ON LOWER BOUNDS FOR THE VARIANCE OF FUNCTIONS OF RANDOM VARIABLES

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Abstract. In this paper, we obtain lower bounds for the variance of a function of random variables in terms of measures of reliability and entropy. Also based on the obtained characterization via the lower bounds for the variance of a function of random variable X, we find a characterization of the weighted function corresponding to density function $f(x)$, in terms of Chernoff-type inequalities. Subsequently, we obtain monotonic relationships between variance residual life and dynamic cumulative residual entropy and between variance past lifetime and dynamic cumulative past entropy. Moreover, we find lower bounds for the variance of functions of weighted random variables with specific weight functions applicable in reliability under suitable conditions.

Keywords: variance bound; Chernoff inequality; size-biased distribution; reliability measure; dynamic cumulative residual entropy; dynamic cumulative past entropy

MSC 2020: 60E15

1. Introduction

During the last four decades, several papers have been prepared regarding upper bounds for functions of the random variables, based on the Chernoff type inequality. Let Z be a standard normal random variable and $g\colon \mathbb{R}\to\mathbb{R}$ any absolutely continuous function with derivative g' such that $\mathrm{E}[g'(Z)]<\infty$. Chernoff [8], using Hermite polynomials, proved that

$$\operatorname{Var}[g(Z)] \leqslant \mathrm{E}[g'(Z)]^2. \tag{1.1}$$

Chen [7] proved (1.1) using the integral representation of g and the Cauchy-Schwarz inequality. The equality in (1.1) holds if and only if g is a linear function. Cacoullos [2] and Cacoullos and Papathanasiou [4] obtained lower bounds for the variance of

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functions of arbitrary random variables. The papers [5] and [15] established that if X is a continuous random variable with support an interval (a, b), $-\infty \leqslant a < b \leqslant \infty$, mean μ, finite variance σ^2, and density function f, then the following general covariance identity holds:

$$\operatorname{Cov}(h(X), g(X)) = \mathrm{E}(z(X)g'(X)), \tag{1.2}$$

where g is an absolutely continuous function with $\mathrm{E}|z(X)g'(X)| < \infty$, $h(x)$ is a given function and

$$z(x) = \frac{1}{f(x)} \int_a^x (\mathrm{E}[h(X)] - h(t))f(t)\,\mathrm{d}t. \tag{1.3}$$

(For $g(x) = x$, (1.2) yields $\mathrm{E}[z(X)] = \operatorname{Cov}(h(X), X)$.)

They also established that if there are functions $h(x)$ and $z(x)$ such that (1.2) holds for every differentiable g, then $h(x)$, $z(x)$ and the density f are related through (1.3).

If $h(x) = x$, then $z(x) = \sigma^2 w(x)$ and (1.2) reduces to the following simple covariance identity:

$$\operatorname{Cov}(X, g(X)) = \sigma^2 \mathrm{E}[w(X)g'(X)], \tag{1.4}$$

where the $w(\cdot)$-function is defined by

$$\sigma^2 w(x) f(x) = \int_a^x (\mu - t) f(t)\,\mathrm{d}t. \tag{1.5}$$

It is trivial that (1.3) uniquely determines $f(x)$ as

$$f(x) = \frac{1}{z(x)} \exp\left\{ \int_a^x \frac{\mathrm{E}[h(X)] - h(t)}{z(t)}\,\mathrm{d}t \right\}, \tag{1.6}$$

and thus (1.2) characterizes (1.6).

Furthermore, [14] produced characterization of continuous exponential families through a representation for a survival function in terms of covariance identities. Cacoullos and Papathanasiou [6] showed that under the conditions of identity (1.2), for every absolutely continuous function $h(x)$ with $h'(x) > 0$,

$$\operatorname{Var}[g(X)] \geqslant \frac{\mathrm{E}^2[z(X)g'(X)]}{\mathrm{E}[z(X)h'(X)]} \tag{1.7}$$

with equality if and only if $g(x) = c_1 h(x) + c_2$, where

$$z(x)f(x) = \int_a^x (\mathrm{E}[h(X)] - h(t))f(t)\,\mathrm{d}t.$$

They also established that if there are functions h and z such that inequality (1.7) holds for every differentiable g and equality $g(x) = c_1 h(x) + c_2$ holds, then h, z and f are related through (1.3).

Now, assume that the failure rate of the distribution is defined as

$$r(t) = -\frac{\mathrm{d}}{\mathrm{d}t} \ln \overline{F}(t) = \frac{f(t)}{\overline{F}(t)}$$

for $t < b$. The mean residual life function $m(t)$ of X is defined as

$$m(t) = \mathrm{E}(X - t \mid X > t) = \frac{\int_t^\infty \overline{F}(x)\,\mathrm{d}x}{\overline{F}(t)},$$

for $t < b$, and variance residual life of X is given by

$$\sigma^2(t) = \mathrm{Var}(X - t \mid X > t) = \frac{2}{\overline{F}(t)} \int_t^\infty \overline{F}(x) m(x)\,\mathrm{d}x - m^2(t). \tag{1.8}$$

The MRL function is usually of interest for a non-negative random variable. For instance, if X is thought of as the lifetime of a device, then for every $t \geqslant 0$, $m(t)$ expresses the conditional expected residual life of the device at time t given that the device is still alive at time t. Hence, we assume $F(t) = 0$ for $t = 0$ (i.e. $a = 0$, $b = \infty$).

Furthermore, Glaser's function (also known as eta-function) $\eta(t)$ for a random variable X is defined as

$$\eta(t) = -\frac{f'(t)}{f(t)},$$

and the aging intensity function is defined as

$$L(t) = \frac{tr(t)}{\int_0^t r(x)\,\mathrm{d}x} = \frac{-tf(t)}{\overline{F}(t) \ln \overline{F}(t)}.$$

Definition 1.1. Let X be a non-negative absolutely continuous random variable:

(a) F is DFR (IFR) [decreasing failure rate (increasing failure rate)] if $\overline{F}(x \mid t) = \overline{F}(x + t)/\overline{F}(t)$ is increasing (decreasing) in $0 \leqslant t < \infty$ for each $x \geqslant 0$.
(b) F is NWU [new worse than used] if $\overline{F}(x \mid t) \geqslant \overline{F}(x)$ for each $x, t \geqslant 0$.
(c) F is IMRL [increasing mean residual life] if $m(t)$ is increasing in $t \geqslant 0$.
(d) F is IVRL (DVRL) [increasing variance residual life (decreasing variance residual life)] if $\sigma^2(t)$ is increasing (decreasing) in $t \geqslant 0$.

Besides, the reversed hazard rate function of X is given by $\overline{r}(t) = f(t)/F(t)$ for $t > a$. Also the mean past lifetime function of X is defined by

$$k(t) = \mathrm{E}(t - X \mid X < t) = \frac{1}{F(t)} \int_a^t F(x)\,\mathrm{d}x.$$

In analogy with the variance of the residual life function defined in (1.8), the *variance past lifetime* (VPL) can be introduced as

$$\overline{\sigma}^2(t) = \mathrm{Var}(t - X \mid X < t) = \frac{2}{F(t)} \int_a^t k(x)F(x)\,\mathrm{d}x - k^2(t).$$

For further details on definitions and also terms used in the text, see [23].

Nair and Sudheesh [17] characterized the class of continuous distributions given in (1.6) by the relationships the conditional variance has with the truncated expectations and/or failure rate as well as the bound to the conditional variance.

Consider now a random variable X with density function $f(x)$ and distribution function $F(x)$. Let $\delta(x)$ be a non-negative function with finite non-zero expectation. Define a random variable X^* with density function

$$f_{X^*}(x) = \frac{\delta(x)f(x)}{\mathrm{E}[\delta(X)]}, \quad x \in \mathbb{R}. \tag{1.9}$$

The variable X^* is called the *weighted random variable corresponding to* X and its distribution is called the *weighted distribution corresponding to* $f(x)$. Though the concept of weighted distribution is due to [10], but it was Rao [21] who studied the weighted distributions in a unified way.

