AN INSTANTANEOUS SEMI-LAGRANGIAN APPROACH FOR BOUNDARY CONTROL OF A MELTING PROBLEM

Youness Mezzan, Moulay Hicham Tber, Marrakech

Received February 6, 2020. Published online April 1, 2021.

Abstract. In this paper, a sub-optimal boundary control strategy for a free boundary problem is investigated. The model is described by a non-smooth convection-diffusion equation. The control problem is addressed by an instantaneous strategy based on the characteristics method. The resulting time independent control problems are formulated as function space optimization problems with complementarity constraints. At each time step, the existence of an optimal solution is proved and first-order optimality conditions with regular Lagrange multipliers are derived for a penalized-regularized version. The performance of the overall approach is illustrated by numerical examples.

Keywords: free boundary problem; sub-optimal boundary control; characteristics method; complementarity constraint; penalization-regularization

MSC 2020: 35R35, 65M25, 49K20, 90C33

1. Introduction

Heat transfer processes involving phase change are relevant to many engineering disciplines including casting of metals, thermal storage, power systems, microelectronics, etc., see [10]. Enhancing the thermal performance of systems using such processes requires a proper control of the temperature profile and the associated phase change interface.

Our motivation in this paper is to design an optimization strategy for a melting process that might be affected by a convection in the liquid phase. We focus on two-phase materials with sharp interface and we adopt a single domain approach, where the Stefan condition is automatically satisfied across the free boundary. More precisely, we consider a source-based method in which the total enthalpy is split into a specific heat and a latent heat acting as a source term in the energy equation [5]. Our goal is to control the temperature profile using the heat flux on a part of the

DOI: 10.21136/AM.2021.0028-20

boundary. This task is, nevertheless, quite challenging even for simple geometries. In fact, the liquid-solid free boundary changes sharply with respect to the temperature. Furthermore, from a numerical point of view, solutions may exhibit non-physical oscillations for convection dominated flows. Finally, the related optimal control problem is very demanding in terms of computational time and storage.

Optimal control problems in the context of Stefan-like models have attracted a lot of attention since the eighties of the last century. We refer, in particular, to the monograph [13] and the references therein. However, most used models were generally based on simplified assumptions on the free boundary, and therefore describe roughly the phase-change process. Subsequent studies [16], [17], [7], [6] have considered two-phase Stefan problems with a focus on numerical aspects. Recently, some existence and differentiability results are established in [1], [2], [3] for one-dimensional problems.

To accommodate the problem, our strategy here exploits a semi-Lagrangian scheme [20] in the context of an instantaneous control approach [9], [8]. The time derivative and the convection terms are combined as a directional derivative along the characteristics. We show that the time-discrete state equation satisfies a maximum principle. Then, at each time step we cast the time-discrete optimal control problem—which only depends on the state at the previous time—as an optimization problem with a complemantarity constraint between the temperature and solid fraction. However, due to the structure of the feasible set, standard numerical algorithms cannot be applied directly to solve such optimization problems (see for instance [14]). Here, we propose a regularization-penalization technique, where we first regularize the constraint on the temperature variable, then we incorporate the related complementarity into the objective functional via an ℓ_1-penalty approach [15]. For the resulting regularized-penalized problems we show an existence and consistency result and further we derive first-order necessary optimality conditions that enjoy regular Lagrange multipliers. The overall approach leads, naturally, to sub-optimal solutions. Nevertheless, a good performance is achieved in the numerical experiments.

2. State equation

Mathematical model. We consider the melting of a finite slab of a pure substance. The model is described by the non-dimensional source-based Stefan equation

$$\frac{\partial y}{\partial t} + \vec{v} \cdot \nabla y - \nabla \cdot (\kappa \nabla y) = \frac{\partial}{\partial t}\xi + \vec{v} \cdot \nabla \xi \quad \text{in } \Omega \times (t_0, t_f),$$

where $\kappa = \kappa(x,t)$ is the thermal conductivity and $\vec{v} = \vec{v}(x,t)$ is a convection velocity. The solid fraction $\xi = \xi(x,t)$ and temperature distribution $y = y(x,t)$ are related through the relation

$$\xi \in \mathcal{H}(y) := \begin{cases} 0 & \text{if } y > 0, \\ [0,1] & \text{if } y = 0, \\ 1 & \text{if } y < 0. \end{cases}$$

Here the phase-change processes are assumed to be isothermal. The model domain Ω is an open bounded subdomain of $\mathbb{R}^d$ $(d = 1, 2)$ with a smooth boundary Γ corresponding to both solid and liquid regions (see Fig. 1). On Γ we distinguish three parts: the system is insulated on Γ_N, a fixed temperature $y_D = 0$ is maintained on Γ_D and a non-negative heat flux control $u = u(x,t)$ is applied on Γ_C. The normal is denoted by ν. The substance is initially at the melting/freezing point

$$y(x, t_0) = 0, \quad \xi(x, t_0) = \xi_0(x) \in [0,1] \quad \text{for } x \in \Omega.$$

The complete model equation reads

$$(\mathcal{M}^t) \qquad \begin{cases} \dfrac{\partial y}{\partial t} + \vec{v} \cdot \nabla y - \nabla \cdot (\kappa \nabla y) = \dfrac{\partial}{\partial t}\xi + \vec{v} \cdot \nabla \xi & \text{in } \Omega \times (t_0, t_f), \\ \xi \in \mathcal{H}(y) & \text{in } \Omega \times (t_0, t_f), \\ \dfrac{\partial y}{\partial \nu} = 0 & \text{in } \Gamma_N \times (t_0, t_f), \\ \dfrac{\partial y}{\partial \nu} = u & \text{in } \Gamma_C \times (t_0, t_f), \\ y = 0 & \text{in } \Gamma_D \times (t_0, t_f), \\ y(t_0) = 0, \xi(t_0) = \xi_0 & \text{in } \Omega. \end{cases}$$

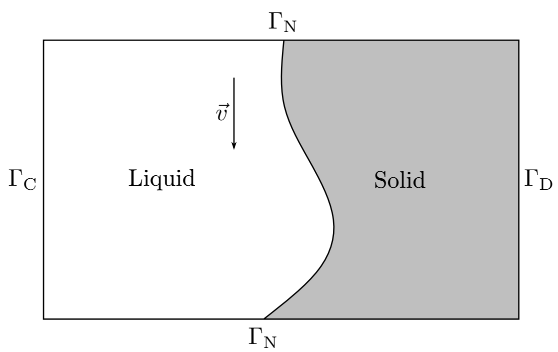

Figure 1. Problem configuration.

Time discretization. Due to the hyperbolic character of the state equation, the numerical solutions may exhibit undesired oscillations for dominated convection terms. One approach to deal with this issue consists in writing $\partial\varphi/\partial t + \vec{v} \cdot \nabla\varphi$

as $\mathrm{D}\varphi/\mathrm{D}t$ the material derivative of a given function φ in the direction of $\vec{v}$. The corresponding characteristic curves are defined by

$$\begin{cases} \dfrac{\mathrm{d}X(x,t;s)}{\mathrm{d}s} = \vec{v}(x,t), \\ X(x,t;t) = x \end{cases}$$

with $X(x,t;s)$ being the position of a particle at time s, which was at x at time t.

Now for a given uniform time step size $\tau = (t_f - t_0)/N > 0$ we can get an approximate value of X at time $t^{n-1} = t_0 + (n-1)\tau$ by

$$X^n(x) := X(x, t^n; t^{n-1}) = x - \tau\vec{v}(x, t^n), \quad n = 1, \dots, N.$$

Using a fully-implicit scheme, we obtain the semi-discrete form of $(\mathcal{M}^t)$

$$(\mathcal{M}^\tau) \qquad \begin{cases} y^n - \tau\nabla\cdot(\kappa^n\nabla y^n) = \xi^n + \overline{y}^{n-1} - \overline{\xi}^{n-1} & \text{in } \Omega, \\ \xi^n \in \mathcal{H}(y^n) & \text{in } \Omega, \\ \dfrac{\partial y^n}{\partial\nu} = 0 & \text{on } \Gamma_N, \\ \dfrac{\partial y^n}{\partial\nu} = u^n & \text{on } \Gamma_C, \\ y^n = 0 & \text{on } \Gamma_D, \\ \xi^0 = \xi_0, \quad y^0 = 0 & \text{in } \Omega, \end{cases}$$

where $n = 1, \dots, N$, $\varphi^n(\cdot) := \varphi(\cdot, t^n)$ and $\overline{\varphi}^{n-1} := \varphi^{n-1} \circ X^n$. To avoid technical difficulties, it is assumed that X^n maps Ω to itself. Formulation $(\mathcal{M}^\tau)$ has the advantage of not being restricted by a CFL condition and large time steps may be used [19].

