NONTRIVIAL SOLUTIONS TO BOUNDARY VALUE PROBLEMS FOR SEMILINEAR Δ_γ-DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the existence of nontrivial weak solutions for the following boundary value problem:

$$-\Delta_\gamma u = f(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $\Omega \cap \{x_j = 0\} \neq \emptyset$ for some j, Δ_γ is a subelliptic linear operator of the type

$$\Delta_\gamma := \sum_{j=1}^{N} \partial_{x_j}(\gamma_j^2 \partial_{x_j}), \quad \partial_{x_j} := \frac{\partial}{\partial x_j}, \quad N \geqslant 2,$$

where $\gamma(x) = (\gamma_1(x), \gamma_2(x), \dots, \gamma_N(x))$ satisfies certain homogeneity conditions and degenerates at the coordinate hyperplanes and the nonlinearity $f(x,\xi)$ is of subcritical growth and does not satisfy the Ambrosetti-Rabinowitz (AR) condition.

Keywords: Δ_γ-Laplace problem; Cerami condition; variational method; weak solution; Mountain Pass Theorem

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1. Introduction

In the last decades, the boundary value problem for semilinear elliptic equations

$$-\Delta u = f(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

has been studied by many authors. The following AR condition, introduced in [2],

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(AR) For some $\theta > 2$, and $R > 0$, we have

$$\theta F(x,\xi) \leqslant f(x,\xi)\xi \quad \forall\, |\xi| \geqslant R,\ \forall\, x \in \Omega,$$

where $F(x,\xi) = \int_0^{\xi} f(x,\tau)\,\mathrm{d}\tau$, plays an important role in their studies. With this (AR) condition, one can use the classical version of the Mountain Pass Theorem of Ambrosetti and Rabinowitz to study the existence of solutions (see, for example, [1], [2], [28] and references therein). Although the (AR) condition is quite natural and important, there are many problems where the nonlinear term $f(x,\xi)$ does not satisfy the (AR) condition, for example,

$$f(x,\xi) = \xi \ln(1 + |\xi|).$$

For this reason, in recent years, some authors have studied problem (1.1) by trying to drop the (AR) condition, for instance, Schechter and Zou [29], Miyagaki and Souto [27], Lam and Lu [14], [15], Liu [18], Liu and Wang [19].

Boundary value problems for nonlinear degenerate elliptic differential equations were treated in [10] and subsequently in [8], [11]. In [33], the critical exponent phenomenon was observed for a model of the Grushin-type operators. The results were then generalized in [30] for a large class of semilinear degenerate elliptic differential equations. Recently, in [31], Thuy and Tri have extended the research to a more complicated class of nonlinear degenerate elliptic differential operators. Very recently, in [12] the authors investigated the Δ_γ-Laplace operator under the additional assumption that the operator is homogeneous of degree two with respect to a semigroup of dilations in $\mathbb{R}^N$.

In this paper, we study the existence of nontrivial weak solutions to the following problem:

$$-\Delta_\gamma u = f(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.2}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $\Omega \cap \{x_j = 0\} \neq \emptyset$ for some j, and Δ_γ is a subelliptic operator of the form

$$\Delta_\gamma := \sum_{j=1}^{N} \partial_{x_j}(\gamma_j^2 \partial_{x_j}),\ \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_N)\colon\ \mathbb{R}^N \to \mathbb{R}^N.$$

Here, the functions $\gamma_j\colon\ \mathbb{R}^N \longrightarrow \mathbb{R}$ are assumed to be continuous, different from zero and of class C^1 in $\mathbb{R}^N \setminus \Pi$, where

$$\Pi := \Big\{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N\colon\ \prod_{j=1}^{N} x_j = 0\Big\}.$$

Moreover, we assume the following properties:

(i) There exists a semigroup of dilations $\{\delta_t\}_{t>0}$ such that

$$\delta_t\colon\ \mathbb{R}^N \longrightarrow \mathbb{R}^N,$$
$$(x_1,\dots,x_N) \longmapsto \delta_t(x_1,\dots,x_N) = (t^{\varepsilon_1}x_1,\dots,t^{\varepsilon_N}x_N),$$

where $1=\varepsilon_1 \leqslant \varepsilon_2 \leqslant \ldots \leqslant \varepsilon_N$, such that γ_j is δ_t-homogeneous of degree $\varepsilon_j - 1$, i.e.,

$$\gamma_j(\delta_t(x)) = t^{\varepsilon_j-1}\gamma_j(x) \quad \forall\, x\in\mathbb{R}^N,\ \forall t>0,\ j=1,\dots,N.$$

The number

$$\widetilde{N} := \sum_{j=1}^{N} \varepsilon_j$$

is called the *homogeneous dimension* of $\mathbb{R}^N$ with respect to $\{\delta_t\}_{t>0}$.

(ii)

$$\gamma_1 = 1,\ \gamma_j(x) = \gamma_j(x_1,x_2,\dots,x_{j-1}), \quad j=2,\dots,N.$$

(iii) There exists a constant $\varrho \geqslant 0$ such that

$$0 \leqslant x_k\partial_{x_k}\gamma_j(x) \leqslant \varrho\gamma_j(x) \quad \forall\, k\in\{1,2,\dots,j-1\}\ \forall\, j=2,\dots,N,$$

and for every $x\in\overline{\mathbb{R}}_+^N := \{(x_1,\dots,x_N)\in\mathbb{R}^N\colon\ x_j\geqslant 0 \quad \forall\, j=1,2,\dots,N\}$.

(iv) Equalities $\gamma_j(x)=\gamma_j(x^*)$ $(j=1,2,\dots,N)$ are satisfied for every $x\in\mathbb{R}^N$, where

$$x^* = (|x_1|,\dots,|x_N|) \quad \text{if} \quad x=(x_1,x_2,\dots,x_N).$$

The class of Δ_γ-operators includes many degenerate elliptic operators such as the Grushin-type operator

$$G_\alpha := \Delta_x + |x|^{2\alpha}\Delta_y, \quad \alpha\geqslant 0,$$

where (x,y) denotes a point of $\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}$ (see [9]), and the operator of the form

$$P_{\alpha,\beta} := \Delta_x + \Delta_y + |x|^{2\alpha}|y|^{2\beta}\Delta_z, \quad (x,y,z)\in\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}\times\mathbb{R}^{N_3},$$

where α,β are non-negative real numbers (see [31]). Many aspects of the theory of degenerate elliptic differential operators are presented in monographs [35], [36] (see also some recent results in [3], [7], [13], [20], [21], [22], [23], [24], [26], [32], [34]).