In particular, if F is defined on $[0, \infty)$, then the weighted distribution with weight function $\delta(x) = x$ is called the size-biased or length-biased distribution. Indeed, since the distribution of X^* is weighted by the value or size of X, we say that X^* has the X-size biased distribution.

2. Main results

In this section, we give a lower bound for the variance of the function of given random variables in text in terms of measures of reliability. We also study the monotonic behavior of variance residual life with respect to dynamic cumulative residual entropy and variance past lifetime with respect to dynamic cumulative past entropy. Besides, we obtain a lower bound for the variance of $g(X^*)$ by using characteristics of the associated random variable X.

Proposition 2.1. *Let the random variable X belong to the exponential family*

$$f(x;\theta) = \exp[P(x)Q(\theta) + T(x) + S(\theta)] \tag{2.1}$$

with $P(x)$ and $T(x)$ being real valued measurable functions and $Q(\theta)$ and $S(\theta)$ being real functions with continuous non-vanishing derivatives. Also, let g be an absolutely continuous function with derivative g'. Then assuming differentiation under the integration sign, we have

$$\operatorname{Var}[g(X)] \geqslant \frac{1}{Q'(\theta)}\Big(\frac{\partial}{\partial\theta}\mathrm{E}\Big[\frac{g'(X)}{r(X)}\Big]\Big)^2\Big(\frac{\partial}{\partial\theta}\mathrm{E}\Big[\frac{P'(X)}{r(X)}\Big]\Big)^{-1}. \tag{2.2}$$

The equality holds if and only if $g(x)$ is a linear function of $P(x)$.

P r o o f. By considering $h(x) = P(x)$ in inequality (1.7), we can obtain inequality (2.2). We observe from [16] that for the exponential family (2.1),

$$\mathrm{E}(P(X) \mid X > x) = \mathrm{E}(P(X)) + \frac{1}{Q'(\theta)}\frac{\partial \log \overline{F}(x)}{\partial\theta},$$

and thus compute $z(x)f(x)$ as follows:

$$\begin{aligned} z(x)f(x) = \int_x^\infty \{P(t) - \mathrm{E}(P(X))\}f(t)\,\mathrm{d}t &= \int_x^\infty P(t)f(t)\,\mathrm{d}t - \mathrm{E}(P(X))\overline{F}(x) \\ &= \overline{F}(x)\{\mathrm{E}(P(X) \mid X > x) - \mathrm{E}(P(X))\} \\ &= \overline{F}(x)\Big(\frac{1}{Q'(\theta)}\frac{\partial \log \overline{F}(x)}{\partial\theta}\Big). \end{aligned}$$

Now, assuming the differentiation under the integral sign,

$$\begin{aligned} \mathrm{E}[z(X)P'(X)] &= \int_{-\infty}^\infty \overline{F}(x)\Big(\frac{1}{Q'(\theta)}\frac{\partial \log \overline{F}(x)}{\partial\theta}\Big)P'(x)\,\mathrm{d}x \\ &= \frac{1}{Q'(\theta)}\frac{\partial}{\partial\theta}\int_{-\infty}^\infty \overline{F}(x)P'(x)\,\mathrm{d}x \\ &= \frac{1}{Q'(\theta)}\frac{\partial}{\partial\theta}\mathrm{E}\Big[\frac{P'(X)}{r(X)}\Big]. \end{aligned} \tag{2.3}$$

Similarly, we can show that

$$\mathrm{E}[z(X)g'(X)] = \frac{1}{Q'(\theta)}\frac{\partial}{\partial\theta}\mathrm{E}\Big[\frac{g'(X)}{r(X)}\Big]. \tag{2.4}$$

Finally, by substituting (2.3) and (2.4) into (1.7), the result is obtained.

The equality is also obvious in view of (1.7). □

E x a m p l e 2.2. Let X have a Pareto distribution with probability density function $f(x)=\alpha\beta^\alpha/x^{\alpha+1}$, $x>\beta$, where $\alpha,\beta>0$, and β is constant. Hence, we can consider that $Q(\alpha)=-\alpha+1$ and $P(x)=\ln(x)$. Now, since $r(x)=\alpha/x$, by considering $g(x)=-\ln f(x)$ and using (2.2), we can easily obtain that $\operatorname{Var}[-\ln f(X)]\geqslant 1/\alpha^2$.

Notice that if X has an exponential family distribution and $d(X)=P(X)$ be an unbiased estimator of $\gamma(\theta)=\mathrm{E}_\theta[P(X)]$, then the variance of $d(X)$ is equal to Cramer Rao lower bound. On the other hand, since $g(x)=P(x)$, the lower bound given in (2.2) is equivalent to Cramer Rao lower bound. Thus, we can easily show that

$$\operatorname{Var}[P(X)]=\frac{1}{Q'(\theta)}\frac{\partial}{\partial\theta}\mathrm{E}\Big[\frac{P'(X)}{r(X)}\Big]=\frac{Q''(\theta)S'(\theta)-S''(\theta)Q'(\theta)}{[Q'(\theta)]^3}.$$

R e m a r k 2.3. Let X be a random variable with the probability density function given by

$$f(x;\boldsymbol{\theta})=\exp\Big\{\sum_{i=1}^k P_i(x)Q_i(\boldsymbol{\theta})+T(x)+S(\boldsymbol{\theta})\Big\}, \tag{2.5}$$

where $\boldsymbol{\theta}=(\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_k)$. The family $\{f(x;\boldsymbol{\theta})\}$ is called a *k-parameter exponential family.* Now if $Q_i(\boldsymbol{\theta})=\theta_i$, $h(x)=\sum\limits_{i=1}^k\alpha_iP_i(x)$, where $0<\alpha_i<1$ such that $\sum\limits_{i=1}^k\alpha_i=1$ and g is an absolutely continuous function with derivative g', then we can obtain

$$\operatorname{Var}[g(X)]\geqslant\Big(\sum_{i=1}^k\alpha_i\frac{\partial}{\partial\theta_i}\mathrm{E}\Big[\frac{g'(X)}{r(X)}\Big]\Big)^2\Big(\sum_{j=1}^k\sum_{i=1}^k\alpha_i\alpha_j\frac{\partial}{\partial\theta_i}\mathrm{E}\Big[\frac{P_j'(X)}{r(X)}\Big]\Big)^{-1}.$$

Proposition 2.4. *Let X be an absolutely continuous non-negative random variable. If g is an absolutely continuous function with derivative g', then*

$$\operatorname{Var}[g(X)]\geqslant\mathrm{E}^2\Big[\frac{Xg'(X)}{L(X)}\Big]. \tag{2.6}$$

The equality holds if and only if $g(x)$ is a linear function of $-\ln\overline{F}(x)$.

P r o o f. By using inequality (1.7), let $h(x)=-\ln\overline{F}(x)$, then $h'(x)=f(x)/\overline{F}(x)$ and

$$\begin{aligned}z(x)f(x)&=\int_0^x(\mathrm{E}[-\ln\overline{F}(X)]+\ln\overline{F}(t))f(t)\,\mathrm{d}t=\int_0^x(1+\ln\overline{F}(t))f(t)\,\mathrm{d}t\\&=F(x)+\int_0^x\ln\overline{F}(t)f(t)\,\mathrm{d}t=-\overline{F}(x)\ln\overline{F}(x).\end{aligned}$$

Now since

$$\mathrm{E}[z(X)h'(X)] = \int_0^\infty -\overline{F}(X)\ln\overline{F}(X)\frac{f(x)}{\overline{F}(x)}\,\mathrm{d}x = 1 \tag{2.7}$$

and

$$\begin{aligned}\mathrm{E}[z(X)g'(X)] &= \int_0^\infty -\overline{F}(x)\ln\overline{F}(x)g'(x)\,\mathrm{d}x \\ &= \int_0^\infty \frac{-\overline{F}(x)\ln\overline{F}(x)}{xf(x)}xg'(x)f(x)\,\mathrm{d}x = \mathrm{E}\Big[\frac{Xg'(X)}{L(X)}\Big],\end{aligned} \tag{2.8}$$

substituting (2.7) and (2.8) into (1.7) yields the desired result.