Variational formulation. In the following, standard notations for Lebesgue and Sobolev spaces are employed (see e.g. [12], Chap. 5). The $L^2(\Omega)$ norm for either vector-valued or real-valued functions is denoted by $\|\cdot\|$. The $L^2(\Gamma_C)$ norm is specified by $\|\cdot\|_{\Gamma_C}$. To define a variational formulation for the semi-discrete problem we introduce the space

$$\mathcal{V} := \{\varphi \in H^1(\Omega);\ \varphi = 0 \text{ on } \Gamma_D\}$$

endowed with the $H^1(\Omega)$ norm $\|\cdot\|_{H^1(\Omega)}$.

At a specific time step t^n the variational formulation of the semi-discrete state equation consists in finding $(y^n, \xi^n) \in \mathcal{V} \times L^2(\Omega)$ such that

$$(\mathcal{WF}^n) \qquad \begin{cases} Ay^n = \xi^n + Bu^n + \overline{y}^{n-1} - \overline{\xi}^{n-1} & \text{in } \mathcal{V}', \\ \xi^n \in \mathcal{H}(y^n) & \text{a.e. in } \Omega \end{cases}$$

for given $u^n \in L^2(\Gamma_C)$, $\xi^{n-1} \in L^2(\Omega)$ and $y^{n-1} \in \mathcal{V}$. Here $B\colon L^2(\Gamma_C) \mapsto \mathcal{V}'$ and $A\colon \mathcal{V} \mapsto \mathcal{V}'$ standing for the linear bounded operator defined by

$$\begin{aligned} \langle Bv, \varphi\rangle &:= \tau(v, \gamma_0\varphi)_{\Gamma_C} \quad \forall\,(v,\varphi) \in L^2(\Gamma_C) \times \mathcal{V}, \\ \langle A\psi, \varphi\rangle &:= (\psi, \varphi) + \tau(\kappa^n \nabla\psi, \nabla\varphi) \quad \forall\,(\psi,\varphi) \in \mathcal{V} \times \mathcal{V}, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the pairing between $\mathcal{V}$ and its dual $\mathcal{V}'$. The inner products in $L^2(\Omega)$ and $L^2(\Gamma_C)$ are indicated by $(\cdot,\cdot)$ and $(\cdot,\cdot)_{\Gamma_C}$, respectively. Moreover, γ_0 is the trace operator in $H^1(\Omega)$ and $\kappa^n \in L^\infty(\Omega)$ is such that the operator A is uniformly coercive with a constant $\underline{\kappa}$.

Regarding the solvability of $(\mathcal{WF}^n)$ we state the following theorem whose proof is deferred to Appendix A.

Theorem 2.1. *Let $u^n \in L^2(\Gamma_C)$, $y^{n-1} \in \mathcal{V}$ and $\xi^{n-1} \in L^2(\Omega)$ such that $u^n \geqslant 0$ a.e. in Γ_C, $y^{n-1} \geqslant 0$ a.e. in Ω and $0 \leqslant \xi^{n-1} \leqslant 1$ a.e. in Ω. Problem $(\mathcal{WF}^n)$ has one and only one solution $(y^n, \xi^n) \in \mathcal{V} \times L^2(\Omega)$ that is given by the solution of*

$$(\mathcal{CS}^n) \qquad \begin{cases} Ay^n = \xi^n + Bu^n + \overline{y}^{n-1} - \overline{\xi}^{n-1} & \text{in } \mathcal{V}', \\ y^n \geqslant 0, \quad \xi^n \geqslant 0 & \text{a.e. in } \Omega, \\ (y^n, \xi^n) = 0. \end{cases}$$

3. Sub-optimal control problem

In the following we aim to steer the system to a desired configuration, by acting on the heat flux u at the boundary Γ_C. We adopt an instantaneous optimal control concept: at each time step t^n, given the previous temperature and solid fraction profiles y^{n-1} and ξ^{n-1}, we solve a time-independent optimal control problem. Regarding the previous theorem, we consider the following PDE-constrained optimization problems

$$(\mathcal{O}^n) \qquad \begin{cases} \text{min} \quad J(y^n, \xi^n, u^n) = \dfrac{1}{2}\|y^n - y_d^n\|^2_{H^1(\Omega)} + \dfrac{1}{2}\|\xi^n - \xi_d^n\|^2 + \dfrac{\nu}{2}\|u^n\|^2_{\Gamma_C}, \\ \text{over} \quad (y^n, \xi^n, u^n) \in \mathcal{V} \times L^2(\Omega) \times L^2(\Gamma_C), \\ \text{s.t.} \quad Ay^n = \xi^n + Bu^n + \overline{y}^{n-1} - \overline{\xi}^{n-1} \quad \text{in } \mathcal{V}', \\ \qquad\quad y^n \geqslant 0 \qquad \text{a.e. in } \Omega, \\ \qquad\quad \xi^n \geqslant 0 \qquad \text{a.e. in } \Omega, \\ \qquad\quad u^n \geqslant 0 \qquad \text{a.e. on } \Gamma_C, \\ \qquad\quad (\xi^n, y^n) = 0, \end{cases}$$

where $y_d^n \in H^1(\Omega)$ and $\xi_d^n \in L^2(\Omega)$ correspond to a desired state at time t^n and ν is a regularization parameter.

The next lemma serves as a tool to establish some results of this paper. Its proof is straightforward.

Lemma 3.1. *Let $(y_k, \xi_k, u_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{V} \times L^2(\Omega) \times L^2(\Gamma_C)$ such that $(\xi_k, u_k)_{k\in\mathbb{N}}$ is bounded in $L^2(\Omega) \times L^2(\Gamma_C)$ and*

$$Ay_k = Bu_k + \xi_k + \overline{y}^{n-1} - \overline{\xi}^{n-1} \quad \text{in } \mathcal{V}', \tag{3.1}$$

$$y_k \geqslant 0, \quad \xi_k \geqslant 0 \quad \text{a.e. in } \Omega, \tag{3.2}$$

$$u_k \geqslant 0 \quad \text{a.e. on } \Gamma_C. \tag{3.3}$$

Then there exists a sub-sequence still denoted by $(y_k, u_k, \xi_k)_{k\in\mathbb{N}}$ such that

$$u_k \rightharpoonup u \quad \text{in } L^2(\Gamma_C), \tag{3.4}$$

$$\xi_k \rightharpoonup \xi \quad \text{in } L^2(\Omega), \tag{3.5}$$

$$y_k \to y \quad \text{in } L^2(\Omega), \tag{3.6}$$

$$y_k \rightharpoonup y \quad \text{in } \mathcal{V} \tag{3.7}$$

with (y, ξ, u) being an element of $\mathcal{V} \times L^2(\Omega) \times L^2(\Gamma_C)$ satisfying

$$Ay = Bu + \xi + \overline{y}^{n-1} - \overline{\xi}^{n-1} \quad \text{in } \mathcal{V}', \tag{3.8}$$

$$y \geqslant 0, \quad \xi \geqslant 0 \quad \text{a.e. in } \Omega, \tag{3.9}$$

$$u \geqslant 0 \quad \text{a.e. on } \Gamma_C. \tag{3.10}$$

Further,

$$\lim_{k\to\infty} (\xi_k, y_k) = (\xi, y). \tag{3.11}$$

In particular, if $(\xi_k, y_k) = 0$, then $(\xi, y) = 0$.

