\int_0^1 \mathrm{e}^{-\tau\sigma} z(x,\sigma,t)z_\sigma(x,\sigma,t)\,\mathrm{d}\sigma\,\mathrm{d}x\\
&= -\int_0^1\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\sigma}[\mathrm{e}^{-\tau\sigma}z^2(x,\sigma,t)]\,\mathrm{d}\sigma\,\mathrm{d}x - \tau\int_0^1\int_0^1 \mathrm{e}^{-\tau\sigma}z^2(x,\sigma,t)\,\mathrm{d}\sigma\,\mathrm{d}x\\
&= -\int_0^1 [\mathrm{e}^{-\tau}z^2(x,1,t) - z^2(x,0,t)]\,\mathrm{d}x - \tau\int_0^1\int_0^1 \mathrm{e}^{-\tau\sigma}z^2(x,\sigma,t)\,\mathrm{d}\sigma\,\mathrm{d}x\\
&= -\int_0^1 \mathrm{e}^{-\tau}z^2(x,1,t)\,\mathrm{d}x + \int_0^1 w_t^2\,\mathrm{d}x - \tau\int_0^1\int_0^1 \mathrm{e}^{-\tau\sigma}z^2(x,\sigma,t)\,\mathrm{d}\sigma\,\mathrm{d}x.\end{aligned}$$

Next, exploiting the inequality $\mathrm{e}^{-\tau}\leqslant \mathrm{e}^{-\sigma\tau}\leqslant 1$ for any $\sigma\in(0,1)$, for some $m_1=\mathrm{e}^{-\tau}$, we arrive at the estimate (3.19). □

4. Exponential stability

Our next task is to define a Lyapunov functional $\mathcal{L}$ and show that it is equivalent to the energy functional E.

Lemma 4.1. *For $N>0$, $N_k>0$ $(k=1,\dots,5)$, the Lyapunov functional defined by*

$$\mathcal{L}(t) := NE(t) + \sum_{k=1}^{5} N_kF_k(t),\quad t\geqslant 0, \tag{4.1}$$

satisfies the equivalence relation ($\mathcal{L}\sim E$)

$$c_1E(t)\leqslant \mathcal{L}(t)\leqslant c_2E(t)\quad \forall\, t\geqslant 0 \tag{4.2}$$

for some positive constants c_1 and c_2.

P r o o f.

$$\begin{aligned}|\mathcal{L}(t)-NE(t)| \leqslant\ & \varrho N_1\int_0^1 |ww_t|\,\mathrm{d}x+\varrho N_1\int_0^1\left|w_t\int_0^x\psi(y)\,\mathrm{d}y\right|\mathrm{d}x\\ &+I_\varrho N_2\int_0^1|(3s_t-\psi_t)(3s-\psi)|\,\mathrm{d}x+3I_\varrho N_3\int_0^1|s_ts|\,\mathrm{d}x\\ &+2\beta N_3\int_0^1 s^2\,\mathrm{d}x+N_4\int_0^1|(3s_t-\psi_t)w_x|\,\mathrm{d}x\\ &+N_4\int_0^1|(3s_x-\psi_x)w_t|\,\mathrm{d}x+3N_4\int_0^1|(3s_t-\psi_t)s|\,\mathrm{d}x\\ &+\tau N_5\int_0^1\int_0^1\mathrm{e}^{-\sigma\tau}z^2(x,\sigma,t)\,\mathrm{d}\sigma\,\mathrm{d}x.\end{aligned}$$

Furthermore, using Young's, Poincaré's, Cauchy-Schwarz inequalities, the facts that $\psi=-(3s-\psi)+3s$, $w_x=-(\psi-w_x)-(3s-\psi)+3s$, and $\mathrm{e}^{-\sigma\tau}\leqslant 1$ for all $\sigma\in(0,1)$, we obtain

$$\begin{aligned}|\mathcal{L}(t)-NE(t)|\leqslant b&\int_0^1[w_t^2+(3s_t-\psi_t)^2+(3s_x-\psi_x)^2+s_t^2+s_x^2+s^2+(\psi-w_x)^2]\,\mathrm{d}x\\ &+b\int_0^1\int_0^1 z^2(x,\sigma,t)\,\mathrm{d}\sigma\,\mathrm{d}x,\end{aligned}$$