To study problem (1.2), we make the following assumptions:

(A1) f is a real Carathéodory function on $\Omega \times \mathbb{R}$ such that there exist $q_1 \in (2, 2^*_\gamma)$ (where $2^*_\gamma := 2\widetilde{N}/(\widetilde{N}-2)$), $\omega \in \mathcal{K}_{2,q_1}$ (see Definition 2.2) and some positive constant C_0 such that

$$|f(x,\xi)| \leqslant |\omega(x)\xi|^{q_1-1} + C_0 \quad \forall (x,\xi) \in \Omega \times \mathbb{R};$$

(A2) there exist $C \in [0,\infty)$ and $h \in L^1(\Omega)$ such that $|f(x,\xi)| \leqslant h(x)$ for every x in Ω and $|\xi| \leqslant C$;

(A3) there exists a non-positive function k in $L^{q_2}(\Omega)$, $q_2 > \widetilde{N}/2$ such that $k(x) \leqslant f(x,\xi)/\xi$ for every $(x,\xi) \in \Omega \times \mathbb{R}$;

(A4) $f(x,0) = 0$ for every x in Ω and the following limit holds uniformly for a.e. $x \in \Omega$:

$$\lim_{\xi \to 0} \frac{f(x,\xi)}{\xi} = 0;$$

(A5) $\lim\limits_{\xi\to\infty} f(x,\xi)/\xi = \infty$ or $\lim\limits_{\xi\to-\infty} f(x,\xi)/\xi = \infty$ a.e. in Ω;

(A6) $f(x,\xi)/\xi$ is increasing in $\xi \geqslant C$ and decreasing in $\xi \leqslant -C$ for every x in Ω.

Our main result is given by the following theorem.

Theorem 1.1. *Suppose that f satisfies* (A1)–(A6). *Then the boundary value problem* (1.2) *has a nontrivial weak solution.*

Remark 1.2. Suppose that f is continuous on $\overline{\Omega} \times \mathbb{R}$ and satisfies the following conditions:

(A1′) There exist $q_3 \in (2, 2^*_\gamma)$ and a positive real number C_1 such that

$$|f(x,\xi)| \leqslant C_1(1 + |\xi|^{q_3-1}) \quad \forall (x,\xi) \in \Omega \times \mathbb{R}.$$

(A4′) $f(x,0) = 0$ for every x in Ω and $\lim\limits_{\xi\to 0} f(x,\xi)/\xi = 0$ uniformly in Ω.

(A5′) $\lim\limits_{\xi\to\infty} f(x,\xi)/\xi = \infty$ or $\lim\limits_{\xi\to-\infty} f(x,\xi)/\xi = \infty$ uniformly in Ω.

Then f satisfies (A1)–(A5). Therefore, our theorem improves the corresponding results in [31], [35], [36].

This article is organized as follows. In Section 2 we present some definitions and preliminary results. In Section 3 we give the proof of our results.

2. Preliminary results

2.1. Function spaces and embedding theorem. For a function u of class C^1, we put

$$\nabla_\gamma u := (\gamma_1 \partial_{x_1} u, \gamma_2 \partial_{x_2} u, \ldots, \gamma_N \partial_{x_N} u), \quad |\nabla_\gamma u| := \left(\sum_{j=1}^{N} |\gamma_j \partial_{x_j} u|^2 \right)^{1/2}.$$

If Ω is a bounded domain in $\mathbb{R}^N$, we denote by

$$S^2_{\gamma,0}(\Omega)$$

the closure of $C^1_0(\Omega)$ with respect to the norm

$$\|u\|_{S^2_{\gamma,0}(\Omega)} := \left(\int_\Omega |\nabla_\gamma u|^2 \, \mathrm{d}x \right)^{1/2}.$$

From Proposition 3.2 and Theorem 3.3 in [12], we have the following embedding result.

Proposition 2.1. *Assume Ω is a bounded domain in $\mathbb{R}^N$, $\widetilde{N} > 2$. Then the embedding $S^2_{\gamma,0}(\Omega) \hookrightarrow L^p(\Omega)$ is continuous for any $p \in [1, 2^*_\gamma]$, i.e., there exists $C_p > 0$ such that $\|u\|_{L^p(\Omega)} \leqslant C_p \|u\|_{S^2_{\gamma,0}(\Omega)}$ for all $u \in S^2_{\gamma,0}(\Omega)$. Moreover, $S^2_{\gamma,0}(\Omega)$ is compactly embedded into $L^p(\Omega)$ only for $p \in [1, 2^*_\gamma)$, where $L^p(\Omega)$ denotes the Lebesgue space with the standard norm $\|\cdot\|_{L^p(\Omega)}$.*

Definition 2.2. Let σ be a measurable function on Ω. We put

$$T_\sigma u = \sigma u \quad \forall\, u \in S^2_{\gamma,0}(\Omega).$$

We say that

(i) σ is of class $\mathcal{C}_{2,p}$, if T_σ is a continuous mapping from $S^2_{\gamma,0}(\Omega)$ into $L^p(\Omega)$,
(ii) σ is of class $\mathcal{K}_{2,p}$, if T_σ is a compact mapping from $S^2_{\gamma,0}(\Omega)$ into $L^p(\Omega)$.

We have the following results.

Theorem 2.3. *Let us assume that $\omega_1 \in \mathcal{C}_{2,p_1}$, $\omega_2 \in \mathcal{C}_{2,p_2}$ such that ω_1 and ω_2 are non-negative, where $p_1, p_2 \in [1, \infty)$, $p_1 < p_2$. Put*

$$\omega = \omega_1^{p_1(p_2-p)/(p(p_2-p_1))} \omega_2^{p_2(p-p_1)/(p(p_2-p_1))} \quad \textit{for any } p \in (p_1, p_2).$$

Then $\omega \in \mathcal{C}_{2,p}$.

