SOLUTION OF OPTION PRICING EQUATIONS USING ORTHOGONAL POLYNOMIAL EXPANSION

Falko Baustian, Rostock, Kateřina Filipová, Jan Pospíšil, Plzeň

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Abstract. We study both analytic and numerical solutions of option pricing equations using systems of orthogonal polynomials. Using a Galerkin-based method, we solve the parabolic partial differential equation for the Black-Scholes model using Hermite polynomials and for the Heston model using Hermite and Laguerre polynomials. We compare the obtained solutions to existing semi-closed pricing formulas. Special attention is paid to the solution of the Heston model at the boundary with vanishing volatility.

Keywords: orthogonal polynomial expansion; Hermite polynomial; Laguerre polynomial; Heston model; option pricing

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1. Introduction

One of the fundamental tasks in financial mathematics is the pricing of derivatives, in particular option pricing. An *option* is a contract between two parties which gives the holder the right (but not the obligation) to buy or sell the underlying asset under certain conditions on or before a specified future date. The price that is paid for the underlying asset when the option is exercised is called the *strike* price and the last day on which the option may be exercised is called the *expiration date* or *maturity date*. Whether the holder has the right to buy or sell the underlying asset depends on the type of option to which the contract is signed. There is either a *call option* which allows the holder to buy the asset at a stated price within a specific time-frame, or a *put option* which allows the holder to sell the asset. In this article we will restrict ourselves to European options that can be exercised only on the expiration date.

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In their Nobel-prize winning paper, Black and Scholes [9] proposed a model for evaluating the fair value of the European call option that gives the right to buy a single share of common stock and derived a semi-closed formula for the option price, the so-called Black-Scholes formula. For the model they have assumed a frictionless market with ideal conditions like the absence of arbitrage and the possibility to borrow and lend any amount of money and to buy and sell any amount of stock. Volatility in the Black-Scholes (BS) model is assumed to be constant, which has later become its most discussed feature. Constant volatility matches poorly with the observed implied volatility surface for real market data. Especially for out-of-the-money options the market prices are significantly higher than what the model suggests. This phenomenon is widely known as the volatility smile. For a better fit to the data, Hull and White [27] proposed to model volatility as another stochastic process. There are various stochastic volatility models from Hull and White [27], Stein and Stein [41], Heston [24], and many others. Later on, additional jump components were included into the models, e.g. Bates [6].

Up to this day, the Heston model is quite popular among economists and practitioners. Heston [24] modelled the volatility using the mean-reverting Cox, Ingersoll, and Ross [13] process (CIR), which allowed arbitrary correlation between volatility and spot asset returns. Heston also derived a semi-closed formula close to the BS formula. Both in the BS and Heston model, one can derive the pricing partial differential equation (PDE) in several different ways, for example Wilmott [50], Rouah [40], Hull [26] using arbitrage arguments with self-financing trading strategies, approaches with martingale measures or the Fokker-Planck equation for the transition probability density function. Although semi-closed formulas have been widely used in practice for a long time, only recently Daněk and Pospíšil [14] showed that for certain values of model parameters these formulas can bring serious numerical difficulties, especially in evaluation of the integrands in these formulas, and its implementation therefore sometimes requires a demanding high-precision arithmetic to be adopted.

Many different numerical methods can be used to solve option pricing problems, such as Monte Carlo methods (including the Quasi Monte Carlo), Fourier-based methods (including the Fast Fourier Transform method, Fourier method with Gauss-Laguerre quadrature, cosine series method), finite differences methods (with different time-stepping schemes, different grid refinements including the adaptive refinement or discontinuous Galerkin method), finite element methods (including the method with NURBS basis functions introduced by Pospíšil and Švígler [37]) or, for example, radial basis function methods (RBF). We refer the reader to the references in the BENCHOP project report written by von Sydow et al. [48], who implemented fifteen different numerical methods with the help of different advanced specialized techniques, and who compared all the methods for different benchmark problems and consequently discussed advantages and disadvantages of each method.

The aim of this paper is to solve the pricing PDEs for both the BS and Heston model using orthogonal polynomial expansions that are motivated by the Galerkin method. The expansion approach offers several advantages as we approximate the solution by smooth functions. Therefore, it gives more insight into how parameters influence prices and to what extent, and hence give a better understanding of the solution than the semi-closed form or other approximation method, especially for the Heston model. For the sake of clarity of the method we omit application of specialized techniques that could further improve the proposed method. Among the other mentioned methods, only FEM with smooth basis functions and RBF approximate the solution by smooth functions. One advantage of the orthogonal polynomial expansion is hence the independence of the space variable discretization (finite elements) or spacial node locations (RBF).

Aubin in [5] studied Galerkin-type methods and their convergence for elliptic partial differential equations and Birkhoff, Schultz, and Varga [8] used piecewise Hermite polynomials for this problem. Time-dependent equations were investigated with the usage of the Galerkin method by Schwartz and Wendroff [42]. The initial value problem for a general parabolic equation of second order was first studied by Douglas and Dupont [16]. They used Galerkin-type methods, both continuous and discrete in time, and established a priori estimates to control the error. These articles initiated several other papers by Dupont [17], Fix and Nassif [20], Wheeler [49], Bramble and Thomée [11], Bramble, Schatz, Thomée, and Wahlbin [10], and Thomée [45]. Most of the a priori estimates are formulated with regard to the L^2 norm, but Bramble, Schatz, Thomée, and Wahlbin [10] offer estimates for the maximum norm as well. Nonlinear parabolic equations were covered by Wheeler [49]. A survey of results can be found in Thomée [46] and in the monograph Thomée [47].

The application of orthogonal polynomial expansions in option pricing was to our knowledge for the first time suggested by Jarrow and Rudd [28], who pioneered the use of Edgeworth expansions for valuation of derivative securities. Later Corrado and Su [12] introduced the Gram-Charlier expansions. In the recent past, Hermite polynomial expansion approaches have been used in some interesting articles regarding different aspects of the option pricing problem.

Xiu [51] studied a closed-form series expansion of European call option prices in the time variable and this series expansion was derived using the Hermite polynomials. Xiu introduced two approaches on vanilla option and binary option. The first one has been a bottom-up Hermite polynomial approach and the second one has been a top-down lucky guess approach. As the benchmark model he has chosen the BS model but stated that square-root (SQR) models for the volatility like Heston [24], quadratic volatility (QV) models, constant elasticity of variance (CEV) models, which introduce as an additional parameter the elasticity of variance, or several jump-diffusion models can be considered, see for example a recent monograph by Lewis [35].

Heston and Rossi [25] showed that Edgeworth expansions for option valuation are equivalent to approximating the option payoff using Hermite polynomials and logistic polynomials. Consequently, the value of an option is equal to the value of an infinite series of replicating polynomials. Heston and Rossi provide efficient alternative moment-based formulas to express option values in terms of skewness, kurtosis and higher moments.

Polynomial expansions with Hermite and Laguerre polynomials play also a substantial role in Alziary and Takáč [3]. The authors rigorously formulate the Cauchy problem connected to the Heston model as a parabolic PDE with a special focus on the boundary conditions, which are often neglected in the literature. Alziary and Takáč provide the real analyticity of the solution which is directly connected to the problem of market completeness studied in Davis and Obłój [15]. The polynomial expansions are used in the proof of the main results of the article. Further investigations of the boundary conditions can be found in the forthcoming work Alziary and Takáč [4].

Very recently, option pricing with orthogonal polynomial expansions has been studied by Ackerer and Filipović [2], who derived option prices series representation by expansion of the characteristic functions rather than by solving the pricing PDE.

The structure of the paper is the following. In Section 2 we introduce the system of orthogonal polynomials, studied models, as well as other necessary terms and fundamental properties. In Section 3 we solve the Black-Scholes and Heston PDE using the orthogonal polynomial expansion. To solve the BS PDE we use Hermite polynomials and to solve the Heston PDE we use a combination of Hermite and Laguerre polynomials. In Section 4 we present all numerical results, especially the comparison to the existing semi-closed form solutions. We conclude in Section 5.

2. Preliminaries and notation

2.1. Orthogonal polynomials. Standard theory for parabolic PDEs requires initial data in a Lebesgue space. In the PDE pricing approach for European-type derivatives, the initial value corresponds to the payoff function of the contract but unfortunately the payoff of many European options, e.g., the European call option, is unbounded and not Lebesgue-integrable. For this reason we consider weighted Lesbesgue spaces with a positive *weight function* w as studied in Kufner [31], Kufner and Sändig [32], and Funaro [22].

The *weighted Lebesgue space* $L^2(\mathbb{R}, w\,\mathrm{d}x)$ is the space of all measurable functions f for which

$$\|f\|_w := \left(\int_{\mathbb{R}} |f(x)|^2 w(x)\,\mathrm{d}x\right)^{1/2} < \infty.$$

As usual, we consider representatives of classes of functions which are equal almost everywhere. We can also define weighted Sobolev spaces $H^k(\mathbb{R}, w\,\mathrm{d}x)$ for $k \geqslant 1$. Again, we refer the reader to Kufner [31], Kufner and Sändig [32], and Funaro [22], for details on such spaces.

We consider sequences (F_n) of real polynomials in $L^2(\mathbb{R}, w\,\mathrm{d}x)$ which are pairwise orthogonal with respect to the *inner product* defined by

$$\langle f, g\rangle_w := \int_{\mathbb{R}} f(x)g(x)w(x)\,\mathrm{d}x \quad \text{for } f, g \in L^2(\mathbb{R}, w\,\mathrm{d}x). \tag{2.1}$$

It can be shown that for given $F_0(x)$ and $F_1(x)$ that are not both identically zero, there exist functions $\alpha(n, x)$ and $\beta(n, x)$ such that the system of orthogonal polynomials satisfies the so-called *three-term recurrence relation*

$$F_{n+1}(x) = \alpha(n, x)F_n(x) + \beta(n, x)F_{n-1}(x), \quad n \in \mathbb{N}. \tag{2.2}$$

Relation (2.2) is arguably the single most important piece of information for the constructive and computational use of orthogonal polynomials. For more details on general systems of orthogonal polynomials and on the proof of the recurrence relation we refer the reader to the book by Gautschi [23].

Throughout the paper we will work especially with Hermite and Laguerre polynomials. Their properties are rather extensively described in many monographs, we refer the readers for example to the books by Abramowitz and Stegun [1], Chapter 22; Lebedev [33], Chapter 4; Szëgo [43], Chapter 5; Thangavelu [44] and Olver, Lozier, Boisvert, and Clark [36], Chapter 18, to name a few. The definition and basic properties of Hermite and Laguerre polynomials can be found in all of these monographs.