Theorem 3.1. *Problem $(\mathcal{O}^n)$ has at least one solution.*

P r o o f. Let $(y_k^n, \xi_k^n, u_k^n)_{k\in\mathbb{N}} \in \mathcal{V} \times L^2(\Omega) \times L^2(\Gamma_C)$ be a minimizing sequence of J over the feasible set of $(\mathcal{O}^n)$. Then $(\xi_k^n, u_k^n)_{k\in\mathbb{N}}$ is bounded in $L^2(\Omega) \times L^2(\Gamma_C)$. From Lemma 3.1 there exists a feasible element (y^n, ξ^n, u^n) such that up to a sub-sequence, $(y_k^n, \xi_k^n, u_k^n)_{k\in\mathbb{N}}$ converges weakly to (y^n, ξ^n, u^n) in $\mathcal{V} \times L^2(\Omega) \times L^2(\Gamma_C)$. It is immediate to verify that J is weakly lower semi-continuous, which proves that (y^n, ξ^n, u^n) is a solution of $(\mathcal{O}^n)$. □

4. Penalized-regularized optimal control problem

In this section we propose a function space approach to solve the problem $(\mathcal{O}^n)$. Inspired by [15], we process a series of sub-problems $(\mathcal{O}^n_\gamma)_{\gamma>0}$ defined by

$$
(\mathcal{O}^n_\gamma) \qquad \begin{cases} \min\ J_\gamma(y^n,\xi^n,u^n) := J(y^n,\xi^n,u^n) + \gamma(\xi^n, y^n+\varepsilon_\gamma\xi^n), \\ \text{over } (y^n,\xi^n,u^n)\in\mathcal{V}\times L^2(\Omega)\times L^2(\Gamma_C), \\ \text{s.t.}\quad Ay^n = Bu^n + \xi^n + \overline{y}^{n-1} - \overline{\xi}^{n-1} \quad \text{in } \mathcal{V}', \\ \qquad y^n + \varepsilon_\gamma\xi^n \geqslant 0 \qquad \text{a.e. in } \Omega, \\ \qquad \xi^n \geqslant 0 \qquad \text{a.e. in } \Omega, \\ \qquad u^n \geqslant 0 \qquad \text{a.e. on } \Gamma_C, \end{cases}
$$

where γ and ε_γ are positive parameters such that $\gamma\to\infty$ and $\varepsilon_\gamma\to 0$. More precisely, we assume that

$$\varepsilon_\gamma\gamma \underset{\gamma\to\infty}{\to} 0.$$

The complementarity constraint will be increasingly satisfied by letting $\gamma\to\infty$, which provides a path-following method for the solution of the original control problems $(\mathcal{O}^n)$. Further we will show that Lagrange multipliers for $(\mathcal{O}^n_\gamma)$ are regular functions, so that using, for instance, a conforming finite elements discretization in numerical experiments is justified.

Here and in the following, C is a generic constant not depending on γ.

Solvability and consistency of $(\mathcal{O}^n_\gamma)_{\gamma>0}$.

Theorem 4.1. *For every fixed $\gamma>0$ the penalized-regularized problem $(\mathcal{O}^n_\gamma)$ has at least one solution $(y^n_\gamma,\xi^n_\gamma,u^n_\gamma)$ in $\mathcal{V}\times L^2(\Omega)\times L^2(\Gamma_C)$. Furthermore, there exist (y^n_*,ξ^n_*,u^n_*) in $\mathcal{V}\times L^2(\Omega)\times L^2(\Gamma_C)$ and a sub-sequence $(y^n_\gamma,\xi^n_\gamma,u^n_\gamma)_{\gamma>0}$ such that*

$$u^n_\gamma \rightharpoonup u_* \quad \text{in } L^2(\Gamma_C), \tag{4.1}$$

$$\xi^n_\gamma \rightharpoonup \xi_* \quad \text{in } L^2(\Omega), \tag{4.2}$$

$$y^n_\gamma \to y^n_* \quad \text{in } L^2(\Omega), \tag{4.3}$$

$$y^n_\gamma \rightharpoonup y^n_* \quad \text{in } \mathcal{V}, \tag{4.4}$$

and (y^n_*,ξ^n_*,u^n_*) is a solution of $(\mathcal{O}^n)$.

Proof. The existence of a solution $(y^n_\gamma,\xi^n_\gamma,u^n_\gamma)$ to $(\mathcal{O}^n_\gamma)$ follows from Lemma 3.1 and J_γ weak lower semi-continuity applied to a minimizing sequence. We recall that

$$
\begin{aligned}
J_\gamma(y^n_\gamma,\xi^n_\gamma,u^n_\gamma) &:= J(y^n_\gamma,\xi^n_\gamma,u^n_\gamma) + \gamma(\xi^n_\gamma, y^n_\gamma+\varepsilon_\gamma\xi^n_\gamma),\\
&= J(y^n_\gamma,\xi^n_\gamma,u^n_\gamma) + \gamma(\xi^n_\gamma,y^n_\gamma) + \gamma\varepsilon_\gamma\|\xi^n_\gamma\|^2.
\end{aligned}
$$

On the other hand,

$$J_\gamma(y_\gamma^n, \xi_\gamma^n, u_\gamma^n) \leqslant J_\gamma(\tilde{y}, \tilde{\xi}, \tilde{u}) \leqslant J(\tilde{y}, \tilde{\xi}, \tilde{u}) + \gamma\varepsilon_\gamma \|\tilde{\xi}\|^2 \leqslant J(\tilde{y}, \tilde{\xi}, \tilde{u}) + C\|\tilde{\xi}\|^2 \tag{4.5}$$

for all $(\tilde{y}, \tilde{\xi}, \tilde{u})$ in $\mathcal{F}^n$. Notice that $\mathcal{F}^n \subseteq \mathcal{F}_\gamma^n$ for all $\gamma > 0$ with $\mathcal{F}^n$ and $\mathcal{F}_\gamma^n$ being the feasible sets of $(\mathcal{O}^n)$ and $(\mathcal{O}_\gamma^n)$, respectively.

Therefore, there exists a constant C not depending on γ such that

$$\|\xi_\gamma^n\| \leqslant C, \quad \|u_\gamma^n\| \leqslant C, \quad 0 \leqslant (\xi_\gamma^n, y_\gamma^n + \varepsilon_\gamma \xi_\gamma^n) \leqslant \frac{C}{\gamma} \quad \forall\, \gamma > 0. \tag{4.6}$$

Then, by Lemma 3.1, there exist sub-sequences still denoted by $(y_\gamma, \xi_\gamma, u_\gamma) \in \mathcal{F}_\gamma$ and (y_*^n, ξ_*^n, u_*^n) in $\mathcal{V} \times L^2(\Omega) \times L^2(\Gamma_C)$ such that (4.1)–(4.4) hold and

$$\begin{aligned}
&Ay_*^n = Bu_*^n + \xi_*^n + \overline{y}^{n-1} - \overline{\xi}^{n-1} && \text{in } \mathcal{V}',\\
&y_*^n \geqslant 0 && \text{a.e. in } \Omega,\\
&\xi_*^n \geqslant 0 && \text{a.e. in } \Omega,\\
&u_*^n \geqslant 0 && \text{a.e. on } \Gamma_C.
\end{aligned}$$

From (4.2), (4.3), (4.6) and $\lim\limits_{\gamma\to\infty} \varepsilon_\gamma = 0$ we have

$$\lim_{\gamma\to\infty} (\xi_\gamma^n, y_\gamma^n + \varepsilon_\gamma \xi_\gamma^n) = \lim_{\gamma\to\infty} \varepsilon_\gamma \|\xi_\gamma^n\|^2 + \lim_{\gamma\to\infty} (\xi_\gamma^n, y_\gamma^n) = (\xi_*^n, y_*^n)$$

additionally, (4.6) yields that

$$\lim_{\gamma\to\infty} (\xi_\gamma^n, y_\gamma^n + \varepsilon_\gamma \xi_\gamma^n) = 0.$$

Hence, $(\xi_*^n, y_*^n) = 0$ and $(y_*^n, \xi_*^n, u_*^n) \in \mathcal{F}^n$.

