EXISTENCE RESULTS AND ITERATIVE METHOD FOR FULLY THIRD ORDER NONLINEAR INTEGRAL BOUNDARY VALUE PROBLEMS

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Abstract. We consider the boundary value problem

$$u'''(t) = f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1,$$
$$u(0) = u'(0) = 0, \quad u(1) = \int_0^1 g(s)u(s)\,\mathrm{d}s,$$

where $f\colon [0,1] \times \mathbb{R}^3 \to \mathbb{R}^+$, $g\colon [0,1] \to \mathbb{R}^+$ are continuous functions. The case when $f = f(u(t))$ was studied in 2018 by Guendouz et al. Using the fixed-point theory on cones they established the existence of positive solutions. Here, by the method developed by ourselves very recently, we establish the existence, uniqueness and positivity of the solution under easily verified conditions and propose an iterative method for finding the solution. Some examples demonstrate the validity of the obtained theoretical results and the efficiency of the iterative method.

Keywords: fully third order nonlinear differential equation; integral boundary condition; positive solution; iterative method

MSC 2020: 34B15, 34B27

1. Introduction

Recently, boundary value problems for nonlinear differential equations with integral boundary conditions have attracted attention from many researchers. They constitute a very interesting and important class of problems, because they arise in many applied fields, such as heat conduction, chemical engineering, underground

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water flow, thermoelasticity, and plasma physics. It is worth mentioning some works concerning the problems with integral boundary conditions for second order equations such as [2], [4], [3], [11], [16], and some works for fourth order equations such as [1], [15], [17], [18], [20]. There are also many papers devoted to the third order equations with integral boundary conditions.

Below we will be concerned only with third order equations. The first work we would mention is of Boucherif et al. [5] in 2009. It is about the problem

$$\begin{gathered} u'''(t) = f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1, \\ u(0) = 0, \\ u'(0) - au''(0) = \int_0^1 h_1(u(s), u'(s)) \, \mathrm{d}s, \\ u'(1) + bu''(1) = \int_0^1 h_2(u(s), u'(s)) \, \mathrm{d}s, \end{gathered} \tag{1.1}$$

where a, b are positive real numbers, and f, h_1, h_2 are continuous functions. Based on a priori bounds and a fixed-point theorem for a sum of two operators, one a compact operator and the other a contraction, the authors established the existence of solutions to the problem under complicated conditions on the functions f, h_1, h_2. Independently from the above work, in 2010 Sun and Li [19] considered the problem

$$\begin{gathered} u'''(t) + f(t, u(t), u'(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \quad u'(1) = \int_0^1 g(t)u'(t) \, \mathrm{d}t. \end{gathered} \tag{1.2}$$

By using the Krasnoselskii fixed-point theorem, some sufficient conditions are obtained for the existence and nonexistence of monotone positive solutions to the above problem.

Next, in 2012 Guo, Liu and Liang [13] studied the boundary value problem with second derivative

$$\begin{gathered} u'''(t) + f(t, u(t), u''(t)) = 0, \quad 0 < t < 1, \\ u(0) = u''(0) = 0, \quad u(1) = \int_0^1 g(t)u(t) \, \mathrm{d}t. \end{gathered} \tag{1.3}$$

The authors obtained sufficient conditions for the existence of positive solutions by using the fixed-point index theory in a cone and spectral radius of a linear operator. No examples of the functions f and g satisfying the conditions of existence are shown.

In another paper, in 2013 Guo and Yang [14] considered a problem with other boundary conditions, namely, the problem

$$\begin{aligned} u'''(t) &= f(t,u(t),u'(t)), \quad 0<t<1, \\ u(0) = u''(0) = 0, &\quad u(1) = \int_0^1 g(t)u(t)\,\mathrm{d}t. \end{aligned} \tag{1.4}$$

Based on the Krasnoselskii fixed-point theorem on a cone, the authors established the existence of positive solutions of the problem under very complicated and artificial growth conditions posed on the nonlinearity $f(t,x,y)$.

Very recently, in [12] Guendouz et al. studied the problem

$$\begin{aligned} u'''(t) + f(u(t)) &= 0, \quad 0<t<1, \\ u(0) = u'(0) = 0, &\ u(1) = \int_0^1 g(t)u(t)\,\mathrm{d}t. \end{aligned} \tag{1.5}$$

By applying Krasnoselskii's fixed-point theorem on cones they established the existence results of positive solutions of the problem. This technique was used also by Benaicha and Haddouchi in [1] for an integral boundary problem for a fourth order nonlinear equation.

