A NEW ENERGY CONSERVATIVE SCHEME FOR REGULARIZED LONG WAVE EQUATION

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Received March 6, 2020. Published online April 30, 2021.

Abstract. An energy conservative scheme is proposed for the regularized long wave (RLW) equation. The integral method with variational limit is used to discretize the spatial derivative and the finite difference method is used to discretize the time derivative. The energy conservation of the scheme and existence of the numerical solution are proved. The convergence of the order $O(h^2+\tau^2)$ and unconditional stability are also derived. Numerical examples are carried out to verify the correctness of the theoretical analysis.

Keywords: regularized long wave equation; integral method with variational limit; finite difference method; Lagrange interpolation; energy conservation scheme

MSC 2020: 65M06, 65M12

1. Introduction

Consider the initial boundary value problem for the regularized long wave (RLW) equation

$$u_t + u_x - u_{xxt} + uu_x = 0 \tag{1.1}$$

with the initial condition

$$u(x,0) = u_0(x), \quad x \in [x_l, x_r], \tag{1.2}$$

and boundary condition

$$u(x_l,t) = u(x_r,t) = 0, \quad t \in [0,T]. \tag{1.3}$$

This research was supported by the National Natural Science Foundation of China (No. 11801116), the Fundamental Research Funds for the Central Universities, and Shandong Province Natural Science Foundation (ZR2019BA018).

DOI: 10.21136/AM.2021.0066-20

The physical boundary requires $u \to 0$ as $|x| \to \infty$. So, if $-x_l \gg 0$ and $x_r \gg 0$, problem (1.1)–(1.3) is in accordance with the Cauchy problem for equation (1.1).

The RLW initial boundary value problem (1.1)–(1.3) has the conservation law

$$E(t) \doteq \int_{x_l}^{x_r} (u(x,t) + u_x(x,t))\,\mathrm{d}x = \text{const.}, \tag{1.4}$$

where $E(t)$ is a positive constant which relates to the initial condition.

The RLW was first proposed by Peregrine [24], [25]. It is a representative form of a nonlinear long wave which can describe a lot of important physical phenomena, such as shallow waves and ionic waves, etc. [31], [28].

There is some theoretical work available for the regularized long wave equation. In [6], Chertovskih et al. consider space-periodic evolutionary and travelling-wave solutions to the regularized long wave equation with damping and forcing. Existence, uniqueness and smoothness of the evolutionary solutions for smooth initial conditions are established. In [1], the locally and globally well-posed initial value problem for the symmetric regularized long wave equation is studied. The existence and nonlinear stability of periodic travelling wave solutions are also proved in [1]. In [12], [30], [9], the existence and uniqueness of the solution and the existence of weak attractors are proved for the symmetric regularized long wave equation with a damping term.

It is usually impossible to find analytical solutions for the RLW equation, especially when the nonlinear terms are involved [19], [24]. Therefore, finding its numerical solutions is of practical importance. Various types of methods have been used to solve (1.1), like the H1-Galerkin mixed finite element method [11], the B-spline finite element method [27], [10], [17], [18], finite difference methods [31], [37], the distributed approximating functional method [26], and the Haar wavelet combined with the finite difference method [23]. Moreover, [8], [2], [34], [32], [5], [7], [36], [4] also display interesting numerical results for the RLW equation. In recent research, Lin Bin et al. solved the RLW equation using the parametric spline method in [20] and non-polynomial splines method in [21]. Hammad derived a Chebyshev spectral collocation scheme for the RLW equation in [13]. In [35], energy and momentum preserving schemes were proposed and designed using the discrete variational derivative method and the finite volume method to solve the modified RLW equation. In [29], a collocation method depending on the cubic trigonometric B-spline approach based on a finite difference scheme was suggested to solve the modified RLW equation. In [15], [16], the B-spline Galerkin finite element space discretization with different time discretization was used to solve the RLW equation numerically. In [3], a numerical scheme for the equation was developed and analyzed by the Petrov-Galerkin method for the RLW equation in which the element shape functions are cubic and weight functions are quadratic B-splines. In [33] two conservative and fourth-order compact finite difference schemes were proposed and analyzed for solving the RLW equation.

Taking these backgrounds into account, in this paper we propose a new conservative scheme using the integral method with variational limit for the RLW equation which has the second-order accuracy both in space and time and conserves energy.

This paper is organized as follows. In Section 2, we present the discretization scheme. The scheme is obtained by using the integral method with variational limit to discretize the space and the finite difference method to discretize the time. In Section 3, we discuss discrete conservation laws. In Section 4, we prove the scheme is solvable and present an a priori estimate for the solution of the scheme. The convergence and stability for the proposed scheme are proved in Section 5. In Section 6, we give some numerical experiments to test the convergence precision and energy conservation, to calculate the three invariants and to simulate two waves collision. Finally, the conclusions are drawn in Section 7.

2. Construction of the conservative scheme by the integral method with variational limit

2.1. Brief introduction to the integral method with variational limit. The integral method with variational limit is a new numerical method for solving differential equations. It has been successfully applied to solve the Klein-Gordon equation in [22]. Eliminating the derivative by integrating is the main idea of the integral method with variational limit.

For example, when the highest order derivative in space is the second order one, we integrate each term of the equation,

$$\int_{xx} u(x) \overset{\text{def}}{=} \int_{x_j}^{x_j+\varepsilon_2} \mathrm{d}x_b \int_{x_j-\varepsilon_1}^{x_j} \mathrm{d}x_a \int_{x_a}^{x_b} u(x)\,\mathrm{d}x,$$

where ε_1 and ε_2 are undetermined coefficients. Then we can eliminate all the derivatives in x directions of order less than second (see [22]).

The integral method with variational limit transforms a partial differential equation into an integral equation and provides an effective method to solve differential equations. By using the “weighted” information at all points in the region around the mesh point, and not just only the information at the mesh points, the integral method with variational limit can be used to design many interesting schemes with excellent properties by adjusting bounds for parameters and choosing an appropriate interpolation function.

2.2. Some notations. For positive integers J and N, let $h = (x_r - x_l)/J$ be the space step $x_j = x_l + jh$, $j = 0, 1, \ldots, J-1, J$, and let $\tau = T/N$ be the time step $t_n = n\tau$, $n = 0, 1, \ldots, N$. Define $u_j^n = u(x_j, t_n)$, $\Omega_h = \{u_j;\ u_j = u(x_j),\ j = 0, 1, \ldots, J,\ u_0 = u_J = 0\}$.

First we introduce the notations

$$\delta_x^+ u_j^n = \frac{u_{j+1}^n - u_j^n}{h}, \quad \delta_x^- u_j^n = \frac{u_j^n - u_{j-1}^n}{h}, \quad \delta_x u_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2h},$$
$$\delta_x^2 u_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}, \quad \delta_t u_j^n = \frac{u_j^{n+1} - u_j^{n-1}}{2\tau}, \quad \bar{u}_j^n = \frac{u_j^{n-1} + u_j^{n+1}}{2}.$$

For any two grid functions $u, v \in \Omega_h$, we define the discrete inner product as

$$(u, v) = h \sum_{j=1}^{J-1} u_j v_j.$$

The discrete L^2-norm and infinity norm are defined as

$$\|v\| = \sqrt{(v, v)}, \qquad \|v\|_\infty = \max_j |v_j|.$$

In this paper, we denote by C a general positive constant, which may take different values at different occurrences and is independent of h and τ.