Now from the weak lower semi-continuity of J we have

$$J(y_*^n, \xi_*^n, u_*^n) \leqslant \liminf_{\gamma\to\infty} J(y_\gamma^n, \xi_\gamma^n, u_\gamma^n).$$

Since $J \leqslant J_\gamma$, $\mathcal{F}^n \subseteq \mathcal{F}_\gamma^n$ and $\varepsilon_\gamma \gamma \underset{\gamma\to\infty}{\to} 0$, it follows that

$$\begin{aligned}
J(y_*^n, \xi_*^n, u_*^n) &\leqslant \liminf_{\gamma\to\infty} J_\gamma(y_\gamma^n, \xi_\gamma^n, u_\gamma^n) \leqslant \liminf_{\gamma\to\infty} J_\gamma(\tilde{y}, \tilde{\xi}, \tilde{u})\\
&\leqslant J(\tilde{y}, \tilde{\xi}, \tilde{u}) + \lim_{\gamma\to\infty} \gamma\varepsilon_\gamma \|\tilde{\xi}\|^2 \leqslant J(\tilde{y}, \tilde{\xi}, \tilde{u})
\end{aligned}$$

for any $(\tilde{y}, \tilde{\xi}, \tilde{u})$ in $\mathcal{F}$. Consequently, (y_*^n, ξ_*^n, u_*^n) is a solution to the limit optimal control problem $(\mathcal{O}^n)$. $\square$

First order optimality conditions for $(\mathcal{O}_\gamma^n)_{\gamma>0}$**.** In order to derive the first order optimality system for the regularized-penalized problems $(\mathcal{O}_\gamma^n)_{\gamma>0}$ we check the Zowe-Kurcyusz constraints qualification [22], [21] which we recall in Appendix B. In our context it requires the existence of $(c_y, c_\xi, c_u, \zeta, \lambda)$ in $\mathcal{V} \times L^2(\Omega) \times L^2(\Gamma_C) \times L^2(\Omega) \times \mathbb{R}$ such that the following system holds:

$$(\mathcal{CQ}) \qquad \begin{cases} Ac_y = Bc_u + c_\xi + z' & \text{in } \mathcal{V}', \\ c_y + \varepsilon_\gamma c_\xi - \zeta + \lambda(y_\gamma^n + \varepsilon_\gamma \xi_\gamma^n) = z & \text{a.e. in } L^2(\Omega), \\ c_\xi \geqslant 0, \quad \zeta \geqslant 0 & \text{a.e. in } L^2(\Omega), \\ c_u \geqslant 0 & \text{a.e. in } L^2(\Gamma_C), \\ \lambda \geqslant 0 & \end{cases}$$

for a given $(z', z) \in \mathcal{V}' \times L^2(\Omega)$. First, we pose

$$\lambda = 0, \quad c_u = 0, \quad c_\xi = c_{\xi,1} + c_{\xi,2}, \quad \zeta = \zeta_1 + \zeta_2$$

with

$$c_{\xi,2} = \frac{1}{\varepsilon_\gamma} \max(0, z), \quad \zeta_2 = \max(0, -z).$$

Then, we choose $\zeta_1 \in \mathcal{V}$ such that the system

$$A\zeta_1 = c_{\xi,2} + z' + \Lambda \quad \text{in } \mathcal{V}', \tag{4.7}$$

$$\Lambda \geqslant 0, \quad \zeta_1 \geqslant 0, \quad \langle \Lambda, \zeta_1 \rangle = 0, \tag{4.8}$$

holds for some $\Lambda \in \mathcal{V}'$. We mention that (4.7)–(4.8) is well-posed by the theory of variational inequalities [18]. Finally, we assign to $c_{\xi,1}$ the solution of the following elliptic partial differential equation

$$\varepsilon_\gamma A c_{\xi,1} + c_{\xi,1} = \Lambda \quad \text{in } \mathcal{V}'. \tag{4.9}$$

Observe that $c_{\xi,1} \geqslant 0$ by a standard maximum principle [12], p. 327. Now for

$$c_y = \zeta_1 - \varepsilon_\gamma c_{\xi,1}$$

we obtain

$$\begin{aligned} c_y + \varepsilon_\gamma c_\xi - \zeta &= c_y + \varepsilon_\gamma c_{\xi,1} - \zeta_1 + \varepsilon_\gamma c_{\xi,2} - \zeta_2 \\ &= \varepsilon_\gamma c_{\xi,2} - \zeta_2 = \max(0, z) - \max(0, -z) = z \end{aligned}$$

and

$$Ac_y = A\zeta_1 - \varepsilon_\gamma A c_{\xi,1} = c_{\xi,2} + z' + \Lambda - \varepsilon_\gamma A c_{\xi,1} = c_{\xi,2} + z' + c_{\xi,1} = c_\xi + z'.$$

Here we have used (4.7)–(4.8) and (4.9). Therefore, problem $(\mathcal{O}_\gamma^n)$ constraints are qualified and the next proposition holds true.

Proposition 4.1. *Let $(y^n_\gamma, \xi^n_\gamma, u^n_\gamma)$ be a solution for the problem $(\mathcal{O}^n_\gamma)$. Then there exists a Lagrange multiplier vector $(p^n_\gamma, \lambda^n_\gamma)$ in $\mathcal{V} \times L^2(\Gamma_C)$ such that the following first order optimality system holds*

$$Ay^n_\gamma - Bu^n_\gamma - \xi^n_\gamma - \overline{y}^{n-1}_\gamma + \overline{\xi}^{n-1}_\gamma = 0 \quad \text{in } \mathcal{V}', \tag{4.10}$$

$$Ap^n_\gamma + y^n_\gamma - y^n_d + \gamma\xi^n_\gamma - \lambda^n_\gamma = 0 \quad \text{in } \mathcal{V}', \tag{4.11}$$

$$y^n_\gamma + \varepsilon_\gamma \xi^n_\gamma \geqslant 0, \quad \lambda^n_\gamma \geqslant 0, \quad (y^n_\gamma + \varepsilon_\gamma\xi^n_\gamma, \lambda^n_\gamma) = 0, \tag{4.12}$$

$$\xi^n_\gamma \geqslant 0, \quad (\xi^n_\gamma - \xi^n_d + 2\gamma\varepsilon_\gamma\xi^n_\gamma + \gamma y^n_\gamma - p^n_\gamma - \varepsilon_\gamma\lambda^n_\gamma, \tau - \xi^n_\gamma) \geqslant 0, \tag{4.13}$$

$$u^n_\gamma \geqslant 0, \quad (\alpha u^n_\gamma - \tau B^* p^n_\gamma, v - u^n_\gamma)_{\Gamma_C} \geqslant 0, \tag{4.14}$$

where τ and v are two non-negative arbitrary functions in $L^2(\Omega)$ and $L^2(\Gamma_C)$, respectively, and B^* is the adjoint operator of B.

Remark 4.1. (i) Conditions (4.13) and (4.14) correspond to the projection of $(1 + 2\gamma\varepsilon_\gamma)^{-1}(\xi^n_d + \lambda^n_\gamma)$ and $\tau\alpha^{-1}B^*p^n_\gamma$ over the non-negative cones in $L^2(\Omega)$ and $L^2(\Gamma_C)$, respectively:

$$u^n_\gamma = \max\Big(0, \frac{\tau}{\alpha}B^*p^n_\gamma\Big), \quad \xi^n_\gamma = \max\Big(0, \frac{1}{1+2\gamma\varepsilon_\gamma}(p^n_\gamma + \xi^n_d + \varepsilon_\gamma\lambda^n_\gamma - \gamma y^n_\gamma)\Big).$$

(ii) The step function u^τ_γ, which is equal to the time discrete controls u^n_γ on each interval $[(n-1)\tau, n\tau]$ for $n = 1, \dots, N$, comprises a sub-optimal solution to a time-continuous optimal control problem related to $(\mathcal{M}^t)$. A convergence analysis—with respect to time step size τ—of u^τ_γ and the corresponding state and Lagrange multipliers is of interest. In this respect one may investigate adapting the arguments presented in [4] and the references therein. This analysis will be discussed in some upcoming work.

5. Numerical experiments

In this section we present two preliminary numerical experiments to assess the validity of the above developed theoretical procedure. At each time step $t^n = n\tau$ we solve a discrete version of the optimization problem $(\mathcal{O}^n)_{\gamma_k}$ for a sequence of penalty parameters $(\gamma_k)_{k\in\mathbb{N}}$ with $\gamma_k = 10^{-3} \times 1.5^k$ and $k = 1, \dots, 40$. We select a regularization parameter $\varepsilon_{\gamma_k} = (10^3 + \gamma_k^4)^{-1}$. The parameter for the cost of the control is taken $\nu = 10^{-4}$. The thermal conductivity is set to 1 in the two examples. All functions are discretized by continuous piecewise linear finite elements. The fully discretized penalized-regularized control problems corresponding to $(\mathcal{O}^n)_{\gamma_k}$ are then solved numerically using the fmincon Matlab function.