P r o o f. From $\omega_1 \in \mathcal{C}_{2,p_1}$, $\omega_2 \in \mathcal{C}_{2,p_2}$, hence there exists a positive real number C_2 such that

$$\left(\int_\Omega \omega_i^{p_i} |u|^{p_i}\,\mathrm{d}x\right)^{1/p_i} \leqslant C_2 \|u\|_{S^2_{\gamma,0}(\Omega)} \quad \forall\, u \in S^2_{\gamma,0}(\Omega),\ i = 1,2. \tag{2.1}$$

By $p = p_1(p_2 - p)/(p_2 - p_1) + p_2(p - p_1)/(p_2 - p_1)$, applying Hölder's inequality and (2.1), we get

$$\begin{aligned}\left(\int_\Omega \omega^p |u|^p\,\mathrm{d}x\right)^{1/p} &= \biggl(\int_\Omega \omega_1^{p_1(p_2-p)/(p_2-p_1)} |u|^{p_1(p_2-p)/(p_2-p_1)} \\ &\qquad \times \omega_2^{p_2(p-p_1)/(p_2-p_1)} |u|^{p_2(p-p_1)/(p_2-p_1)p_2}\,\mathrm{d}x\biggr)^{1/p} \\ &\leqslant \left\{\left(\int_\Omega \omega_1^{p_1}|u|^{p_1}\,\mathrm{d}x\right)^{(p_2-p)/(p_2-p_1)} \left(\int_\Omega \omega_2^{p_2}|u|^{p_2}\,\mathrm{d}x\right)^{(p-p_1)/(p_2-p_1)}\right\}^{1/p} \\ &\leqslant C_2\|u\|_{S^2_{\gamma,0}(\Omega)} \quad \forall\, u \in S^2_{\gamma,0}(\Omega).\end{aligned}$$

The proof of Theorem 2.3 is complete. □

Theorem 2.4. *Let us assume that $\omega \in \mathcal{C}_{2,p}$ and θ are measurable functions on Ω such that $\omega \geqslant 0$ and $|\theta| \leqslant \omega^\alpha$, where $p \in [1, 2^*_\gamma)$, $\alpha \in (0,1)$. Then θ is of class $\mathcal{K}_{2,p}$.*

P r o o f. Since ω is of class $\mathcal{C}_{2,p}, T_\omega$ is continuous from $S^2_{\gamma,0}(\Omega)$ into $L^p(\Omega)$ and by Proposition 2.1, we have

$$\left(\int_\Omega |u|^p \omega^p\,\mathrm{d}x\right)^{1/p} \leqslant C_p \|u\|_{S^2_{\gamma,0}(\Omega)} \quad \forall\, u \in S^2_{\gamma,0}(\Omega). \tag{2.2}$$

By $\omega^\alpha(x) \leqslant 1 + \omega(x)$ for every x in Ω and $1, \omega \in \mathcal{C}_{2,p}$ hence ω^α belongs to $\mathcal{C}_{2,p}$. Thus θ is of class $\mathcal{C}_{2,p}$. Let M be a positive real number and $\{u_n\}_{n=1}^\infty$ be a sequence in $S^2_{\gamma,0}(\Omega)$, such that $\|u_n\|_{S^2_{\gamma,0}(\Omega)} \leqslant M$ for all n. By Proposition 2.1, we have (by passing to a subsequence if necessary)

$$\begin{aligned} &u_n \rightharpoonup u \text{ weakly in } S^2_{\gamma,0}(\Omega) \text{ as } n \to \infty, \\ &u_n \to u \text{ strongly in } L^p(\Omega) \text{ as } n \to \infty. \end{aligned} \tag{2.3}$$

Therefore

$$\|u\|_{S^2_{\gamma,0}(\Omega)} \leqslant \liminf_{n\to\infty} \|u_n\|_{S^2_{\gamma,0}(\Omega)} \leqslant M.$$

We shall prove that $\{T_\theta(u_n)\}_{n=1}^\infty$ converges to $T_\theta(u)$ in $L^p(\Omega)$. Let ε be a positive real number. Choose a positive real number δ such that

$$(2C_pM)^p \delta^{(\alpha-1)p} < \frac{\varepsilon^p}{2}. \tag{2.4}$$

Put $\Omega' = \{x \in \Omega\colon\ \omega(x) > \delta\}$. From (2.2) and (2.4), we get that

$$\begin{aligned}
(2.5)\qquad \int_\Omega |\theta(u_n-u)|^p\,\mathrm{d}x &= \int_\Omega |u_n-u|^p|\theta|^p\,\mathrm{d}x \\
&\leqslant \int_{\Omega'} |u_n-u|^p\omega^{\alpha p}\,\mathrm{d}x + \int_{\Omega\setminus\Omega'} |u_n-u|^p\omega^{\alpha p}\,\mathrm{d}x \\
&\leqslant \delta^{(\alpha-1)p}\int_{\Omega'} |u_n-u|^p\omega^p\,\mathrm{d}x + \delta^{\alpha p}\int_{\Omega\setminus\Omega'} |u_n-u|^s\,\mathrm{d}x \\
&\leqslant \delta^{(\alpha-1)p}\int_{\Omega} |u_n-u|^p\omega^p\,\mathrm{d}x + \delta^{\alpha p}\int_{\Omega} |u_n-u|^p\,\mathrm{d}x \\
&\leqslant \delta^{(\alpha-1)p}(C_p\|u_n-u\|_{S^2_{\gamma,0}(\Omega)})^p + \delta^{\alpha p}\int_{\Omega} |u_n-u|^p\,\mathrm{d}x \\
&\leqslant \delta^{(\alpha-1)p}(2C_pM)^p + \delta^{\alpha p}\int_{\Omega} |u_n-u|^p\,\mathrm{d}x \\
&\leqslant \frac{\varepsilon^p}{2} + \delta^{p\alpha}\int_{\Omega} |u_n-u|^p\,\mathrm{d}x.
\end{aligned}$$

By (2.3), there exists an integer n_0 such that

$$(2.6)\qquad \int_\Omega |u_n-u|^p\,\mathrm{d}x \leqslant \delta^{-\alpha p}\frac{\varepsilon^p}{2}\quad \forall\, n\geqslant n_0.$$

Combining (2.5) and (2.6), we complete the proof. □

Corollary 2.5. *Let $p\in[1,2^*_\gamma)$, $\eta\in(2p\widetilde{N}/(2\widetilde{N}-p(\widetilde{N}-2)),\infty)$ and $\theta\in L^\eta(\Omega)$. Then θ is of class $\mathcal{K}_{2,p}$.*