The equality is trivial by inequality (1.7). □

Corollary 2.5. *In inequality* (2.6), *if* $g(x) = x$, *then*

$$\mathrm{Var}(X) \geqslant \Big(\int_0^\infty -\overline{F}(x)\ln\overline{F}(x)\,\mathrm{d}x\Big)^2,$$

and thus we conclude that

$$\mathrm{E}[m(X)] \leqslant \sqrt{\mathrm{Var}(X)}. \tag{2.9}$$

The equality holds if and only if F *has two-parameter exponential distribution.*

It is shown by [22] *that in general when* X *is a non-negative random variable,*

$$\mathrm{E}[m(X)] \leqslant \frac{\mathrm{E}(X^2)}{2\mathrm{E}(X)}. \tag{2.10}$$

It is easy to see that $\sqrt{\mathrm{Var}(X)} \leqslant \mathrm{E}(X^2)/(2\mathrm{E}(X))$, *hence the upper bound given in* (2.9) *is sharper than the upper bound in* (2.10).

Example 2.6. Let X be distributed as $\mathrm{Gamma}(\alpha,\beta)$ with probability density function

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}\mathrm{e}^{-x/\beta}, \quad x>0, \quad \alpha>0, \quad \beta>0.$$

Then the lower bound (2.9) is given by $\mathrm{E}[m(X)] \leqslant \beta\sqrt{\alpha}$.

Also based on the upper bound given in (2.10), we have $\mathrm{E}[m(X)] \leqslant \beta(1+\alpha)/2$. We note that always $\sqrt{\alpha} \leqslant (1+\alpha)/2$.

Remark 2.7. The survival function of $X - t$ given that $X > t$, is

$$\overline{F}_t(x) = P(X - t > x \mid X > t) = \frac{\overline{F}(t+x)}{\overline{F}(t)}, \tag{2.11}$$

and thus the probability density function of the residual life random variable $X_t = (X - t \mid X > t)$ is given by

$$f_t(x) = \begin{cases} \dfrac{f(t+x)}{\overline{F}(t)}, & 0 < x < \infty, \\ 0, & \text{otherwise}. \end{cases} \tag{2.12}$$

Now if

$$h(x) = -\ln \frac{\overline{F}(x+t)}{\overline{F}(t)},$$

then $h'(x) = f(x+t)/\overline{F}(x)$ and similarly to Proposition 2.4 we can show that

$$\operatorname{Var}[g(X_t)] \geqslant \left[\int_t^{\infty} -\frac{\overline{F}(x)}{\overline{F}(t)} \ln \frac{\overline{F}(x)}{\overline{F}(t)} g'(x-t)\,\mathrm{d}x\right]^2. \tag{2.13}$$

The equality holds if and only if

$$g(x) = -c_1 \ln \frac{\overline{F}(t+x)}{\overline{F}(t)} + c_2,$$

where c_1 and c_2 are constant. Also if $g(x) = x$, then

$$\begin{aligned} \sigma^2(t) = {} & \operatorname{Var}[X_t] = \operatorname{Var}[X - t \mid X > t] \\ & \geqslant \left[\int_t^{\infty} -\frac{\overline{F}(x)}{\overline{F}(t)} \ln \frac{\overline{F}(x)}{\overline{F}(t)}\,\mathrm{d}x\right]^2 = (\mathscr{E}(X;t))^2, \end{aligned} \tag{2.14}$$

where $\mathscr{E}(X;t)$ is dynamic cumulative residual entropy.

In inequality (2.14), the equality holds if and only if

$$\frac{f(t+x)}{\overline{F}(t)} = \frac{1}{c_1}\mathrm{e}^{-(x-c_2)/c_1}, \quad x > c_2,$$

where c_1 and c_2 are constant. On the other hand, F has a two-parameter exponential distribution.

The concept of variability is a basic one in statistics, probability, and many other related areas. The simplest way of comparing the variability of two distributions is by comparison of the standard deviations. However, the comparison of numerical measures is not always sufficiently informative. Shaked and Shanthikumar [23] presented discussion on excess wealth transform, as a measure of spread. Let X be a non-negative random variable having distribution function $F(x)$. The quantile function $F^{-1}(p)$ of $F(x)$ is defined by $F^{-1}(p) = \inf\{x \mid F(x) \geqslant p\}$, $p \in (0,1)$ and $F^{-1}(0)$ and $F^{-1}(1)$ are defined as the left and right extremes of the support, respectively. Note that the excess wealth function is defined as

$$W(p;F) = \mathrm{E}[(X - F^{-1}(p))^+] = \int_{F^{-1}(p)}^{\infty} \overline{F}(x)\,\mathrm{d}x = \int_p^1 (F^{-1}(q) - F^{-1}(p))\,\mathrm{d}q, \tag{2.15}$$

where $(Z)^+ = \max\{Z, 0\}$. The excess wealth function can be considered as a measure of spread to the right of every quantile $F^{-1}(x)$. This function is also related to its mean residual life function by the relationship

$$\mathrm{E}[(X - F^{-1}(p))^+] = (1-p)m(F^{-1}(p)).$$

One can obtain a representation for the dynamic cumulative residual entropy in terms of the mean residual life function as follows:

$$\mathscr{E}(X;t) = \mathrm{E}(m(X) \mid X > t), \tag{2.16}$$

hence

$$\begin{aligned} \mathscr{E}(X;F^{-1}(p_0)) &= \frac{1}{1-p_0}\int_{F^{-1}(p_0)}^{\infty} m(x)f(x)\,\mathrm{d}x = \frac{1}{1-p_0}\int_{p_0}^{1} m(F^{-1}(p))\,\mathrm{d}p \\ &= \frac{1}{1-p_0}\int_{p_0}^{1} \frac{W(p;F)}{1-p}\,\mathrm{d}p. \end{aligned} \tag{2.17}$$

This function is called the *dynamic cumulative residual quantile entropy* (DCRQE) of X. To illustrate with real data, it is necessary to introduce the non-parametric estimator of excess wealth function (see [13] for details).

Example 2.8. For $x \geqslant a > 0$ and $k > 0$, let X have the Pareto distribution with density function and distribution function $f(x) = ka^k/x^{k+1}$ and $F(x) = 1 - (a/x)^k$, respectively. Then by using (2.15) and (2.17), we have respectively

$$W(p;F) = \frac{a(1-p)^{-1/k+1}}{k-1} \quad \text{for } k > 1,$$

and

$$\mathscr{E}(X;F^{-1}(p_0)) = \frac{ka}{(k-1)^2(1-p_0)^{1/k}} \quad \text{for } k > 1.$$

It is obvious that for fixed values of a and k, DCRQE is increasing in $p_0 \in [0,1)$.

R e m a r k 2.9. If X is increasing dynamic cumulative residual entropy (IDCRE), then X is IVRL, that is, IDCRE $\Rightarrow$ IVRL.

In fact, since $\sigma(t) \geqslant \mathscr{E}(X;t)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Var}[X-t \mid X>t] &= r(t)(\sigma^2(t)-m^2(t)) = r(t)(\sigma(t)-m(t))(\sigma(t)+m(t)) \\ &\geqslant r(t)(\mathscr{E}(X;t)-m(t))(\sigma(t)+m(t)),\end{aligned}$$

and hence, if $\mathscr{E}(X;t) \geqslant m(t)$ or in other words, X is increasing dynamic cumulative residual entropy (IDCRE) (see [1]), then $\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Var}[X-t \mid X>t] \geqslant 0$ and thus X is IVRL.