2.1.1. Hermite polynomials. Hermite polynomials are orthogonal polynomials on the real line. There exists two types of Hermite polynomials that differ slightly in the choice of weight function and that are called probabilists' (weight function $\mathrm{e}^{-x^2/2}$) and physicists' (weight function e^{-x^2}) Hermite polynomials. Those two types can be easily converted into each other and we will consider physicists' polynomials only.

Definition 2.1. The *system of Hermite polynomials* is defined by the Rodrigues formula

$$H_m(x) := (-1)^m \mathrm{e}^{x^2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \mathrm{e}^{-x^2}, \quad m \in \mathbb{N}_0.$$

The three-term recurrence (2.2) for Hermite polynomials reads

$$H_{m+1}(x) = 2xH_m(x) - 2mH_{m-1}(x), \quad m \geqslant 1. \tag{2.3}$$

The Hermite polynomials form a complete orthogonal system in the weighted Lebesgue space $L^2(\mathbb{R}, \mathrm{e}^{-x^2}\,\mathrm{d}x)$ with $\langle H_m, H_n\rangle_w = 2^n n!\sqrt{\pi} \cdot \delta_{m,n}$, where $\delta_{m,n}$ is the Kronecker delta, as well as an orthogonal set in the weighted Sobolev space $H^k(\mathbb{R}, \mathrm{e}^{-x^2}\,\mathrm{d}x)$ for $k \geqslant 1$. See Lebedev [33], Section 4.14 for the orthogonality and Szegö [43], Section 5.7 for the completeness of the system.

In the following lemma we state several useful simplifications of integral terms that are consequences of Definition 2.1 and the three-term recurrence (2.3).

Lemma 2.2. *For all* $m, n \in \mathbb{N}_0$,

$$\frac{1}{2^n n!\sqrt{\pi}} \int_{-\infty}^{\infty} H_m'(x) H_n'(x) \mathrm{e}^{-x^2}\,\mathrm{d}x = 2m\delta_{m,n}, \tag{2.4}$$

$$\frac{1}{2^n n!\sqrt{\pi}} \int_{-\infty}^{\infty} H_m(x) H_n'(x) \mathrm{e}^{-x^2}\,\mathrm{d}x = \delta_{m+1,n}, \tag{2.5}$$

$$\frac{1}{2^n n!\sqrt{\pi}} \int_{-\infty}^{\infty} xH_m'(x) H_n(x) \mathrm{e}^{-x^2}\,\mathrm{d}x = 2(n+1)m\delta_{m,n+2} + m\delta_{m,n}, \tag{2.6}$$

$$\frac{1}{2^n n!\sqrt{\pi}} \int_{-\infty}^{\infty} xH_m(x) H_n(x) \mathrm{e}^{-x^2}\,\mathrm{d}x = \frac{1}{2}\delta_{m+1,n} + m\delta_{m-1,n}. \tag{2.7}$$

A proof can be found in the thesis Filipová [19], Chapter 2, Lemmas 2.5–2.9.

2.1.2. Laguerre polynomials. The volatility process in the Heston model is strictly positive provided that the Feller condition is satisfied. Hence, we need a system of orthogonal polynomials on the positive part of the real line for the expansion in the volatility variable. With the weight function $w\colon \mathbb{R}^+ \to \mathbb{R}$, $w(v) = \mathrm{e}^{-v}$, on $\mathbb{R}^+ = (0, \infty)$ such a system is given by the Laguerre polynomials.

Definition 2.3. The *system of Laguerre polynomials* is defined by

$$L_n(v) := \frac{\mathrm{e}^v}{n!}\frac{\mathrm{d}^n}{\mathrm{d}v^n}(\mathrm{e}^{-v}v^n), \quad n \in \mathbb{N}_0.$$

The three-term recurrence (2.2) for the Laguerre polynomials is

$$L_{n+1}(v) = \frac{1}{n+1}[(-v+2n+1)L_n(v) - nL_{n-1}(v)], \quad n \geqslant 1. \tag{2.8}$$

The Laguerre polynomials form a complete orthonormal system in the weighted Lebesgue space $L^2(\mathbb{R}^+, \mathrm{e}^{-v}\,\mathrm{d}v)$. The orthonormality of the system is studied in Lebedev [33], Section 4.21 and the completeness in Szegö [43], Section 5.7.

We use Definition 2.3 and the three-term recurrence (2.8) to obtain some simplifications.

Lemma 2.4. *For all* $m, n \in \mathbb{N}_0$,

$$\int_0^\infty vL_m(v)L_n(v)\mathrm{e}^{-v}\,\mathrm{d}v = (2m+1)\delta_{m,n} - m\delta_{m-1,n} - (m+1)\delta_{m+1,n}, \tag{2.9}$$

$$\int_0^\infty vL'_m(v)L_n(v)\mathrm{e}^{-v}\,\mathrm{d}v = m(\delta_{m,n} - \delta_{m-1,n}), \tag{2.10}$$

$$\int_0^\infty vL'_m(v)L'_n(v)\mathrm{e}^{-v}\,\mathrm{d}v = m\delta_{m,n}, \tag{2.11}$$

$$\int_0^\infty L'_m(v)L_n(v)\mathrm{e}^{-v}\,\mathrm{d}v = -\sum_{a=0}^{m-1}\delta_{a,n}. \tag{2.12}$$

A proof can be found in the thesis Filipová [19], Chapter 2, Lemmas 2.12–2.15. It is worth mentioning that the formulas in Lemma 2.2 and Lemma 2.4 are not stated in any of the monographs listed above.

2.1.3. Finite-dimensional projections. In the following, we study orthogonal projections of functions in weighted Lebesgue spaces into finite-dimensional subspaces spanned by Hermite and Laguerre polynomials. See Funaro [22] for details of the projection operators.

First, we consider the weight function $w(x) = \mathrm{e}^{-x^2}$ on the real line $\mathbb{R}$ and denote by S_M^{H} the vector space spanned by the first $M+1$ Hermite polynomials. The orthogonal projector $\Pi_M^{\mathrm{H}}\colon\ L^2(\mathbb{R}, w\,\mathrm{d}x) \to S_M^{\mathrm{H}}$ with

$$\Pi_M^{\mathrm{H}} f = \sum_{i=0}^{M} \frac{\langle f, H_i\rangle_w}{\|H_i\|_w^2} H_i = \sum_{i=0}^{M} \frac{\langle f, H_i\rangle_w}{2^i i!\sqrt{\pi}} H_i \quad \text{for } f \in L^2(\mathbb{R}, w\,\mathrm{d}x) \tag{2.13}$$

satisfies

$$\|f - \Pi_M^{\mathrm{H}} f\|_w = \inf_{\phi \in S_M^{\mathrm{H}}} \|f - \phi\|_w \quad \text{and} \quad \lim_{M\to\infty} \|f - \Pi_M^{\mathrm{H}} f\|_w = 0$$

for every $f \in L^2(\mathbb{R}, w\,\mathrm{d}x)$. Moreover, for each $k \in \mathbb{N}_0$ there exists a constant $C = C(k) > 0$ such that

$$\|f - \Pi_M^{\mathrm{H}} f\|_w \leqslant C M^{-k/2} \Big\| \frac{\mathrm{d}^k f}{\mathrm{d}x^k} \Big\|_w \quad \forall\, M > k \tag{2.14}$$

and for every $f \in H^k(\mathbb{R}, w\,\mathrm{d}x)$, see Funaro [22], Theorem 6.2.6. We will later use the orthogonal projector Π_M^{H} defined in (2.13) to study the Black-Scholes model.

Next, we consider the weight function $w(v) = \mathrm{e}^{-v}$ on $\mathbb{R}^+$ and denote by S_N^{L} the vector space spanned by the first $N+1$ Laguerre polynomials. The orthogonal projector $\Pi_N^{\mathrm{L}} \colon L^2(\mathbb{R}^+, w\,\mathrm{d}v) \to S_N^{\mathrm{L}}$ with

$$\Pi_N^{\mathrm{L}} f = \sum_{j=0}^{N} \langle f, L_j \rangle_w L_j \quad \text{for } f \in L^2(\mathbb{R}^+, w\,\mathrm{d}v)$$

satisfies the same approximation properties as Π_M^{H} and for each $k \in \mathbb{N}_0$ we have

$$\|f - \Pi_N^{\mathrm{L}} f\|_w \leqslant C N^{-k/2} \Big\| x^{k/2} \frac{\mathrm{d}^k f}{\mathrm{d}x^k} \Big\|_w \quad \forall\, N > k \tag{2.15}$$

for every f with $\frac{\mathrm{d}^m f}{\mathrm{d}x^m} x^{m/2} \in L^2(\mathbb{R}^+, w\,\mathrm{d}v)$, $0 \leqslant m \leqslant k$, and a constant $C = C(k) > 0$, see Funaro [22], Theorem 6.2.5.

To treat models with non-constant volatility, we use a weighted Lebesgue space in two variables. A *weighted Lebesgue space* $L^2(\mathbb{R} \times \mathbb{R}^+, w\,\mathrm{d}x\,\mathrm{d}v)$ with the *weight function* $w \colon \mathbb{R} \times \mathbb{R}^+ \to (0, \infty)$ is the space of all measurable functions g for which

$$\|g\|_w := \Big(\int_{\mathbb{R}} \int_{\mathbb{R}^+} |g(x,v)|^2 w(x,v)\,\mathrm{d}v\,\mathrm{d}x \Big)^{1/2} < \infty.$$

The inner product is defined in accordance with (2.1). For the Heston model, we will consider the weighted Lebesgue space $L^2(\mathbb{R} \times \mathbb{R}^+, w\,\mathrm{d}x\,\mathrm{d}v)$ with the weight $w(x,v) = \mathrm{e}^{-x^2 - v}$. Due to Reed and Simon [39], Section II.4, the products of Hermite and Laguerre polynomials $P_{i,j}(x,v) = H_i(x) L_j(v)$ for $i, j \in \mathbb{N}_0$ with $\langle P_{k,l}, P_{i,j} \rangle_w = 2^i i! \sqrt{\pi} \cdot \delta_{k,i} \cdot \delta_{l,j}$ for $k, l, i, j \in \mathbb{N}_0$ form a complete orthogonal set in $L^2(\mathbb{R} \times \mathbb{R}^+, w\,\mathrm{d}x\,\mathrm{d}v)$. Let $S_{M,N}$ denote the vector space spanned by the products of the first $M+1$ Hermite polynomials and the first $N+1$ Laguerre polynomials. The orthogonal projector $\Pi_{M,N} \colon L^2(\mathbb{R} \times \mathbb{R}^+, w\,\mathrm{d}x\,\mathrm{d}v) \to S_{M,N}$ defined by

$$\Pi_{M,N} f = \sum_{i=0}^{M} \sum_{j=0}^{N} \frac{\langle f, P_{i,j} \rangle_w}{2^i i! \sqrt{\pi}} P_{i,j} \quad \text{for } f \in L^2(\mathbb{R} \times \mathbb{R}^+, w\,\mathrm{d}x\,\mathrm{d}v) \tag{2.16}$$

inherits the approximation properties from the projection operators Π_M^{H} and Π_N^{L}.