E x a m p l e 5.1. We consider a one-dimensional free convection problem with a known analytical solution [11]:

$$y_{\mathrm{ex}}(x,t) = \begin{cases} \exp(t-x)-1 & \text{if } 0 \leqslant x \leqslant t, \\ 0 & \text{if } t \leqslant x \leqslant x_{\max}, \end{cases} \qquad \xi_{\mathrm{ex}}(x,t) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant t, \\ 1 & \text{if } t \leqslant x \leqslant x_{\max}. \end{cases}$$

Our aim here is to apply a heat flux on the left boundary, $\Gamma_C = \{0\}$, to get temperature and solid fraction profiles as close as possible to the exact solution. For the instantaneous boundary control problem we choose a fixed time step $\tau = 0.01$ and we set

$$y_d^n = y_{\mathrm{ex}}(x, n\tau), \quad \xi_d^n = \xi_{\mathrm{ex}}(x, n\tau) \quad \text{for } n = 1, \ldots, N = 300.$$

For the computational domain we choose a uniform grid of size $h = 0.01$ with $x_{\max} = 4$. The analytical control $u_{\mathrm{ex}}(t) = \exp(t)$ is very well reconstructed up to the first few iterations as shown in Fig. 2. An excellent agreement has been found between the analytical and controlled temperature profiles as depicted in Fig. 3. The complementarity condition between the temperature and solid fraction are respected, as shown in Fig. 4 for the sample instant $t = 3$, which emphasizes the relevance of the developed regularization-penalization approach.

E x a m p l e 5.2. Here we consider a two-dimensional problem where the computational domain $\Omega = (0,2) \times (0,4)$ is discretized using a 50×100 uniform grid. The time step is taken $\tau = 0.1$ and a constant convection velocity $\vec{v} = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}$ is used. No-heat flux condition is applied on the right boundary and temperature $y = 0$ is held at the top and bottom. We apply the heat flux control on $\Gamma_C := \{x \in \Omega;\ x_1 = 0\}$ to govern the system toward the following time-independent desired state:

$$y_d^n(x) = y_d(x) = \begin{cases} \exp(\frac{1}{4}(4-x_2)x_2 - x_1) - 1 & \text{if } x_1 < \frac{1}{4}(4-x_2)x_2, \\ 0 & \text{if } x_1 \geqslant \frac{1}{4}(4-x_2)x_2, \end{cases}$$
$$\xi_d^n(x) = \xi_d(x) = \begin{cases} 0 & \text{if } x_1 < \frac{1}{4}(4-x_2)x_2, \\ 1 & \text{if } x_1 \geqslant \frac{1}{4}(4-x_2)x_2. \end{cases}$$

Figs. 5–6 show the evolution of temperature y and solid fraction ξ driven by the sub-optimal controls towards the desired state. A fairly good approximation is obtained. The significant reduction of the cost functional value is achieved during the first five time steps and almost stagnates up to $t \approx 1$ as shown in Fig. 7. In Fig. 8 we present the computed sub-optimal control at sample instances. We observe a strong control at the first time step getting inactive near the boundaries. The controls shape is consistent with the desired state one.

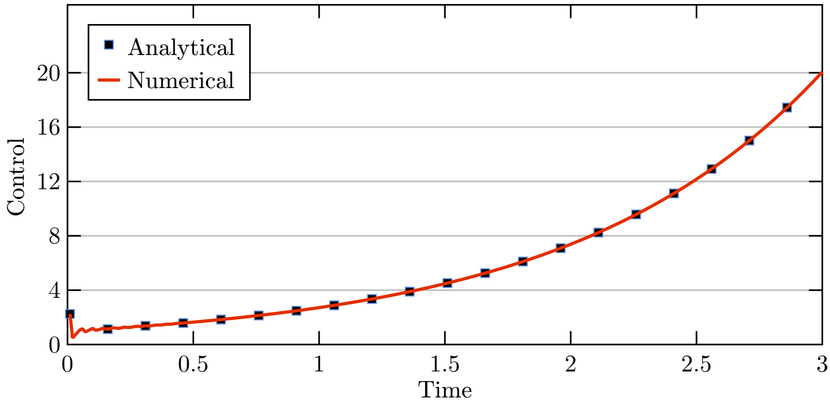

Figure 2. Analytical and computed sub-optimal controls.

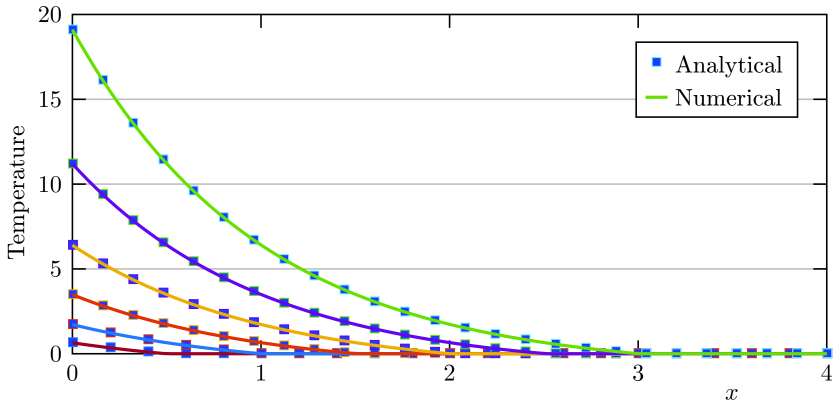

Figure 3. Analytical and computed temperature profiles at different instants.

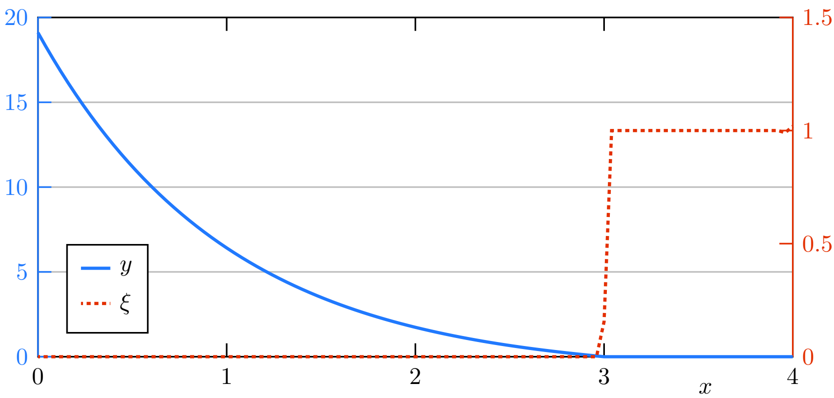

Figure 4. Computed temperature and solid fraction profiles at time $t = t_f = 3$.