Proof. Let $\alpha\in(0,1)$ such that $\alpha\eta = 2p\widetilde{N}/(2\widetilde{N}-p(\widetilde{N}-2))$ and $\omega=|\theta|^{1/\alpha}$. Then ω is in $L^{2p\widetilde{N}/(2\widetilde{N}-p(\widetilde{N}-2))}(\Omega)$. By

$$\frac{2\widetilde{N}-p(\widetilde{N}-2)}{2\widetilde{N}} + \frac{p(\widetilde{N}-2)}{2\widetilde{N}} = 1,$$

applying Hölder's inequality, we obtain

$$\begin{aligned}
\int_\Omega |\omega u|^p\,\mathrm{d}x &\leqslant \Big(\int_\Omega |\omega|^{2p\widetilde{N}/(2\widetilde{N}-p(\widetilde{N}-2))}\,\mathrm{d}x\Big)^{2\widetilde{N}-p(\widetilde{N}-2)/2\widetilde{N}} \\
&\quad\times\Big(\int_\Omega |u|^{2\widetilde{N}/(\widetilde{N}-2)}\,\mathrm{d}x\Big)^{p(\widetilde{N}-2)/2\widetilde{N}}\qquad \forall\, u\in S^2_{\gamma,0}(\Omega),
\end{aligned}$$

which implies that T_ω is continuous at 0 in $S^2_{\gamma,0}(\Omega)$. Thus T_ω is a linear continuous map from $S^2_{\gamma,0}(\Omega)$ into $L^p(\Omega)$. By Theorem 2.4, we have θ is of class $\mathcal{K}_{2,p}$. □

E x a m p l e 2.6. Let $\gamma = (1, 1, |x_1x_2|)$, $\Omega = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3\colon\ |x| < 1\}$ and $p = 3$. Then $N = 3$, $\widetilde{N} = 5$; $2p\widetilde{N}/(2\widetilde{N} - p(\widetilde{N} - 2)) = 2 \cdot 3 \cdot 5/(5 \cdot 2 - 3(5 - 2)) = 30 < 40$. Put $\omega_0 = |x|^{-1/50} \sin(2017|x|)$, then ω_0 is in $L^{40}(\Omega)$. Thus by Corollary 2.5, ω_0 is of class $\mathcal{K}_{2,3}$.

Corollary 2.7. *Let $p \in [1, 2^*_\gamma)$, α be in $(0, 1)$ and $\eta \in \mathcal{C}_{2,2}$. Then*

$$\theta = \eta^{\alpha 2(2^*_\gamma - p)/(p(2^*_\gamma - 2))}$$

is of class $\mathcal{K}_{2,p}$.

P r o o f. Put $\omega_1 = \eta$, $\omega_2 = 1$, $p_1 = 2$, $p_2 = 2^*_\gamma$. By Proposition 2.1, $\omega_2 \in \mathcal{C}_{2,2^*_\gamma}$. By Theorem 2.3, we see that $\eta^{2(2^*_\gamma - p)/(p(2^*_\gamma - 2))} \in \mathcal{C}_{2,p}$. Thus by Theorem 2.4, $\eta^{\alpha 2(2^*_\gamma - p)/(p(2^*_\gamma - 2))}$ is of class $\mathcal{K}_{2,p}$. □

E x a m p l e 2.8. Let $\gamma = (1, 1, |x_1x_2|)$, $\Omega = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3\colon\ |x| < 1\}$, $p = 3$, $\alpha = \frac{3}{4}$ and $\eta(x) = (1 - |x|^2)^{-1/2}$ for every x in Ω, hence $\eta \in \mathcal{C}_{2,2}$, $\widetilde{N} = 5$. Note that $2^*_\gamma = 2\widetilde{N}/(\widetilde{N} - 2) = \frac{10}{3}$ and

$$\alpha \frac{2(2^*_\gamma - p)}{p(2^*_\gamma - 2)} = 2.$$

Put $\theta(x) = (1 - |x|^2)^{-1}$. Then $\theta \in \mathcal{K}_{2,3}$.

E x a m p l e 2.9. Let $\gamma = (1, 1, |x_1x_2|)$, $\Omega = \{x \in \mathbb{R}^3\colon\ |x| < 1\}$, $p = 3$, $\alpha = \frac{3}{4}$ and $\varrho(x) = (\frac{1}{2} - |x|^2)^2(1 - |x|^2)^{-1}$ for every x in Ω. By Example 2.8, $\varrho \in \mathcal{K}_{2,3}$. Put $a(x) = \varrho(x)^2 = (\frac{1}{2} - |x|^2)^4(1 - |x|^2)^{-2}$. Thus a is not integrable on Ω.

Theorem 2.10. *Let $p \in [1, 2^*_\gamma)$, ω be in $\mathcal{K}_{2,p}$, a function $g \in L^{p/(p-1)}(\Omega)$ and f be a Carathéodory function from $\Omega \times \mathbb{R}$ into $\mathbb{R}$. Assume*

$$|f(x, \xi)| \leqslant |\omega(x)|^{p-1}|\xi|^{p-1} + g(x) \quad \forall\, (x, \xi) \in \Omega \times \mathbb{R}.$$

Then $\Phi_1(u) \in C^1(S^2_{\gamma,0}(\Omega), \mathbb{R})$ and

$$\langle \Phi_1'(u), v\rangle = \int_\Omega f(x, u(x))v(x)\,\mathrm{d}x$$

for all $v \in S^2_{\gamma,0}(\Omega)$, where

$$\Phi_1(u) = \int_\Omega F(x, u(x))\,\mathrm{d}x,$$

and $F(x, \xi) = \int_0^\xi f(x, \tau)\,\mathrm{d}\tau$.

P r o o f. With a slight modification, the proof of this lemma is similar to the one of Lemma 2.3 in [25]. We omit the details. □

Define the Euler-Lagrange functional associated with problem (1.2) as follows:

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla_\gamma u|^2 \,\mathrm{d}x - \int_\Omega F(x, u(x)) \,\mathrm{d}x.$$

From Theorem 2.10 and the fact that f satisfies (A1), we have that Φ is well-defined on $S^2_{\gamma,0}(\Omega)$ and $\Phi \in C^1(S^2_{\gamma,0}(\Omega), \mathbb{R})$ with

$$\langle \Phi'(u), v\rangle = \int_\Omega \nabla_\gamma u \cdot \nabla_\gamma v \,\mathrm{d}x - \int_\Omega f(x, u(x))v(x) \,\mathrm{d}x$$

for all $v \in S^2_{\gamma,0}(\Omega)$.

Recall that a function $u \in S^2_{\gamma,0}(\Omega)$ is called a weak solution of problem (1.2) if

$$\int_\Omega \nabla_\gamma u \cdot \nabla_\gamma v \,\mathrm{d}x = \int_\Omega f(x, u(x))v(x) \,\mathrm{d}x \quad \forall\, v \in S^2_{\gamma,0}(\Omega).$$

Hence, the weak solutions of problem (1.2) are critical points of the functional Φ.