For practical reasons, we have to evaluate the finite-dimensional projections (2.13) and (2.16) numerically, where Clenshaw's algorithm (see Press, Teukolsky, Vetterling, and Flannery [38], Section 5.4) will be of use. To evaluate the Fourier coefficients

$$\text{(2.17)} \qquad \text{(i) } c_i = \frac{\langle f, H_i\rangle_w}{2^i i! \sqrt{\pi}} \quad \text{and} \quad \text{(ii) } c_{i,j} = \frac{\langle f, P_{i,j}\rangle_w}{2^i i! \sqrt{\pi}}$$

in (2.13) and (2.16) precisely, it is necessary to choose the appropriate quadrature. Here we consider the Gauss-Hermite and Gauss-Laguerre quadratures, see for example in the books Abramowitz and Stegun [1], Section 25.4, Szegö [43], Section 14.5–14.7, Olver, Lozier, Boisvert, and Clark [36], Section 3.5 or Press, Teukolsky, Vetterling, and Flannery [38], Section 4.6.

2.2. Option pricing models. Since options are frequently traded contracts, the derivation of the option prices is an important task in mathematical finance. There exist several models for option pricing in an arbitrage-free setting. The prices that can be provided by these models give us an idea how the real market prices should behave. We will consider option pricing in the classical models by Black and Scholes [9] with constant volatility and by Heston [24] with a mean-reverting stochastic volatility process.

In this article, we restrict ourselves to the pricing of European call options, since the price of the corresponding European put options can be obtained by the put-call parity. A European option contract is characterized by two parameters, maturity T and strike price K. We introduce also a variable $\gamma > 0$ that is sometimes called *moneyness* and that measures a relative position of the price S of an underlying asset (typically a stock) with respect to the strike price, i.e., $S = \gamma K$. If $\gamma = 1$, we say that the option is at-the-money (ATM), for $\gamma > 1$ the call option is in-the-money (ITM) and for $\gamma < 1$ it is out-of-the-money (OTM). For put options it is clearly the reverse.

In both models the money market is represented by a risk-free bond

$$\mathrm{d}B_t = rB_t \,\mathrm{d}t,$$

with constant interest rate $r > 0$.

2.2.1. Black-Scholes model. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a fixed filtration $(\mathcal{F}_t)$ generated by a standard Wiener process W_t^S. In the BS model, the stock price process S_t is modelled as a positive continuous semimartingale with respect to $(\mathcal{F}_t)$ and satisfies the stochastic differential equation

$$\text{(2.18)} \qquad \mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t^S,$$

where the drift $\mu \in \mathbb{R}$ and volatility $\sigma > 0$ are constant. The fair price $V_t = V(S_t, t)$ of a European call option with maturity T and strike price K is defined by the risk-free pricing formula

$$V_t = \mathrm{e}^{-r(T-t)}\mathbb{E}^*[(S_T - K)^+ \mid \mathcal{F}_t], \tag{2.19}$$

where the conditional expectation is considered under the unique equivalent martingale measure $\mathbb{P}^*$ provided that the function V is continuous. The equivalent measure $\mathbb{P}^*$ can be obtained from (2.18) by replacing μ by $\mu^* = r$ and keep $\sigma^* = \sigma$. It can be shown that V also satisfies the Black-Scholes partial differential equation

$$\frac{\partial}{\partial t}V + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

for $(S, t) \in (0, \infty) \times (0, T)$ with the terminal condition $V(S, T) = (S - K)^+$. There exist several approaches to obtain the PDE-like replication of the derivative with a self-financing portfolio or delta hedging. For more details on replication strategies we refer to Karatzas and Shreve [30], Chapter 5.8.B. We introduce new variables $\tau = T - t$ and $x = \ln S$, for the time till maturity and the logarithm of the stock price, respectively. For the function $u(x, \tau) = V(S, t)$ we obtain the parabolic Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial \tau}u(x,\tau) = \mathcal{L}^{\mathrm{BS}}u(x,\tau) & \text{for } (x,\tau) \in \mathbb{R} \times (0,T), \\ u(x,0) = (\mathrm{e}^x - K)^+ & \text{for } x \in \mathbb{R}, \end{cases} \tag{BS}$$

with the Black-Scholes operator

$$\mathcal{L}^{\mathrm{BS}}u := \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}u + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial x}u - ru.$$

Black and Scholes [9] formula for the fair price of a European call option reads

$$u^{\mathrm{BS}}(x,\tau) = \mathrm{e}^x N(d_1) - K\mathrm{e}^{-r\tau}N(d_2), \tag{2.20}$$

where

$$\begin{aligned} d_1 &= \frac{x - \ln K + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \\ d_2 &= d_1 - \sigma\sqrt{\tau}, \end{aligned}$$

and $N(\cdot)$ denotes the cumulative distribution function of the standard normal distribution.

2.2.2. Heston model. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $(\mathcal{F}_t)$ and let W_t^S and W_t^v be two standard Wiener processes with respect to the filtration that are correlated by a factor $\varrho \in [-1, 1]$. In contrast to the BS model, in the Heston model the volatility is modelled as the square-root of a mean-reverting stochastic process v_t. Both the stock price process S_t and v_t are continuoussemi-martingales with respect to $(\mathcal{F}_t)$, these processes are assumed to be positive. The model dynamics are

$$
\begin{aligned}
\mathrm{d}S_t &= \mu S_t \,\mathrm{d}t + \sqrt{v_t} S_t \,\mathrm{d}W_t^S, \\
\mathrm{d}v_t &= \kappa(\theta - v_t)\,\mathrm{d}t + \widetilde{\sigma}\sqrt{v_t}\,\mathrm{d}W_t^v, \\
\varrho \,\mathrm{d}t &= \mathrm{d}W_t^S \,\mathrm{d}W_t^v.
\end{aligned} \tag{2.21}
$$

The drift $\mu \in \mathbb{R}$ and parameter (also called volatility of volatility) $\widetilde{\sigma} > 0$ are constant. The mean-reverting stochastic variance process v_t, also referred to as the Cox, Ingersoll, and Ross [13] process, with constant rate of mean reversion κ and long-run mean level θ, both positive, is strictly positive provided that the so-called Feller's condition $2\kappa\theta > \widetilde{\sigma}^2$ holds. Again, if the function $V_t = V(S_t, v_t, t)$ of the option price is continuous, then it is given by the pricing formula (2.19) for an equivalent martingale measure $\mathbb{P}^*$ that we get from (2.21) by replacing μ, κ, θ by $\mu^* = r$, $\kappa^* = \kappa + \lambda > 0$, $\theta^* = \kappa\theta/\kappa^*$, respectively, and keeping $\widetilde{\sigma}^* = \widetilde{\sigma}$ and $\varrho^* = \varrho$. The parameter $\lambda \in [0, \infty)$ is referred to as the price of volatility risk. If the price function V is continuous on $[0, \infty) \times \mathbb{R}^2$, it is given uniquely by (2.19).

If it is additionally in $C^{1,2}$ on $(0, \infty) \times \mathbb{R}^2$, then after a transformation of variables we can make use of the Feynman-Kac theorem (see Kallenberg [29], Theorem 24.1) to show that it solves the partial differential equation

$$
\begin{aligned}
\frac{\partial}{\partial t} V + \frac{1}{2} v S^2 \frac{\partial^2}{\partial S^2} V + \varrho \widetilde{\sigma} v S \frac{\partial^2}{\partial S \partial v} V + \frac{1}{2} \widetilde{\sigma}^2 v \frac{\partial^2}{\partial v^2} V& \\
+ r S \frac{\partial}{\partial S} V + [\kappa(\theta - v) - \lambda v] \frac{\partial}{\partial v} V - r V &= 0
\end{aligned}
$$

for $(S, v, t) \in (0, \infty)^2 \times (0, T)$ with the terminal condition $V(S, v, t) = (S - K)^+$. A heuristic derivation of the PDE with the help of a replicating self-financing portfolio is given for example in (see Fouque, Papanicolaou, and Sircar [21], Section 2.4), where an Ohrnstein-Uhlenbeck process is used instead of the CIR process. Without loss of generality, we set $\lambda = 0$ by using standard transformation techniques Heston [24], Section 3.