Moreover, if X is DVRL, then X is decreasing dynamic cumulative residual entropy (DDCRE), that is, DVRL $\Rightarrow$ DDCRE.

Because in this case, $\sigma(t) \leqslant m(t)$ and furthermore $\sigma(t) \geqslant \mathscr{E}(X;t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(X;t) = r(t)(\mathscr{E}(X;t)-m(t)) \leqslant r(t)(\sigma(t)-m(t))) \leqslant 0 \tag{2.18}$$

and thus X is decreasing dynamic cumulative residual entropy (DDCRE).

E x a m p l e 2.10. If X has a Rayleigh distribution with probability density function $f(x) = 2x\mathrm{e}^{-x^2}$, $x>0$, then X is IFR and thus DVRL. Therefore, we conclude that X is DDCRE (see Figure 1).

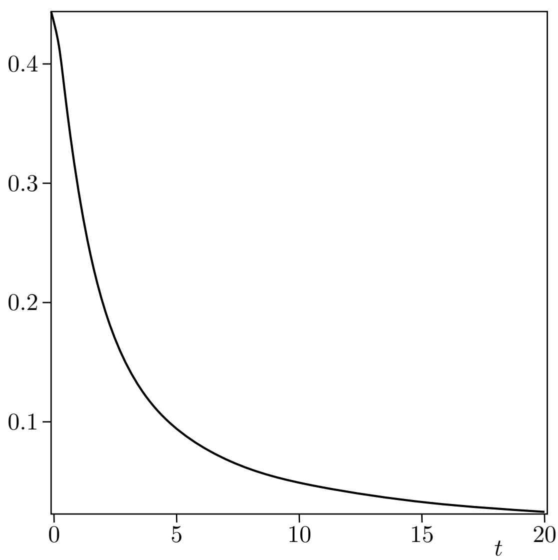

Figure 1. DCRE function of the weibull distribution.

E x a m p l e 2.11. If X has a Pareto distribution with probability density function $f(x) = ka^k/x^{k+1}$, $x \geqslant a$ for $a > 0$ and $k > 2$, then after simple calculation, we obtain $\mathscr{E}(X;t) = kt/(k-1)^2$ and hence X is IDCRE. So X is IVRL. Moreover, we can obtain $\mathscr{E}(X;t) = km(t)/(k-1)$ and therefore X is IMRL.

E x a m p l e 2.12. If X has a Burr type XII distribution with parameters $c = 1.5$ and $k = 2$, the survival function of X is given by

$$\overline{F}_X(t) = (1+t^{3/2})^{-2} \quad \text{for } t > 0. \tag{2.19}$$

Navarro et al. [19] showed that X is IDCRE and thus by using Remark 2.9, we conclude that X is IVRL (see Figure 2).

One of the most widely used measures of uncertainty or information in sciences is varentropy. If we view the entropy as a measure of the extent a probability is concentrated or dispersed, the varentropy $\mathrm{Var}[-\ln f(X)]$ measures the intrinsic shape of a distribution. If we assume $g(x) = -\ln f(x)$ in (2.6), then a straightforward computation shows

$$\mathrm{Var}[-\ln f(X)] \geqslant \mathrm{E}^2\Big[-\ln \overline{F}(X)\frac{\eta(X)}{r(X)}\Big], \tag{2.20}$$

where equality holds if and only if F has a two-parameter exponential distribution.

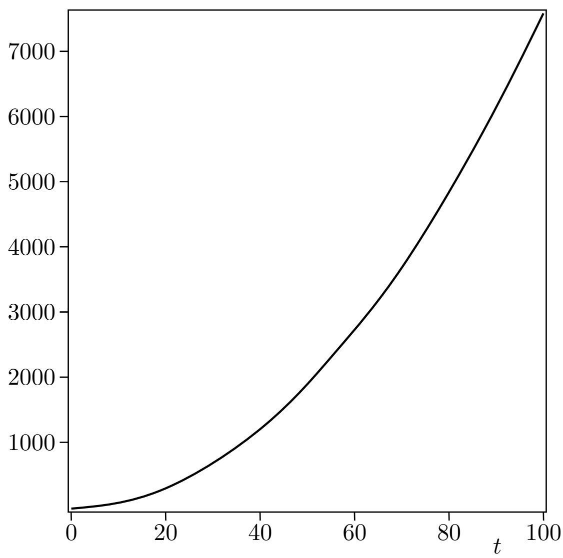

Figure 2. IVRL function of the Burr type XII distribution.

Example 2.13. Let X have beta distribution with parameters $a = 2$ and $b = 1$. Goodarzi et al. [12] obtained a lower bound for the varentropy of random variable X. They had shown that $\mathrm{Var}[-\ln f(X)] \geqslant \frac{2}{9}$, whereas by using (2.20), we can obtain lower bound 0.103985 for varentropy. The new lower bound is not sharper than the lower bound [12], however its calculation is very straightforward.

Goodarzi et al. [13] defined the variance residual entropy as

$$\mathrm{Var}[-\ln f_t(X_t)] = \int_t^{\infty} \frac{f(x)}{\overline{F}(t)} \Big(\log \frac{f(x)}{\overline{F}(t)}\Big)^2 \,\mathrm{d}x - \Big(\int_t^{\infty} \frac{f(x)}{\overline{F}(t)} \log \frac{f(x)}{\overline{F}(t)} \,\mathrm{d}x\Big)^2.$$

Later it is called residual variance entropy by [9]. [13] obtained an upper bound for variance residual entropy. Now we obtain a lower bound for it. In inequality (2.13), if we take $g(x) = -\ln f_t(x)$, then we have

$$\mathrm{Var}[-\ln f_t(X_t)] \geqslant \Big[\int_t^{\infty} -\frac{\overline{F}(x)}{\overline{F}(t)} \ln \frac{\overline{F}(x)}{\overline{F}(t)} \eta(x) \,\mathrm{d}x\Big]^2. \tag{2.21}$$

The equality holds if and only if

$$\ln \frac{f(x+t)}{\overline{F}(t)} = c_1 \ln \frac{\overline{F}(x+t)}{\overline{F}(t)} + c_2,$$

where c_1 and c_2 are constant.

Notice that for the uniform distribution on the interval $(0, \theta)$, exponential distribution with mean $1/\lambda$ and beta distribution with $a = 1$ and $b = 2$, variance residual entropy is equal to the lower bound (2.21) and equal to 0, 1 and $\frac{1}{4}$, respectively.

Remark 2.14. Let X be an absolutely continuous non-negative random variable. If g is an absolutely continuous function with derivative g' and we have $h(x) = m(x)$ in inequality (1.7), then by straightforward calculations it can be shown that

$$\mathrm{Var}[g(X)] \geqslant \frac{\big(\int_0^{\infty} \overline{F}(x)[\mathscr{E}(X;x) - \mathscr{E}(X)] g'(x) \,\mathrm{d}x\big)^2}{\int_0^{\infty} \overline{F}(x)[\mathscr{E}(X;x) - \mathscr{E}(X)] m'(x) \,\mathrm{d}x}.$$

It is trivial that $\mathrm{Var}[m(X)] = \int_0^{\infty} \overline{F}(x)[\mathscr{E}(X;x) - \mathscr{E}(X)] m'(x) \,\mathrm{d}x$.