It should be emphasized that in all of the above-mentioned works the authors could only (even could not) show examples of the nonlinear terms satisfying required sufficient conditions, but *no exact solutions are shown. Moreover, the known results are of purely theoretical characteristics concerning the existence of solutions but are not methods for finding solutions.*

Motivated by the above facts, in this paper, we consider the boundary value problem

$$u'''(t) = f(t,u(t),u'(t),u''(t)), \quad 0<t<1, \tag{1.6}$$

$$u(0) = u'(0) = 0, \quad u(1) = \int_0^1 g(s)u(s)\,\mathrm{d}s, \tag{1.7}$$

where $f\colon [0,1]\times\mathbb{R}^3\to\mathbb{R}^+$, $g\colon [0,1]\to\mathbb{R}^+$.

This problem is a natural generalization of the problem (1.5), when $f(u(t))$ is replaced by the fully nonlinear term $f(t,u(t),u'(t),u''(t))$. By the method of reducing BVPs to the operator equation for right-hand sides developed in [9], [8], [10], [7], we establish the existence, uniqueness and positivity of a solution and propose an iterative method for finding the solution. Some examples demonstrate the validity of the obtained theoretical results and the efficiency of the iterative method. Especially, one example of exact solution of the problem is constructed so that the functions f and g satisfy the required conditions.

2. Existence results

To investigate the problem (1.6)–(1.7) we associate it with an operator equation as follows.

First, we denote the space of pairs $w = (\varphi, \alpha)^\top$, where $\varphi \in C[0,1]$, $\alpha \in \mathbb{R}$, by $\mathcal{B}$, i.e., set $\mathcal{B} = C[0,1] \times \mathbb{R}$, and equip it with the norm

$$\|w\|_{\mathcal{B}} = \max(\|\varphi\|, k|\alpha|), \tag{2.1}$$

where $\|\varphi\| = \max\limits_{0\leqslant t\leqslant 1} |\varphi(t)|$, k is a number, $k \geqslant 1$. The constant k will have a significance in the conditions for the existence and uniqueness of the solution. Later, in examples the selection of it will depend on particular cases.

Further, define the operator $A\colon \mathcal{B} \to \mathcal{B}$ by the formula

$$Aw = \begin{pmatrix} f(t, u(t), u'(t), u''(t)) \\ \int_0^1 g(s)u(s)\,\mathrm{d}s \end{pmatrix}, \tag{2.2}$$

where $u(t)$ is the solution of the problem

$$u'''(t) = \varphi(t), \quad 0 < t < 1, \tag{2.3}$$

$$u(0) = u'(0) = 0, \quad u(1) = \alpha. \tag{2.4}$$

It is easy to verify the following lemma.

Lemma 2.1. *If $w = (\varphi, \alpha)^\top$ is a fixed point of the operator A in the space $\mathcal{B}$, i.e., is a solution of the operator equation*

$$Aw = w \tag{2.5}$$

in $\mathcal{B}$, then the function $u(t)$ defined from the problem (2.3)–(2.4) *is a solution of the original problem* (1.6)–(1.7).

Conversely, if $u(t)$ is a solution of (1.6)–(1.7), *then the pair $(\varphi, \alpha)^\top$, where*

$$\varphi(t) = f(t, u(t), u'(t), u''(t)), \tag{2.6}$$

$$\alpha = \int_0^1 g(s)u(s)\,\mathrm{d}s, \tag{2.7}$$

is a solution of the operator equation (2.5).

Thus, by this lemma, the problem (1.6)–(1.7) is reduced to the fixed-point problem for A.

Remark that the above operator A, which is defined on pairs of functions $\varphi(t)$, $t\in[0,1]$ and boundary values α of $u(t)$ at $t=1$, is similar to the mixed boundary-domain operator introduced in [6] for studying biharmonic type equations.

Now we study the properties of A. For this purpose, notice that the problem (2.3)–(2.4) has a unique solution representable in the form

$$u(t)=\int_0^1 G_0(t,s)\varphi(s)\,\mathrm{d}s+\alpha t^2,\quad 0<t<1, \tag{2.8}$$

where

$$G_0(t,s)=\begin{cases}-\frac12 s(1-t)(2t-ts-s), & 0\leqslant s\leqslant t\leqslant 1,\\ -\frac12(1-s)^2t^2, & 0\leqslant t\leqslant s\leqslant 1,\end{cases}$$

is the Green function of the operator $u'''(t)$ associated with the homogeneous boundary conditions $u(0)=u'(0)=u(1)=0$.