As above, we introduce the new variables $\tau = T - t$ and $x = \ln S$. For the function $u(x,v,\tau) = V(S,v,t)$ we obtain the initial value problem

$$
\text{(H)} \qquad \begin{cases} \dfrac{\partial}{\partial \tau} u(x,v,\tau) = \mathcal{L}^{\mathrm{H}} u(x,v,\tau) & \text{for } (x,v,\tau) \in \mathbb{R} \times (0,\infty) \times (0,T), \\ u(x,v,0) = (\mathrm{e}^x - K)^+ & \text{for } (x,v) \in \mathbb{R} \times (0,\infty), \end{cases}
$$

with the partial differential operator

$$
\mathcal{L}^{\mathrm{H}} u := \frac{1}{2} v \frac{\partial^2 u}{\partial x^2} + \varrho \widetilde{\sigma} v \frac{\partial^2 u}{\partial x \partial v} + \frac{1}{2} \widetilde{\sigma}^2 v \frac{\partial^2 u}{\partial v^2} + \Big(r - \frac{1}{2} v\Big) \frac{\partial u}{\partial x} + \kappa(\theta - v) \frac{\partial u}{\partial v} - ru.
$$

In the book by Lewis [34], the author presents the so-called fundamental transform approach for the solution of the initial value problem (H). We present here only the pricing formula that has, among others, one numerical advantage in the sense that we have to calculate only one numerical integral for each price of the option (compared to the two-integrals formula by Heston). The price of the European call option can be expressed as the so-called Heston-Lewis formula

$$
u^{\mathrm{H}}(x,v,\tau) = \mathrm{e}^x - K \mathrm{e}^{-r\tau} \frac{1}{\pi} \int_{0+\mathrm{i}/2}^{\infty+\mathrm{i}/2} \mathrm{e}^{-\mathrm{i}kX} \frac{\widehat{H}(k,v,\tau)}{k^2 - \mathrm{i}k} \, \mathrm{d}k, \tag{2.22}
$$

where $X = x - \ln(K) + r\tau$ and

$$
\widehat{H}(k,v,\tau) = \exp\Big(\frac{2\kappa\theta}{\widetilde{\sigma}^2} \Big[q\, g - \ln\Big(\frac{1 - h\mathrm{e}^{-\xi q}}{1-h} \Big) \Big] + vg \Big(\frac{1 - \mathrm{e}^{-\xi q}}{1 - h\mathrm{e}^{-\xi q}} \Big) \Big),
$$

where

$$
g = \frac{b - \xi}{2}, \quad h = \frac{b - \xi}{b + \xi}, \quad q = \frac{\widetilde{\sigma}^2 \tau}{2},
$$
$$
\xi = \sqrt{b^2 + \frac{4(k^2 - \mathrm{i}k)}{\widetilde{\sigma}^2}}, \quad b = \frac{2}{\widetilde{\sigma}^2} \Big(\mathrm{i}k\varrho\widetilde{\sigma} + \kappa \Big).
$$

To show that the original Heston [24] pricing formula and (2.22) are equivalent, we refer to the paper by Baustian, Mrázek, Pospíšil, and Sobotka [7], where the authors also extended Lewis's approach to models with jumps.

3. Methodology

In this section we present our main results, in particular we introduce our Galerkin-based method. First, we establish the weak formulation of the Black-Scholes equation in a weighted Lebesgue space and show how we can solve the equation in finite-dimensional subspaces spanned by Hermite polynomials. The smooth solutions in the finite-dimensional subspaces approximate the weak solution of the Black-Scholes equation. Although the Black-Scholes model has already been studied in detail, Section 3.1 gives us a good understanding how the method should work for the more complicated Heston model. Second, we establish the method for the Heston model and study the equation for vanishing volatility.

As we have pointed out in the introduction, the Galerkin method for parabolic equations and its convergence properties were widely studied in the past. Even so, our applications are special in the sense that we have an unbounded domain and unbounded initial data. Most numerical schemes for unbounded domains just cut the domain at a certain point. Contrary to this, we use an orthogonal base on the whole unbounded domain. To treat the unbounded initial condition we consider weighted Lebesgue spaces.

3.1. Solution of the Black-Scholes PDE. Let us now consider the parabolic Cauchy problem for the function $u(x,\tau) = V(S,t)$ introduced in Section 2.2.1

$$\begin{cases} \dfrac{\partial}{\partial \tau} u(x,\tau) = \mathcal{L}^{\mathrm{BS}} u(x,\tau) & \text{for } (x,\tau) \in \mathbb{R} \times (0,T), \\ u(x,0) = (\mathrm{e}^x - K)^+ & \text{for } x \in \mathbb{R}, \end{cases} \tag{3.1}$$

with the Black-Scholes operator

$$\mathcal{L}^{\mathrm{BS}} u := \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} u + \Big(r - \frac{1}{2}\sigma^2\Big)\frac{\partial}{\partial x} u - ru.$$

The initial data is obviously not in $L^2(\mathbb{R})$, but in the weighted Lebesgue space $L^2(\mathbb{R}, w\,\mathrm{d}x)$ with the weight function $w(x) = \mathrm{e}^{-x^2}$ and even in the weighted Sobolev space $H^1(\mathbb{R}, w\,\mathrm{d}x)$. We want to obtain a weak formulation of the problem in the weighted space. Therefore, we multiply the partial differential equation (3.1) by a test function $\phi \in C_0^\infty(\mathbb{R})$ and the weight function w. If we integrate over $\mathbb{R}$, then integration by parts yields

$$\int_{-\infty}^{\infty} \frac{\partial u}{\partial \tau} \phi w \,\mathrm{d}x + \int_{-\infty}^{\infty} \Big[\frac{1}{2}\sigma^2 \frac{\partial u}{\partial x}\frac{\partial \phi}{\partial x} - \Big(r - \frac{1}{2}\sigma^2 + \sigma^2 x\Big)\frac{\partial u}{\partial x}\phi + ru\phi\Big] w\,\mathrm{d}x = 0.$$

Following the standard procedure described for example in Evans [18], p. 296, we define the bilinear form

$$\mathcal{B}(\varphi,\psi) := \int_{-\infty}^{\infty} \Big[\frac{1}{2}\sigma^2 \frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x} + \Big(\frac{1}{2}\sigma^2 - r - \sigma^2 x\Big)\frac{\partial\varphi}{\partial x}\psi + r\varphi\psi\Big] w \,\mathrm{d}x \tag{3.2}$$

for $\varphi, \psi \in H^1(\mathbb{R}, w\,\mathrm{d}x)$. We call

$$u \in L^2((0,T) \to H^1(\mathbb{R}, w\,\mathrm{d}x)) \quad \text{with} \quad \frac{\mathrm{d}}{\mathrm{d}\tau}u \in L^2((0,T) \to H^{-1}(\mathbb{R}, w\,\mathrm{d}x))$$

a weak solution of (3.1) if

$$\Big\langle \frac{\mathrm{d}}{\mathrm{d}\tau}u, \phi\Big\rangle_w + \mathcal{B}(u,\phi) = 0 \tag{3.3}$$

for each test function $\phi \in H^1(\mathbb{R}, w\,\mathrm{d}x)$ and a.e. time $0 \leqslant \tau \leqslant T$, and $u(0) = (\mathrm{e}^x - K)^+$. Here, H^{-1} is the dual space of the Sobolev space H^1 and can be canonically identified with it by the Riesz representation theorem. The existence of the unique weak solution in the weighted space can be obtained by modifying the proof of the Galerkin method in Evans [18], Chapter 7.1, p. 349.

Following the Galerkin method, we want to approximate the weak solution u with solutions u_M of the Cauchy problem (3.1) in the finite-dimensional subspace S_M^{H}, i.e., we look for a solution u_M in the form

$$u_M(x,\tau) = \sum_{k=0}^{M} c_k(\tau) H_k(x) \tag{3.4}$$

with a given initial condition

$$u_M(x,0) = \sum_{k=0}^{M} c_k(0) H_k(x), \tag{3.5}$$

where $c(\tau)$ is a column vector of Fourier coefficients $c(\tau) = [c_0(\tau), c_1(\tau), \ldots, c_M(\tau)]^\top$, where $\top$ denotes the transposition (not to be confused with time T).

The natural choice for the initial condition is the orthogonal projection of the payoff function $\Pi_M^{\mathrm{H}} u(x,0)$ defined in (2.13). For instance, the coefficients in the initial condition (3.5) satisfy

$$c_k(0) = c_{0,k} := \int_{-\infty}^{\infty} (\mathrm{e}^x - K)^+ H_k(x) \mathrm{e}^{-x^2}\,\mathrm{d}x, \quad k = 0,1,\ldots,M, \tag{3.6}$$

or in vector form $c(0) = c_0 = [c_{0,0}, c_{0,1}, \ldots, c_{0,M}]^\top$.

Let us now substitute $\varphi = H_i(x)$ and $\psi = H_j(x)$ into the bilinear form (3.2). In view of Lemma 2.2 we can simplify the term $\mathcal{B}(H_i, H_j)$ and obtain the explicit form

$$\frac{1}{2^j j!\sqrt{\pi}}\mathcal{B}(H_i, H_j) = i(\sigma^2 - 2r)\delta_{i,j+1} - 2i\sigma^2(j+1)\delta_{i,j+2} + r\delta_{i,j}. \tag{3.7}$$

We plug (3.4) into (3.3) and choose the Hermite polynomial H_j as the test function

$$\left\langle \sum_{k=0}^{M} c_k'(\tau)H_k(x), H_j(x) \right\rangle_w + \sum_{k=0}^{M} c_k(\tau)\mathcal{B}(H_k(x), H_j(x)) = 0.$$

We make use of the orthogonality of the Hermite polynomials to obtain a system of ODEs

$$c_j'(\tau) + \frac{1}{2^j j!\sqrt{\pi}}\sum_{k=0}^{M} c_k(\tau)\mathcal{B}(H_k(x), H_j(x)) = 0, \quad j = 0, 1, \ldots, M, \tag{3.8}$$

that possesses a unique solution to the initial data (3.6) by standard existence theory.

Let us introduce a matrix $B = [B_{k,j}]$, $k, j = 0, 1, \ldots, M$, with elements

$$B_{k,j} := \frac{1}{2^j j!\sqrt{\pi}}\mathcal{B}(H_k, H_j) \tag{3.9}$$

and denote by $B^\top$ the transposed matrix[1]. From (3.7) we can easily see that $B^\top$ is a three-diagonal matrix with entries on the main diagonal and two superdiagonals. With the matrix $B^\top$ we can rewrite (3.8) in the matrix form as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}c(\tau) + B^\top c(\tau) = 0, \quad c(0) = c_0. \tag{3.10}$$

We can write the solution in terms of the matrix exponential as

$$c(\tau) = \mathrm{e}^{-B^\top\tau}c_0. \tag{3.11}$$

3.2. Solution of the Heston PDE. Let us now consider the Heston [24] model with stochastic volatility. As above, we can use the Hermite polynomials for the polynomial expansion in the variable connected to the logarithm of the stock price. However, for the volatility variable we prefer Laguerre polynomials due to the fact

[1] In the implementation, one can easily swap the arguments of the bilinear form in (3.9) in order to get an already transposed matrix. However, in the text we prefer the *natural ordering* and hence the transposition in the formulas below is needed.