Psarrakos and Navarro [20] defined the generalized cumulative residual entropy (GCRE) of X as

$$\mathscr{E}_n(X) = \int_0^{\infty} \overline{F}(x) \frac{[-\ln \overline{F}(x)]^n}{n!} \,\mathrm{d}x$$

for $n = 1, 2, \ldots$ Analogously, they also considered the dynamic version of the GCRE, that is, the GCRE of the residual lifetime $X_t = (X - t \mid X > t)$ given by

$$\mathscr{E}_n(X;t) = \mathscr{E}_n(X_t) = \frac{1}{n!}\int_t^\infty \frac{\overline{F}(x)}{\overline{F}(t)}\Big[-\ln\frac{\overline{F}(x)}{\overline{F}(t)}\Big]^n \,\mathrm{d}x.$$

Now with considering

$$h(x) = \frac{1}{n!}\Big[-\ln\frac{\overline{F}(x+t)}{\overline{F}(t)}\Big]^n,$$

we can extend Remark 2.7 as follows:

$$\begin{aligned}\text{(2.22)}\qquad \operatorname{Var}[g(X_t)] \geqslant{}& \frac{n((n-1)!)^2}{2(2n-1)! - n((n-1)!)^2}\\ &\times\Big[\sum_{i=0}^{n-1}\int_t^\infty \frac{1}{(n-i)!}\frac{\overline{F}(x)}{\overline{F}(t)}\Big(-\ln\frac{\overline{F}(x)}{\overline{F}(t)}\Big)^{n-i} g'(x-t)\,\mathrm{d}x\Big]^2.\end{aligned}$$

In particular, for $g(x) = x$ we have

$$\text{(2.23)}\qquad \operatorname{Var}[X_t] \geqslant \frac{n((n-1)!)^2}{2(2n-1)! - n((n-1)!)^2}\Big[\sum_{i=0}^{n-1}\mathscr{E}_{n-i}(X_t)\Big]^2.$$

Remark 2.15. Let X be a non-negative random variable. The survival function $t - X$ given that $X < t$, is

$$\text{(2.24)}\qquad F_{(t)}(x) = P(t - X < x \mid X < t) = 1 - \frac{F(t-x)}{F(t)},\quad 0 < x < t,$$

and thus the probability density function of $X_{(t)} = (t - X \mid X < t)$ is given by

$$\text{(2.25)}\qquad f_{(t)}(x) = \begin{cases}\dfrac{f(t-x)}{F(t)}, & 0 < x < t,\\ 0, & \text{otherwise.}\end{cases}$$

In fact, $X_{(t)}$ shows the time elapsed from the failure of a component given that its lifetime is less than or equal to t. Now if

$$h(x) = -\ln\frac{F(t-x)}{F(t)},$$

then $h'(x) = -f(t-x)/F(x)$ and therefore, we have

$$\text{(2.26)}\qquad z(x)f(x) = \int_x^t \Big(-\ln\frac{F(t-y)}{F(t)} - \mathrm{E}\Big[-\ln\frac{F(t-X_{(t)})}{F(t)}\Big]\Big)\frac{f(t-y)}{F(t)}\,\mathrm{d}y.$$

On the other hand,

$$\mathrm{E}\Big[-\ln\frac{F(t-X_{(t)})}{F(t)}\Big]=-\int_0^t\frac{f(t-x)}{F(t)}\ln\frac{F(t-x)}{F(t)}\,\mathrm{d}y=-\int_0^1\ln u\,\mathrm{d}u=1$$

and thus

$$\begin{aligned}z(x)f(x)&=\int_x^t\Big(-\ln\frac{F(t-y)}{F(t)}-1\Big)\frac{f(t-y)}{F(t)}\,\mathrm{d}y\\&=-\int_0^{F(t-x)/F(t)}\ln u\,\mathrm{d}u-\frac{F(t-x)}{F(t)}=-\frac{F(t-x)}{F(t)}\ln\frac{F(t-x)}{F(t)}.\end{aligned}$$

Now since

$$\begin{aligned}\mathrm{E}[z(X_{(t)})h'(X_{(t)})]&=-\int_0^t\frac{F(t-x)}{F(t)}\Big(\ln\frac{F(t-x)}{F(t)}\Big)\frac{f(t-x)}{F(t-x)}\,\mathrm{d}x\\&=-\int_0^1\ln u\,\mathrm{d}u=1\end{aligned}\tag{2.27}$$

and

$$\mathrm{E}[z(X_{(t)})g'(X_{(t)})]=-\int_0^t\frac{F(t-x)}{F(t)}\Big(\ln\frac{F(t-x)}{F(t)}\Big)g'(x)\,\mathrm{d}x,$$

we obtain

$$\mathrm{Var}[g(X_{(t)})]\geqslant\Big[-\int_0^t\frac{F(x)}{F(t)}\Big(\ln\frac{F(x)}{F(x)}\Big)g'(t-x)\,\mathrm{d}x\Big]^2.$$

The equality holds if and only if $g(x)=-c_1\ln(F(t-x)/F(t))+c_2$, where c_1 and c_2 are constant. Also if $g(x)=x$, then

$$\begin{aligned}\overline{\sigma}^2(t)=&\ \mathrm{Var}[X_{(t)}]=\mathrm{Var}[t-X\mid X<t]\\&>\Big[-\int_0^t\frac{F(x)}{F(t)}\ln\frac{F(x)}{F(t)}\,\mathrm{d}x\Big]^2=(\overline{\mathscr{E}}(X;t))^2,\end{aligned}\tag{2.28}$$

where $\overline{\mathscr{E}}(X;t)$ is dynamic cumulative past entropy and it can be shown that $\overline{\mathscr{E}}(X;t)=E[k(X)\mid X<t]$. In inequality (2.28), for a non-negative random variable X, the equality does not hold.

Remark 2.16. If X is increasing variance past lifetime (IVPL), then X is increasing dynamic cumulative past entropy (IDCPE), that is, IVPL $\Rightarrow$ IDCPE.

This implication holds, because

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Var}(t-X\mid X<t)=\overline{r}(t)(k^2(t)-\overline{\sigma}^2(t))=\overline{r}(t)(k(t)-\overline{\sigma}(t))(k(t)+\overline{\sigma}(t))\geqslant 0$$

and thus $\overline{r}(t)(k(t)-\overline{\sigma}(t))\geqslant 0$ and thereby by using (2.28), we have $\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathscr{E}}(X;t)=\overline{r}(t)(k(t)-\overline{\mathscr{E}}(X;t))\geqslant 0$ and therefore, X is IDCPE.

E x a m p l e 2.17. Let X be a continuous random variable with distribution function

$$
F_X(t) = \begin{cases} \dfrac{t^2}{16}, & 0 \leqslant t < 1, \\ \dfrac{t^4 - 2t + 2}{16}, & 1 \leqslant t < a, \\ 1, & t \geqslant a, \end{cases} \tag{2.29}
$$

where $a \approx 2.06338$ is the unique positive root of the equation $a^4 - 2a - 14 = 0$. Nanda et al. [18] has shown that VPL of X is an increasing function, that is, X is IVPL and thus we conclude that X is IDCPE.

We must notice that [19] proved that if X is a non-negative non-degenerate random variable, then $\overline{\mathscr{E}}(X;t)$ cannot be a decreasing function of t for all values of t.

R e m a r k 2.18. Let the reliability function for the weighted random variable X^* associated to X and δ be given by

$$
\overline{F}_{X^*}(t) = \frac{\mathrm{E}(\delta(X) \mid X > t)}{\mathrm{E}(\delta(X))} \overline{F}_X(t).
$$

Similarly to Remark 2.7 we can show that

$$
\operatorname{Var}(X_t^*) \geqslant (\mathscr{E}(X^*;t))^2. \tag{2.30}
$$

Now if $\mathrm{E}(\delta(X) \mid X > t)$ is increasing in t and X^* is IMRL, then [19] proved that $\mathscr{E}(X^*;t) \geqslant \mathscr{E}(X;t)$ for all t and thus we conclude that

$$
\operatorname{Var}(X_t^*) \geqslant (\mathscr{E}(X;t))^2. \tag{2.31}
$$

To obtain a lower bound for $\operatorname{Var}(g(X^*))$, we first obtain $w(\cdot)$-function of random variable X^* (i.e. $w^*(\cdot)$), according to $w(\cdot)$-function of random variable X.