2.3. Constructing an energy conservative scheme for the RLW equation. We construct an energy conservative scheme for the RLW equation by the integral method with variational limit in this subsection.

Firstly, integrating each term of (1.1) from x_a to x_b with respect to x, we get

$$\int_{x_a}^{x_b} \frac{\partial u}{\partial t}\,\mathrm{d}x + u(x_b, t) - u(x_a, t) - \frac{\partial^2 u(x_b, t)}{\partial x_b \partial t} + \frac{\partial^2 u(x_a, t)}{\partial x_a \partial t} + \frac{1}{2}((u(x_b, t))^2 - (u(x_a, t))^2) = 0. \tag{2.1}$$

Secondly, notice that

$$\int_{x_j}^{x_j+\varepsilon_2} \int_{x_j-\varepsilon_1}^{x_j} \Big(\frac{\partial^2 u(x_b, t)}{\partial x_b \partial t} - \frac{\partial^2 u(x_a, t)}{\partial x_a \partial t} \Big)\,\mathrm{d}x_a\,\mathrm{d}x_b = \varepsilon_1 \frac{\partial u(x_j + \varepsilon_2, t)}{\partial t} - (\varepsilon_1 + \varepsilon_2) \frac{\partial u(x_j, t)}{\partial t} + \varepsilon_2 \frac{\partial u(x_j - \varepsilon_1, t)}{\partial t}, \tag{2.2}$$

then integrate each term of equation (2.1) from $x_j-\varepsilon_1$ to x_j with respect to x_a and from x_j to $x_j+\varepsilon_2$ with respect to x_b , and (2.1) can be rewritten as

$$\begin{aligned}(2.3)\qquad &\int_{x_j}^{x_j+\varepsilon_2}\int_{x_j-\varepsilon_1}^{x_j}\int_{x_a}^{x_b}\frac{\partial u(x,t)}{\partial t}\,\mathrm{d}x\,\mathrm{d}x_a\,\mathrm{d}x_b\\
&+\int_{x_j}^{x_j+\varepsilon_2}\int_{x_j-\varepsilon_1}^{x_j}(u(x_b,t)-u(x_a,t))\,\mathrm{d}x_a\,\mathrm{d}x_b\\
&-\varepsilon_1\frac{\partial u(x_j+\varepsilon_2,t)}{\partial t}-\varepsilon_2\frac{\partial u(x_j-\varepsilon_1,t)}{\partial t}+(\varepsilon_2+\varepsilon_1)\frac{\partial u(x_j,t)}{\partial t}\\
&+\frac{1}{2}\int_{x_j}^{x_j+\varepsilon_2}\int_{x_j-\varepsilon_1}^{x_j}((u(x_b,t))^2-(u(x_a,t))^2)\,\mathrm{d}x_a\,\mathrm{d}x_b=0,\end{aligned}$$

where ε_1 and ε_2 are undetermined coefficients.

Thirdly, the following Lagrange interpolation is used to approximate $u(x)$ in equation (2.3) near the point $x=x_j$,

$$(2.4)\qquad u(x,t)=\sum_{p=j-1}^{j+1}\Big(\prod_{\substack{r=j-1\\ r\neq p}}^{j+1}\frac{x-x_r}{x_p-x_r}\Big)u(x_p,t)+R(x,t),$$

where $R(x,t)$ is the truncation error and there exists ξ between x and x_j such that

$$R(x,t)=\frac{1}{6}\frac{\partial^3}{\partial x^3}u(\xi,t)(x-x_{j-1})(x-x_j)(x-x_{j+1}).$$

Letting $\varepsilon_1=\varepsilon_2=\varepsilon$ and substituting (2.4) into (2.3), each term of (2.3) can be written as

$$\begin{aligned}(2.5)\qquad &\int_{x_j}^{x_j+\varepsilon_2}\int_{x_j-\varepsilon_1}^{x_j}\int_{x_a}^{x_b}\frac{\partial u(x,t)}{\partial t}\,\mathrm{d}x\,\mathrm{d}x_a\,\mathrm{d}x_b\\
&=\frac{\varepsilon^5}{12h^2}\frac{\partial u(x_{j-1},t)}{\partial t}+\Big(-\frac{\varepsilon^5}{6h^2}+\varepsilon^3\Big)\frac{\partial u(x_j,t)}{\partial t}+\frac{\varepsilon^5}{12h^2}\frac{\partial u(x_{j+1},t)}{\partial t}+R_1,\end{aligned}$$

$$\begin{aligned}(2.6)\qquad &\int_{x_j}^{x_j+\varepsilon_2}\int_{x_j-\varepsilon_1}^{x_j}(u(x_b,t)-u(x_a,t))\,\mathrm{d}x_a\,\mathrm{d}x_b\\
&=-\frac{\varepsilon^3}{2h}u(x_{j-1},t)+\frac{\varepsilon^3}{2h}u(x_{j+1},t)+R_2,\end{aligned}$$

$$\begin{aligned}(2.7)\qquad &\varepsilon_1\frac{\partial u(x_j+\varepsilon_2,t)}{\partial t}-(\varepsilon_1+\varepsilon_2)\frac{\partial u(x_j,t)}{\partial t}+\varepsilon_2\frac{\partial u(x_j-\varepsilon_1,t)}{\partial t}\\
&=\frac{\varepsilon^3}{h^2}\frac{\partial u(x_{j-1},t)}{\partial t}-\frac{2\varepsilon^3}{h^2}\frac{\partial u(x_j,t)}{\partial t}+\frac{\varepsilon^3}{h^2}\frac{\partial u(x_{j+1},t)}{\partial t}+R_3,\end{aligned}$$

and

$$\int_{x_j}^{x_j+\varepsilon_2}\int_{x_j-\varepsilon_1}^{x_j}((u(x_b,t))^2-(u(x_a,t))^2)\,\mathrm{d}x_a\,\mathrm{d}x_b \tag{2.8}$$
$$=\frac{\varepsilon^5}{4h^3}((u(x_{j+1},t))^2-(u(x_{j-1},t))^2)$$
$$+\Big(\frac{\varepsilon^3}{h}-\frac{\varepsilon^5}{2h^3}\Big)(u(x_{j+1},t)-u(x_{j-1},t))u(x_j,t)+R_4.$$

Dividing (2.5)–(2.8) by the integral factor $\int_{x_j}^{x_j+\varepsilon_2}\int_{x_j-\varepsilon_1}^{x_j}\int_{x_a}^{x_b}1\,\mathrm{d}x\,\mathrm{d}x_a\,\mathrm{d}x_b=\varepsilon^3$, which is introduced by integration, we obtain

$$\frac{\varepsilon^2}{12h^2}((u_{j-1})_t+(u_{j+1})_t)+\Big(1-\frac{\varepsilon^2}{6h^2}\Big)(u_j)_t+\frac{1}{2h}(u_{j+1}-u_{j-1}) \tag{2.9}$$
$$-\frac{1}{h^2}((u_{j-1})_t+(u_{j+1})_t)+\frac{2}{h^2}(u_j)_t+\frac{\varepsilon^2}{8h^3}((u_{j+1})^2-(u_{j-1})^2)$$
$$+\Big(\frac{1}{2h}-\frac{\varepsilon^2}{4h^3}\Big)(u_{j+1}-u_{j-1})u_j+R_1'+R_2'-R_3'+\frac{R_4'}{2}=0,$$

where $R_i'=R_i/\varepsilon^3$, $i=1,2,3,4$.