that the volatility is strictly positive. The Cauchy problem connected to the model of Heston [24] is

$$\begin{cases} \dfrac{\partial}{\partial\tau}u(x,v,\tau) = \mathcal{L}^{\mathrm{H}}u(x,v,\tau) & \text{for } (x,v,\tau)\in\mathbb{R}\times(0,\infty)\times(0,T),\\ u(x,v,0) = (\mathrm{e}^x - K)^+ & \text{for } x\in\mathbb{R}\times(0,\infty), \end{cases} \tag{3.12}$$

with the Heston operator

$$\mathcal{L}^{\mathrm{H}}u := \frac{1}{2}v\frac{\partial^2 u}{\partial x^2} + \varrho\widetilde{\sigma}v\frac{\partial^2 u}{\partial x\partial v} + \frac{1}{2}\widetilde{\sigma}^2 v\frac{\partial^2 u}{\partial v^2} + \Big(r - \frac{1}{2}v\Big)\frac{\partial u}{\partial x} + \kappa(\theta - v)\frac{\partial u}{\partial v} - ru.$$

To obtain a weak formulation of the solution we multiply (3.12) by a test function $\phi\in C_0^\infty(\mathbb{R}\times\mathbb{R}^+)$ and the weight function $w(x,v)=\mathrm{e}^{-x^2-v}$. Integration over the domain $\mathbb{R}\times(0,\infty)$ and application of Gauss's theorem then yields the variational formulation of the problem

$$\int_0^\infty\int_{-\infty}^\infty \frac{\partial u}{\partial\tau}\phi w\,\mathrm{d}x\,\mathrm{d}v + \widetilde{\mathcal{B}}(u,\phi) = 0$$

with the bilinear form $\widetilde{\mathcal{B}}$ defined by

$$\begin{aligned} \widetilde{\mathcal{B}}(\varphi,\psi) = {} & \int_0^\infty\int_{-\infty}^\infty \frac{1}{2}v\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x}w\,\mathrm{d}x\,\mathrm{d}v + \int_0^\infty\int_{-\infty}^\infty \frac{1}{2}\varrho\widetilde{\sigma}v\frac{\partial\varphi}{\partial v}\frac{\partial\psi}{\partial x}w\,\mathrm{d}x\,\mathrm{d}v\\ & + \int_0^\infty\int_{-\infty}^\infty \frac{1}{2}\varrho\widetilde{\sigma}v\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial v}w\,\mathrm{d}x\,\mathrm{d}v + \int_0^\infty\int_{-\infty}^\infty \frac{1}{2}\widetilde{\sigma}^2 v\frac{\partial\varphi}{\partial v}\frac{\partial\psi}{\partial v}w\,\mathrm{d}x\,\mathrm{d}v\\ & + \int_0^\infty\int_{-\infty}^\infty \Big(-xv - \frac{1}{2}\varrho\widetilde{\sigma}v + \frac{1}{2}\varrho\widetilde{\sigma} - r + \frac{1}{2}v\Big)\frac{\partial\varphi}{\partial x}\psi w\,\mathrm{d}x\,\mathrm{d}v\\ & + \int_0^\infty\int_{-\infty}^\infty \Big(-xv\varrho\widetilde{\sigma} - \frac{1}{2}\widetilde{\sigma}^2 v + \frac{1}{2}\widetilde{\sigma}^2 - \kappa(\theta - v)\Big)\frac{\partial\varphi}{\partial v}\psi w\,\mathrm{d}x\,\mathrm{d}v\\ & + \int_0^\infty\int_{-\infty}^\infty r\varphi\psi w\,\mathrm{d}x\,\mathrm{d}v \end{aligned} \tag{3.13}$$

for all $\varphi,\psi\in H^1(\mathbb{R}\times\mathbb{R}^+, w\,\mathrm{d}x\,\mathrm{d}v)$.

Similarly as for the BS model, we substitute the elements of the complete orthogonal set $\varphi = P_{i,j}(x,v) = H_i(x)L_j(v)$ and $\psi = P_{k,l}(x,v) = H_k(x)L_l(v)$ into the bilinear form (3.13). For clarity, we study all seven integral terms separately. In particular, let

$$\frac{1}{2^k k!\sqrt{\pi}}\widetilde{\mathcal{B}}(P_{i,j}(x,v),P_{k,l}(x,v)) := \frac{1}{2^k k!\sqrt{\pi}}\sum_{r=1}^{7}\widetilde{\mathcal{B}}_r(P_{i,j}(x,v),P_{k,l}(x,v)), \tag{3.14}$$

where each $\widetilde{\mathcal{B}}_r(P_{i,j}(x,v),P_{k,l}(x,v))$, $r=1, 2,\dots, 7$, represents an individual integral term.

Theorem 3.1. *The integrals in* (3.14) *satisfy*

$$\frac{1}{2^k k!\sqrt{\pi}}\widetilde{\mathcal{B}}_1(P_{i,j},P_{k,l}) = i\delta_{i,k}((2j+1)\delta_{j,l} - j\delta_{j-1,l} - (j+1)\delta_{j+1,l}),$$

$$\frac{1}{2^k k!\sqrt{\pi}}\widetilde{\mathcal{B}}_2(P_{i,j},P_{k,l}) = \frac{1}{2}\varrho\widetilde{\sigma}\delta_{i+1,k}j(\delta_{j,l} - \delta_{j-1,l}),$$

$$\frac{1}{2^k k!\sqrt{\pi}}\widetilde{\mathcal{B}}_3(P_{i,j},P_{k,l}) = i\varrho\widetilde{\sigma}\delta_{i,k+1}l(\delta_{j,l} - \delta_{j,l-1}),$$

$$\frac{1}{2^k k!\sqrt{\pi}}\widetilde{\mathcal{B}}_4(P_{i,j},P_{k,l}) = \frac{1}{2}\widetilde{\sigma}^2 j\delta_{i,k}\delta_{j,l},$$

$$\begin{aligned}\frac{1}{2^k k!\sqrt{\pi}}\widetilde{\mathcal{B}}_5(P_{i,j},P_{k,l}) &= 2i\Big(\frac{1}{2}\varrho\widetilde{\sigma} - r\Big)\delta_{i,k+1}\delta_{j,l}\\ &\quad + [-2i(k+1)\delta_{i,k+2} - i\delta_{i,k}][(2j+1)\delta_{j,l} - j\delta_{j-1,l} - (j+1)\delta_{j+1,l}]\\ &\quad + [i(1-\varrho\widetilde{\sigma})\delta_{i,k+1}][(2j+1)\delta_{j,l} - j\delta_{j-1,l} - (j+1)\delta_{j+1,l}]\end{aligned}$$

$$\begin{aligned}\frac{1}{2^k k!\sqrt{\pi}}\widetilde{\mathcal{B}}_6(P_{i,j},P_{k,l}) &= \Big(\frac{1}{2}\widetilde{\sigma}^2 - \kappa\theta\Big)\delta_{i,k}\Big(-\sum_{a=0}^{j-1}\delta_{a,l}\Big)\\ &\quad + \Big[-\varrho\widetilde{\sigma}\Big(\frac{1}{2}\delta_{i+1,k} + i\delta_{i-1,k}\Big) + \Big(\kappa - \frac{1}{2}\widetilde{\sigma}^2\Big)\delta_{i,k}\Big]j(\delta_{j,l} - \delta_{j-1,l}),\end{aligned}$$

$$\frac{1}{2^k k!\sqrt{\pi}}\widetilde{\mathcal{B}}_7(P_{i,j},P_{k,l}) = r\delta_{i,k}\delta_{j,l}$$

for all $0 \leqslant i,\ k \leqslant M$ *and all* $0 \leqslant j,\ l \leqslant N$.

P r o o f. For the calculation of $\widetilde{\mathcal{B}}_1$ we apply (2.4) and (2.9). For $\widetilde{\mathcal{B}}_2$ we use (2.5) and (2.10). $\widetilde{\mathcal{B}}_3$ is derived with the help of a modification of (2.5)

$$\frac{1}{2^n n!\sqrt{\pi}}\int_{-\infty}^{\infty} H_m'(x)H_n(x)\mathrm{e}^{-x^2}\,\mathrm{d}x = 2m\delta_{m,n+1} \tag{3.15}$$

and (2.10). For $\widetilde{\mathcal{B}}_4$ we need (2.11). In the calculation of $\widetilde{\mathcal{B}}_5$ we make use of (2.6), (2.9), and (3.15). For $\widetilde{\mathcal{B}}_6$ we need the same equations as for $\widetilde{\mathcal{B}}_5$ and (2.12). The integral $\widetilde{\mathcal{B}}_7$ is trivial. More detailed calculations can be found in the thesis Filipová [19], Section 3.2. □

Analogously to the BS case, we say that $u \in L^2((0,T) \to H^1(\mathbb{R}\times\mathbb{R}^+, w\,\mathrm{d}x\,\mathrm{d}v))$ with $\frac{\mathrm{d}}{\mathrm{d}\tau}u \in L^2((0,T) \to H^{-1}(\mathbb{R}\times\mathbb{R}_0^+, w\,\mathrm{d}x\,\mathrm{d}v))$ is a weak solution of (3.12) if

$$\Big\langle\frac{\mathrm{d}}{\mathrm{d}\tau}u,\phi\Big\rangle_w + \widetilde{\mathcal{B}}(u,\phi) = 0 \tag{3.16}$$

for each test function $\phi \in H^1(\mathbb{R}\times\mathbb{R}^+, w\,\mathrm{d}x\,\mathrm{d}v)$ and a.e. time $0 \leqslant \tau \leqslant T$, and $u(0,v) = (\mathrm{e}^x - K)^+$ for all $v > 0$. The existence and uniqueness of the weak solution is given by the Galerkin method in Evans [18], Chapter 7.1, p. 349. A detailed proof of the existence of the unique solution in a convenient weighted space considering the boundary conditions can be found in Alziary and Takáč [3].

We study the solutions of the Cauchy problem (3.12) in finite-dimensional subspaces $S_{M,N}$, i.e., we look for the solution $u_{M,N}$ in the form

$$u_{M,N}(x,v,\tau) = \sum_{i=0}^{M}\sum_{j=0}^{N} c_{i,j}(\tau)P_{i,j}(x,v), \tag{3.17}$$

where

$$P_{i,j}(x,v) = H_i(x)L_j(v), \quad i,j \in \mathbb{N}_0$$

and $c_{i,j}(\tau)$, $i = 0,1,\dots,M$; $j = 0,1,\dots,N$; are (yet unknown) Fourier coefficients.