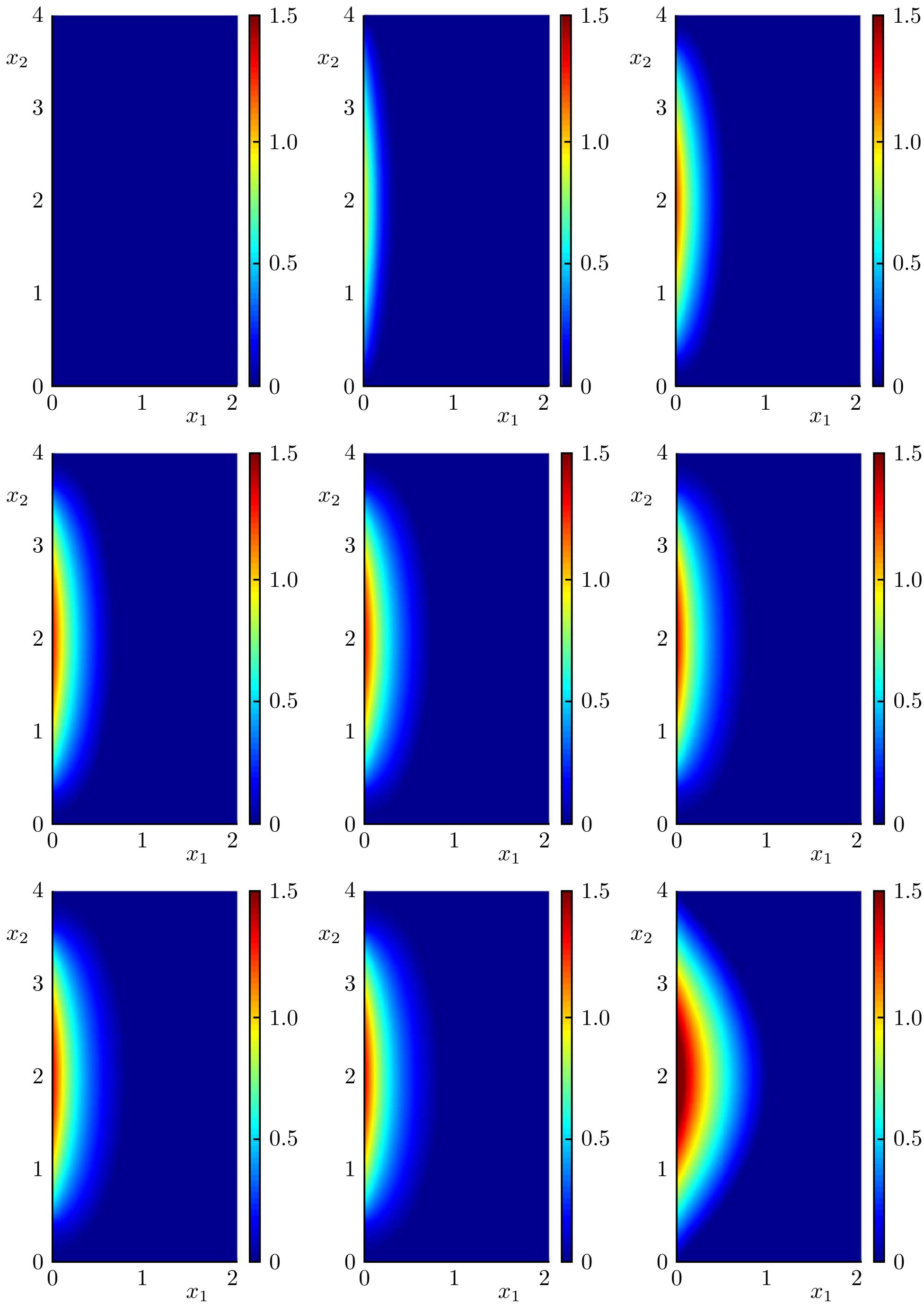

Figure 5. Computed temperatures at $t = 0, 0.1, 0.3, 0.5, 0.7, 0.9, 1.1, 1.4$ and the desired temperature profile.

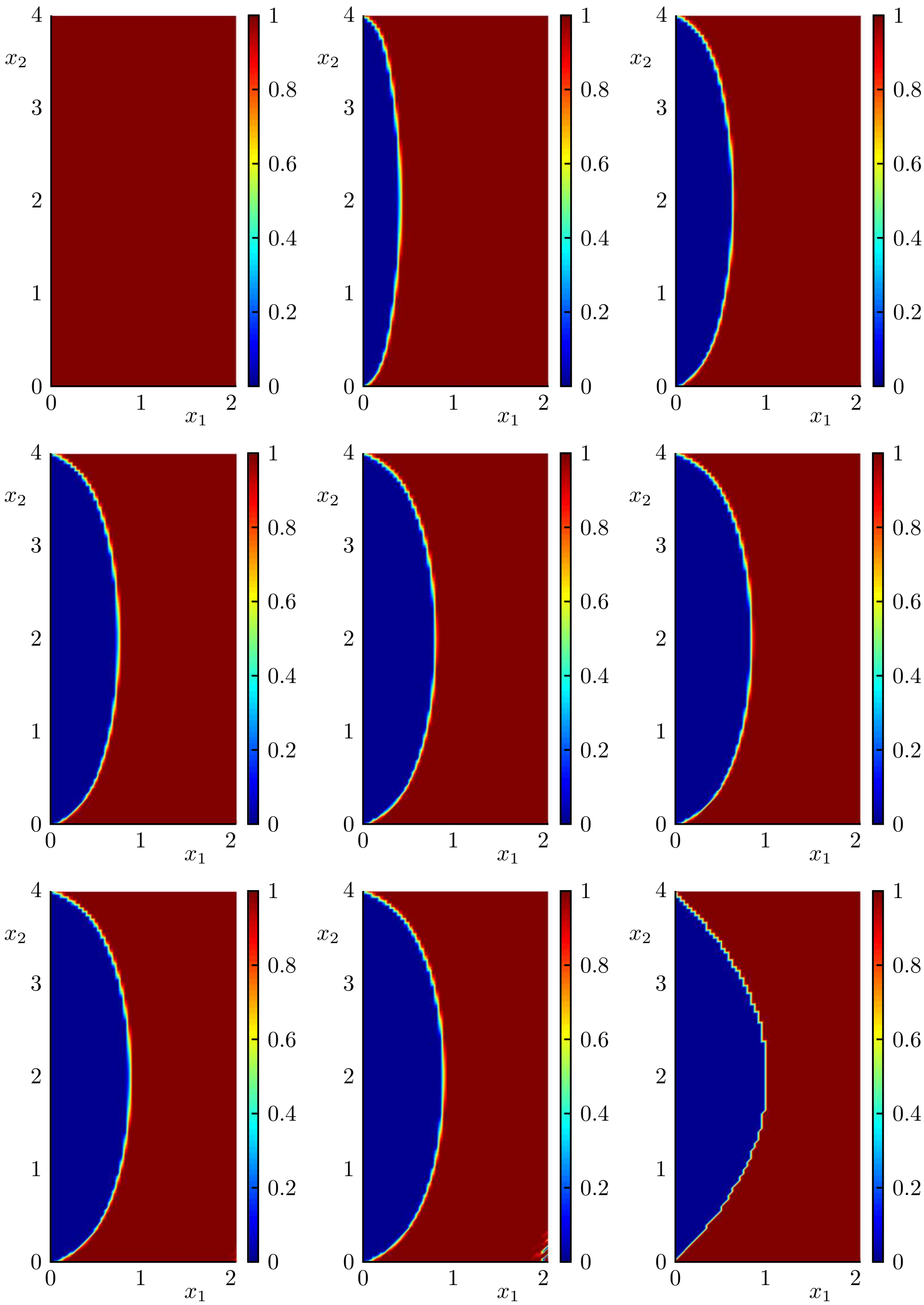

Figure 6. Computed solid fraction at $t = 0, 0.1, 0.3, 0.5, 0.7, 0.9, 1.1, 1.4$ and the desired solid fraction profile.

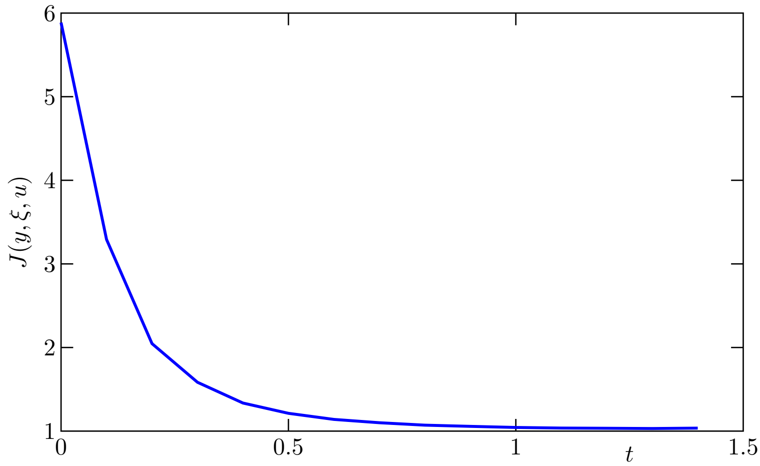

Figure 7. Reduction of the cost functional.

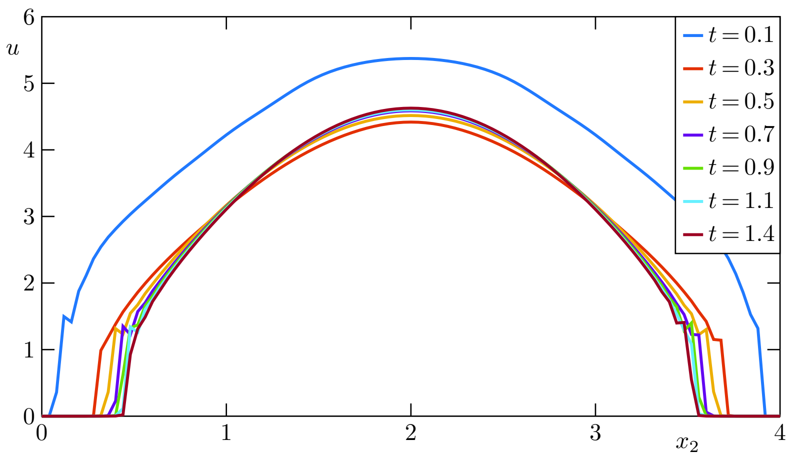

Figure 8. Computed sub-optimal controls at different time steps.

R e m a r k 5.1. To apply the developed approach on more realistic benchmarks, a coupling with momentum and mass conservation equations is required. However, many discretization and algorithmic aspects have to be developed first. Questions related to adaptive mesh refinement, selection of the optimization parameters, solution algorithm and preconditioning will be addressed in a forthcoming study.