From the Taylor formula, we have

$$(u_j^n)_t=\frac{u_j^{n+1}-u_j^{n-1}}{2\tau}+O(\tau^2),\quad u_j^n=\bar{u}_j^n+O(\tau^2). \tag{2.10}$$

Then equation (2.9) can be rewritten as

$$\frac{\varepsilon^2}{12}\delta_x^2\delta_t u_j^n+\delta_t u_j^n+\delta_x\bar{u}_j^n-\delta_x^2\delta_t u_j^n+\frac{\varepsilon^2}{8h^3}((\bar{u}_{j+1}^n)^2-(\bar{u}_{j-1}^n)^2) \tag{2.11}$$
$$+\Big(\frac{1}{2h}-\frac{\varepsilon^2}{4h^3}\Big)(\bar{u}_{j+1}^n-\bar{u}_{j-1}^n)\bar{u}_j^n+R_1'+R_2'-R_3'+\frac{R_4'}{2}+O(\tau^2)=0.$$

Omitting R_1', R_2', R_3', R_4' and $O(\tau^2)$ in (2.11), and letting $\varepsilon=2h/\sqrt{3}$, we obtain the scheme

$$\Big(\frac{h^2}{9}-1\Big)\delta_x^2\delta_t U_j^n+\delta_t U_j^n+\delta_x\overline{U}_j^n+\frac{1}{3}(\overline{U}_j^n\delta_x\overline{U}_j^n+\delta_x(\overline{U}_j^n)^2)=0. \tag{2.12}$$

Remark 2.1. The purpose of letting $\varepsilon=2h/\sqrt{3}$ is to design appropriate nonlinear terms to obtain a numerical scheme that can preserve the conservation of energy. See the conservation of energy theorem (Theorem 3.2).

R e m a r k 2.2. The proposed numerical scheme (2.12) looks like a finite difference scheme with an additional regularization term $\frac{1}{9}h^2\delta_x^2\delta_t U_j^n$. In fact, it comes from the variable limit integral of the u_t term. It is easy to see $\frac{1}{9}h^2\delta_x^2\delta_t U_j^n + \delta_t U_j^n = \frac{1}{9}\delta_t(U_{j-1}^n + 7U_j^n + U_{j+1}^n)$. This makes the variable limit integration method use more space information for the discretization of u_t and the introduction of this term may make the numerical scheme more stable. The same phenomenon is more obvious in the higher order variable limit integral scheme, see [22].

The matrix form of the above scheme (2.12) is

$$\Big(\frac{h^2}{9} - 1\Big)\delta_x^2\delta_t U^n + \delta_t U^n + \delta_x\overline{U}^n + \frac{1}{3}(\overline{U}^n\delta_x\overline{U}^n + \delta_x(\overline{U}^n)^2) = 0, \tag{2.13}$$

where $U^n = (U_1^n, U_2^n, \dots, U_{J-1}^n)^\top$, $n = 0, 1, \dots, N$.

R e m a r k 2.3. Since (2.13) is a three level scheme, the first two layers are required to start this scheme. For the first level, taking initial conditions (1.2) into consideration, we have

$$U^0 = (u_0(x_1), u_0(x_2), \dots, u_0(x_{J-1}))^\top. \tag{2.14}$$

And from the Taylor formula, we obtain the second level as

$$U_j^1 = U_j^0 + \tau(U_j^0)_t + \frac{\tau^2}{2}(U_j^0)_{tt} + O(\tau^3), \tag{2.15}$$

where $(U_j^0)_t$ and $(U_j^0)_{tt}$ are the numerical values of $\partial u(x_j,0)/\partial t$ and $\partial^2 u(x_j,0)/\partial t^2$. Noticing that $\varepsilon = 2h/\sqrt{3}$ and from (2.9), we can use the formulas

$$\Big(\frac{h^2}{9} - 1\Big)\delta_x^2(U_j^0)_t + (U_j^0)_t = -\delta_x U_j^0 - \frac{1}{3}(U_j^0\delta_x U_j^0 + \delta_x(U_j^0)^2)$$

and

$$\Big(\frac{h^2}{9} - 1\Big)\delta_x^2(U_j^0)_{tt} + (U_j^0)_{tt} = -\delta_x(U_j^0)_t - \frac{1}{3}((U_j^0)_t\delta_x U_j^0 + U_j^0\delta_x(U_j^0)_t + 2\delta_x(U_j^0(U_j^0)_t))$$

to calculate $(U_j^0)_t$ and $(U_j^0)_{tt}$ in (2.15).

3. Conservation of energy

Before giving the conservation law, we introduce the following lemma:

Lemma 3.1 (see [38], [14]). *For any two mesh functions $u, v \in \Omega_h$, we have*

$$(\delta_x^2 u, v) = (\delta^+\delta^- u, v) = -(\delta^+ u, \delta^+ v), \quad (\delta u, v) = -(u, \delta v), \quad \|\delta_x u\|^2 \leqslant \|\delta_x^+ u\|^2.$$

Theorem 3.2. *The solution of scheme* (2.13)–(2.15) *satisfies the energy conservation law*

$$E^n \doteq \Big(1 - \frac{h^2}{9}\Big)(\|\delta_x^+ U^{n+1}\|^2 + \|\delta_x^+ U^n\|^2) + \|U^{n+1}\|^2 + \|U^n\|^2 = C,$$

where $n = 0, 1, \ldots, N-1$.