2.2. Mountain Pass Theorem.

Definition 2.11. Let $\mathbf{X}$ be a real Banach space with its dual space $\mathbf{X}^*$ and $\Phi \in C^1(\mathbf{X}, \mathbb{R})$. For $c \in \mathbb{R}$ we say that Φ satisfies the $(C)_c$ *condition* if for any sequence $\{u_n\}_{n=1}^\infty \subset \mathbf{X}$ with

$$\Phi(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|_{\mathbf{X}})\|\Phi'(u_n)\|_{\mathbf{X}^*} \to 0,$$

there exists a subsequence $\{u_{n_k}\}_{k=1}^\infty$ that converges strongly in $\mathbf{X}$.

We will use the following version of the Mountain Pass Theorem:

Lemma 2.12 (see [5], [6]). *Let $\mathbf{X}$ be a real Banach space and let $\Phi \in C^1(\mathbf{X}, \mathbb{R})$ satisfy the $(C)_c$ condition for any $c \in \mathbb{R}$, $\Phi(0) = 0$ and*

(i) *there exist constants $\varrho, \alpha > 0$ such that $\Phi(u) \geqslant \alpha$ for all $u \in \mathbf{X}$, $\|u\|_{\mathbf{X}} = \varrho$;*
(ii) *there exists a $u_1 \in \mathbf{X}$, $\|u_1\|_{\mathbf{X}} \geqslant \varrho$ such that $\Phi(u_1) \leqslant 0$.*

Then $\beta := \inf\limits_{\lambda\in\Lambda} \max\limits_{0\leqslant t\leqslant 1} \Phi(\lambda(t)) \geqslant \alpha$ is a critical value of Φ, where

$$\Lambda := \{\lambda \in C([0; 1], \mathbf{X})\colon\ \lambda(0) = 0,\ \lambda(1) = u_1\}.$$

3. Proof of the main theorem

We prove Theorem 1.1 by verifying that all conditions of Lemma 2.12 are satisfied. First, we check condition (i) in Lemma 2.12.

Lemma 3.1. *Assume that f satisfies conditions* (A1) *and* (A4). *Then there exist* ϱ, $\alpha > 0$ *such that*

$$\Phi(u) \geqslant \alpha \quad \forall\, u \in S^2_{\gamma,0}(\Omega), \quad \|u\|_{S^2_{\gamma,0}(\Omega)} = \varrho.$$

P r o o f. Suppose by contradiction that

$$\inf\Big\{\Phi(u)\colon\ u \in S^2_{\gamma,0}(\Omega), \|u\|_{S^2_{\gamma,0}(\Omega)} = \frac{1}{n}\Big\} \leqslant 0 \quad \forall\, n \in \mathbb{N}.$$

Then, there exists a sequence $\{u_n\}_{n=1}^{\infty}$ in $S^2_{\gamma,0}(\Omega)$ such that $\|u_n\|_{S^2_{\gamma,0}(\Omega)} = 1/n$ and $\Phi(u_n) < 1/n^3$. Hence, we have

$$\frac{1}{n} > \frac{\Phi(u_n)}{\|u_n\|^2_{S^2_{\gamma,0}(\Omega)}} = \frac{1}{2} - \int_\Omega \frac{F(x,u_n(x))}{\|u_n\|^2_{S^2_{\gamma,0}(\Omega)}}\,\mathrm{d}x,$$

and thus

$$\int_\Omega \frac{F(x,u_n(x))}{\|u_n\|^2_{S^2_{\gamma,0}(\Omega)}}\,\mathrm{d}x > \frac{1}{2} - \frac{1}{n}. \tag{3.1}$$

By (A4), for each $\varepsilon > 0$, we can find a number $\delta > 0$ such that

$$|F(x,\xi)| \leqslant \varepsilon\xi^2 \quad \text{for } |\xi| \leqslant \delta. \tag{3.2}$$

From (A1), we deduce that

$$|F(x,\xi)| \leqslant \frac{1}{q_1}|\omega(x)|^{q_1-1}|\xi|^{q_1} + C(\delta)|\xi|^{q_1} \quad \text{for} |\xi| \geqslant \delta. \tag{3.3}$$

It follows from (3.2), (3.3), Proposition 2.1 and Hölder's inequality that

$$\begin{aligned}\Big|\int_\Omega F(x,u_n(x))\,\mathrm{d}x\Big| &\leqslant \varepsilon\int_\Omega |u_n(x)|^2\,\mathrm{d}x + \frac{1}{q_1}\int_\Omega |\omega(x)|^{q_1-1}|u_n(x)|^{q_1}\,\mathrm{d}x \\ &\quad + C(\delta)\int_\Omega |u_n(x)|^{q_1}\,\mathrm{d}x \\ &\leqslant \frac{1}{q_1}\Big(\int_\Omega |\omega(x)u_n(x)|^{q_1}\,\mathrm{d}x\Big)^{(q_1-1)/q_1}\Big(\int_\Omega |u_n(x)|^{q_1}\,\mathrm{d}x\Big)^{1/q_1} \\ &\quad + \varepsilon\|u_n\|^2_{L^2(\Omega)} + C(\delta)\|u_n\|^{q_1}_{L^{q_1}(\Omega)} \\ &\leqslant \varepsilon C_2^2\|u_n\|^2_{S^2_{\gamma,0}(\Omega)} + \frac{C^{q_1}_{q_1}}{q_1}\|u_n\|^{q_1}_{S^2_{\gamma,0}(\Omega)} + C(\delta)C^{q_1}_{q_1}\|u_n\|^{q_1}_{S^2_{\gamma,0}(\Omega)},\end{aligned}$$

hence

$$\left|\int_\Omega \frac{F(x,u_n(x))}{\|u_n\|^2_{S^2_{\gamma,0}(\Omega)}}\,\mathrm{d}x\right| \to 0 \quad \text{as } n\to\infty,$$

which yields a contradiction to (3.1). Lemma 3.1 is proved. □

Next, we check condition (ii) in Lemma 2.12.

