AN IMPROVED REGULARITY CRITERIA FOR THE MHD SYSTEM BASED ON TWO COMPONENTS OF THE SOLUTION

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Abstract. As observed by Yamazaki, the third component b_3 of the magnetic field can be estimated by the corresponding component u_3 of the velocity field in L^λ ($2 \leqslant \lambda \leqslant 6$) norm. This leads him to establish regularity criterion involving u_3, j_3 or u_3, ω_3. Noticing that λ can be greater than 6 in this paper, we can improve previous results.

Keywords: MHD equations; regularity criteria

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1. Introduction

In this paper, we investigate the following three-dimensional (3D) magnetohydro-dynamic (MHD) equations:

$$(1) \quad \begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{b} \cdot \nabla)\boldsymbol{b} - \triangle \boldsymbol{u} + \nabla \Pi = \boldsymbol{0}, \\ \partial_t \boldsymbol{b} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{b} - (\boldsymbol{b} \cdot \nabla)\boldsymbol{u} - \triangle \boldsymbol{b} = \boldsymbol{0}, \\ \nabla \cdot \boldsymbol{u} = 0, \\ \nabla \cdot \boldsymbol{b} = 0, \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \ \boldsymbol{b}|_{t=0} = \boldsymbol{b}_0, \end{cases}$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$ is the fluid velocity field, $\boldsymbol{b} = (b_1, b_2, b_3)$ is the magnetic field, Π is a scalar pressure, and $\boldsymbol{u}_0, \boldsymbol{b}_0$ are the prescribed initial data satisfying $\nabla \cdot \boldsymbol{u}_0 = \nabla \cdot \boldsymbol{b}_0 = 0$ in the distributional sense. From the physical point of view,

(1) governs the dynamics of the velocity and magnetic fields in electrically conducting fluids, such as plasmas, liquid metals, and salt water. Moreover, $(1)_1$ reflects the conservation of momentum, $(1)_2$ is the induction equation, and $(1)_3$ specifies the conservation of mass.

Besides its physical applications, the MHD system (1) is also mathematically significant. Duvaut-Lions [5] (see also Sermange-Temam [14]) showed that (1) possesses at least one global weak solution for initial data with finite energy. However, the issue of regularity and uniqueness of weak solutions remains a challenging open problem in mathematical fluid dynamics. Many interesting and important sufficient conditions (see e.g., [1], [3], [4], [6], [7], [8], [9], [10], [11], [12], [15], [18], [20], [21], [24], [25] and the references therein) were derived to guarantee the regularity of the weak solution. In this paper, we are interested in regularity criteria involving only two components of the solution, in view of the fact that there exists no such result via one component. In 2010, Ji-Lee [7] showed that if

$$(2) \qquad u_1, u_2 \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = \frac{1}{2}, \quad 6 \leqslant q \leqslant \infty,$$

then the solution is smooth on $(0, T)$. The method is to establish first a regularity criterion involving u_1, u_2, b_1, b_2, and then to control b_1, b_2 in terms of u_1, u_2. For later developments, let us denote by

$$\nabla \times \boldsymbol{u} = \boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3), \quad \nabla \times \boldsymbol{b} = \boldsymbol{j} = (j_1, j_2, j_3)$$

the vorticity and the current density respectively. In 2014, Yamazaki [15] found a fine structure of the horizontal convective terms

$$(3) \quad \int [(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}] \cdot \triangle_h \boldsymbol{u} \, \mathrm{d}x - \int [(\boldsymbol{b} \cdot \nabla)\boldsymbol{b}] \cdot \triangle_h \boldsymbol{u} \, \mathrm{d}x$$

$$+ \int [(\boldsymbol{u} \cdot \nabla)\boldsymbol{b}] \cdot \triangle_h \boldsymbol{b} \, \mathrm{d}x - \int [(\boldsymbol{b} \cdot \nabla)\boldsymbol{u}] \cdot \triangle_h \boldsymbol{b} \, \mathrm{d}x$$

$$\leqslant C \int (|u_3| + |b_3|) \cdot |\nabla(\boldsymbol{u}, \boldsymbol{b})| \cdot |\nabla \nabla_h(\boldsymbol{u}, \boldsymbol{b})| \, \mathrm{d}x + C \int |\nabla_h(\boldsymbol{u}, \boldsymbol{b})|^2 \cdot |j_3| \, \mathrm{d}x,$$

where $\triangle_h = \partial_1 \partial_1 + \partial_2 \partial_2$ is the horizontal Laplacian. With (3) and the following bound of b_3 in terms of u_3,