Proof. Computing the inner product of (2.13) with $2\overline{U}^n = U^{n+1} + U^{n-1}$, we have from Lemma 3.1,

$$(3.1) \qquad \begin{aligned}(\delta_x^2\delta_t U^n, U^{n+1} + U^{n-1}) &= -(\delta_x^+\delta_t U^n, \delta_x^+(U^{n+1} + U^{n-1})) \\ &= -\frac{1}{2\tau}(\|\delta_x^+ U^{n+1}\|^2 - \|\delta_x^+ U^{n-1}\|^2),\end{aligned}$$

$$(3.2) \qquad (\delta_t U^n, U^{n+1} + U^{n-1}) = \frac{1}{2\tau}(\|U^{n+1}\|^2 - \|U^{n-1}\|^2)$$

and

$$(3.3) \qquad (\delta_x \overline{U}^n, U^{n+1} + U^{n-1}) = \frac{1}{2}(\delta_x(U^{n+1} + U^{n-1}), U^{n+1} + U^{n-1}) = 0.$$

Notice that

$$(\overline{U}^n\delta_x\overline{U}^n, \overline{U}^n) = h\sum_{j=1}^{J-1}\overline{U}_j^n\delta_x\overline{U}_j^n\overline{U}_j^n = h\sum_{j=1}^{J-1}\delta_x\overline{U}_j^n(\overline{U}_j^n)^2 = (\delta_x\overline{U}^n, (\overline{U}^n)^2).$$

Then we obtain

$$(3.4) \qquad (\overline{U}^n\delta_x\overline{U}^n + \delta_x(\overline{U}^n)^2, \overline{U}^n) = (\overline{U}^n\delta_x\overline{U}^n, \overline{U}^n) - ((\overline{U}^n)^2, \delta_x\overline{U}^n) = 0.$$

From (3.2)–(3.4), we obtain

$$(3.5) \qquad \Big(1 - \frac{h^2}{9}\Big)(\|\delta_x^+ U^{n+1}\|^2 - \|\delta_x^+ U^{n-1}\|^2) + \|U^{n+1}\|^2 - \|U^{n-1}\|^2 = 0$$

for all $n = 1, 2, \ldots, N$. Define E^n by

$$(3.6) \qquad E^n = \Big(1 - \frac{h^2}{9}\Big)(\|\delta_x^+ U^{n+1}\|^2 + \|\delta_x^+ U^n\|^2) + \|U^{n+1}\|^2 + \|U^n\|^2,$$

then equation (3.5) can be rewritten as $E^n = E^{n-1}$ for all $n = 1, 2, \ldots, N-1$.

This completes the proof. □

R e m a r k 3.3. When the space step $h \to 0$, the energy conservation law (3.6) can more strictly reflect energy conservation property (1.4) of original partial differential equation (2.1)–(2.4).

4. Solvability and a priori estimate

Lemma 4.1 (see [33], Lemma 2.12; [38]). *Let H be a finite dimensional inner product space. Suppose that $g: H \to H$ is a continuous operator and there exists $\alpha > 0$ such that $(g(z), z) > 0$ for all $z \in H$ with $\|z\| = \alpha$. Then there exists $z^* \in H$ such that $g(z^*) = 0$ and $\|z^*\| \leqslant \alpha$.*

Lemma 4.2 (Discrete Sobolev inequality [31], Lemma 2; [38]). *For any discrete function u_h and for any given $\varepsilon > 0$, there exists a constant $K(\varepsilon, n)$, depending only ε and n, such that*

$$\|u^n\|_\infty \leqslant \varepsilon\|\delta_x^+ u^n\| + K(\varepsilon, n)\|u^n\|.$$

Theorem 4.3. *The solution of scheme* (2.13) *exists.*

P r o o f. From (2.14)–(2.15), the solution exists for $n = 0$ and $n = 1$. Suppose that there exist $U^0, U^1, \ldots, U^n$ satisfying scheme (2.14)–(2.15) for $0 \leqslant n \leqslant N-1$. Next we prove that there exists U^{n+1} satisfying scheme (2.13)–(2.15). Let ω be an operator defined on Ω_h by

$$\omega(\zeta) = \Big(\frac{h^2}{9} - 1\Big)\delta_x^2(\zeta - U^{n-1}) + (\zeta - U^{n-1}) + \tau\delta_x\zeta + \frac{\tau}{3}(\zeta\delta_x\zeta + \delta_x(\zeta)^2).$$

Taking the inner product of $\omega(\zeta)$ with ζ, similarly to the proof of Theorem 3.2, by the Cauchy-Schwarz inequality, and when h is small enough, we have

$$\begin{aligned}(\omega(\zeta), \zeta) &= \Big(1 - \frac{h^2}{9}\Big)(\|\delta_x^+\zeta\|^2 - (\delta_x^+ U^{n-1}, \delta_x^+\zeta)) + \|\zeta\|^2 - (U^{n-1}, \zeta) \\ &\geqslant \frac{9-h^2}{9}\|\delta_x^+\zeta\|^2 - \frac{9-h^2}{18}(\|\delta_x^+ U^{n-1}\|^2 + \|\delta_x^+\zeta\|^2) \\ &\quad + \|\zeta\|^2 - \frac{1}{2}(\|U^{n-1}\|^2 + \|\zeta\|^2) \\ &\geqslant \frac{1}{2}\|\zeta\|^2 - \frac{1}{2}(\|\delta_x^+ U^{n-1}\|^2 + \|U^{n-1}\|^2).\end{aligned} \tag{4.1}$$

Therefore, for all $\zeta \in \Omega_h$ and $\|\zeta\|^2 = \|\delta_x^+ U^{n-1}\|^2 + \|U^{n-1}\|^2 + 1$, we have $(\omega(\zeta), \zeta) > 0$. From Lemma 4.1, there exists ζ^* such that $\omega(\zeta^*) = 0$. Let $U^{n+1} = 2\zeta^* - U^{n-1}$ then U^{n+1} is the solution of scheme (2.13)–(2.15). $\square$

Next, we present an a priori estimate for the solution of scheme (2.13)–(2.15).

Theorem 4.4. *Assume that* $u_0 \in H_0^1[x_l, x_r]$. *Then*

$$\|U^n\| \leqslant C, \quad \|\delta_x^+ U^n\| \leqslant C, \quad \|U^n\|_\infty \leqslant C,$$

where C *is a generic positive constant depending on the initial condition* $u_0(x)$ *and independent of* h *and* τ.

P r o o f. From (2.15) and (3.6), when h is small enough we have

$$\begin{aligned} E^n = E^0 &= \Big(1 - \frac{h^2}{9}\Big)(\|\delta_x^+ U^1\|^2 + \|\delta_x^+ U^0\|^2) + \|U^1\|^2 + \|U^0\|^2 \\ &\leqslant \|\delta_x^+ U^1\|^2 + \|\delta_x^+ U^0\|^2 + \|U^1\|^2 + \|U^0\|^2 = C. \end{aligned} \tag{4.2}$$

When h is small enough, we have $1 - h^2/9 > 0$, thus from (4.2) and the definition of E^n we have

$$\Big(1 - \frac{h^2}{9}\Big)(\|\delta_x^+ U^n\|^2) \leqslant C, \quad \|U^n\|^2 \leqslant C.$$

For any h small enough, there exists the same positive constant C_1 such that $9/(9-h^2) < C_1$, and then

$$\|\delta_x^+ U^n\| \leqslant \frac{9}{9-h^2} C \leqslant C.$$

From Lemma 4.2 we can prove that $\|U^n\|_\infty \leqslant C$. □

5. Convergence and stability

Lemma 5.1 (see [38]). *Suppose that the discrete function* w_h *satisfies the recurrence formula*

$$w_n - w_{n-1} \leqslant A\tau w_n + B\tau w_n + C_n\tau,$$

where A, B, C $(n = 1, \ldots, N)$ *are nonnegative constants. Then*

$$\|w_n\| \leqslant \Big(w_0 + \tau \sum_{k=1}^{N} C_k\Big) \mathrm{e}^{2(A+B)\tau}$$

where τ *is small, such that* $(A+B)\tau \leqslant \frac{1}{2}(N-1)/N$ *for* $N > 1$.