Let $c(\tau) = [c_a(\tau)]^\top$, $a = 0,1,\dots,(M+1)(N+1)$, be a column vector of these coefficients, where $a = i(N+1)+j$, $i = 0,1,\dots,M$; $j = 0,1,\dots,N$; i.e.,

$$c(\tau) = [c_{0,0}(\tau),\dots,c_{0,N}(\tau),c_{1,0}(\tau),\dots,c_{1,N}(\tau),\dots,c_{M,0}(\tau),\dots,c_{M,N}(\tau)]^\top.$$

For the initial data we choose the orthogonal projection of the payoff function $\Pi_{M,N}u(x,v,0)$, where for $i = 0,1,\dots,M$, $j = 0,1,\dots,N$

$$c_{i,j}(0) = c_{0,i,j} := \int_0^\infty \int_{-\infty}^\infty (\mathrm{e}^x - K)^+ P_{i,j}(x,v)\mathrm{e}^{-x^2}\mathrm{e}^{-v}\,\mathrm{d}x\,\mathrm{d}v, \tag{3.18}$$

or in vector form

$$c(0) = c_0 = [c_{0,0,0},\dots,c_{0,0,N},c_{0,1,0},\dots,c_{0,1,N},\dots,c_{0,M,0},\dots,c_{0,M,N}]^\top.$$

We use (3.16) with $u_{M,N}$ of the form (3.17) and the test function $P_{k,l}$. Thanks to the orthogonality of the polynomials we obtain

$$c'_{k,l}(\tau) + \sum_{i=0}^{M}\sum_{j=0}^{N} c_{i,j}(\tau)\frac{1}{2^k k!\sqrt{\pi}}\widetilde{\mathcal{B}}(P_{i,j}(x,v),P_{k,l}(x,v)) = 0.$$

Let us introduce a matrix $\widetilde{B} = [\widetilde{B}_{a,b}]$, $a,b = 0,1,\dots,(M+1)(N+1)$, defined as

$$\widetilde{B}_{a,b} = \frac{1}{2^k k!\sqrt{\pi}}(\widetilde{\mathcal{B}}(P_{i,j}(x,v),P_{k,l}(x,v))), \tag{3.19}$$

where $a = i(N+1)+j$; $b = k(N+1)+l$; $i,k = 0,\dots,M$; $j,l = 0,\dots,N$. Using this assembly[2] it can be shown (by using Theorem 3.1) that the transposed matrix $\widetilde{B}^\top$

[2] Swapping the arguments in the bilinear form in (3.19) can again easily produce an already transposed matrix.

is an upper triangular matrix with elements on the main diagonal and $2N+3$ superdiagonals if $N > 0$, and 2 superdiagonals in the degenerate case $N = 0$ like in the BS case. It is worth to mention that the BS PDE is not a special case of the Heston PDE. The superdiagonal $2N+3$ is a contribution of the term $\widetilde{\mathcal{B}}_5$.

As above, we obtain a system of ODEs

$$\frac{\mathrm{d}}{\mathrm{d}\tau}c(\tau) + \widetilde{B}^\top c(\tau) = 0, \quad c(0) = c_0. \tag{3.20}$$

The solution can also be written in terms of the matrix exponential as

$$c(\tau) = \mathrm{e}^{-\widetilde{B}^\top \tau} c_0. \tag{3.21}$$

3.2.1. Solution behaviour analysis near $v = 0$. We are interested in the behaviour of the solution of the Heston PDE for small volatility, especially at the boundary $v = 0$. Motivated by Alziary and Takáč [4], we study the partial differential equation for $v \to 0_+$.

The solution $u = u(x, v, \tau)$ satisfies the Heston PDE

$$\frac{\partial}{\partial \tau}u = \frac{1}{2}v\frac{\partial^2 u}{\partial x^2} + \varrho\widetilde{\sigma}v\frac{\partial^2 u}{\partial x \partial v} + \frac{1}{2}\widetilde{\sigma}^2 v\frac{\partial^2 u}{\partial v^2} + \Big(r - \frac{1}{2}v\Big)\frac{\partial u}{\partial x} + \kappa(\theta - v)\frac{\partial u}{\partial v} - ru$$

in $\mathbb{R} \times \mathbb{R}_0^+ \times (0, T)$ and can be rewritten as

$$\frac{\partial u}{\partial \tau} = v\Big(\frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \varrho\widetilde{\sigma}\frac{\partial^2 u}{\partial x \partial v} + \frac{1}{2}\widetilde{\sigma}^2\frac{\partial^2 u}{\partial v^2}\Big) + \Big(r - \frac{1}{2}v\Big)\frac{\partial u}{\partial x} + \kappa(\theta - v)\frac{\partial u}{\partial v} - ru.$$

For $v \to 0_+$ the equation degenerates to the first order equation as shown in Alziary and Takáč [4], Corollary 4.3,

$$\frac{\partial u}{\partial \tau} = r\frac{\partial u}{\partial x} + \kappa\theta\frac{\partial u}{\partial v} - ru.$$

Since we want to study the problem for vanishing volatility, we replace the derivative with respect to v by the differential quotient $\frac{1}{h}(u(x, h, \tau) - u(x, 0, \tau))$, where $h > 0$ denotes a small distance to the boundary. By doing this, we obtain an initial value problem on the boundary

$$\begin{cases} \mathcal{L}^{\mathrm{B}}u(x, 0, \tau) = \dfrac{\kappa\theta}{h}u(x, h, \tau) & \text{for } (x, \tau) \in \mathbb{R} \times (0, T), \\ u(x, 0, 0) = (\mathrm{e}^x - K)^+ & \text{for } x \in \mathbb{R}, \end{cases} \tag{3.22}$$

with the unknown function $u(x, 0, \tau)$ for fixed volatility $v = 0$ and with the differential operator

$$\mathcal{L}^{\mathrm{B}}u = \frac{\partial u}{\partial \tau} - r\frac{\partial u}{\partial x} + \Big(r + \frac{\kappa\theta}{h}\Big)u.$$

We can derive a solution of the Cauchy problem depending on the inhomogeneity which consists of values of the solution of the Heston equation away from the boundary.

We introduce a new variable $y = x + r\tau$ and the function $\widetilde{u}(y,\tau) = u(x,0,\tau)$ that satisfies the inhomogeneous *transport equation*

$$\frac{\partial \widetilde{u}}{\partial \tau} + \Big(\frac{\kappa\theta}{h} + r\Big)\widetilde{u} = \frac{\kappa\theta}{h} u(y - r\tau, h, \tau)$$

with the initial condition $\widetilde{u}(y,0) = (\mathrm{e}^{y-r\tau} - K)^+$. Following the standard procedure, we define the function

$$U(y,\tau) = \mathrm{e}^{(\kappa\theta/h+r)\tau}\widetilde{u}(y,\tau)$$

and obtain

$$\text{(3.23)} \qquad \frac{\partial U}{\partial \tau} = \frac{\kappa\theta}{h}\mathrm{e}^{(\kappa\theta/h+r)\tau} u(y - r\tau, h, \tau) \quad \text{with} \quad U(y,0) = (\mathrm{e}^{y-r\tau} - K)^+.$$

We integrate (3.23) with respect to the time variable

$$U(y,\tau) = (\mathrm{e}^{y-r\tau} - K)^+ + \frac{\kappa\theta}{h}\int_0^\tau \mathrm{e}^{(\kappa\theta/h+r)\xi} u(y - r\xi, h, \xi)\,\mathrm{d}\xi.$$

Hence, we get the *boundary solution* that we denote as

$$u^{\mathrm{B}}(x,0,\tau) = \mathrm{e}^{-(\kappa\theta/h+r)\tau}\Big[(\mathrm{e}^x - K)^+ + \frac{\kappa\theta}{h}\int_0^\tau \mathrm{e}^{(\kappa\theta/h+r)\xi} u(x + r(\tau - \xi), h, \xi)\,\mathrm{d}\xi\Big]$$

and in particular

$$\text{(3.24)} \qquad u^{\mathrm{B}}(x,0,T) = \mathrm{e}^{-(\kappa\theta/h+r)T} \times \Big[(\mathrm{e}^x - K)^+ + \frac{\kappa\theta}{h}\int_0^T \mathrm{e}^{(\kappa\theta/h+r)\xi} u(x + r(T - \xi), h, \xi)\,\mathrm{d}\xi\Big].$$

These formulas contain an integral over the finite interval $[0,T]$. For the values of u for $h > 0$ we could make use of the polynomial expansion of the solution obtained in Section 3.2.

4. Results

In this section we present numerical results for several particular examples. All supporting codes are implemented in MATLAB. Parameter values in the considered examples are chosen consistently with other cited resources in order to demonstrate the functionality of the proposed method. To provide a thorough analysis of the numerical solution for all possible parameter value combinations goes beyond the scope of the present paper. When we refer to the L^2 error it is the error with respect to the norm of the weighted Lebesgue spaces $L^2(\mathbb{R}, \mathrm{e}^{-x^2}\,\mathrm{d}x)$ and $L^2(\mathbb{R}\times\mathbb{R}^+, \mathrm{e}^{-x^2-v}\,\mathrm{d}x\,\mathrm{d}v)$, respectively. For convenience, the point-wise error is calculated for several selected nodes as well as the average absolute and relative error. We compare the newly proposed solution to the existing closed formula (2.20) for the BS model and semi-closed formula (2.22) for the Heston model.

4.1. Black-Scholes model. In the setting for the BS model, the parameters are chosen as follows:

- ▷ volatility $\sigma = 0.03$,
- ▷ risk-free interest rate $r = 0.1$,

and option parameters are the following:

- ▷ maturity $T = 1$,
- ▷ strike price $K = 100$,
- ▷ stock price $S \in [0; 2K]$,

and we impose $x = \ln(S)$. In the case of the BS model, we choose Hermite polynomials as the complete orthogonal system of polynomials and focus on solving the Black-Scholes PDE. Our numerical solution u_M of the BS PDE (3.1) is considered in the form (3.4). Fourier coefficients for $\tau = T$ are obtained by solving the system of ODEs (3.10) with B given by (3.7) and (3.9). We use MATLAB ODE solver `ode45` to solve this system since it leads to smaller values of L^2 errors then the numerical calculation of the matrix exponential in (3.11) using MATLAB procedure `expm`. See Filipová [19], Section 4.1 for a comparison of these two methods.

The BS formula u^{BS} and solutions u_M obtained by `ode45` for $M = 20$ and $M = 120$ are shown in Figure 1 on the left. For convenience, the solution is plotted only for $x \geqslant 0$. The three vertical dashed lines represent moneyness $\gamma \in \{0.7, 1, 1.3\}$. On the right we can see the behaviour of the absolute error in this region.