$$(4) \quad \sup_{t_1 \leqslant t \leqslant t_2} \|b_3(t)\|_{L^\lambda}^2$$

$$\leqslant \|b_3(t_1)\|_{L^2}^2 + C \int_{t_1}^{t_2} \|\nabla \boldsymbol{b}(t)\|_{L^2}^2 \|u_3(t)\|_{L^{6\lambda/(6-\lambda)}}^2 \, \mathrm{d}t \quad (2 \leqslant \lambda \leqslant 6),$$

he was able to prove the following regularity condition

$$u_3 \in L^p(0,T;L^q(\mathbb{R}^3)), \quad j_3 \in L^r(0,T;L^s(\mathbb{R}^3)),$$

(5)

$$\frac{2}{p} + \frac{3}{q} = \frac{1}{3} + \frac{1}{2q}, \quad \frac{15}{2} < q \leqslant \infty; \quad \frac{2}{r} + \frac{3}{s} = 2, \quad \frac{3}{2} < s \leqslant \infty.$$

By treating (4) via the multiplicative Sobolev inequality

(6)

$$\|f\|_{L^6} \leqslant C \|\partial_1 f\|_{L^2}^{1/3} \|\partial_2 f\|_{L^2}^{1/3} \|\partial_3 f\|_{L^2}^{1/3},$$

and estimating the nonlinear terms in a flexible way, Zhang [21], [22] refined (5) to be

$$u_3 \in L^p(0,T;L^q(\mathbb{R}^3)), \quad j_3 \in L^r(0,T;L^s(\mathbb{R}^3)),$$

(7)

$$\frac{2}{p} + \frac{3}{q} = \frac{4}{9}, \quad \frac{27}{4} \leqslant q \leqslant \infty; \quad \frac{2}{r} + \frac{3}{s} = 2, \quad \frac{3}{2} < s \leqslant \infty.$$

In 2016, Yamazaki [18] observed one another new nice structure

(8)
$$\int [(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}] \cdot \triangle_h \boldsymbol{u} \, \mathrm{d}x - \int [(\boldsymbol{b} \cdot \nabla)\boldsymbol{b}] \cdot \triangle_h \boldsymbol{u} \, \mathrm{d}x$$

$$+ \int [(\boldsymbol{u} \cdot \nabla)\boldsymbol{b}] \cdot \triangle_h \boldsymbol{b} \, \mathrm{d}x - \int [(\boldsymbol{b} \cdot \nabla)\boldsymbol{u}] \cdot \triangle_h \boldsymbol{b} \, \mathrm{d}x$$

$$\leqslant C \int (|u_3| + |b_3|) \cdot |\nabla(\boldsymbol{u},\boldsymbol{b})| \cdot |\nabla \nabla_h(\boldsymbol{u},\boldsymbol{b})| \, \mathrm{d}x$$

$$+ C \int |(-\triangle_h)^{-1} \nabla_h^2 \omega_3| \cdot |\nabla_h \boldsymbol{b}|^2 \, \mathrm{d}x$$

$$+ C \int |(-\triangle_h)^{-1} \nabla_h^2 u_3| \cdot |\nabla_h \boldsymbol{b}| \cdot |\nabla \nabla_h \boldsymbol{b}| \, \mathrm{d}x,$$

whose proof relies on the fact that u_1, u_2 (resp. b_1, b_2) can be expressed explicitly in terms of ω_3 and $\partial_3 u_3$ (resp. j_3 and $\partial_3 b_3$), see also [13], [23] (and [2], [19], [17] for its ultility in the well-posedness theory). This structure (8) enabled Yamazaki to show the following regularity criterion:

$$u_3 \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \omega_3 \in L^r(0,T;L^s(\mathbb{R}^3)),$$

(9)

$$\frac{2}{p} + \frac{3}{q} = \frac{4}{9} - \frac{1}{3q}, \quad \frac{15}{2} \leqslant q \leqslant \infty; \quad \frac{2}{r} + \frac{3}{s} = 2, \quad \frac{3}{2} < s \leqslant \infty.$$

The scaling $\frac{4}{9} - \frac{1}{3}q^{-1}$ has been improved to be $\frac{4}{9}$ in [22].