for some constant $b>0$. Using (2.5), we obtain

$$|\mathcal{L}(t)-NE(t)|\leqslant b_1E(t), \tag{4.3}$$

where $b_1\geqslant\frac{1}{2}\max\{2b,\varrho,3I_\varrho,3D,4\gamma,G,\tau|\mu_2|\}$. Inequality (4.3) yields

$$(N-b_1)E(t)\leqslant\mathcal{L}(t)\leqslant(N+b_1)E(t).$$

Taking N is sufficiently large, the estimate (4.2) follows accordingly. □

At this point, we are ready to state and prove the first part of our main results.

Theorem 4.1. *Let* (w,ψ,s,z) *be a solution of* (2.3). *Then the energy functional* (2.5) *satisfies, for all* $t\geqslant 0$,

$$E(t)\leqslant k_0\mathrm{e}^{-k_1t}\quad \textit{if}\ \ \frac{G}{\varrho}=\frac{D}{I_\varrho}, \tag{4.4}$$

where k_0 *and* k_1 *are positive constants.*

P r o o f. We begin by differentiating (4.1), and then substitute the estimates (3.6), (3.12), (3.15), (3.17), and (3.19) to obtain

$$
\begin{aligned}
\mathcal{L}'(t) \leqslant\; & -\Big[m_0 N - cN_1\Big(1+\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\Big) - cN_4 - N_5\Big]\int_0^1 w_t^2 \,\mathrm{d}x \\
& - 3DN_3\int_0^1 s_x^2 \,\mathrm{d}x - 3\gamma N_3 \int_0^1 s^2 \,\mathrm{d}x \\
& - \Big[4\beta N - \varepsilon_2 N_1 - 3I_\varrho N_3 - \frac{9N_4}{\varepsilon_3}\Big]\int_0^1 s_t^2 \,\mathrm{d}x \\
& - [I_\varrho N_2 - \varepsilon_1 N_1 - \varepsilon_3 N_4]\int_0^1 (3s_t-\psi_t)^2 \,\mathrm{d}x \\
& - \Big[\frac{GN_1}{2} - cN_2 - cN_3 - cN_4\Big]\int_0^1 (\psi - w_x)^2 \,\mathrm{d}x \\
& - \Big[\frac{DN_4}{2I_\varrho} - \frac{3DN_2}{2}\Big]\int_0^1 (3s_x-\psi_x)^2 \,\mathrm{d}x \\
& - [m_1N_5 - cN_1 - cN_4]\int_0^1 z^2(x,1,t)\,\mathrm{d}x - m_1\tau N_5\int_0^1\int_0^1 z^2(x,\sigma,t)\,\mathrm{d}\sigma\,\mathrm{d}x \\
& - N_4\Big(\frac{D}{I_\varrho} - \frac{G}{\varrho}\Big)\int_0^1 (3s_x-\psi_x)(\psi-w_x)_x \,\mathrm{d}x.
\end{aligned}
$$

Next, we carefully choose our constants. We begin by setting

$$N_2 = N_3 = \varepsilon_2 = 1, \quad \varepsilon_1 = \frac{I_\varrho}{3N_1}, \quad \varepsilon_3 = \frac{I_\varrho}{3N_4} \quad \text{and} \quad N_4 = 4I_\varrho,$$

to arrive at

$$
\begin{aligned}
\mathcal{L}'(t) \leqslant\; & - [m_0N - c_3N_1(1+N_1) - c_3 - N_5]\int_0^1 w_t^2 \,\mathrm{d}x \\
& - [4\beta N - N_1 - 3I_\varrho]\int_0^1 s_t^2 \,\mathrm{d}x - \Big[\frac{GN_1}{2} - c_3\Big]\int_0^1 (\psi-w_x)^2 \,\mathrm{d}x \\
& - 3D\int_0^1 s_x^2 \,\mathrm{d}x - \frac{I_\varrho}{3}\int_0^1 (3s_t - \psi_t^2 \,\mathrm{d}x - 3\gamma\int_0^1 s^2 \,\mathrm{d}x - \frac{D}{2}\int_0^1 (3s_x-\psi_x)^2 \,\mathrm{d}x \\
& - [m_1N_5 - c_3N_1 - c_3]\int_0^1 z^2(x,1,t)\,\mathrm{d}x - m_1\tau N_5\int_0^1\int_0^1 z^2(x,\sigma,t)\,\mathrm{d}\sigma\,\mathrm{d}x \\
& - N_4\Big(\frac{D}{I_\varrho} - \frac{G}{\varrho}\Big)\int_0^1 (3s_x-\psi_x)(\psi-w_x)_x \,\mathrm{d}x,
\end{aligned}
$$