We measure the following errors. First and foremost we compute the L^2 error using the Gauss-Hermite quadrature with 251 Hermite points. In the second column of Table 2 we list the corresponding L^2 error for different polynomial orders. Convergence of the L^2 error is shown in Figure 2 on the left. The set of nodes used on the right side of Figure 2 and in Table 2 will be introduced below.

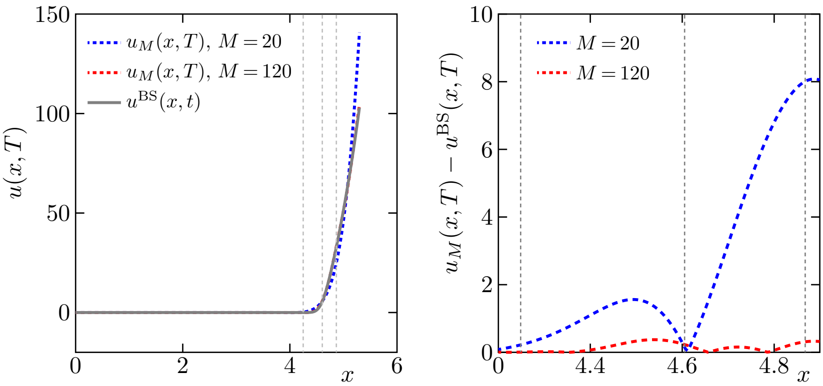

Figure 1. Solution u_M of the BS PDE for $M = 20$ and $M = 120$ together with the BS formula u^{BS} is depicted on the left and the absolute error on the right. Vertical grid lines are plotted at $\gamma \in \{0.7, 1, 1.3\}$.

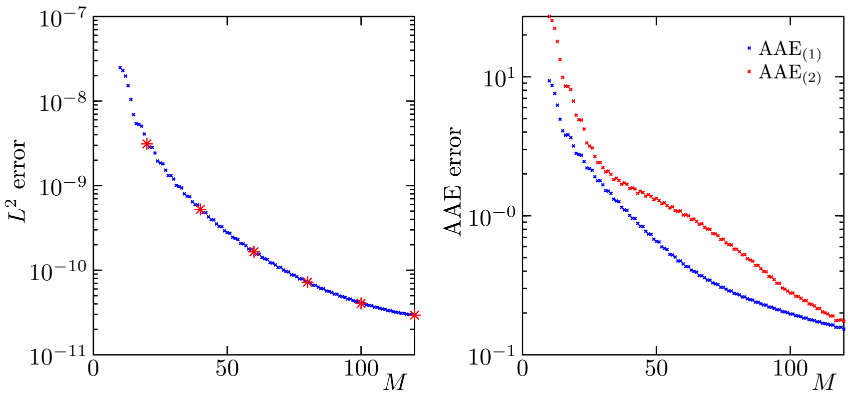

Figure 2. Convergence of the L^2 error on the left. Red asterisks indicate values that are listed in the second column of Table 2. On the right we can see the convergence of the average absolute error calculated for two different sets of nodes. The scale at the vertical axis is logarithmic in both graphs.

M	AE(0.7)	RE(0.7)	AE(1)	RE(1)	AE(1.3)	RE(1.3)
20	0.22055	185.605	0.152195	0.027266	8.00953	0.243
40	0.0991825	83.4675	0.287501	0.0515061	1.38974	0.0421632
60	0.0875455	73.6744	0.249526	0.0447028	0.196272	0.00595466
80	0.0476676	40.1149	0.225168	0.0403392	0.542587	0.0164615
100	0.0209973	17.6704	0.229207	0.0410627	0.456833	0.0138598
120	0.00662291	5.57354	0.230279	0.0412547	0.312095	0.00946862

Table 1. Errors comparison for BS model for different Hermite polynomial orders: absolute error AE and relative error RE at several selected nodes are listed.

M	L^2 error	$\mathrm{AAE}_{(1)}$	$\mathrm{ARE}_{(1)}$	$\mathrm{AAE}_{(2)}$	$\mathrm{ARE}_{(2)}$
20	3.11957e − 09	2.81507	11.9282	5.29885	0.190209
40	5.23407e − 10	1.00759	3.23288	1.57162	0.0630167
60	1.65443e − 10	0.451035	3.10148	1.01331	0.0370747
80	7.21744e − 11	0.274018	1.91442	0.575191	0.0220953
100	4.05356e − 11	0.196393	1.02488	0.277925	0.0127912
120	2.93889e − 11	0.152986	0.473851	0.17448	0.00880548

Table 2. Error comparison for the BS model for different Hermite polynomial orders: L^2 error, average absolute error AAE and average relative error ARE are listed for two sets of points (1) and (2).

Next we measure the pointwise absolute and relative errors at selected nodes and their average. In particular, by $\mathrm{AE}(\gamma)$ we denote the absolute error with respect to the Black-Scholes formula (2.20) at the point $S = \gamma K$,

$$\mathrm{AE}(\gamma) = |u_M(\ln(\gamma K), T)) - u^{\mathrm{BS}}(\ln(\gamma K), T)|,$$

where $\gamma > 0$ is the moneyness introduced in Section 2.2. Similarly we measure the relative error

$$\mathrm{RE}(\gamma) = \left|1 - \frac{u_M(\ln(\gamma K), T))}{u^{\mathrm{BS}}(\ln(\gamma K), T)}\right|.$$

In Table 1, we list the values of both absolute and relative error at three different moneyness nodes ($\gamma \in \{0.7, 1, 1.3\}$) for different polynomial orders. The relative error for $\gamma < 1$ is high, because the option price is close to zero. For small values of M, when the approximation is not optimal, the errors do not have to be strictly decreasing in M, which is expected.

For convenience, in Table 2 we list also (arithmetic) averages of both AE (denoted AAE) and RE (denoted ARE) for two different sets of nodes:

(1) $\gamma_i \in [0.7; 1.3]$, $i = 1, \dots, 61$, taken with the equidistant step $\Delta\gamma = 0.01$,

(2) $\gamma_i \in [1; 1.5]$, $i = 1, \dots, 11$, taken with the equidistant step $\Delta\gamma = 0.05$.

Convergence of AAE of both sets is shown in Figure 2.

4.2. Heston model. We consider the following setting of the Heston model. The parameters are chosen as in many examples in the book by Rouah [40]:

- ▷ initial variance $v_0 = 0.05$,
- ▷ variance $v \in [0; 0.5]$,
- ▷ mean reversion rate $\kappa = 5$,
- ▷ long-run variance $\theta = 0.05$,
- ▷ volatility of volatility $\widetilde{\sigma} = 0.5$,
- ▷ correlation $\varrho = -0.8$,

▷ the price of volatility risk $\lambda = 0$,

▷ risk-free interest rate $r = 0.03$,

and the parameters of the options are the same as for the BS model (Section 4.1). We also impose $x = \ln(S)$. Combinations of Hermite and Laguerre polynomials are chosen for the orthogonal polynomial expansion. Our numerical solution $u_{M,N}$ of the Heston PDE (3.12) is considered in the form (3.17). Fourier coefficients for $\tau = T$ are obtained by solving the system of ODEs (3.20) with $\widetilde{B}$ given by (3.14) and (3.19). For consistency, we use MATLAB ODE solver `ode45` to solve this system, that again leads to smaller values of L^2 error, although its speed is now much lower than for the numerical calculation of the matrix exponential in (3.21) using MATLAB procedure `expm`, see Filipová [19], Section 4.1.

In order to evaluate

$$u_{M,N}(x,v,T) = \sum_{m=0}^{M}\sum_{n=0}^{N} c_{m,n}(T)H_m(x)L_n(v) = \sum_{n=0}^{N} L_n(v)\left(\sum_{m=0}^{M} c_{m,n}(T)H_m(x)\right),$$

we repeatedly apply Clenshaw's recurrence formula as indicated. To numerically evaluate the L^2 error, we make use of the Gauss-Hermite (with 251 Hermite points) and Gauss-Laguerre (with 201 Laguerre points) quadratures. For pointwise comparison we make use of the same set of nodes (1) and (2) as in Section 4.1 with $v = v_0$.

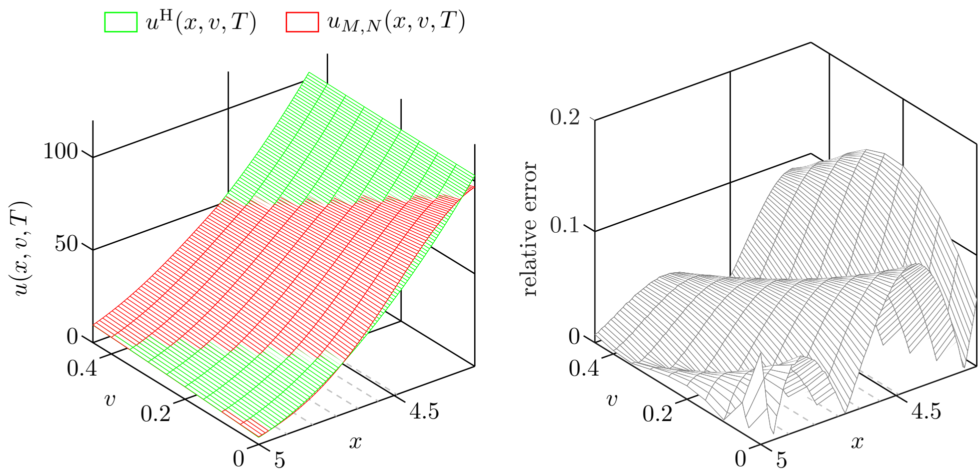

Figure 3. Heston-Lewis formula $u^{\mathrm{H}}(x,v,T)$ and the PDE solution $u_{M,N}(x,v,T)$ for $M = 35$ and $N = 30$ on the left, relative error $|1 - u_{M,N}/u^{\mathrm{H}}|$ on the right. The five dashed grid lines at the xv plane are plotted at $\gamma \in \{1.1, 1.2, 1.3, 1.4, 1.5\}$.

In Figure 3 on the left, we can see the Heston-Lewis formula u^{H} and the numerical solution $u_{M,N}$ for $M = 35$ and $N = 30$ zoomed to the ITM region. The chosen combinations of polynomial orders present the anticipated behaviour of the solution.

The five dashed grid lines at the xv plane are plotted at $\gamma \in \{1.1, 1.2, 1.3, 1.4, 1.5\}$. On the right we plot the relative error $|1 - u_{M,N}/u^{\mathrm{H}}|$. For the same polynomial orders we plot the absolute error $|u_{M,N} - u^{\mathrm{H}}|$ and relative error $|1 - u_{M,N}/u^{\mathrm{H}}|$ for different values of v in Figure 4.