Lemma 5.2. *Suppose $u(x,t)$ is a smooth enough function, and $\partial^4 u(x,t)/(\partial x^3\partial t)$, $\partial^3 u(x,t)/\partial x^3$, $\partial^5 u(x,t)/(\partial x^4\partial t)$ and $u(x,t)$ are bounded by C, then the truncation error of scheme* (2.13) *is $O(h^2+\tau^2)$, that means*

$$R'_1 + R'_2 - R'_3 + \tfrac{1}{2}R'_4 + O(\tau^2) \leqslant C(h^2+\tau^2).$$

Proof. From (2.4) and (2.5) and noticing that $\varepsilon_1 = \varepsilon_2 = 2h/\sqrt{3}$ we have

$$R_1 = \int_{xx} R_t \leqslant \int_{xx}\Big|\frac{1}{3!}\frac{\partial^4 u(\xi,t)}{\partial x^3\partial t}\Big||(x-x_{j-1})(x-x_j)(x-x_{j+1})| \leqslant Ch^6. \tag{5.1}$$

For R_2, first we have

$$\begin{aligned}
&\int_{x_j}^{x_j+\varepsilon_4}\int_{x_j-\varepsilon_3}^{x_j} R_t(x_b,t)\,\mathrm{d}x_a\,\mathrm{d}x_b \\
&= \frac{1}{6}\int_{x_j}^{x_j+\varepsilon_4}\int_{x_j-\varepsilon_3}^{x_j}\frac{\partial^4 u(\xi_{2b},t)}{\partial x^3\partial t}(x_b-x_{j-1})(x_b-x_j)(x_b-x_{j+1})\,\mathrm{d}x_a\,\mathrm{d}x_b \\
&\leqslant \int_{x_j}^{x_j+\varepsilon_4}\int_{x_j-\varepsilon_3}^{x_j}\Big|\frac{\partial^4 u(\xi_{2b},t)}{\partial x^3\partial t}\Big||(x_b-x_{j-1})(x_b-x_j)(x_b-x_{j+1})|\,\mathrm{d}x_a\,\mathrm{d}x_b \leqslant Ch^5,
\end{aligned} \tag{5.2}$$

where $\xi_{2b}\in(x_{j-1},x_{j+1})$. Similarly we can prove

$$-\int_{x_j}^{x_j+\varepsilon_4}\int_{x_j-\varepsilon_3}^{x_j} R_t(x_a,t)\,\mathrm{d}x_a\,\mathrm{d}x_b \leqslant Ch^5.$$

Then we obtain

$$R_2 = \int_{x_j}^{x_j+\varepsilon_4}\int_{x_j-\varepsilon_3}^{x_j}(R_t(x_b,t)-R_t(x_a,t))\,\mathrm{d}x_a\,\mathrm{d}x_b \leqslant Ch^5.$$

Next, noticing that $R_t(x_j,t)=0$ and the mean value theorem, we have

$$\begin{aligned}
R_3 &= \varepsilon_1(R_t(x_j+\varepsilon_2,t)-R_t(x_j,t)) - \varepsilon_2(R_t(x_j,t)-R_t(x_j-\varepsilon_1,t)) \\
&= \varepsilon_1 R_t(x_j+\varepsilon_2,t) + \varepsilon_2 R_t(x_j-\varepsilon_1,t) \\
&= \frac{\varepsilon_1}{3!}\frac{\partial^4 u(\xi_{32},t)}{\partial x^3\partial t}(x_j+\varepsilon_2-x_{j-1})(x_j+\varepsilon_2-x_i)(x_j+\varepsilon_2-x_{j+1}) \\
&\quad + \frac{\varepsilon_2}{3!}\frac{\partial^4 u(\xi_{31},t)}{\partial x^3\partial t}(x_j-\varepsilon_1-x_{j-1})(x_j-\varepsilon_1-x_j)(x_j-\varepsilon_1-x_{j+1}) \\
&= \frac{\varepsilon^2(\varepsilon-h)(\varepsilon+h)}{3!}\frac{\partial^4 u(\xi_{32},t)}{\partial x^3\partial t} - \frac{\varepsilon^2(\varepsilon-h)(\varepsilon+h)}{3!}\frac{\partial^4 u(\xi_{31},t)}{\partial x^3\partial t} \\
&= \frac{\varepsilon^2(\varepsilon-h)(\varepsilon+h)}{3!}\Big(\frac{\partial^4 u(\xi_{32},t)}{\partial x^3\partial t} - \frac{\partial^4 u(\xi_{31},t)}{\partial x^3\partial t}\Big) \\
&= \frac{\varepsilon^2(\varepsilon-h)(\varepsilon+h)(\xi_{32}-\xi_{31})}{3!}\frac{\partial^5 u(\xi_3,t)}{\partial x^4\partial t} \leqslant Ch^5,
\end{aligned} \tag{5.3}$$

where $\xi_{31},\xi_{32},\xi_3\in(x_{j-1},x_{j+1})$.

For R_4, first we know that $|R(x_a,t)| \leqslant Ch^3$, $|R(x_b,t)| \leqslant Ch^3$,

$$(5.4)\qquad \begin{aligned} |u(x_b,t)| &\leqslant \frac{1}{h^2}|u(x_{j-1},t)(x_b-x_j)(x_b-x_{j+1})| \\ &\quad + \frac{1}{h^2}|u(x_j,t)(x_b-x_{j-1})(x_b-x_{j+1})| \\ &\quad + \frac{1}{h^2}|u(x_{j+1},t)(x_b-x_{j-1})(x_b-x_j)| \\ &\leqslant C(|u(x_{j-1},t)|+|u(x_j,t)|+|u(x_{j+1},t)|) \leqslant C \end{aligned}$$

and similarly $|u(x_a,t)| \leqslant C$. Then we obtain

$$(5.5)\qquad \begin{aligned} R_4 &= \int_{x_j}^{x_j+\varepsilon_4}\int_{x_j-\varepsilon_3}^{x_j}(2u(x_b,t)R(x_b,t)+(R(x_b,t))^2)\,\mathrm{d}x_a\,\mathrm{d}x_b \\ &\quad - \int_{x_j}^{x_j+\varepsilon_4}\int_{x_j-\varepsilon_3}^{x_j}(2u(x_a,t)R(x_a,t)+(R(x_a,t))^2)\,\mathrm{d}x_a\,\mathrm{d}x_b \leqslant Ch^5. \end{aligned}$$

Noticing the integrating factor $\varepsilon^3 = 8\sqrt{3}h^3/9$, then $|R_i'| = |R_i/\varepsilon^3| \leqslant h^2$, $i=1,2,3,4$.

This completes the proof. $\square$

Remark 5.3. Lemma 5.2 gives the truncation error of the numerical scheme. In fact, the error comes from using interpolation polynomials to approximate instead of unknown functions to integrate. The truncation error of the interpolation polynomial determines the truncation error of the numerical scheme.