Lemma 3.2. *Let ϱ be as in Lemma 3.1 and assume that f satisfies conditions* (A3) *and* (A5). *Then there exists u_1 in $S^2_{\gamma,0}(\Omega)\setminus B(0,\varrho)$ such that $\Phi(u_1)<0$.*

Proof. We limit ourselves to consider the case

$$\lim_{\xi\to\infty}\frac{f(x,\xi)}{\xi}=\infty. \tag{3.4}$$

Take a point $u\in S^2_{\gamma,0}(\Omega)$ such that $\|u\|_{S^2_{\gamma,0}(\Omega)}=1$, $u>0$ and $\|u\|_{L^2(\Omega)}\neq 0$. Then, for any constant $R>0$

$$\Phi(Ru)=\frac{R^2}{2}-\int_\Omega F(x,Ru(x))\,\mathrm{d}x.$$

Since we are assuming (3.4), there exists a number $M>0$ such that $f(x,\xi)\geqslant 4\xi/\|u\|^2_{L^2(\Omega)}$ for $\xi\geqslant M$; moreover, from (A3) we get

$$\begin{aligned}F(x,\xi)&\geqslant\int_0^M k(x)\tau\,\mathrm{d}\tau=k(x)\frac{M^2}{2}\quad\text{for } 0\leqslant\xi\leqslant M,\\ F(x,\xi)&=\int_0^M f(x,\tau)\,\mathrm{d}\tau+\int_M^\xi f(x,\tau)\,\mathrm{d}\tau\\ &\geqslant k(x)\frac{M^2}{2}+\frac{2\xi^2}{\|u\|^2_{L^2(\Omega)}}-\frac{2M^2}{\|u\|^2_{L^2(\Omega)}}\quad\text{for } \xi\geqslant M.\end{aligned}$$

As $0\leqslant Ru\leqslant M$ on $\Omega^u_{M/R}:=\{x\in\Omega\colon\ |u(x)|\leqslant M/R\}$, we have

$$\begin{aligned}\int_\Omega F(x,Ru(x))\,\mathrm{d}x&=\int_{\Omega^u_{M/R}}F(x,Ru(x))\,\mathrm{d}x+\int_{\Omega\setminus\Omega^u_{M/R}}F(x,Ru(x))\,\mathrm{d}x\\ &\geqslant\int_{\Omega\setminus\Omega^u_{M/R}}\frac{2R^2u^2(x)}{\|u\|^2_{L^2(\Omega)}}\,\mathrm{d}x+M^2\int_\Omega k(x)\,\mathrm{d}x-\frac{2M^2}{\|u\|^2_{L^2(\Omega)}}\operatorname{meas}(\Omega),\end{aligned}$$

where $\operatorname{meas}(\cdot)$ denotes the Lebesgue measure of a set in $\mathbb{R}^N$. By Lebesgue's theorem, there exists a number R_0 such that

$$\int_{\Omega\setminus\Omega^u_{M/R_0}}u^2(x)\,\mathrm{d}x\geqslant\frac{\|u\|^2_{L^2(\Omega)}}{2}.$$

Therefore, if $R \geqslant R_0$, then

$$\int_\Omega F(x, Ru(x))\,\mathrm{d}x \geqslant R^2 + M^2 \int_\Omega k(x)\,\mathrm{d}x - \frac{2M^2}{\|u\|^2_{L^2(\Omega)}} \operatorname{meas}(\Omega).$$

Consequently, if

$$R > \max\left\{2\sqrt{-M^2 \int_\Omega k(x)\,\mathrm{d}x + \frac{2M^2}{\|u\|^2_{L^2(\Omega)}} \operatorname{meas}(\Omega)}, R_0\right\},$$

then

$$\Phi(Ru) \leqslant -M^2 \int_\Omega k(x)\,\mathrm{d}x + \frac{2M^2}{\|u\|^2_{L^2(\Omega)}} \operatorname{meas}(\Omega) - \frac{R^2}{2} < 0.$$

Thus, Φ satisfies the condition (ii) in Lemma 2.12. □

Lemma 3.3. *Assume that f satisfies conditions* (A2) *and* (A6). *Then there exists a positive real number C_3 such that*

$$f(x,s)s - 2F(x,s) \leqslant f(x,t)t - 2F(x,t) + C_3 h(x) \quad \forall\, x \in \Omega,\ |s| \leqslant |t|.$$

P r o o f. Since (A6) holds, by Lemma 2.3 in [18], we have that, for any $x \in \Omega$,

$$\xi \mapsto f(x,\xi)\xi - 2F(x,\xi)$$

is increasing in $\xi \geqslant C$ and decreasing in $\xi \leqslant -C$. Hence,

$$f(x,s)s - 2F(x,s) \leqslant f(x,t)t - 2F(x,t) \quad \forall\, x \in \Omega,\ C \leqslant s \leqslant t.$$

Let $x \in \Omega$ and $\xi \in [-C, C]$. By (A2), we get that

$$|f(x,\xi)| \leqslant h(x), \quad |F(x,\xi)| \leqslant \int_0^\xi h(x)\,\mathrm{d}\tau \leqslant Ch(x).$$

Thus,

$$\begin{aligned} f(x,s)s - 2F(x,s) \leqslant& f(x,t)t - 2F(x,t) + 6Ch(x) \qquad \forall\, x \in \Omega,\ 0 \leqslant s \leqslant t \leqslant C,\\ f(x,s)s - 2F(x,s) \leqslant& f(x,C)C - 2F(x,C) + 6Ch(x)\\ \leqslant& f(x,t)t - 2F(x,t) + 6Ch(x) \qquad \forall\, x \in \Omega,\ 0 \leqslant s \leqslant C \leqslant t. \end{aligned}$$

Thus we get the lemma for $0 \leqslant s \leqslant t$. Similarly we obtain it for $t \leqslant s \leqslant 0$. The proof of Lemma 3.3 is complete. □

We now show the main lemma of this paper.

Lemma 3.4. *Assume that f satisfies conditions* (A1)–(A3), (A5) *and* (A6). *Then Φ satisfies the $(C)_c$ condition for all $c \in \mathbb{R}$.*

P r o o f. Let $\{u_n\}_{n=1}^{\infty} \subset S^2_{\gamma,0}(\Omega)$ be a $(C)_c$ sequence, i.e.,

$$\Phi(u_n) \to c \quad \text{as } n \to \infty, \quad \lim_{n\to\infty} (1 + \|u_n\|_{S^2_{\gamma,0}(\Omega)})\|\Phi'(u_n)\|_{(S^2_{\gamma,0}(\Omega))^*} = 0, \tag{3.5}$$

hence,

$$\begin{aligned}\lim_{n\to\infty} \int_{\Omega} \Big(\frac{1}{2} f(x, u_n(x))u_n(x) - F(x, u_n(x))\Big)\,\mathrm{d}x \\ = \lim_{n\to\infty} \Big(\Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle\Big) = c.\end{aligned} \tag{3.6}$$