Differentiating both sides of (2.8) gives

$$u'(t)=\int_0^1 G_1(t,s)\varphi(s)\,\mathrm{d}s+2\alpha t, \tag{2.9}$$

$$u''(t)=\int_0^1 G_2(t,s)\varphi(s)\,\mathrm{d}s+2\alpha, \tag{2.10}$$

where

$$G_1(t,s)=\begin{cases}-s(st-2t+1), & 0\leqslant s\leqslant t\leqslant 1,\\ -(1-s)^2t, & 0\leqslant t\leqslant s\leqslant 1,\end{cases}$$

$$G_2(t,s)=\begin{cases}-s(s-2), & 0\leqslant s\leqslant t\leqslant 1,\\ -(1-s)^2, & 0\leqslant t\leqslant s\leqslant 1.\end{cases}$$

It is easily seen that $G_0(t,s)\leqslant 0$ in $Q=[0,1]^2$, and

$$\begin{aligned} M_0&=\max_{0\leqslant t\leqslant 1}\int_0^1|G_0(t,s)|\,\mathrm{d}s=\frac{2}{81},\\ M_1&=\max_{0\leqslant t\leqslant 1}\int_0^1|G_1(t,s)|\,\mathrm{d}s=\frac{1}{18},\\ M_2&=\max_{0\leqslant t\leqslant 1}\int_0^1|G_2(t,s)|\,\mathrm{d}s=\frac{2}{3}.\end{aligned} \tag{2.11}$$

Therefore, from (2.8), (2.9), (2.10) and (2.11) we obtain

$$\|u\|\leqslant M_0\|\varphi\|+|\alpha|,\quad \|u'\|\leqslant M_1\|\varphi\|+2|\alpha|,\quad \|u''\|\leqslant M_2\|\varphi\|+2|\alpha|. \tag{2.12}$$

Now for any number $M > 0$ define the domain

$$\mathcal{D}_M = \Big\{(t,x,y,z) \mid 0 \leqslant t \leqslant 1,\ |x| \leqslant \Big(M_0 + \frac{1}{k}\Big)M,\\ |y| \leqslant \Big(M_1 + \frac{2}{k}\Big)M,\ |z| \leqslant \Big(M_2 + \frac{2}{k}\Big)M\Big\}. \tag{2.13}$$

Next, denote

$$C_0 = \int_0^1 g(s)\,\mathrm{d}s, \quad C_2 = \int_0^1 s^2 g(s)\,\mathrm{d}s. \tag{2.14}$$

Lemma 2.2. *Suppose that the function $f(t,x,y,z)$ is continuous and bounded by M in $\mathcal{D}_M$, i.e.,*

$$|f(t,x,y,z)| \leqslant M \quad \text{in } \mathcal{D}_M \tag{2.15}$$

and

$$q_1 := kC_0M_0 + C_2 \leqslant 1. \tag{2.16}$$

Then the operator A defined by (2.2) *maps the closed ball $B[0,M]$ in $\mathcal{B}$ onto itself.*

Proof. Take any $w = (\varphi,\alpha)^\top \in B[0,M]$. Then $\|\varphi\| \leqslant M$ and $k|\alpha| \leqslant M$. Let $u(t)$ be the solution of the problem (2.3)–(2.4). Then from the estimates (2.12) for the solution $u(t)$ and its derivatives we obtain

$$\|u\| \leqslant \Big(M_0 + \frac{1}{k}\Big)M, \quad \|u'\| \leqslant \Big(M_1 + \frac{2}{k}\Big)M, \quad \|u''\| \leqslant \Big(M_2 + \frac{2}{k}\Big)M.$$

Therefore, $(t,u(t),u'(t),u''(t)) \in \mathcal{D}_M$. Hence, by the assumption (2.15) we have

$$|f(t,u(t),u'(t),u''(t))| \leqslant M.$$

Now estimate $I := k|\int_0^1 g(s)u(s)\,\mathrm{d}s|$. In view of representation (2.8) we obtain

$$\begin{aligned} I &\leqslant k\int_0^1 g(s)\Big|\int_0^1 G_0(s,y)\varphi(y)\,\mathrm{d}y\Big|\,\mathrm{d}s + k|\alpha|\int_0^1 g(s)s^2\,\mathrm{d}s \\ &\leqslant kC_0M_0M + C_2M = (kC_0M_0 + C_2)M \leqslant M. \end{aligned} \tag{2.17}$$

The inequalities on the above line occur due to (2.11), (2.14) and the assumption (2.16).