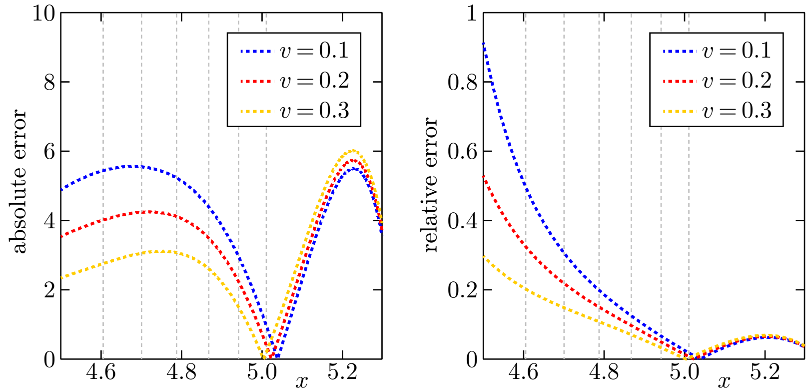

Figure 4. Absolute and relative errors of the PDE solution $u_{M,N}(x, v, T)$ for $M = 35$ and $N = 30$ plotted for different values of v. The five dashed vertical grid lines are plotted at $\gamma \in \{1.1, 1.2, 1.3, 1.4, 1.5\}$.

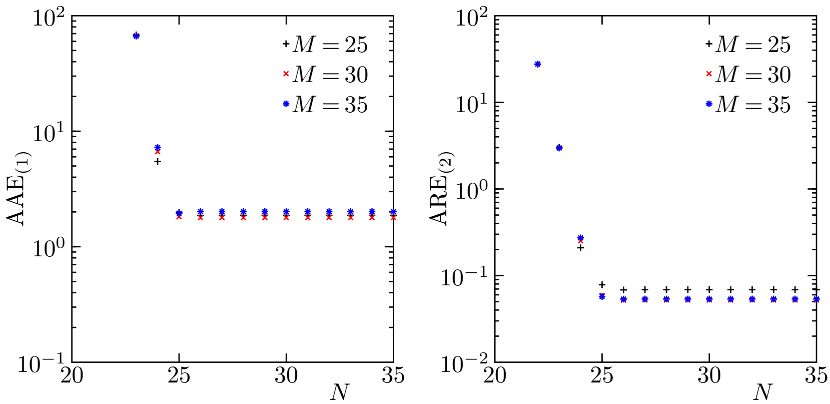

Figure 5. Convergence of the average absolute error AAE and average relative error ARE for the set of nodes (2).

Similarly as in the BS case, we measure the L^2 error, absolute error $\mathrm{AE}(\gamma)$ and relative error $\mathrm{RE}(\gamma)$ calculated at a given point $x = \ln(\gamma K)$ and $v = v_0$, i.e., now

$$\mathrm{AE}(\gamma) = |u_{M,N}(\ln(\gamma K), v_0, T) - u^{\mathrm{H}}(\ln(\gamma K), v_0, T)|,$$
$$\mathrm{RE}(\gamma) = |1 - u_{M,N}(\ln(\gamma K), v_0, T)/u^{\mathrm{H}}(\ln(\gamma K), v_0, T)|,$$

and also average errors AAE and ARE for the two set of nodes (1) and (2) as described above. All results are summarized in Tables 3 and 4. We can see that the influence of increasing N is not significant for AE and RE, which can be seen also in Figure 5 where we depicted the AAE for the ITM set of nodes (2). On the contrary, the L^2 error seems to be N-sensitive. More detailed numerical analysis goes beyond the scope of the present paper and opens space for further research in this area. From the numerical analysis point of view it is also interesting to mention that very few Laguerre points used in the numerical quadrature lie within the considered region $v \in [0; 0.5]$ and experiments showed that the contribution from the majority of the remaining points can be neglected.

M	N	L^2 error	$\mathrm{AAE}_{(1)}$	$\mathrm{ARE}_{(1)}$	$\mathrm{AAE}_{(2)}$	$\mathrm{ARE}_{(2)}$
25	26	0.238643	1.09888	0.390292	1.86799	0.0683568
25	28	1.1757e − 06	1.1014	0.382943	1.87171	0.0686957
25	30	8.39452e − 07	1.1014	0.382948	1.8717	0.0686954
30	26	0.238643	0.547496	0.288746	1.78832	0.0520266
30	28	1.18026e − 06	0.547673	0.281325	1.7877	0.0522312
30	30	8.44246e − 07	0.547673	0.28133	1.7877	0.052231
35	26	0.238642	0.493863	0.244035	2.01489	0.0535306
35	28	1.17799e − 06	0.491256	0.236509	2.01206	0.0536491
35	30	8.42292e − 07	0.491258	0.236514	2.01206	0.0536491

Table 3. Error comparison for the Heston model for different polynomial orders: L^2 error, average absolute error AAE and average relative error ARE are listed for two sets of points (1) and (2).

M	N	AE(0.7)	RE(0.7)	AE(1)	RE(1)	AE(1.3)	RE(1.3)
25	26	0.511921	3.51694	0.92327	0.0909802	0.844315	0.024334
25	28	0.497216	3.41591	0.936251	0.0922593	0.85614	0.0246748
25	30	0.497225	3.41598	0.936242	0.0922585	0.856132	0.0246746
30	26	0.386106	2.65258	0.539511	0.053164	0.939491	0.027077
30	28	0.371401	2.55155	0.552491	0.0544432	0.927665	0.0267362
30	30	0.37141	2.55162	0.552483	0.0544423	0.927673	0.0267364
35	26	0.331769	2.27928	0.418323	0.0412221	1.90743	0.0549739
35	28	0.317063	2.17825	0.431304	0.0425013	1.8956	0.0546331
35	30	0.317073	2.17831	0.431296	0.0425004	1.89561	0.0546333

Table 4. Error comparison for the Heston model for different polynomial orders: absolute error AE and relative error RE at several selected nodes are listed.

Following Section 3.2.1, we now analyse the solution close to the boundary $v = 0$. We consider $h = 0.005$ and polynomial orders $M = 35$ and $N = 30$. In Figure 6 on the left, we can see the Heston-Lewis formula u^{H} for $v = 0$, PDE solution $u_{M,N}$

at $v = 0$ and the boundary solution u^{B} gained by the application of the theory in Section 3.2.1 for $h = 0.005$. The vertical dashed grid lines are plotted again at $\gamma \in \{0.7, 1, 1.3\}$. On the right we can see how the transport equation solution differs from the two remaining ones.

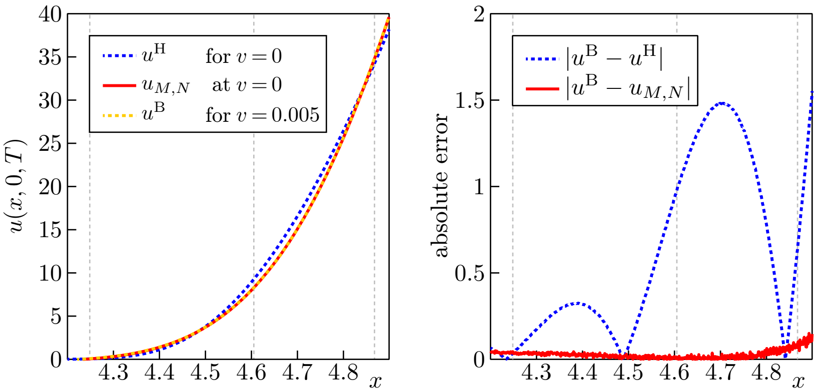

Figure 6. Comparison of the solution behaviour near the boundary $v = 0$ with zoom to the ATM region. The vertical grid lines are plotted at $\gamma \in \{0.7, 1, 1.3\}$.

5. Conclusion

The analyticity of the solution of the Heston model has been shown in the recent paper of Alziary and Takáč [3]. A crucial step in their proof is the approximation of the payoff by a sequence of entire functions, in particular Hermite and Laguerre functions (see Alziary and Takáč [3], Section 11.1) with the Galerkin method (see Alziary and Takáč [3], Section 11.2). The aim of our paper is to make use of these theoretical results to study an alternative method for the option pricing problem for the Black and Scholes [9] model and the Heston [24] model. Moreover, we were interested in the behaviour of the solution near the zero volatility boundary and considered the equation for vanishing volatility. This approach was also motivated and theoretically justified by results of Alziary and Takáč [4].

By the numerical implementation of Galerkin's method in weighted Sobolev spaces we found an alternative representation of the solution to both the BS and Heston models. The obtained representation is a smooth approximation of the solution that does not share the serious numerical difficulties of existing semi-closed formulas as they were presented by Daněk and Pospíšil [14]. The presented approach is also independent of the space variable discretization (used for example by Galerkin finite element methods) or spacial node locations (needed by radial basis function methods).

The presented experiments give a first insight into the performance of the method but thorough numerical analysis has to be performed in order to properly understand the behaviour of the solutions for higher polynomial orders. There are different possibilities how one could try to improve the method, for example to use other procedures to solve the system of ordinary differential equations, especially such that take into consideration the specific triangular form of the matrix. A detailed error analysis and the application of additional procedures were beyond the scope of this paper and are left as an open issue.

A considerable advantage of the presented approach is that it can be easily adapted to other stochastic volatility models by following the steps at the beginning of Section 3.2 and using the calculations of Theorem 3.1. Aside from that, different payoff functions can be used as long as they are in the weighted Lebesgue space, which applies to most of the generally used payoffs.

A c k n o w l e d g e m e n t s. This work is based on the Master's thesis Filipová [19] titled *Solution of option pricing equations using orthogonal polynomial expansion* that was written by Kateřina Filipová and supervised by Jan Pospíšil. The thesis was also advised by Falko Baustian during the two months internship of Kateřina Filipová at the University of Rostock.

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Authors' addresses: *Falko Baustian*, Department of Mathematics, University of Rostock, Ulmenstr. 69, 18057 Rostock, Germany, e-mail: `falko.baustian@uni-rostock.de`; *Kateřina Filipová*, *Jan Pospíšil* (corresponding author), NTIS – New Technologies for the Information Society, Faculty of Applied Sciences, University of West Bohemia, Univerzitní 2732/8, 301 00 Plzeň, Czech Republic, e-mail: `filipovk@ntis.zcu.cz`, `honik@ntis.zcu.cz`.