We first show that $\{u_n\}_{n=1}^{\infty}$ is bounded in $S^2_{\gamma,0}(\Omega)$ by a contradiction argument. By passing to a subsequence if necessary, we can assume that $\|u_n\|_{S^2_{\gamma,0}(\Omega)} > 1$ and

$$\|u_n\|_{S^2_{\gamma,0}(\Omega)} \to \infty \quad \text{as } n \to \infty. \tag{3.7}$$

Setting

$$w_n = \frac{u_n}{\|u_n\|_{S^2_{\gamma,0}(\Omega)}},$$

we get $\|w_n\|_{S^2_{1,0}(\Omega)} = 1$, so we can extract a subsequence relabelled $\{w_n\}_{n=1}^{\infty}$ such that $\{w_n\}_{n=1}^{\infty}$ converges weakly to w in $S^2_{\gamma,0}(\Omega)$. Since Ω is bounded, Proposition 2.1 implies that

$$\begin{aligned} &w_n \to w \text{ strongly in } L^p(\Omega),\ 1 \leqslant p < 2^*_\gamma \text{ as } n \to \infty, \\ &w_n \to w \text{ a.e. in } \Omega \text{ as } n \to \infty. \end{aligned} \tag{3.8}$$

Now, we consider two possible cases: $w = 0$ or $w \neq 0$.

Case 1: If $w = 0$, then for any $n \in \mathbb{N}$ there exists $t_n \in [0, 1]$ such that

$$\Phi(t_n u_n) = \max\{\Phi(s u_n) \colon s \in [0, 1]\}. \tag{3.9}$$

Fix a positive integer m and put $v_n = (4m)^{1/2} w_n$ for every positive integer n. Therefore, we have

$$\begin{aligned} &v_n \rightharpoonup 0 \text{ weakly in } S^2_{\gamma,0}(\Omega) \text{ as } n \to \infty, \\ &v_n \to 0 \text{ a.e. in } \Omega \text{ as } n \to \infty, \\ &v_n \to 0 \text{ strongly in } L^p(\Omega),\ 1 \leqslant p < 2^*_\gamma \text{ as } n \to \infty, \\ &\omega v_n \to 0 \text{ strongly in } L^{q_1}(\Omega) \text{ as } n \to \infty. \end{aligned} \tag{3.10}$$

By (A1), applying Hölder's inequality, we get

$$\tag{3.11} \begin{aligned} \left|\int_\Omega F(x,v_n(x))\,\mathrm{d}x\right| &\leqslant \int_\Omega |\omega(x)v_n(x)|^{q_1-1}|v_n(x)|\,\mathrm{d}x + C_0\int_\Omega |v_n(x)|\,\mathrm{d}x \\ &\leqslant \left(\int_\Omega |\omega(x)v_n(x)|^{q_1}\,\mathrm{d}x\right)^{(q_1-1)/q_1}\left(\int_\Omega |v_n(x)|^{q_1}\,\mathrm{d}x\right)^{1/q_1} \\ &\quad + C_0\int_\Omega |v_n(x)|\,\mathrm{d}x. \end{aligned}$$

From (3.10) and (3.11), we have

$$\left|\int_\Omega F(x,v_n(x))\,\mathrm{d}x\right| \to 0 \quad \text{as } n\to\infty,$$

hence,

$$\lim_{n\to\infty}\int_\Omega F(x,v_n(x))\,\mathrm{d}x = 0.$$

Since $\lim\limits_{n\to\infty}(4m)^{1/2}\|u_n\|^{-1}_{S^2_{\gamma,0}(\Omega)} = 0$, there exists an integer N_m such that

$$\Phi(t_nu_n) \geqslant \Phi(v_n) = 2m - \int_\Omega F(x,v_n(x))\,\mathrm{d}x \geqslant m \quad \forall\, n\geqslant N_m,$$

that is, $\lim\limits_{n\to\infty}\Phi(t_nu_n) = \infty$. Since $\Phi(0)=0$ and $\lim\limits_{n\to\infty}\Phi(u_n) = c$, it implies $t_n\in(0,1)$ for any sufficiently large n and

$$\begin{aligned} \int_\Omega |\nabla_\gamma(t_nu_n)|^2\,\mathrm{d}x &- \int_\Omega f(x,t_nu_n(x))t_nu_n(x)\,\mathrm{d}x \\ &= \langle\Phi'(t_nu_n),t_nu_n\rangle = t_n\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_n}\Phi(tu_n) = 0. \end{aligned}$$

Therefore, by Lemma 3.3, we get

$$\begin{aligned} \int_\Omega \Big(\frac12 f(x,u_n(x))u_n(x) &- F(x,u_n(x))\Big)\,\mathrm{d}x \\ &\geqslant \int_\Omega\Big(\frac12 f(x,t_nu_n(x))t_nu_n(x) - F(x,t_nu_n(x))\Big)\,\mathrm{d}x - C_3\|h\|_{L^1(\Omega)} \\ &= \int_\Omega\Big(\frac12|\nabla_\gamma t_nu_n|^2 - F(x,t_nu_n(x))\Big)\,\mathrm{d}x - C_3\|h\|_{L^1(\Omega)} \\ &= \Phi(t_nu_n) - C_3\|h\|_{L^1(\Omega)} \to \infty, \end{aligned}$$

which contradicts (3.6).

Case 2: If $w \neq 0$, the Lebesgue measure of the set $\Theta = \{x \in \Omega\colon w(x) \neq 0\}$ is positive. We have $\lim_{n\to\infty} |u_n(x)| = \infty$ for every x in Θ. Hence, by (A5) we deduce

$$\frac{f(x, u_n(x))}{u_n(x)} \to \infty \quad \text{as } n \to \infty. \tag{3.12}$$