Therefore, by the definition of the norm in the space $\mathcal{B}$ we have

$$\|Aw\|_{\mathcal{B}} \leqslant M,$$

which means that the operator A maps the closed ball $B[0,M]$ in $\mathcal{B}$ onto itself. The lemma is proved. $\square$

Lemma 2.3. *The operator A is a compact operator in $B[0,M]$.*

P r o o f. The compactness of A follows from the compactness of the integral operators (2.8), (2.9), (2.10), the continuity of the function $f(t,x,y,z)$ and the compactness of the integral operator $\int_0^1 g(s)u(s)\,\mathrm{d}s$. $\square$

Theorem 2.4 (Existence of solution)**.** *Suppose the conditions of Lemma* 2.2 *are satisfied. Then the problem* (1.6)–(1.7) *has a solution.*

P r o o f. By Lemma 2.2 and Lemma 2.3, the operator A is a compact operator mapping the closed ball $B[0,M]$ in the Banach space $\mathcal{B}$ onto itself. Therefore, according to the Schauder fixed point theorem, the operator A has a fixed point in $B[0,M]$. This fixed point corresponds to a solution of the problem (1.6)–(1.7). $\square$

In order to establish the existence of positive solutions of (1.6)–(1.7), let us introduce the domain

$$\begin{aligned}\mathcal{D}_M^+ = \Big\{(t,x,y,z) \mid 0 \leqslant t \leqslant 1, \quad 0 \leqslant x \leqslant \Big(M_0 + \frac{1}{k}\Big)M,\\ |y| \leqslant \Big(M_1 + \frac{2}{k}\Big)M,\ |z| \leqslant \Big(M_2 + \frac{2}{k}\Big)M\Big\},\end{aligned} \tag{2.18}$$

and the strip

$$S_M = \{w = (\varphi,\alpha)^\top \mid -M \leqslant \varphi \leqslant 0,\ 0 \leqslant k\alpha \leqslant M\} \tag{2.19}$$

in the space $\mathcal{B}$.

Theorem 2.5 (Positivity of solution)**.** *Suppose the function $f(t,x,y,z)$ is continuous and*

$$-M \leqslant f(t,x,y,z) \leqslant 0 \quad \text{in } \mathcal{D}_M^+, \tag{2.20}$$

and the condition (2.16) *is satisfied. Then the problem* (1.6)–(1.7) *has a nonnegative solution. Moreover, if $f(t,0,0,0) \not\equiv 0$, then this solution is positive.*

P r o o f. It is easy to verify that under the conditions of the theorem, the operator A maps S_M into itself. Indeed, for any $w \in S_M$, $w = (\varphi,\alpha)^\top$, $-M \leqslant \varphi \leqslant 0$, $0 \leqslant k\alpha \leqslant M$. Since $G_0(t,s) \leqslant 0$, from (2.8), (2.9), and (2.10) we have

$$0 \leqslant u(t) \leqslant \Big(M_0 + \frac{1}{k}\Big)M, \quad |u'(t)| \leqslant \Big(M_1 + \frac{2}{k}\Big)M, \quad |u''(t)| \leqslant \Big(M_2 + \frac{2}{k}\Big)M$$

for $0 \leqslant t \leqslant 1$. Therefore, for the solution $u(t)$ of (2.3)–(2.4) we have

$$(t,u(t),u'(t),u''(t)) \in \mathcal{D}_M^+,$$

and by the condition (2.20) we obtain

$$-M \leqslant f(t, u(t), u'(t), u''(t)) \leqslant 0.$$

As in the proof of Theorem 2.4 we also have the estimate

$$0 \leqslant k \int_0^1 g(s)u(s)\,\mathrm{d}s \leqslant M.$$

Hence, $(f(t, u(t), u'(t), u''(t)), \int_0^1 g(s)u(s)\,\mathrm{d}s)^\top \in S_M$, i.e. $A\colon S_M \to S_M$.