Theorem 2.19. *Let X be a continuous random variable with density function $f(x)$ and X^* be a weighted random variable with density function given by* (1.9) *and variance ${\sigma^*}^2$. Then $w^*(\cdot)$-function of random variable X^* satisfies the equation*

$$
{\sigma^*}^2 w^*(x)\delta(x) = \Big\{ z_2(x) - z_1(x) \frac{\mathrm{E}[X\delta(X)]}{\mathrm{E}[\delta(X)]} \Big\}, \tag{2.32}
$$

where

$$
z_1(x) = \frac{1}{f(x)} \int_a^x (\mathrm{E}[\delta(X)] - \delta(t)) f(t) \,\mathrm{d}t \tag{2.33}
$$

and

$$z_2(x) = \frac{1}{f(x)} \int_a^x (\mathrm{E}[X\delta(X)] - t\delta(t)) f(t)\,\mathrm{d}t. \tag{2.34}$$

Proof. For the proof see [11]. □

In the special case that X^* has the X-size biased distribution, [11] showed that

$$f(x)\Big(x\sigma^{*2}w^*(x) + \frac{\mathrm{E}(X^2)}{\mathrm{E}(X)}w(x)\sigma^2\Big) = \mathrm{E}(X^2)F(x) - \int_a^x t^2 f(t)\,\mathrm{d}t, \tag{2.35}$$

where

$$\sigma^{*2} = \frac{\mathrm{E}(X^3)}{\mathrm{E}(X)} - \Big(\frac{\mathrm{E}(X^2)}{\mathrm{E}(X)}\Big)^2.$$

Example 2.20. In (1.9), let X have an exponential distribution with density function $f(x) = \mathrm{e}^{-x/\theta}/\theta$, $x > 0$ and $\delta(x) = x$. By using (1.5), [3] showed that $w(x) = x/\theta$. Now since

$$\int_0^x \frac{1}{\theta} t^2 \mathrm{e}^{-t/\theta}\,\mathrm{d}t = -x^2\mathrm{e}^{-x/\theta} - 2x\theta\mathrm{e}^{-x/\theta} - 2\theta^2\mathrm{e}^{-x/\theta} + 2\theta^2, \tag{2.36}$$

with substituting the moments of distribution and equation (2.36) into (2.35), we can compute $w^*(x) = x/(2\theta)$. It should be noted that weighted distribution is a gamma distribution with shape parameter 2 and scale parameter θ. Recursively, we can also obtain $w(\cdot)$-function of the Erlang distribution.

Remark 2.21. If X is a continuous random variable and X^* has the weighted distribution given in (1.9), then

$$\mathrm{Var}[g(X^*)] \geqslant \frac{1}{\sigma^{*2}}\Big(\frac{1}{\mathrm{E}[\delta(X)]}\mathrm{E}\{z_2(X)g'(X)\} - \frac{\mathrm{E}[X\delta(X)]}{\mathrm{E}^2[\delta(X)]}\mathrm{E}\{z_1(X)g'(X)\}\Big)^2, \tag{2.37}$$

where $g(\cdot)$ is an absolutely continuous function, and the equality holds if and only if g is linear.

If X is a non-negative, integer-valued random variable with probability mass function $P(X = j)$ and X^* has the weighted distribution given by

$$P(X^* = j) = \frac{\delta(j)}{\mathrm{E}[\delta(X)]}P(X = j), \quad j = 0, 1, \ldots,$$

then replacing integrals by sums and $g'(x)$ by $\Delta g(x) = g(x+1) - g(x)$, we arrive at the discrete version of (2.32) and (2.35) and the discrete version of (2.37) is denoted as

$$\mathrm{Var}[g(X^*)] \geqslant \frac{1}{\sigma^{*2}}\Big(\frac{1}{\mathrm{E}[\delta(X)]}\mathrm{E}[z_2(X)\Delta g(X)] - \frac{\mathrm{E}[X\delta(X)]}{\mathrm{E}^2[\delta(X)]}\mathrm{E}[z_1(X)\Delta g(X)]\Big)^2, \tag{2.38}$$

where the equality holds if and only if g is linear.

E x a m p l e 2.22. Let X have a binomial distribution with parameters p and n. If the weight function is $\delta(x)=\mathrm{e}^{ax}$ for constant a, then X^* has a weighted binomial distribution with $\mathrm{E}[\delta(X)]=(p\mathrm{e}^a+q)^n$ and $\mathrm{E}[X\delta(X)]=np\mathrm{e}^a(p\mathrm{e}^a+q)^{n-1}$, where $q=1-p$. Now by using (2.32), we get

$$\begin{aligned}(2.39)\quad &\sigma^{*2}w^*(x)\mathrm{e}^{ax}\binom{n}{x}p^x(1-p)^{n-x}\\ &= np\mathrm{e}^a(p\mathrm{e}^a+q)^{n-1}\sum_{k=0}^{x}\binom{n}{k}p^kq^{n-k}-np\mathrm{e}^a\sum_{k=0}^{x-1}\binom{n-1}{k}(p\mathrm{e}^a)^kq^{n-1-k}\\ &\quad-\Big((p\mathrm{e}^a+q)^n\sum_{k=0}^{x}\binom{n}{k}p^kq^{n-k}-\sum_{k=0}^{x}\binom{n}{k}(p\mathrm{e}^a)^kq^{n-k}\Big)\frac{np\mathrm{e}^a}{p\mathrm{e}^a+q}\\ &=\frac{np\mathrm{e}^a}{p\mathrm{e}^a+q}\sum_{k=0}^{x}\binom{n}{k}(p\mathrm{e}^a)^kq^{n-k}-np\mathrm{e}^a\sum_{k=0}^{x-1}\binom{n-1}{k}(p\mathrm{e}^a)^kq^{n-1-k},\end{aligned}$$

and consequently,

$$\begin{aligned}(2.40)\quad &\sigma^{*2}w^*(x)\frac{\mathrm{e}^{ax}}{(p\mathrm{e}^a+q)^n}\binom{n}{x}p^xq^{n-x}\\ &=\frac{np\mathrm{e}^a}{p\mathrm{e}^a+q}\Big\{\sum_{k=0}^{x}\binom{n}{k}\Big(\frac{p\mathrm{e}^a}{p\mathrm{e}^a+q}\Big)^k\Big(\frac{q}{p\mathrm{e}^a+q}\Big)^{n-k}\\ &\quad-\sum_{k=0}^{x-1}\binom{n-1}{k}\Big(\frac{p\mathrm{e}^a}{p\mathrm{e}^a+q}\Big)^k\Big(\frac{q}{p\mathrm{e}^a+q}\Big)^{n-k}\Big\}\\ &=\frac{np\mathrm{e}^a}{p\mathrm{e}^a+q}\Big\{\frac{p\mathrm{e}^a}{p\mathrm{e}^a+q}\sum_{k=0}^{x-1}\binom{n-1}{k}\Big(\frac{p\mathrm{e}^a}{p\mathrm{e}^a+q}\Big)^k\Big(\frac{q}{p\mathrm{e}^a+q}\Big)^{n-1-k}\\ &\quad+\frac{q}{p\mathrm{e}^a+q}\sum_{k=0}^{x}\binom{n-1}{k}\Big(\frac{p\mathrm{e}^a}{p\mathrm{e}^a+q}\Big)^k\Big(\frac{q}{p\mathrm{e}^a+q}\Big)^{n-1-k}\\ &\quad-\sum_{k=0}^{x-1}\binom{n-1}{k}\Big(\frac{p\mathrm{e}^a}{p\mathrm{e}^a+q}\Big)^k\Big(\frac{q}{p\mathrm{e}^a+q}\Big)^{n-1-k}\Big\}\\ &=\frac{npq\mathrm{e}^a}{(p\mathrm{e}^a+q)^2}\binom{n-1}{x}\Big(\frac{p\mathrm{e}^a}{p\mathrm{e}^a+q}\Big)^x\Big(\frac{q}{p\mathrm{e}^a+q}\Big)^{n-1-x},\end{aligned}$$

which implies $w^*(x)=(1-x/n)(1+p\mathrm{e}^a/q)$.