for some $c_3 > 0$. Now, we choose N_1 large enough such that

$$\frac{GN_1}{2} - c_3 > 0.$$

Once N_1 is fixed, we proceed to choose N_5 large enough such that

$$m_1 N_5 - c_3 N_1 - c_3 > 0.$$

Lastly, we choose N so large that (4.2) remains valid, and furthermore,

$$m_0 N - c_3 N_1(1+N_1) - N_5 - c_3 > 0 \quad \text{and} \quad 4\beta N - N_1 - 3I_\varrho > 0.$$

Hence, for some $\alpha_0 > 0$, we end up with

(4.5)
$$\begin{aligned}\mathcal{L}'(t) \leqslant & -\alpha_0 \int_0^1 [w_t^2 + s_t^2 + (3s_t - \psi_t)^2 + (3s_x - \psi_x)^2 + s_x^2 + s^2 + (\psi - w_x)^2]\,\mathrm{d}x \\ & - \alpha_0 \int_0^1 \int_0^1 z^2(x,\sigma,t)\,\mathrm{d}\sigma\,\mathrm{d}x \\ & - 4I_\varrho \alpha_0 \Big(\frac{D}{I_\varrho} - \frac{G}{\varrho}\Big) \int_0^1 (3s_x - \psi_x)(\psi - w_x)_x\,\mathrm{d}x.\end{aligned}$$

Hence, from (2.5) and the fact that $G/\varrho = D/I_\varrho$, we arrive at

$$\mathcal{L}'(t) \leqslant -\alpha_1 E(t) \quad \forall\, t > 0 \tag{4.6}$$

for some $\alpha_1 > 0$. It follows directly from (4.2) and (4.6) that

$$\mathcal{L}'(t) \leqslant -k_1 \mathcal{L}(t) \quad \forall\, t > 0, \tag{4.7}$$

where $k_1 = \alpha_1/c_2$. A simple integration of (4.7) over $(0,t)$ yields

$$\mathcal{L}(t) \leqslant \mathcal{L}(0)\mathrm{e}^{-k_1 t} \quad \forall\, t > 0. \tag{4.8}$$

Consequently, the relation (4.4) follows from (4.8) and (4.2) with $k_0 = c_2 E(0)/c_1$.
□

5. Polynomial stability

In this section, we consider the case of non-equal wave speeds and establish a polynomial stability result.

Theorem 5.1. *Let (w,ψ,s,z) be a strong solution of* (2.3). *If $G/\varrho \neq D/I_\varrho$, then the energy functional* (2.5) *satisfies*

$$E(t) \leqslant \frac{k_2}{t} \quad \forall\, t > 0, \tag{5.1}$$

where k_2 is a positive constant.

P r o o f. To prove estimate (5.1), we require the following second-order energy functional.

(5.2)

$$\mathcal{E}(t)=\frac{1}{2}\int_0^1[\varrho w_{tt}^2+I_\varrho(3s_{tt}-\psi_{tt})^2+D(3s_{xt}-\psi_{xt})^2+3I_\varrho s_{tt}^2+3Ds_{xt}^2+4\gamma s_t^2]\,\mathrm{d}x$$
$$+\frac{1}{2}\int_0^1[G(\psi_t-w_{xt})^2+\tau|\mu_2|\int_0^1 z_t^2(x,\sigma,t)\,\mathrm{d}\sigma]\,\mathrm{d}x.$$

As in Lemma 3.1, it follows that $\mathcal{E}$ satisfies the relation

$$\mathcal{E}'(t)\leqslant -m_0\int_0^1 w_{tt}^2\,\mathrm{d}x-4\beta\int_0^1 s_{tt}^2\,\mathrm{d}x\quad \forall\, t\geqslant 0. \tag{5.3}$$