Theorem 5.4. *Suppose that $u_0 \in H_0^2[x_l,x_r]$, $u(x,t)$ is the exact solution of equations* (1.1)–(1.3) *and U^n is the solution of scheme* (2.13). *Then*

$$\begin{aligned} \|u^n - U^n\| &\leqslant O(\tau^2+h^2), \\ \|u^n - U^n\|_\infty &\leqslant O(\tau^2+h^2). \end{aligned}$$

Proof. Let $u_j^n = u(x_j,t_n)$, $u^n = (u_1^n,u_2^n,\dots,u_{J-1}^n)^\top$, $e_j^n = u_j^n - U_j^n$, $e^n = (e_1^n,e_2^n,\dots,e_{J-1}^n)^\top$. From (2.9) and (2.13), the truncation error is

$$(5.6)\qquad R^n = \Big(\frac{h^2}{9}-1\Big)\delta_x^2\frac{e^{n+1}-e^{n-1}}{2\tau} + \delta_t e^n + \delta_x\bar{e}^n + \varphi(\bar{u}^n) - \varphi(\overline{U}^n),$$

where $\varphi(\bar{u}^n) = \frac{1}{3}(\bar{u}^n\delta_x\bar{u}^n + \delta_x(\bar{u}^n)^2)$.

Computing the inner product of each term of (5.6) with $e^{n-1} + e^{n+1} = 2\bar{e}^n$, similarly to the proof of Theorem 3.2 we have

$$(R^n, e^{n-1} + e^{n+1}) \leqslant \|R^n\|^2 + C(\|e^{n+1}\|^2 + \|e^{n-1}\|^2), \tag{5.7}$$

$$\Big(\delta_x^2 \frac{e^{n+1} - e^{n-1}}{2\tau}, e^{n-1} + e^{n+1}\Big) = -\frac{1}{2\tau}(\|\delta_x^+ e^{n+1}\|^2 - \|\delta_x^+ e^{n-1}\|^2), \tag{5.8}$$

$$(\delta_t e^n, e^{n-1} + e^{n+1}) = \frac{1}{2\tau}(\|e^{n+1}\|^2 - \|e^{n-1}\|^2), \tag{5.9}$$

$$\begin{aligned}
&(\varphi(\bar{u}^n) - \varphi(\overline{U}^n), \bar{e}^n) \\
&\quad = \frac{1}{3}h\sum_{j=1}^{J-1}(\bar{u}_j^n\delta_x\bar{u}_j^n + \delta_x(\bar{u}_j^n)^2 - \overline{U}_j^n\delta_x\overline{U}_j^n - \delta_x(\overline{U}_j^n)^2)\bar{e}_j^n \\
&\quad = \frac{1}{3}h\sum_{j=1}^{J-1}(\bar{u}_j^n\delta_x\bar{u}_j^n - \bar{u}_j^n\delta_x\overline{U}_j^n + \bar{u}_j^n\delta_x\overline{U}_j^n - \overline{U}_j^n\delta_x\overline{U}_j^n)\bar{e}_j^n \\
&\qquad - \frac{1}{3}h\sum_{j=1}^{J-1}(\bar{u}_j^{n\,2} - \bar{u}_j^n\overline{U}_j^n + \bar{u}_j^n\overline{U}_j^n - (\overline{U}_j^n)^2)\delta_x\bar{e}_j^n \\
&\quad = \frac{1}{3}h\sum_{j=1}^{J-1}(\bar{u}_j^n\delta_x\bar{e}_j^n + \bar{e}_j^n\delta_x\overline{U}_j^n)\bar{e}_j^n - \frac{1}{3}h\sum_{j=1}^{J-1}(\bar{u}_j^n\bar{e}_j^n + \bar{e}_j^n\overline{U}_j^n)\delta_x\bar{e}_j^n \\
&\quad \leqslant \frac{1}{3}Ch\sum_{j=1}^{J-1}(|\delta_x\bar{e}_j^n| + |\bar{e}_j^n|)|\bar{e}_j^n| + \frac{1}{3}Ch\sum_{j=1}^{J-1}|\bar{e}_j^n||\delta_x\bar{e}| \\
&\quad \leqslant C(\|\delta_x\bar{e}^n\|^2 + \|\bar{e}^n\|^2) \leqslant C(\|\delta_x e^{n-1}\|^2 + \|\delta_x e^{n+1}\|^2 + \|e^{n-1}\|^2 + \|e^{n+1}\|^2) \\
&\quad \leqslant C(\|\delta_x^+ e^{n-1}\|^2 + \|\delta_x^+ e^{n+1}\|^2 + \|e^{n-1}\|^2 + \|e^{n+1}\|^2).
\end{aligned} \tag{5.10}$$

From (5.7)–(5.10), we get

$$\begin{aligned}
&\Big(1 - \frac{h^2}{9}\Big)(\|\delta_x^+ e^{n+1}\|^2 - \|\delta_x^+ e^{n-1}\|^2) + (\|e^{n+1}\|^2 - \|e^{n-1}\|^2) \\
&\qquad \leqslant 2\tau\|R^n\|^2 + C\tau(\|\delta_x^+ e^{n-1}\|^2 + \|\delta_x^+ e^{n+1}\|^2 + \|e^{n-1}\|^2 + \|e^{n+1}\|^2).
\end{aligned} \tag{5.11}$$

For any h small enough, there exists the same positive constant K (independent of h) such that $1 \leqslant 9/(9 - h^2) < K$. That means that $K(1 - h^2/9) > 1$ holds. Then for the RHS of inequality (5.11), we have

$$\begin{aligned}
&2\tau\|R^n\|^2 + C\tau(\|\delta_x^+ e^{n-1}\|^2 + \|\delta_x^+ e^{n+1}\|^2 + \|e^{n-1}\|^2 + \|e^{n+1}\|^2) \\
&\quad \leqslant 2\tau\|R^n\|^2 + C\tau K(\|e^{n-1}\|^2 + \|e^{n+1}\|^2) \\
&\qquad + C\tau K\Big(1 - \frac{h^2}{9}\Big)(\|\delta_x^+ e^{n-1}\|^2 + \|\delta_x^+ e^{n+1}\|^2) \\
&\quad \leqslant C\tau\|R^n\|^2 + C\tau K(\|e^{n-1}\|^2 + 2\|e^n\|^2 + \|e^{n+1}\|^2) \\
&\qquad + C\tau K\Big(1 - \frac{h^2}{9}\Big)(\|\delta_x^+ e^{n-1}\|^2 + 2\|\delta_x^+ e^n\|^2 + \|\delta_x^+ e^{n+1}\|^2).
\end{aligned} \tag{5.12}$$

Thus, inequality (5.11) can be written as

$$\begin{aligned}(5.13)\qquad &\Big(1-\frac{h^2}{9}\Big)(\|\delta_x^+ e^{n+1}\|^2-\|\delta_x^+ e^{n-1}\|^2)+(\|e^{n+1}\|^2-\|e^{n-1}\|^2)\\ &\quad\leqslant C\tau\|R^n\|^2+C\tau K(\|e^{n-1}\|^2+2\|e^n\|^2+\|e^{n+1}\|^2)\\ &\qquad+C\tau K\Big(1-\frac{h^2}{9}\Big)(\|\delta_x^+ e^{n-1}\|^2+2\|\delta_x^+ e^n\|^2+\|\delta_x^+ e^{n+1}\|^2).\end{aligned}$$