On the other hand, by applying (2.38), we get a lower bound for $\mathrm{Var}[g(X^*)]$. Since

$$\begin{aligned}(2.41)\quad &\frac{1}{\mathrm{E}[\delta(X)]}\Big\{z_2(x)-\frac{\mathrm{E}[X\delta(X)]}{\mathrm{E}[\delta(X)]}z_1(x)\Big\}\binom{n}{x}p^xq^{n-x}\\ &=\frac{npq\mathrm{e}^a}{(p\mathrm{e}^a+q)^2}\binom{n-1}{x}\Big(\frac{p\mathrm{e}^a}{p\mathrm{e}^a+q}\Big)^x\Big(\frac{q}{p\mathrm{e}^a+q}\Big)^{n-1-x},\end{aligned}$$

thus

$$\operatorname{Var}[g(X^*)] \geqslant \frac{npq\mathrm{e}^a}{(p\mathrm{e}^a+q)^{2n}}(\mathrm{E}_{n-1}[\mathrm{e}^{aX}\Delta g(X)])^2, \tag{2.42}$$

where E_{n-1} denotes expectation when the parameters are p and $n-1$.

In the following proposition, we give a weighted distribution in which probability density function has the general form $f(\cdot)$.

Proposition 2.23. *Let $X_1,\ldots,X_n$ be a random sample of size n from a continuous distribution with distribution function F, and Y_r denote the rth-order statistic of this random sample. Then*

$$\begin{aligned}\operatorname{Var}[g(Y_r)] \geqslant \frac{1}{\sigma^{*2}}\Big(&\frac{n!}{(r-1)!(n-r)!}\mathrm{E}\{z_2(X)g'(X)\}\\ &-\Big[\frac{n!}{(r-1)!(n-r)!}\Big]^2\mathrm{E}[X\delta(X)]\mathrm{E}\{z_1(X)g'(X)\}\Big)^2,\end{aligned} \tag{2.43}$$

where

$$z_1(x) = \frac{(r-1)!(n-r)!}{n!}\frac{1}{f(x)}\Big(F(x)-\sum_{k=r}^{n}\binom{n}{k}[F(x)]^k[1-F(x)]^{n-k}\Big), \tag{2.44}$$

and

$$\begin{aligned} z_2(x) = \frac{1}{f(x)}\Big[&F(x)\mathrm{E}\{X[F(X)]^{r-1}[1-F(X)]^{n-r}\}\\ &-\frac{(r-1)!(n-r)!}{n!}\Big\{x\sum_{k=r}^{n}\binom{n}{k}[F(x)]^k[1-F(x)]^{n-k}\\ &-\sum_{k=r}^{n}\binom{n}{k}\int_a^x[F(t)]^k[1-F(t)]^{n-k}\,\mathrm{d}t\Big\}\Big].\end{aligned} \tag{2.45}$$

Proof. Goodarzi et al. [11] obtained $z_1(x)$ and $z_2(x)$. Now, by substituting (2.44) and (2.45) into (2.37), the lower bound for the variance of $g(X^*)$ is obtained. □

Remark 2.24. In Proposition (2.23), let $X_1,\ldots,X_n$ be a random sample from the uniform distribution on $(0,1)$ and $r=n$. Then

$$z_1(x)=\frac{1}{n}x(1-x^{n-1}), \quad z_2(x)=\frac{1}{n+1}x(1-x^n).$$

Thus, the lower bound for variance of $g(Y_n)$ is calculated as

$$\begin{aligned}\operatorname{Var}[g(Y_n)] &\geqslant \frac{(n+1)^2(n+2)}{n}\biggl(n\int_0^1 g'(x)\frac{1}{n+1}x(1-x^n)\,\mathrm{d}x \\ &\qquad\qquad - \frac{n^2}{n+1}\int_0^1 g'(x)\frac{1}{n}x(1-x^{n-1})\,\mathrm{d}x\biggr)^2 \\ &= n(n+2)\{\mathrm{E}[X^n(1-X)g'(X)]\}^2,\end{aligned}$$

and in the general case, the lower bound for the variance of $g(Y_r)$ will be

$$\operatorname{Var}[g(Y_r)] \geqslant \frac{r(n-r+1)}{(n+1)^2(n+2)}\Bigl(\mathrm{E}\Bigl[\frac{(n+2)!}{r!(n-r+1)!}X^r(1-X)^{n-r+1}g'(X)\Bigr]\Bigr)^2.$$

At the end of this section, we consider two weights applicable in reliability. Reliability is closely related to mathematics, especially to statistics, physics, chemistry, mechanics, and electronics.

Proposition 2.25. *Let X be a non-negative absolutely continuous random variable with density function $f(x)$, survival function $\overline{F}(x)$ and hazard rate $r(x)$. Suppose that F is DFR.*

(a) *If $\delta(x) = f(x+t)/f(x)$, then*

$$\operatorname{Var}[g(X^*)] \geqslant \frac{1}{\sigma^{*2}}\Bigl\{\mathrm{E}\Bigl[\frac{X}{r(X)}g'(X)\Bigr]\Bigr\}^2. \tag{2.46}$$

(b) *If $\delta(x) = I(x>t)$, then*

$$\operatorname{Var}[g(X^*)] \geqslant \frac{1}{\sigma^{*2}}\Bigl\{\mathrm{E}\Bigl[\frac{X-t}{r(X)}g'(X) \mid X>t\Bigr]\Bigr\}^2, \tag{2.47}$$

where $\sigma^{*2} = \operatorname{Var}(X-t \mid X>t)$.

Proof. (a) It is well known that $X^* = X_t$, $\mathrm{E}[X^*] = \mathrm{E}[X-t \mid X>t] = m(t)$ and $\sigma^{*2} = \operatorname{Var}[X-t \mid X>t]$. Now, by applying (1.5) and the fact that $\mathrm{E}[X-\mu]=0$, we have

$$\begin{aligned}\int_x^\infty (y-m(t))\frac{\delta(y)}{\mathrm{E}[\delta(X)]}f(y)\,\mathrm{d}y &= \int_x^\infty (y-m(t))\frac{f(y+t)}{\overline{F}(t)}\,\mathrm{d}y \\ &= \frac{1}{\overline{F}(t)}\Bigl\{\int_x^\infty yf(y+t)\,\mathrm{d}y - m(t)\int_x^\infty f(y+t)\,\mathrm{d}y\Bigr\} \\ &= \frac{1}{\overline{F}(t)}\Bigl\{\int_{x+t}^\infty (y-t)f(y)dy - m(t)\int_{x+t}^\infty f(y)\,\mathrm{d}y\Bigr\} \\ &= \overline{F}(x \mid t)\{m(x+t)-m(t)+x\}.\end{aligned}$$

Now, whereas if F is DFR, then F is NWU, and also F is IMRL, that is $m(s) \leqslant m(t)$ for $0 \leqslant s \leqslant t$, we have $m(x+t) \geqslant m(t)$ and consequently,

$$w^*(x)\sigma^{*2}\frac{f(x+t)}{\mathrm{E}[\delta(X)]} = \int_x^\infty (y-m(t))\frac{\delta(y)}{\mathrm{E}[\delta(X)]}f(y)\,\mathrm{d}y \geqslant x\overline{F}(x),$$

which eventually leads to

$$\begin{aligned}\mathrm{Var}[g(X^*)] &\geqslant \frac{1}{\sigma^{*2}}\Big(\int_0^\infty x\overline{F}(x)g'(x)\,\mathrm{d}x\Big)^2 = \frac{1}{\sigma^{*2}}\Big(\int_0^\infty \frac{x}{r(x)}g'(x)f(x)\,\mathrm{d}x\Big)^2\\ &= \frac{1}{\sigma^{*2}}\Big(\mathrm{E}\Big[\frac{X}{r(X)}g'(X)\Big]\Big)^2.\end{aligned}$$