As was shown above, A is a compact operator in S. Therefore, A has a fixed point in S_M, which generates a solution of the problem (1.6)–(1.7). This solution is nonnegative. Moreover, if $f(t,0,0,0) \not\equiv 0$, then $u(t) \equiv 0$ cannot be the solution. Therefore, the solution is positive. $\square$

Theorem 2.6 (Existence and uniqueness). *Suppose that there exist numbers* $M > 0$, $L_0, L_1, L_2 \geqslant 0$ *such that*

(H1) $|f(t,x,y,z)| \leqslant M$ *for all* $(t,x,y,z) \in \mathcal{D}_M$,
(H2) $|f(t,x_2,y_2,z_2) - f(t,x_1,y_1,z_1)| \leqslant L_0|x_2 - x_1| + L_1|y_2 - y_1| + L_2|z_2 - z_1|$ *for all* $(t,x_i,y_i,z_i) \in \mathcal{D}_M$, $i = 1,2$,
(H3) $q := \max\{q_1, q_2\} < 1$, *where* $q_1 = kC_0M_0 + C_2$ *as was defined by* (2.16) *and*

$$q_2 = L_0\Big(M_0 + \frac{1}{k}\Big) + L_1\Big(M_1 + \frac{2}{k}\Big) + L_2\Big(M_2 + \frac{2}{k}\Big). \tag{2.21}$$

Then the problem (1.6)–(1.7) *has a unique solution* $u \in C^3[0,1]$.

P r o o f. To prove the theorem, it suffices to show that the operator A defined by (2.2) is a contractive mapping from the closed ball $B[0,M]$ in $\mathcal{B}$ onto itself. Indeed, under the assumption (H1) and the condition $q_1 < 1$ in the assumption (H2), by Lemma 2.2 the operator A maps $B[0,M]$ into itself.

Now, we show that A is a contraction map. Let $w_i = (\varphi_i, \alpha_i) \in B[0,M]$. We have

$$Aw_2 - Aw_1 = \begin{pmatrix} f(t, u_2(t), u_2'(t), u_2''(t) - f(t, u_1(t), u_1'(t), u_1''(t) \\ \int_0^1 g(s)(u_2(s) - u_1(s))\,\mathrm{d}s \end{pmatrix},$$

where $u_i(t)$ $(i = 1,2)$ is the solution of the problem

$$\begin{cases} u_i'''(t) = \varphi_i(t), & 0 < t < 1, \\ u_i(0) = u_i'(0) = 0, & u_i(1) = \alpha_i. \end{cases}$$

From the proof of Lemma 2.2 it is known that $(t, u_i(t), u_i'(t), u_i''(t)) \in \mathcal{D}_M$. Therefore, by the Lipschitz condition (H2) for f we have

$$
\begin{aligned}
D_1 &:= |f(t, u_2(t), u_2'(t), u_2''(t) - f(t, u_1(t), u_1'(t), u_1''(t)| \\
&\leqslant L_0|u_2(t) - u_1(t)| + L_1|u_2'(t) - u_1'(t)| + L_2|u_2''(t) - u_1''(t)|.
\end{aligned} \tag{2.22}
$$

Since $u_2(t) - u_1(t)$ is the solution of the problem (2.3)–(2.4) with the right-hand sides $\varphi_2(t) - \varphi_1(t)$ and $\alpha_2 - \alpha_1$, we have

$$
\begin{aligned}
\|u_2 - u_1\| &\leqslant M_0\|\varphi_2 - \varphi_1\| + |\alpha_2 - \alpha_1|, \\
\|u_2' - u_1'\| &\leqslant M_1\|\varphi_2 - \varphi_1\| + 2|\alpha_2 - \alpha_1|, \\
\|u_2'' - u_1''\| &\leqslant M_2\|\varphi_2 - \varphi_1\| + 2|\alpha_2 - \alpha_1|.
\end{aligned} \tag{2.23}
$$

As for the element $w = (\varphi, \alpha)^\top \in \mathcal{B}$ we use the norm

$$
\|w\|_{\mathcal{B}} = \max(\|\varphi\|, k|\alpha|), \quad k \geqslant 1,
$$

from (2.22), (2.23) we obtain

$$
\begin{aligned}
D_1 &\leqslant L_0\Big(M_0 + \frac{1}{k}\Big)\|w_2 - w_1\|_{\mathcal{B}} + L_1\Big(M_1 + \frac{2}{k}\Big)\|w_2 - w_1\|_{\mathcal{B}} \\
&\quad + L_2\Big(M_2 + \frac{2}{k}\Big)\|w_2 - w_1\|_{\mathcal{B}} \\
&\leqslant \Big(L_0\Big(M_0 + \frac{1}{k}\Big) + L_1\Big(M_1 + \frac{2}{k}\Big) + L_2\Big(M_2 + \frac{2}{k}\Big)\Big)\|w_2 - w_1\|_{\mathcal{B}} \\
&= q_2\|w_2 - w_1\|_{\mathcal{B}},
\end{aligned} \tag{2.24}
$$

where q_2 is defined by (2.21).