Similarly to Lemma 4.1, we define a Lyapunov functional $\widetilde{\mathcal{L}}$ as follows:

$$\widetilde{\mathcal{L}}(t)=N[E(t)+\mathcal{E}(t)]+\sum_{k=1}^{5}N_kF_k(t), \tag{5.4}$$

where F_k, $k=1,\ldots,5$, and their respective derivatives remain as defined in Lemmas 3.2–3.6. It is important to note that the relation (4.2) does not hold for $\widetilde{\mathcal{L}}$. To this effect, therefore, the major task ahead of us now is to find an estimate for the term

$$\Big(\frac{G}{\varrho}-\frac{D}{I_\varrho}\Big)\int_0^1(3s_x-\psi_x)(\psi-w_x)_x\,\mathrm{d}x$$

in (3.18) and consequently a new estimate for the derivative of the functional F_4. Let us begin by setting $\chi=(G/\varrho-D/I_\varrho)$. Using $(2.3)_1$, we obtain

$$\begin{aligned}\chi\int_0^1(3s_x-\psi_x)(\psi-w_x)_x\,\mathrm{d}x= &-\frac{\varrho\chi}{G}\int_0^1(3s_x-\psi_x)w_{tt}\,\mathrm{d}x\\ &-\frac{\mu_1\chi}{G}\int_0^1(3s_x-\psi_x)w_t\,\mathrm{d}x\\ &-\frac{\mu_2\chi}{G}\int_0^1(3s_x-\psi_x)z(x,1,t)\,\mathrm{d}x.\end{aligned} \tag{5.5}$$

Next, exploiting the Cauchy-Schwarz and Young's inequalities, we estimate the three terms on the right-hand side of (5.5) as follows:

$$-\frac{\varrho\chi}{G}\int_0^1(3s_x-\psi_x)w_{tt}\,\mathrm{d}x\leqslant\delta\int_0^1(3s_x-\psi_x)^2\,\mathrm{d}x+\frac{\varrho^2\chi^2}{4\delta G^2}\int_0^1 w_{tt}^2\,\mathrm{d}x,$$
$$-\frac{\mu_1\chi}{G}\int_0^1(3s_x-\psi_x)w_t\,\mathrm{d}x\leqslant\delta\int_0^1(3s_x-\psi_x)^2\,\mathrm{d}x+\frac{\mu_1^2\chi^2}{4\delta G^2}\int_0^1 w_t^2\,\mathrm{d}x,$$
$$-\frac{\mu_2\chi}{G}\int_0^1(3s_x-\psi_x)z(x,1,t)\,\mathrm{d}x\leqslant\delta\int_0^1(3s_x-\psi_x)^2\,\mathrm{d}x+\frac{\mu_2^2\chi^2}{4\delta G^2}\int_0^1 z^2(x,1,t)\,\mathrm{d}x.$$

Therefore, for some

$$c_4 \geqslant \max\Big\{\frac{\varrho^2\chi^2}{4\delta G^2}, \frac{\mu_1^2\chi^2}{4\delta G^2}, \frac{\mu_2^2\chi^2}{4\delta G^2}\Big\},$$

we have

$$\begin{aligned}(5.6)\qquad \chi\int_0^1 (3s_x-\psi_x)(\psi-w_x)_x\,\mathrm{d}x \leqslant 3\delta\int_0^1 (3s_x-\psi_x)^2\,\mathrm{d}x + c_4\int_0^1 w_t^2\,\mathrm{d}x \\ + c_4\int_0^1 w_{tt}^2\,\mathrm{d}x + c_4\int_0^1 z^2(x,1,t)\,\mathrm{d}x.\end{aligned}$$