Appendix A: Proof of Theorem 2.1

To show that the problem

$$\begin{cases} Ay^n = \xi^n + Bu^n + \overline{y}^{n-1} - \overline{\xi}^{n-1} & \text{in } \mathcal{V}', \\ \xi^n \in \mathcal{H}(y^n), & \text{a.e. in } \Omega \end{cases}$$

has a solution, let $\mathcal{H}_\varepsilon$ be a regularization of the Heaviside operator $\mathcal{H}$ given by

$$\mathcal{H}_\varepsilon(x) = \begin{cases} 0 & \text{if } x \geqslant \varepsilon, \\ 1 - x/\varepsilon & \text{if } 0 \leqslant x \leqslant \varepsilon, \\ 1 & \text{if } x \leqslant 0. \end{cases}$$

Correspondingly, we consider the following regularized problem:

$$(\mathcal{WF}^n_\varepsilon) \qquad \begin{cases} \text{Find } y^n_\varepsilon \in \mathcal{V} \text{ such that } y^n_\varepsilon \geqslant 0 & \text{a.e in } \Omega, \\ Ay^n_\varepsilon = \mathcal{H}_\varepsilon(y^n_\varepsilon) + Bu^n + \overline{y}^{n-1} - \overline{\xi}^{n-1} & \text{in } \mathcal{V}'. \end{cases}$$

Lemma 5.1. *The regularized problem $(\mathcal{WF}^n_\varepsilon)$ has a unique solution y^n_ε. Moreover, there exists a constant C not depending on ε such that*

$$\|y^n_\varepsilon\|_{H^1(\Omega)} \leqslant C. \tag{5.1}$$

Proof. Consider the mapping T which, for any $y^n_\varepsilon \in L^2(\Omega)$, associates $\widetilde{y^n_\varepsilon} = T(y^n_\varepsilon)$ to the solution of the elliptic problem

$$A\widetilde{y^n_\varepsilon} = \mathcal{H}_\varepsilon(y^n_\varepsilon) + Bu^n + \overline{y}^{n-1} - \overline{\xi}^{n-1} \quad \text{in } \mathcal{V}'. \tag{5.2}$$

Problem (5.2) has a unique solution by Lax-Milgram theorem. Moreover, there exists a constant Const not depending on ε such that

$$\|\widetilde{y^n_\varepsilon}\|_{H^1(\Omega)} \leqslant \text{Const}. \tag{5.3}$$

Here, we have used the fact that $(\mathcal{H}_\varepsilon(y^n_\varepsilon))_\varepsilon$ is bounded in $L^\infty(\Omega)$ independently of ε.

The mapping T is then bounded from $L^2(\Omega)$ to $H^1(\Omega)$. From the compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$, it follows that T is completely continuous from $\mathcal{V}$ to $L^2(\Omega)$. Moreover, estimate (5.3) shows that $T(B_{\text{Const}}) \subset B_{\text{Const}}$ with B_{Const} being the $H^1(\Omega)$-ball of radius Const. Schauder's fixed-point theorem yields the existence of a function y^n_ε such that $T(y^n_\varepsilon) = y^n_\varepsilon$ satisfying (5.1) with $C = \text{Const}$.

Next, we claim that $y_\varepsilon^n \geqslant 0$ a.e. in Ω. Let $(y_\varepsilon^n)^- = \min(0, y_\varepsilon^n)$. It is clear that $(y_\varepsilon^n)^- \in \mathcal{V}$. By choosing $\varphi = (y_\varepsilon^n)^-$ in (5.2), we arrive at

$$\langle Ay_\varepsilon^n, (y_\varepsilon^n)^- \rangle = \langle Bu^n, (y_\varepsilon^n)^- \rangle + (\overline{y}^{n-1} + \mathcal{H}(y_\varepsilon^n) - \overline{\xi}^{n-1}, (y_\varepsilon^n)^-).$$

Since $u^n \geqslant 0$ a.e. in Γ_C, $y^{n-1} \geqslant 0$ a.e. in Ω and $0 \leqslant \xi^{n-1} \leqslant 1$ a.e. in Ω and using the fact that $\mathcal{H}_\varepsilon(x) = 1$ for $x \leqslant 0$, we obtain

$$\langle A(y_\varepsilon^n)^-, (y_\varepsilon^n)^- \rangle = \langle Bu^n, (y_\varepsilon^n)^- \rangle + (\overline{y}^{n-1} + 1 - \overline{\xi}^{n-1}, (y_\varepsilon^n)^-) \leqslant 0.$$

The coercivity of A leads to $(y_\varepsilon^n)^- = 0$ a.e. in Ω and then $y_\varepsilon^n \geqslant 0$ a.e. in Ω. Consequently, the solution y_ε^n is a solution of $(\mathcal{WF}_\varepsilon^n)$. □

Now for any $\varepsilon > 0$, let y_ε^n be the solution of the regularized problem $(\mathcal{WF}_\varepsilon^n)$. From (5.1) we can find a subsequence, also denoted $(y_\varepsilon^n)_{\varepsilon>0}$, such that

$$\begin{aligned} y_\varepsilon^n &\rightharpoonup y^n \quad \text{in } H^1(\Omega), \\ y_\varepsilon^n &\to y^n \quad \text{in } L^2(\Omega), \\ \mathcal{H}_\varepsilon(y_\varepsilon^n) &\overset{*}{\rightharpoonup} \xi^n \quad \text{in } L^\infty(\Omega). \end{aligned}$$

By passing to the limit, we deduce that

$$\begin{cases} y^n \geqslant 0 & \text{a.e. in } \Omega, \\ 0 \leqslant \xi^n \leqslant 1 & \text{a.e. in } \Omega, \\ Ay^n = \xi^n + Bu^n + \overline{y}^{n-1} - \overline{\xi}^{n-1} & \text{in } \mathcal{V}'. \end{cases} \tag{5.4}$$

Further, observe that

$$y^n \geqslant 0, \quad \xi^n \in \mathcal{H}(y^n) \Leftrightarrow y^n \geqslant 0, \quad 0 \leqslant \xi^n \leqslant 1, \quad (y^n, \xi^n) = 0. \tag{5.5}$$

Therefore, to complete the proof of existence of a solution for the initial problem, it remains to prove that $(y^n, \xi^n) = 0$. One has

$$(y_\varepsilon^n, \mathcal{H}_\varepsilon(y_\varepsilon^n)) \to (y^n, \xi^n) \tag{5.6}$$

from the $L^2(\Omega)$ strong convergence of y_ε^n to y^n and the $L^\infty(\Omega)$ weak-* convergence of $\mathcal{H}_\varepsilon(y_\varepsilon^n)$ to ξ^n.

On the other hand, from $\mathcal{H}_\varepsilon$ expression we have

$$(y_\varepsilon^n, \mathcal{H}_\varepsilon(y_\varepsilon^n)) \leqslant \varepsilon \operatorname{meas}(\Omega) \to 0.$$

Consequently $(y^n, \xi^n) = 0$.

Now, notice that if $(y^n, \xi^n) \in \mathcal{V} \times L^2(\Omega)$ is a solution to the complementarity problem

$$
(\mathcal{CS}^n) \qquad \begin{cases} \xi^n + Ay^n = Bu^n + \overline{y}^{n-1} - \overline{\xi}^{n-1} & \text{in } \mathcal{V}', \\ y^n \geqslant 0, \quad \xi^n \geqslant 0, \quad (y^n, \xi^n) = 0 & \text{a.e. in } \Omega, \end{cases}
$$

then y^n is a solution to the variational inequality

$$
(\mathcal{VI}^n) \qquad \begin{cases} y^n \in \mathcal{K} := \{q \in \mathcal{V};\ q \geqslant 0 \quad \text{a.e. in } \Omega\}, \\ \langle Ay^n, q - y^n\rangle \geqslant \langle Bu^n, q - y^n\rangle + (\overline{y}^{n-1} - \overline{\xi}^{n-1}, q - y^n) \quad \forall\, q \in \mathcal{K}. \end{cases}
$$

Since $(\mathcal{VI}^n)$ possesses a unique solution in $\mathcal{V}$ by virtue of Stampacchia-Rodriguez, we deduce that y^n is unique.

Finally, the uniqueness of ξ^n follows from the uniqueness of y^n. More precisely, if $(y^n, \xi_1^n) \in \mathcal{V} \times L^2(\Omega)$ and $(y^n, \xi_2^n) \in \mathcal{V} \times L^2(\Omega)$ are two solutions to $(\mathcal{CS}^n)$, then

$$
\xi_1^n = \xi_2^n = Ay^n - Bu^n - \overline{y}^{n-1} + \overline{\xi}^{n-1} \quad \text{in } \mathcal{V}'.
$$

Therefore, $\xi_1^n - \xi_2^n = 0$ in $\mathcal{V}'$. By the density of $\mathcal{V} \supset H_0^1(\Omega)$ in $L^2(\Omega)$ we conclude that $\xi_1^n = \xi_2^n$ in $L^2(\Omega)$.