Now consider

$$
D_2 := k\Big|\int_0^1 g(s)(u_2(s) - u_1(s))\,\mathrm{d}s\Big|.
$$

By analogy with the estimate (2.17) it is easy to have

$$
D_2 \leqslant (kC_0M_0 + C_2)\|w_2 - w_1\|_{\mathcal{B}} = q_1\|w_2 - w_1\|_{\mathcal{B}}. \tag{2.25}
$$

From (2.24) and (2.25) we obtain

$$
\|Aw_2 - Aw_1\|_{\mathcal{B}} \leqslant \max\{q_1, q_2\}\|w_2 - w_1\|_{\mathcal{B}}.
$$

In view of condition (H3) the operator A is a contraction operator in $B[0, M]$. The theorem is proved. $\square$

Theorem 2.7 (Existence and uniqueness of positive solution)**.** *If in Theorem* 2.6 *we replace* $\mathcal{D}_M$ *by* $\mathcal{D}_M^+$ *and the condition* (H1) *by the condition* (2.20) *then the problem* (1.6)–(1.7) *has a unique nonnegative solution* $u(t) \in C^3[0,1]$. *Also, if* $f(t,0,0,0) \not\equiv 0$ *then this solution is positive.*

3. Iterative method

Suppose all the conditions of Theorem 2.6 are met. Then the problem (1.6)–(1.7) has a unique solution. To find it, consider the following iterative method:

(1) Given $w_0 = (\varphi_0, \alpha_0)^\top \in B[0,M]$, for example,

$$\varphi_0(t) = f(t,0,0,0), \ \ \alpha_0 = 0. \tag{3.1}$$

(2) Knowing $\varphi_n(t)$ and $\alpha_n(t)$ $(n = 0,1,\ldots)$, compute

$$u_n(t) = \int_0^1 G(t,s)\varphi_n(s)\,\mathrm{d}s + \alpha_n t^2, \tag{3.2}$$

$$y_n(t) = \int_0^1 G_1(t,s)\varphi_n(s)\,\mathrm{d}s + 2\alpha_n t, \tag{3.3}$$

$$z_n(t) = \int_0^1 G_2(t,s)\varphi_n(s)\,\mathrm{d}s + 2\alpha_n. \tag{3.4}$$

(3) Update

$$\varphi_{n+1}(t) = f(t, u_n(t), y_n(t), z_n(t)), \tag{3.5}$$

$$\alpha_{n+1} = \int_0^1 g(s)u_n(s)\,\mathrm{d}s. \tag{3.6}$$

Theorem 3.1. *Under the assumptions of Theorem* 2.6, *the above iterative method converges, and for the approximate solution* $u_n(t)$ *and its derivatives the following estimates hold:*

$$\|u_n - u\| \leqslant \Big(M_0 + \frac{1}{k}\Big)p_n d, \tag{3.7}$$

$$\|u_n' - u'\| \leqslant \Big(M_1 + \frac{2}{k}\Big)p_n d, \tag{3.8}$$

$$\|u_n'' - u''\| \leqslant \Big(M_2 + \frac{2}{k}\Big)p_n d, \tag{3.9}$$

where $p_n = q^n/(1-q)$, $d = \|w_1 - w_0\|_{\mathcal{B}}$, $w_1 = (\varphi_1, \alpha_1)^\top$.

Proof. In fact, the above iterative method is the successive iterative method for finding the fixed point of operator A. Therefore, it converges with the rate of geometric progression and the following estimate holds:

$$\|w_n - w\|_{\mathcal{B}} \leqslant \frac{q^n}{1-q}\|w_1 - w_0\|_{\mathcal{B}} = p_n d,$$

where $w_n - w = (\varphi_n - \varphi, \alpha_n - \alpha)^{\top}$.

From the definition of the norm in $\mathcal{B}$ and the above estimate it follows that

$$\begin{aligned} \|\varphi_n - \varphi\| &\leqslant \|w_n - w\|_{\mathcal{B}} \leqslant p_n d, \\ \|\alpha_n - \alpha\| &\leqslant \frac{1}{k}\|w_n - w\|_{\mathcal{B}} \leqslant \frac{1}{k} p_n d. \end{aligned}$$

Now, the estimates (3.7)–(3.9) are easily obtained if taking into account the representations (2.8)–(2.10), (3.2)–(3.4), the estimates of the type (2.12) and the above estimates. □

To numerically realize the iterative method (3.1)–(3.6) we cover the interval $[0,1]$ by the uniform grid $\omega_h = \{t_i = ih,\ h = 1/N,\ i = 0, 1, \ldots, N\}$ and use the trapezium formula for computing definite integrals. In all examples in the next section the numerical computations will be performed on the grid with $h = 0.01$ until $\max\{\|\varphi_n - \varphi_{n-1}\|, k|\alpha_n - \alpha_{n-1}|\} \leqslant 10^{-4}$, where k will be defined for each particular example.