Thus, using (5.6), choosing $\delta = D/(12I_\varrho)$ and setting $c_5 = c + c_4$, we note that the derivative of F_4 satisfies the new estimate

$$\begin{aligned}(5.7)\qquad \frac{\mathrm{d}}{\mathrm{d}t}F_4(t) \leqslant & -\frac{D}{4I_\varrho}\int_0^1 (3s_x-\psi_x)^2\,\mathrm{d}x + c_5\int_0^1 w_t^2\,\mathrm{d}x + c\int_0^1 (\psi-w_x)^2\,\mathrm{d}x \\ & + \frac{9}{\varepsilon_3}\int_0^1 s_t^2\,\mathrm{d}x + \varepsilon_3\int_0^1 (3s_t-\psi_t)^2\,\mathrm{d}x \\ & + c_4\int_0^1 w_{tt}^2\,\mathrm{d}x + c_5\int_0^1 z^2(x,1,t)\,\mathrm{d}x.\end{aligned}$$

Similarly, differentiating $\widetilde{\mathcal{L}}(t)$ defined in (5.4) and then substituting the estimates (3.6), (3.12), (3.15) (3.19), and (5.7), we obtain

$$\begin{aligned}\widetilde{\mathcal{L}}'(t) \leqslant & -\Big[4\beta N - \varepsilon_2 N_1 - 3I_\varrho N_3 - \frac{9N_3}{\varepsilon_3}\Big]\int_0^1 s_t^2\,\mathrm{d}x \\ & - 3DN_3\int_0^1 s_x^2\,\mathrm{d}x - 3\gamma N_3\int_0^1 s^2\,\mathrm{d}x - [m_0N - c_4N_4]\int_0^1 w_{tt}^2\,\mathrm{d}x \\ & - \Big[m_0N - c_3N_1\Big(1+\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\Big) - c_5N_4 - N_5\Big]\int_0^1 w_t^2\,\mathrm{d}x \\ & - \Big[\frac{GN_1}{2} - c_3N_2 - c_3N_3 - c_3N_5\Big]\int_0^1 (\psi-w_x)^2\,\mathrm{d}x \\ & - \Big[\frac{DN_4}{4I_\varrho} - \frac{3DN_2}{2}\Big]\int_0^1 (3s_x-\psi_x)^2\,\mathrm{d}x \\ & - [I_\varrho N_2 - \varepsilon_1 N_1 - \varepsilon_3 N_4]\int_0^1 (3s_t-\psi_t)^2\,\mathrm{d}x \\ & - [m_1N_5 - cN_1 - c_5N_4]\int_0^1 z^2(x,1,t)\,\mathrm{d}x \\ & - m_1\tau N_5\int_0^1\int_0^1 z^2(x,\sigma,t)\,\mathrm{d}\sigma\,\mathrm{d}x.\end{aligned}$$

With similar choices of constants N_2, N_3, ε_1, ε_2, and ε_3 as in the proof of Theorem 4.1, and setting $N_4 = 8I_\varrho$ coupled with suitable choices of N_1, N_5, and N, we deduce that

$$\begin{aligned}\widetilde{\mathcal{L}}'(t) \leqslant & -\alpha_2 \int_0^1 [w_t^2 + (\psi - w_x)^2 + (3s_t - \psi_t)^2 + (3s_x - \psi_x)^2 + s_x^2 + s^2 + s_t^2]\,\mathrm{d}x \\ & - \alpha_2 \tau \int_0^1 \int_0^1 z^2(x,\sigma,t)\,\mathrm{d}\sigma\,\mathrm{d}x,\end{aligned}$$

where α_2 is a positive constant. Comparing with (2.5), we have, for some $\alpha_3 > 0$,

$$\widetilde{\mathcal{L}}'(t) \leqslant -\alpha_3 E(t) \quad \forall\, t > 0. \tag{5.8}$$

Since E is non-increasing, integrating (5.8) over $(0,t)$ yields

$$tE(t) \leqslant \int_0^t E(s)\,\mathrm{d}s \leqslant -\frac{1}{\alpha_3}\int_0^t \widetilde{\mathcal{L}}'(s)\,\mathrm{d}s = \frac{1}{\alpha_3}[\widetilde{\mathcal{L}}(0) - \widetilde{\mathcal{L}}(t)] \leqslant \frac{\widetilde{\mathcal{L}}(0)}{\alpha_3}.$$

Finally, for $k_2 = \widetilde{\mathcal{L}}(0)/\alpha_3 = (E(0) + \mathcal{E}(0))/\alpha_3$, we have

$$E(t) \leqslant \frac{k_2}{t} \quad \forall\, t > 0,$$

which concludes the proof. □

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