Appendix B: Mathematical optimization in Banach spaces

Let $\mathcal{X}$ and $\mathcal{Y}$ be real Banach spaces. For

$$
\begin{aligned} F\colon \mathcal{X} \to \mathbb{R} & \quad \text{Fréchet-differentiable functional,} \\ g\colon \mathcal{X} \to \mathcal{Y} & \quad \text{continuously Fréchet-differentiable,} \end{aligned}
$$

we consider the following mathematical program:

$$
(5.7) \qquad \min\{F(x);\ g(x) \in M,\ x \in C\},
$$

where C is a closed convex subset of $\mathcal{X}$ and M a closed cone in $\mathcal{Y}$ with vertex at 0.

We suppose that problem (5.7) has an optimal solution $\hat{x}$, and we introduce the conical hulls of $C - \{\hat{x}\}$ and $M - \{y\}$, respectively, by

$$
\begin{aligned} C(\hat{x}) &= \{x \in \mathcal{X};\ \exists\, \beta \geqslant 0,\ \exists\, c \in C,\ x = \beta(c - \hat{x})\}, \\ M(y) &= \{z \in \mathcal{Y};\ \exists\, \lambda \geqslant 0,\ \exists\, \zeta \in M,\ z = \zeta - \lambda y\}. \end{aligned}
$$

The main result concerning the existence of a Lagrange multiplier for (5.7) is given in the next theorem.

Theorem 5.1 ([22]). *Let $\hat{x}$ be an optimal solution of problem* (5.7) *satisfying the following constraints qualification:*

$$g'(\hat{x}) \cdot C(\hat{x}) - M(g(\hat{x})) = \mathcal{Y}. \tag{5.8}$$

Then there exists a Lagrange multiplier $\mu^* \in \mathcal{Y}^*$ such that

$$\langle \mu^*, z \rangle_{\mathcal{Y}^*,\mathcal{Y}} \geqslant 0 \quad \forall\, z \in M, \tag{5.9}$$

$$\langle \mu^*, g(\hat{x}) \rangle_{\mathcal{Y}^*,\mathcal{Y}} = 0, \tag{5.10}$$

$$F'(\hat{x}) - \mu^* \circ g'(\hat{x}) \in C(\hat{x})_+, \tag{5.11}$$

where $A_+ = \{x^* \in \mathcal{X}^*;\ \langle x^*, a \rangle_{\mathcal{X}^*,\mathcal{X}} \geqslant 0$ for all $a \in A\}$, $\mathcal{Y}^*$ and $\mathcal{X}^*$ are the topological dual spaces of $\mathcal{Y}$ and $\mathcal{X}$, respectively, and $(\mu^* \circ g'(\hat{x}))d = \langle \mu^*, g'(\hat{x})d \rangle_{\mathcal{Y}^*,\mathcal{Y}}$ for all $d \in \mathcal{X}$.

References

[1] *U. G. Abdulla*: On the optimal control of the free boundary problems for the second order parabolic equations. I: Well-posedness and convergence of the method of lines. Inverse Probl. Imaging *7* (2013), 307–340. zbl MR doi

[2] *U. G. Abdulla*: On the optimal control of the free boundary problems for the second order parabolic equations. II: Convergence of the method of finite differences. Inverse Probl. Imaging *10* (2016), 869–898. zbl MR doi

[3] *U. G. Abdulla, J. M. Goldfarb*: Frechet differentability in Besov spaces in the optimal control of parabolic free boundary problems. J. Inverse Ill-Posed Probl. *26* (2018), 211–227. zbl MR doi

[4] *U. G. Abdulla, B. Poggi*: Optimal Stefan problem. Calc. Var. Partial Differ. Equ. *59* (2020), Article ID 61, 40 pages. zbl MR doi

[5] *S. N. Al-Saadi, Z. J. Zhai*: Modeling phase change materials embedded in building enclosure: A review. Renew. Sust. Energy Rev. *21* (2013), 659–673. doi

[6] *B. Baran, P. Benner, J. Heiland, J. Saak*: Optimal control of a Stefan problem fully coupled with incompressible Navier-Stokes equations and mesh movement. An. Ştiinţ. Univ. "Ovidius" Constanţa, Ser. Mat. *26* (2018), 11–40. zbl MR doi

[7] *M. K. Bernauer, R. Herzog*: Optimal control of the classical two-phase Stefan problem in level set formulation. SIAM J. Sci. Comput. *33* (2011), 342–363. zbl MR doi

[8] *H. Choi, M. Hinze, K. Kunisch*: Instantaneous control of backward-facing step flows. Appl. Numer. Math. *31* (1999), 133–158. zbl MR doi

[9] *H. Choi, R. Temam, P. Moin, J. Kim*: Feedback control for unsteady flow and its application to the stochastic Burgers equation. J. Fluid Mech. *253* (1993), 509–543. zbl MR doi

[10] *V. K. Dhir*: Phase change heat transfer—a perspective for the future. Proceedings of Rohsenow Symposium on Future Trends in Heat Transfer. Massachusetts Institute of Technology, Cambridge, 2003, 6 pages; Available at `http://web.mit.edu/hmtl/www/papers/DHIR.pdf`.

[11] *A. Esen, S. Kutluay*: A numerical solution of the Stefan problem with a Neumann-type boundary condition by enthalpy method. Appl. Math. Comput. *148* (2004), 321–329. zbl MR doi

[12] *L. C. Evans*: Partial Differential Equations. Graduate Studies in Mathematics 19. American Mathematical Society, Providence, 1998. zbl MR doi

[13] *N. L. Gol'dman*: Inverse Stefan Problems. Mathematics and Its Applications 412. Kluwer Academic Publishers, Dordrecht, 1997. zbl MR doi
[14] *M. Hintermüller, A. Laurain, C. Löbhard, C. N. Rautenberg, T. M. Surowiec*: Elliptic mathematical programs with equilibrium constraints in function space: Optimality conditions and numerical realization. Trends in PDE Constrained Optimization. International Series of Numerical Mathematics 165. Springer, Cham, 2014, pp. 133–153. zbl MR doi
[15] *M. Hintermüller, C. Löbhard, M. H. Tber*: An ℓ_1-penalty scheme for the optimal control of elliptic variational inequalities. Numerical Analysis and Optimization. Springer Proceedings in Mathematics & Statistics 134. Springer, Cham, 2015, pp. 151–190. zbl MR doi
[16] *M. Hinze, S. Ziegenbalg*: Optimal control of the free boundary in a two-phase Stefan problem. J. Comput. Phys. *223* (2007), 657–684. zbl MR doi
[17] *M. Hinze, S. Ziegenbalg*: Optimal control of the free boundary in a two-phase Stefan problem with flow driven by convection. ZAMM, Z. Angew. Math. Mech. *87* (2007), 430–448. zbl MR doi
[18] *D. Kinderlehrer, G. Stampacchia*: An Introduction to Variational Inequalities and Their Applications. Pure and Applied Mathematics 88. Academic Press, New York, 1980. zbl MR doi
[19] *O. Pironneau*: On the transport-diffusion algorithm and its applications to the Navier-Stokes equations. Numer. Math. *38* (1982), 309–332. zbl MR doi
[20] *O. Pironneau, S. Huberson*: Characteristic-Galerkin and the particle method for the convection-diffusion equation and the Navier-Stokes equations. Lectures in Applied Mathematics 28. Vortex Dynamics and Vortex Methods. American Mathematical Society, Providence, 1991, pp. 547–565. zbl MR
[21] *F. Tröltzsch*: Optimal Control of Partial Differential Equations: Theory, Methods, and Applications. Graduate Studies in Mathematics 112. American Mathematical Society, Providence, 2010. zbl MR doi
[22] *J. Zowe, S. Kurcyusz*: Regularity and stability for the mathematical programming problem in Banach spaces. Appl. Math. Optim. *5* (1979), 49–62. zbl MR doi

Authors' address: *Youness Mezzan*, *Moulay Hicham Tber* (correspoding author), Cadi Ayyad University, Department of Mathematics, F.S.S., B.P. 2390, Av. Prince Moulay Abdellah, Marrakech 40000, Morocco, e-mail: `y.mezzan@gmail.com`, `hicham.tber@uca.ac.ma`.