4. Examples

Consider some examples for confirming the validity of the obtained theoretical results and the efficiency of the proposed iterative method.

Example 4.1 (Example with exact solution). Consider the problem (1.6)–(1.7) with

$$f = f(t,u) = -\frac{1}{2} + \frac{1}{3}\Big(\frac{1}{6}\Big(t^2 - \frac{t^3}{2}\Big)\Big)^2 - u^2, \quad g(s) = \frac{56}{9}s^4.$$

It is possible to verify that the positive function

$$u(t) = \frac{1}{6}\Big(t^2 - \frac{t^3}{2}\Big), \quad 0 \leqslant t \leqslant 1,$$

is the exact solution of the problem.

For the given $g(s)$, simple calculations give $C_0 = \frac{56}{45}$, $C_2 = \frac{56}{63}$. Therefore, with $k = 2$, we obtain $q_1 = 0.9503 < 1$. For this k it is possible to choose $M = 0.6$ such that $-M \leqslant f(t,x) \leqslant 0$ for

$$(t,x) \in \mathcal{D}_M^+ = \Big\{(t,x) \mid 0 \leqslant t \leqslant 1,\ 0 \leqslant x \leqslant \Big(M_0 + \frac{1}{2}\Big)M = 0.5247M\Big\}.$$

Indeed,

$$0 \leqslant -f(t,x) = \frac{1}{2} + x^2 - \frac{1}{3}\Big(\frac{1}{6}\Big(t^2 - \frac{t^3}{2}\Big)\Big)^2 \leqslant \frac{1}{2} + x^2 \leqslant \frac{1}{2} + (0.5247M)^2 \leqslant M.$$

Thus, M must satisfy $0.2753M^2 - M + 0.5 \leqslant 0$. The direct calculation of the left side for $M = 0.6$ gives the value $= -0.0670$. So, the choice of M is justified.

Further, for $f(t,x)$ we have the Lipschitz coefficient with respect to x in $\mathcal{D}_M^+$, $L_0 = 0.3148$. Consequently, $q_2 = L_0(M_0 + \frac{1}{2}) = 0.1652$, and $q = 0.9503$. Also, $f(t,0) \not\equiv 0$. Therefore, by Theorem 2.7, the problem has a unique positive solution. It is the above exact solution.

The computation shows that the iterative method (3.1)–(3.6) converges and the error of the 46th iteration compared with the exact solution is $1.1458\mathrm{e}-04$.

Example 4.2 (Example 4.1 in [12]). Consider the boundary value problem

$$\begin{gathered} u'''(t) = -u^2\mathrm{e}^u, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \quad u(1) = \int_0^1 s^4 u(s)\,\mathrm{d}s. \end{gathered}$$

In this example

$$f(t,x,y,z) = -x^2\mathrm{e}^x, \quad g(s) = s^4.$$

So,

$$C_0 = \int_0^1 g(s)\,\mathrm{d}s = \frac{1}{5}, \quad C_2 = \int_0^1 s^2 g(s)\,\mathrm{d}s = \frac{1}{7}.$$

Choose $k = 2$ in the definition of the norm of the space $\mathcal{B}$ (2.1) and in the definition of $\mathcal{D}_M^+$ by (2.18). Then $q_1 = kC_0M_0 + C_2 = 0.1527$. For $M = 0.4$ it is possible to verify that $-M \leqslant f(t,x) \leqslant 0$ in $\mathcal{D}_M^+$, $|\partial f/\partial x| \leqslant 0.5721$ in $\mathcal{D}_M^+$. Therefore,

$$L_0 = 0.5721, \quad q_2 = L_0\Big(M_0 + \frac{1}{k}\Big) = 0.3002.$$

Hence, by Theorem 2.7, the problem has a unique nonnegative solution. This solution should be $u(t) \equiv 0$, because $u(t) \equiv 0$ solves the problem. The numerical experiments by the iterative method in Section 3 confirm this conclusion.

Note that, in [12], the authors concluded that the problem has at least one positive solution. From our result above, it is clear that their conclusion is not valid.