(b) It is obvious that

$$f_{X^*}(x) = \frac{I(x>t)}{P(X>t)}f(x)$$

and so $\mathrm{E}[X^*] = m(t)+t$ and $\sigma^{*2} = \mathrm{Var}[X \mid X>t] = \mathrm{Var}[X-t \mid X>t]$. Again by using (1.5), we have

$$\begin{aligned}\int_t^x (m(t)+t-y)\frac{f(y)}{\overline{F}(t)}\,\mathrm{d}y &= \frac{m(t)+t}{\overline{F}(t)}(F(x)-F(t))\\ &\quad - \frac{1}{\overline{F}(t)}(\overline{F}(t)(m(t)+t)-\overline{F}(x)(m(x)+x))\\ &= \frac{\overline{F}(x)}{\overline{F}(t)}(m(x)-m(t)+(x-t)),\end{aligned}$$

and since F is IMRL, then

$$\int_t^x (m(t)+t-y)\frac{f(y)}{\overline{F}(t)}\,\mathrm{d}y \geqslant \frac{\overline{F}(x)}{\overline{F}(t)}(x-t),$$

and therefore

$$\mathrm{Var}[g(X^*)] \geqslant \frac{1}{\sigma^{*2}}\Big(\int_t^\infty \frac{\overline{F}(x)}{\overline{F}(t)}(x-t)g'(x)\,\mathrm{d}x\Big)^2 = \frac{1}{\sigma^{*2}}\Big(\mathrm{E}\Big[\frac{X-t}{r(X)}g'(X) \mid X>t\Big]\Big)^2.$$

□

Corollary 2.26. *In Proposition* (2.25) *part* (a), *if F is an exponential distribution, then the lower bound is equal to Chernoff-type lower variance bound given in* [4], *because in this distribution $\overline{F}(x+t) = \overline{F}(x)\overline{F}(t)$ and $m(x+t) = m(t)$.*

E x a m p l e 2.27. Let X have two parameter Weibull distribution with probability density function

$$f(x;\theta,\beta)=\frac{\beta}{\theta}x^{\beta-1}\mathrm{e}^{-x^{\beta}/\theta},\quad x>0,\ \theta,\beta>0.$$

Now for $\theta=1$, since $r(x)=\beta x^{\beta-1}$, the distribution is DFR for $\beta<1$. By using (2.46), we have $\mathrm{Var}[X-t\mid X>t]\geqslant\Gamma(2/\beta)/\beta$ for all $t>0$, however by using (2.31) and for simplicity in computation for $\beta=\frac{1}{3}$, $\mathrm{Var}[X-t\mid X>t]>(12t^{1/3}+18+3t^{2/3})^2$. Here, in order to compare the two bounds, for $\beta=\frac{1}{3}$ assume that $t=3$, hence the first bound is 360, whereas the second bound is 1726.1.

3. Conclusion

Cacoullos and Papathanasiou obtained the lower bounds for the variance of functions of random variables. In this article, we obtained the lower bounds for the variance of functions of random variables used in reliability analysis and entropy in terms of Chernoff-type inequalities. We also obtained the lower bounds for the function of weighted random variables, that can be used in reliability analysis.

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References

[1] *M. Asadi, Y. Zohrevand*: On the dynamic cumulative residual entropy. J. Stat. Plann. Inference *137* (2007), 1931–1941. zbl MR doi
[2] *T. Cacoullos*: On upper and lower bounds for the variance of the function of a random variable. Ann. Probab. *10* (1982), 799–809. zbl MR doi
[3] *T. Cacoullos, V. Papathanasiou*: On upper bounds for the variance of functions of random variables. Stat. Probab. Lett. *3* (1985), 175–184. zbl MR doi
[4] *T. Cacoullos, V. Papathanasiou*: Characterizations of distributions by variance bounds. Stat. Probab. Lett. *7* (1989), 351–356. zbl MR doi
[5] *T. Cacoullos, V. Papathanasiou*: A generalization of covariance identity and related characterization. Math. Methods Stat. *4* (1995), 106–113. zbl MR
[6] *T. Cacoullos, V. Papathanasiou*: Characterizations of distributions by generalizations of variance bounds and simple proofs of the CLT. J. Stat. Plann. Inference *63* (1997), 157–171. zbl MR doi
[7] *L. H. Y. Chen*: An inequality for the multivariate normal distribution. J. Multivariate Anal. *12* (1982), 306–315. zbl MR doi
[8] *H. Chernoff*: A note on an inequality involving the normal distribution. Ann. Probab. *9* (1981), 533–535. zbl MR doi

[9] *A. Di Crescenzo, L. Paolillo*: Analysis and applications of the residual varentropy of random lifetimes. To appear in Probab. Eng. Inf. Sci. doi

[10] *R. A. Fisher*: The effects of methods of ascertainment upon the estimation of frequencies. Ann. Eugenics *6* (1934), 13–25. doi

[11] *F. Goodarzi, M. Amini, G. R. Mohtashami Borzadaran*: On upper bounds for the variance of functions of random variables with weighted distributions. Lobachevskii J. Math. *37* (2016), 422–435. zbl MR doi

[12] *F. Goodarzi, M. Amini, G. R. Mohtashami Borzadaran*: Characterizations of continuous distributions through inequalities involving the expected values of selected functions. Appl. Math., Praha *62* (2017), 493–507. zbl MR doi

[13] *F. Goodarzi, M. Amini, G. R. Mohtashami Borzadaran*: Some results on upper bounds for the variance of functions of the residual life random variables. J. Comput. Appl. Math. *320* (2017), 30–42. zbl MR doi

[14] *G. R. Mohtashami Borzadaran*: A note on continuous exponential families. Thai J. Math. *8* (2010), 555–563. zbl MR

[15] *G. R. Mohtashami Borzadaran, D. N. Shanbhag*: Further results based on Chernoff-type inequalities. Stat. Probab. Lett. *39* (1998), 109–117. zbl MR doi

[16] *N. U. Nair, K. K. Sudheesh*: Characterization of continuous distributions by variance bound and its implications to reliability modeling and catastrophe theory. Commun. Stat., Theory Methods *35* (2006), 1189–1199. zbl MR doi

[17] *N. U. Nair, K. K. Sudheesh*: Characterization of continuous distributions by properties of conditional variance. Stat. Methodol. *7* (2010), 30–40. zbl MR doi

[18] *A. K. Nanda, H. Singh, N. Misra, P. Paul*: Reliability properties of reversed residual lifetime. Commun. Stat., Theory Methods *32* (2003), 2031–2042. zbl MR doi

[19] *J. Navarro, Y. del Aguila, M. Asadi*: Some new results on the cumulative residual entropy. J. Stat. Plann. Inference *140* (2010), 310–322. zbl MR doi

[20] *G. Psarrakos, J. Navarro*: Generalized cumulative residual entropy and record values. Metrika *76* (2013), 623–640. zbl MR doi

[21] *C. R. Rao*: On discrete distributions arising out of methods of ascertainment. Classical and Contagious Discrete Distributions. Statistical Publishing Society, Calcutta, 1965, pp. 320–332. MR

[22] *M. Rao, Y. Chen, B. C. Vemuri, F. Wang*: Cumulative residual entropy: A new measure of information. IEEE Trans. Inf. Theory *50* (2004), 1220–1228. zbl MR doi

[23] *M. Shaked, J. G. Shanthikumar*: Stochastic Orders. Springer Series in Statistics. Springer, New York, 2007. zbl MR doi

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