Let

$$\Phi_n=\Big(1-\frac{h^2}{9}\Big)(\|\delta_x^+ e^n\|^2+\|\delta_x^+ e_j^{n-1}\|^2)+(\|e^n\|^2+\|e^{n-1}\|^2).$$

Then inequality (5.13) turns to be

$$(5.14)\qquad \Phi_{n+1}-\Phi_n\leqslant C\tau\|R^n\|^2+C\tau(\Phi_{n+1}+\Phi_n).$$

From (2.14)–(2.15), we get $\Phi_1\leqslant O(\tau^2+h^2)^2$. Then from Lemma 5.1 we have

$$\Phi_n\leqslant\bigg(\Phi_1+C\tau\sum_{k=1}^{n-1}\|R^k\|^2\bigg)\mathrm{e}^{2C\tau}.$$

It is easy to prove $\Phi_n\leqslant O(\tau^2+h^2)^2$ by mathematical induction. Then similarly to the proof of Theorem 4.4, we obtain

$$\|e^n\|\leqslant O(\tau^2+h^2),\quad \|\delta_x^+ e^n\|\leqslant O(\tau^2+h^2).$$

From Lemma 4.2 we get

$$\|e^n\|_\infty\leqslant O(\tau^2+h^2).$$

□

Theorem 5.5. *Suppose* $u_0\in H_0^1[x_l,x_r]$ *and* $u(x,t)$ *is the solution of* (1.1)–(1.3), *the solution of scheme* (2.13) *is unconditionally stable.*

Proof. The proof of this theorem can follow the proof of Theorem 5.4 and is omitted here to save space. □

6. Numerical experiments

In this section, we give some numerical experiments to support the convergence order and conservation law of the scheme proposed in this paper.

6.1. Propagation of a single solitary wave.

6.1.1. Problem 1. This problem is used to test the order of accuracy and the energy conservation property of our scheme.

Consider initial-boundary value problem (1.1)–(1.3) with the exact solution

$$u(x,t) = \operatorname{sech}^2\Big(\frac{x}{4} - \frac{t}{3}\Big).$$

The initial condition can be obtained from the exact solution.

	$h=\tau=0.2$	$h=\tau=0.1$	$h=\tau=0.05$
$T=10$			
$\|e^n\|_\infty$	1.1623×10^{-2}	4.1338×10^{-3}	1.0384×10^{-3}
c-order	–	1.9728	1.9931
$T=20$			
$\|e^n\|_\infty$	2.5601×10^{-2}	6.5426×10^{-3}	1.6445×10^{-3}
c-order	–	1.9683	1.9922
$T=30$			
$\|e^n\|_\infty$	3.3227×10^{-2}	8.5025×10^{-3}	2.1396×10^{-3}
c-order	–	1.9659	1.9910
$T=40$			
$\|e^n\|_\infty$	1.0976×10^{-2}	7.4724×10^{-3}	4.6938×10^{-3}
c-order	–	1.9631	1.9908

Table 1. Errors and convergence orders at different final times.

Table 1 shows the errors and computational orders of the proposed method with various values of h and τ at different final times with the range $x \in [-40, 100]$. The numerical results of Table 1 confirm that the scheme has the second order accuracy both in temporal and spatial components.

T	$\|E(T)-E(0)\|$	$\|e^n\|_\infty$
5	6.5314×10^{-13}	6.0704×10^{-4}
10	2.3127×10^{-12}	1.7842×10^{-3}
15	8.3124×10^{-12}	1.9463×10^{-3}
20	8.2559×10^{-12}	1.9946×10^{-3}

Table 2. Discrete energy error and $\|e^n\|_\infty$.

Table 2 shows the energy error and L_∞ error for various values of T with $x \in [-40, 100]$ and $h = \tau = 0.05$. From Table 2 we can see that the energy is exactly preserved by our methods, because the energy error is close to the machine error at the level of 10^{-13}. Besides, the numerical error increases slowly with time.

T	Our	[31]Sch1	[31]C-N	[7]	[36]	[4]
$t = 0.2$	1.2332e − 5	0.00056	0.00070	0.00190	1.575e − 5	0.00053
$t = 0.4$	3.0134e − 5	0.00085	0.03331	0.00283	2.625e − 5	0.00113
$t = 0.6$	5.7564e − 5	0.00112	0.06337	0.00403	6.728e − 5	0.00175
$t = 0.8$	8.6343e − 5	0.00141	0.08433	0.00481	1.925e − 4	0.00237
$t = 1.0$	1.1352e − 4	0.00169	0.11287	0.00563	4.789e − 4	0.00299

Table 3. Comparison of numerical errors $\|e^n\|_2$ between our method and different literatures method with $\tau = 0.1$, $h = 0.05$ and different t.

Table 3 shows a comparison of the $\|e^n\|_2$ error of our method and different literature methods with $\tau = 0.1$, $h = 0.05$. Here [31]Sch1, [31]C-N and the method in [36] are different 2nd order finite difference methods. The method in [7] is one of finite element methods within Galerkin methods. The method in [4] is one of multisymplectic numerical methods. Table 3 comes from literature [31], Table 4 except of our data. From Table 3, we can see that our scheme has the least error $\|e^n\|_2$ under the condition of the same step size. That means that the variable limit integral method performs better than the finite difference method and finite element method under same order.

6.1.2. Problem 2. Consider the initial-boundary value problem of the RLW equation (1.1)–(1.3) with the exact solution

$$u(x,t) = 3c\,\mathrm{sech}^2(p(x - vt - x_0)), \quad \text{where } p = \sqrt{\frac{c}{4(c+1)}}, \quad v = 1 + c.$$

According to [13], the RLW equation (1.1) possesses three polynomial invariants related to mass, momentum, and energy, which are given as

$$I_1 = \int_{x_l}^{x_r} u\,\mathrm{d}x, \quad I_2 = \int_{x_l}^{x_r} (u^2 + u_x^2)\,\mathrm{d}x, \quad I_3 = \int_{x_l}^{x_r} (u^3 + 3u^2)\,\mathrm{d}x.$$

These quantities are used to measure the accuracy of the proposed method and we consider the three conservation laws

$$I_1 = h\sum_{i=1}^{J-1} U_j^n, \quad I_2 = h\sum_{j=1}^{J-1} ((U_j^n)^2 + (\delta_x U_j^n)^2), \quad I_3 = h\sum_{j=1}^{J-1} ((U_j^n)^3 + 3(U_j^n)^2).$$

Relative changes in invariants are defined as $\widehat{I}_i = |I_i^{\text{final}} - I_i^{\text{initial}}|/I_i^{\text{final}}$, $i = 1, 2, 3$.

In Table 4 we show the relative changes of the three invariants for Problem 2 with amplitude 0.3 and the analytical values of the invariants are $I_1 = 3.9799266$, $I_2 = 0.8104624$ and $I_3 = 2.5790074$. We choose $c = 0.1$, $x_0 = 0$, $h = \tau = 0.1$ with the range $[-40, 60]$ and the simulations are run to the time $t = 20$.