By (3.8) and Theorem 4.9 in [4], there exists a function $w_0 \in L^p(\Omega)$ such that

$$|w_n| \leqslant w_0 \quad \forall\, n, \text{ a.e. on } \Omega.$$

Now, using condition (A3) we have

$$\frac{f(x, \xi u_n(x))}{\xi u_n(x)} \cdot \frac{\xi |u_n(x)|^2}{\|u_n\|^2_{S^2_{\gamma,0}(\Omega)}} \geqslant \xi k(x) w_0^2(x) \quad \forall\, x \in \Omega,\ \xi \in (0,1);$$

thus, since $kw_0^2 \in L^1(\Omega)$ (as $k \in L^{q_2}(\Omega)$ for some $q_2 > \widetilde{N}/2$), using (3.12), (A3) and the fact that $\Phi(u_n) \to c$, we deduce via the generalized Fatou lemma that

$$\begin{aligned}
\frac{1}{2} &= \liminf_{n\to\infty} \Big[\frac{1}{2} - \frac{\Phi(u_n)}{\|u_n\|^2_{S^2_{\gamma,0}(\Omega)}}\Big] = \liminf_{n\to\infty} \int_\Omega \frac{F(x, u_n(x))}{\|u_n\|^2_{S^2_{\gamma,0}(\Omega)}} \,\mathrm{d}x \\
&= \liminf_{n\to\infty} \int_\Omega \int_0^1 \frac{f(x, \xi u_n(x))}{\xi u_n(x)} \cdot \xi |w_n(x)|^2 \,\mathrm{d}\xi \,\mathrm{d}x \quad (\text{since } w \equiv 0 \text{ on } \Omega \setminus \Theta) \\
&= \liminf_{n\to\infty} \int_\Theta \int_0^1 \frac{f(x, \xi u_n(x))}{\xi u_n(x)} \cdot \xi |w_n(x)|^2 \,\mathrm{d}\xi \,\mathrm{d}x \\
&\geqslant \int_\Theta \int_0^1 \liminf_{n\to\infty} \frac{f(x, \xi u_n(x))}{\xi u_n(x)} \cdot \xi |w_n(x)|^2 \,\mathrm{d}\xi \,\mathrm{d}x = \infty,
\end{aligned}$$

which is impossible. In any case, we obtain a contradiction. Thus, the sequence $\{u_n\}_{n=1}^\infty$ is bounded in $S^2_{\gamma,0}(\Omega)$. Therefore, we can (by passing to a subsequence if necessary) suppose that

$$\begin{aligned}
u_n &\rightharpoonup u \text{ weakly in } S^2_{\gamma,0}(\Omega) \text{ as } n \to \infty, \\
u_n &\to u \text{ a.e. in } \Omega \text{ as } n \to \infty, \\
u_n &\to u \text{ strongly in } L^p(\Omega),\ 1 \leqslant p < 2^*_\gamma \text{ as } n \to \infty, \\
\omega u_n &\to \omega u \text{ strongly in } L^{q_1}(\Omega) \text{ as } n \to \infty.
\end{aligned} \tag{3.13}$$

Thus by (A1), we have

$$\begin{aligned}
\Big|\int_\Omega f(x, u_n(x))(u_n(x) - u(x)) \,\mathrm{d}x\Big| &\leqslant C_0 \int_\Omega |u_n(x) - u(x)| \,\mathrm{d}x \\
&+ \int_\Omega |u_n(x) - u(x)| |\omega(x) u_n(x)|^{p-1} \,\mathrm{d}x \leqslant C_0 \|u_n - u\|_{L^1(\Omega)} \\
&+ \Big(\int_\Omega |u_n(x) - u(x)|^{q_1} \,\mathrm{d}x\Big)^{1/q_1} \Big(\int_\Omega |\omega(x) u_n(x)|^{q_1} \,\mathrm{d}x\Big)^{(q_1-1)/q_1}.
\end{aligned} \tag{3.14}$$

In view of (3.13), we can conclude that

$$\int_\Omega f(x, u_n(x))(u_n(x) - u(x))\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

Thus,

$$\int_\Omega (f(x, u_n(x)) - f(x, u(x)))(u_n(x) - u(x))\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty. \tag{3.15}$$

It follows from $\lim\limits_{n\to\infty} \Phi'(u_n) = 0$ and (3.13) that

$$\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \to 0 \quad \text{as } n \to \infty. \tag{3.16}$$

By (3.15) and (3.16), we obtain

$$\int_\Omega |\nabla_\gamma u_n - \nabla_\gamma u|^2\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

Therefore, we conclude that $u_n \to u$ strongly in $S^2_{\gamma,0}(\Omega)$. The proof of Lemma 3.4 is complete. □

Proof of Theorem 1.1. By Lemmas 3.1, 3.2, and 3.4, all conditions of Lemma 2.12 are satisfied. Thus, problem (1.2) has a nontrivial weak solution □

Example 3.5. Let $\gamma = (1, 1, |x_1x_2|)$, $N = 3$, $\Omega = \{x \in \mathbb{R}^3 \colon |x| < 1\}$, $\widetilde{N} = 5$, $q_1 = 3$,

$$\begin{aligned}
\omega_0(x) &= |x|^{-1/50}\sin(2017|x|) \quad \forall\, x \in \Omega,\\
\omega_1(x) &= \Big(\frac{1}{2} - |x|^2\Big)^2 (1 - |x|^2)^{-1} \quad \forall\, x \in \Omega,\\
\varphi_0(t) &= \begin{cases} t^2(1 - |t|) & \text{if } |t| \leqslant 1,\\ 0 & \text{if } |t| \in \mathbb{R} \setminus [-1, 1],\end{cases}\\
\varphi_1(t) &= \begin{cases} 0 & \text{if } |t| \leqslant 1,\\ t(|t| - 1)\ln(1 + |t|) & \text{if } |t| \in (1, 2],\\ t\ln(1 + |t|) & \text{if } |t| > 2,\end{cases}\\
f(x, t) &= \omega_0^2(x)\varphi_0(t) + \omega_1^2(x)\varphi_1(t) \quad \forall\,(x, t) \in \Omega \times \mathbb{R}.
\end{aligned}$$

Let $\omega = |\omega_0| + \omega_1$, $C = 1$, $h(x) = |x|^{-1/50}$, $k(x) = -h(x)$ for every x in Ω. We see that $h \in L^1(\Omega)$ and $k \in L^{\widetilde{N}/2}(\Omega)$. By Examples 2.6 and 2.8, ω is in $\mathcal{K}_{2,3}$. Thus f satisfies conditions (A1)–(A5).

We have $f(x, t)/t = \omega_1^2(x)(|t| - 1)\ln(1 + |t|)$ for every $t \in [-2, 2] \setminus [-1, 1]$ and $f(x, t)/t = \omega_1^2(x)\ln(1 + |t|)$ for every $t \in \mathbb{R} \setminus [-2, 2]$. Thus f satisfies (A6). Therefore, we can apply Theorem 1.1 to f with $C = 1$. We have that ω^2 is not integrable on Ω. Hence, the results in [2], [3], [12], [16], [17], [18], [22], [25] cannot be applied to solve of problem (1.2) in this case.

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