E x a m p l e 4.3. Consider Example 4.2 with the nonlinear term $f = -(1+u^2)$.

Clearly, $f(u)/u \to -\infty$ as $u \to +0$ and $u \to \infty$. Thus, neither Theorem 3.1 nor Theorem 3.2 in [12] are applicable, so the existence of positive solution is not guaranteed.

Now apply our method. Choose $M = 2$, $k = 3$. Then

$$\mathcal{D}_M^+ = \Big\{(t,x) \mid 0 \leqslant t \leqslant 1,\ 0 \leqslant x \leqslant \Big(M_0 + \frac{1}{k}\Big)M = 0.7160\Big\}.$$

In $\mathcal{D}_M^+$ we have

$$-M \leqslant f \leqslant 0, \quad |f'_u| \leqslant 1.4321 = L_0,$$
$$q_1 = kC_0M_0 + C_2 = 0.1577, \quad q_2 = L_0\Big(M_0 + \frac{1}{3}\Big) = 0.5127.$$

Hence, by Theorem 2.7, the problem has a unique nonnegative solution. Due to $f(t,0) \neq 0$, this solution is positive. The graph of the approximate solution obtained with the given accuracy 10^{-4} after 4 iterations by the iterative method is depicted in Figure 1.

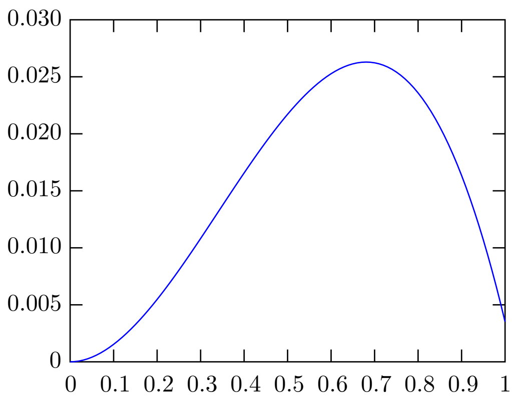

Figure 1. The graph of the approximate solution in Example 4.3.

E x a m p l e 4.4. Consider Example 4.2 with the nonlinear term

$$f = -\Big(u^2 e^u + \frac{1}{5}\sin(u') + \frac{1}{8}\cos(u'') + 1\Big).$$

In this example

$$f(t,x,y,z) = -\Big(x^2 e^x + \frac{1}{5}\sin(y) + \frac{1}{8}\cos(z) + 1\Big).$$

Choose $M = 1.7, k = 4$. It is possible to verify that in $\mathcal{D}_M^+$ we have $-M \leqslant f \leqslant 0$, and the Lipschitz coefficients of f are

$$L_0 = 1.8378, \quad L_1 = \frac{1}{5}, \quad L_2 = \frac{1}{8}.$$

Therefore,

$$q_1 = 0.1626, \quad q_2 = 0.7618.$$

Hence, by Theorem 2.7, the problem has a unique positive solution. The graph of the approximate solution obtained with the given accuracy 10^{-4} after 6 iterations by the iterative method is depicted on Figure 2.

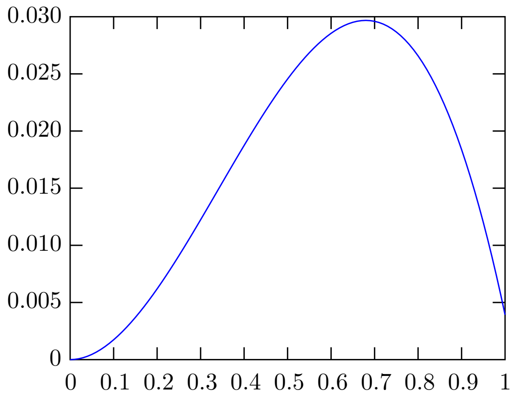

Figure 2. The graph of the approximate solution in Example 4.4.

5. Conclusion

We have proposed a novel method to study the fully third order differential equation with integral boundary conditions. It is based on the reduction of the boundary value problems to a fixed-point problem for an appropriate operator defined on a space of mixed pairs of functions and numbers. In this way, we have established the existence, uniqueness and positivity of solution of the problem under easily verified conditions. Another important result is that we have proposed an effective iterative method for finding the solution. The theoretical results have been demonstrated on some examples, including an example with an exact solution and other examples where the exact solutions are not known. Especially, we have shown that the conclusion on the existence of positive solutions for an example considered before by other authors is not valid.

The proposed method can be applied to problems with other integral boundary conditions for the third and higher order differential equations. This will be the subject of our research in the future.

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