Table 5 gives the relative changes of the three invariants for Problem 2 with the amplitude 0.09. In this second numerical experiment we choose $c = 0.03$ and the analytical values of the invariants are $I_1 = 2.1070468$, $I_2 = 0.1273012$ and $I_3 = 0.3888046$.

For all simulations, the absolute maximum relative changes of I_1, I_2 and I_3 are found to be less than 6.6911×10^{-6}, 3.6215×10^{-9}, 3.2931×10^{-8} in Table 4 and 6.4639×10^{-4}, 2.9905×10^{-6}, 4.04144×10^{-6} in Table 5, respectively. That means the three invariants computed by our scheme are satisfactorily constant.

T	$\widehat{I_1}$	$\widehat{I_2}$	$\widehat{I_3}$
4	6.6911×10^{-6}	1.1997×10^{-10}	5.8274×10^{-9}
8	4.5199×10^{-6}	8.9538×10^{-10}	1.6732×10^{-8}
12	3.2595×10^{-6}	1.8162×10^{-9}	2.4679×10^{-8}
16	6.6638×10^{-6}	2.7453×10^{-9}	2.9758×10^{-8}
20	6.24651×10^{-6}	3.6215×10^{-9}	3.2931×10^{-8}

Table 4. The relative changes of the three invariants $I_1 = 3.9799266$, $I_2 = 0.8104624$ and $I_3 = 2.5790074$ with the solitary wave amplitude $3c = 0.3$ and different final time T.

T	$\widehat{I_1}$	$\widehat{I_2}$	$\widehat{I_3}$
4	6.4639×10^{-4}	2.9905×10^{-6}	4.04144×10^{-6}
8	4.2921×10^{-4}	1.6565×10^{-6}	2.13497×10^{-6}
12	2.4977×10^{-4}	9.5497×10^{-7}	1.18498×10^{-6}
16	1.7969×10^{-6}	2.3041×10^{-9}	1.89877×10^{-8}
20	3.8335×10^{-4}	1.2329×10^{-6}	2.96423×10^{-6}

Table 5. The relative changes of the three invariants $I_1 = 2.1070468$, $I_2 = 0.1273012$ and $I_3 = 0.3888046$ with the solitary wave amplitude $3c = 0.09$ and different final time T.

6.2. Interaction of two solitary waves. In this part, we simulate the phenomenon of the two solitary wave collision.

Consider the initial-boundary value problem of RLW equation (1.1)–(1.3) with the initial condition

$$u(x,0) = u_1 + u_2,$$

where

$$u_i = 3c_i \operatorname{sech}^2(p_i(x - x_i)), \quad c_i = \frac{4p_i^2}{1 - 4p_i^2}, \quad i = 1, 2.$$

For this part, we have chosen $p_1 = 0.4$, $p_2 = 0.3$, $x_1 = 15$, $x_2 = 35$ and $T = 30$ with the range $x \in [0, 120]$. Figure 1 plots the interactions of two solitary waves at different time levels.

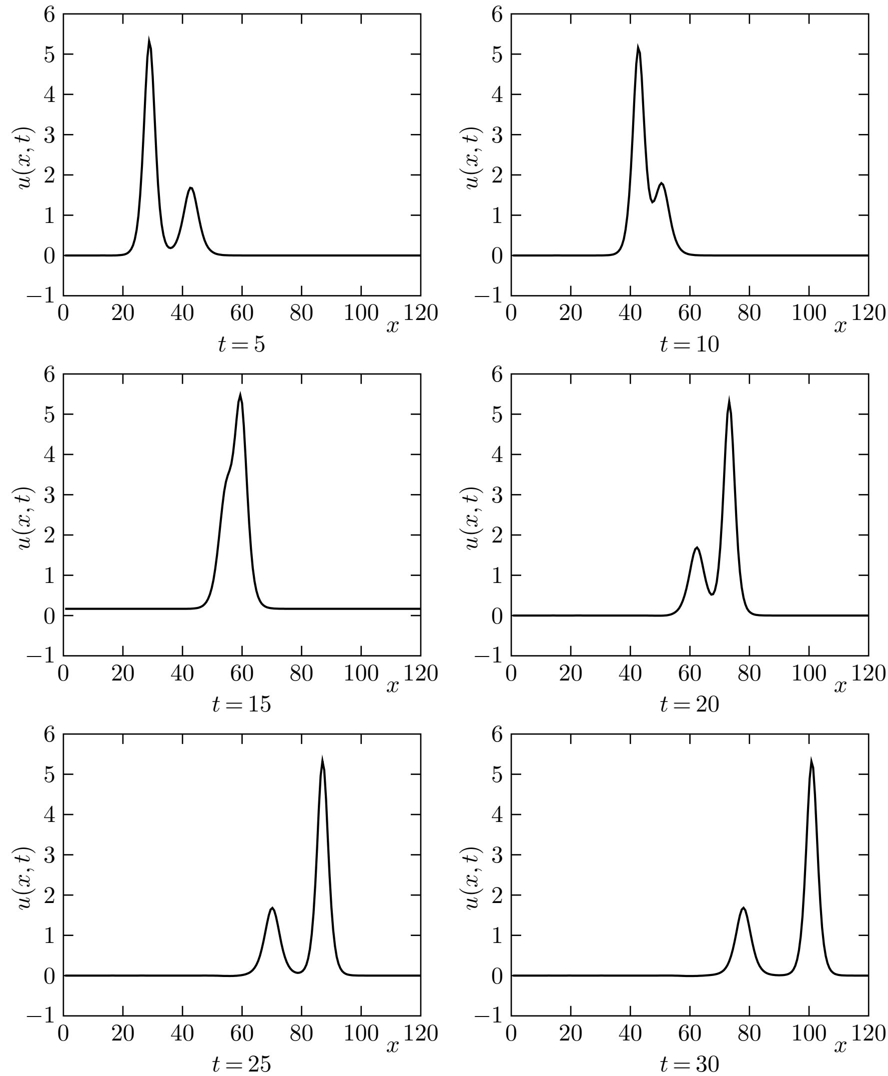

Figure 1. Interaction of two solitary waves at times $t = 5$, $t = 10$, $t = 15$, $t = 20$, $t = 25$ and $t = 30$.

From Figure 1 we can see that the higher amplitude solitary wave passes through the smaller wave, and the amplitude and shape of these two waves do not change significantly before and after the collision. That means our scheme has the capacity to simulate the two waves collision.

7. Conclusion remarks

In this paper, we have presented and analyzed a conservative scheme for the RLW equation. We use the integral method with variational limit to discretize space and the finite difference method to discretize time. The energy conservation of the scheme on discrete levels is discussed and the existence of numerical solutions is shown. Furthermore, we proved that our scheme is stable and $O(h^2+\tau^2)$ convergent. Numerical experiments show that the scheme is of the second order both in space and time. Besides, the error of energy is calculated and by it, we verify the energy conservation property of our scheme. Three invariants computed are satisfactorily constant in our scheme. Two waves test shows that our scheme has the capacity to simulate a collision.

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