The purpose of the present paper is to improve the scaling $\frac{4}{9}$ in (7) a bit further. The key is that we can dominate the L^λ ($\lambda > 6$) norm of b_3 via u_3. Precisely, we have

Theorem 1. *Let* $(\boldsymbol{u}_0, \boldsymbol{b}_0) \in H^2(\mathbb{R}^3)$ *with* $\nabla \cdot \boldsymbol{u}_0 = \nabla \cdot \boldsymbol{b}_0 = 0$, *and* $T > 0$. *Assume that* $(\boldsymbol{u}, \boldsymbol{b})$ *is the unique strong solution of the MHD system* (1) *with initial data* $(\boldsymbol{u}_0, \boldsymbol{b}_0)$ *on* $[0, \Gamma^*)$, *where* Γ^* *is the maximal existence time. If for some* $T > \Gamma^*$ *one of the following conditions holds for the weak solution* $(\boldsymbol{u}, \boldsymbol{b})$,

(i)

$$u_3 \in L^p(0, T; L^q(\mathbb{R}^3)), \quad j_3 \in L^r(0, T; L^s(\mathbb{R}^3)),$$

(10)
$$\frac{2}{p} + \frac{3}{q} = \frac{7}{15}, \quad \frac{45}{7} \leqslant q \leqslant \infty; \quad \frac{2}{r} + \frac{3}{s} = 2, \quad \frac{3}{2} < s \leqslant \infty;$$

(ii)

$$u_3 \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \omega_3 \in L^r(0, T; L^s(\mathbb{R}^3)),$$

(11)
$$\frac{2}{p} + \frac{3}{q} = \frac{7}{15}, \quad \frac{45}{7} \leqslant q \leqslant \infty; \quad \frac{2}{r} + \frac{3}{s} = 2, \quad \frac{3}{2} < s \leqslant \infty,$$

then the strong solution can be smoothly extended beyond time Γ^*.

2. Proof of Theorem 1

In this section, we provide the proof of Theorem 1 under condition (10). The proof of Theorem 1 under condition (11) can be similarly treated via the structure (8), once the following consequence of the boundedness of two-dimensional Riesz transformation in L^α $(1 < \alpha < \infty)$ is noticed:

$$(12) \qquad \|(-\triangle_h)^{-1}\nabla_h^2 f\|_{L^\alpha}^\alpha = \int_{\mathbb{R}^3} |(-\triangle_h)^{-1}\nabla_h^2 f|^\alpha \, \mathrm{d}x$$

$$= \int_{\mathbb{R}} \left[\int_{\mathbb{R}^2} |(-\triangle_h)^{-1}\nabla_h^2 f|^\alpha \, \mathrm{d}x_1 \, \mathrm{d}x_2 \right] \mathrm{d}x_3$$

$$\leqslant C \int_{\mathbb{R}} \left[\int_{\mathbb{R}^2} |f|^\alpha \, \mathrm{d}x_1 \, \mathrm{d}x_2 \right] \mathrm{d}x_3 = C \int_{\mathbb{R}^3} |f|^\alpha \, \mathrm{d}x.$$

As is well-known, it suffices to show that $\|\nabla(\boldsymbol{u}, \boldsymbol{b})(t)\|_{L^2}$ remains bounded as $t \nearrow \Gamma^*$. To this end, for $0 < \varepsilon, \delta \ll 1$ to be determined later on, we may find a $\Gamma \in [0, \Gamma^*)$ such that

$$(13) \quad \nabla(\boldsymbol{u}, \boldsymbol{b})(\Gamma) \in L^2(\mathbb{R}^3), \quad \boldsymbol{b}(\Gamma) \in L^{10}(\mathbb{R}^3), \quad \int_\Gamma^{\Gamma^*} \|\nabla(\boldsymbol{u}, \boldsymbol{b})(\tau)\|_{L^2}^2 \, \mathrm{d}\tau \leqslant \varepsilon,$$

$$\int_\Gamma^{\Gamma^*} \|j_3(\tau)\|_{L^s}^{2s/(2s-3)} \, \mathrm{d}\tau \leqslant \delta.$$

Taking the inner product of $(1)_1$ with $-\triangle_h \boldsymbol{u}$, $(1)_2$ with $-\triangle_h \boldsymbol{b}$ in $L^2(\mathbb{R}^3)$, respectively, we deduce from (3) that

(14)
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 + \|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2$$

$$\leqslant C\int |u_3|\cdot|\nabla(\boldsymbol{u},\boldsymbol{b})|\cdot|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})|\,\mathrm{d}x$$

$$+ C\int |b_3|\cdot|\nabla(\boldsymbol{u},\boldsymbol{b})|\cdot|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})|\,\mathrm{d}x + C\int |\nabla_h(\boldsymbol{u},\boldsymbol{b})|^2\cdot|j_3|\,\mathrm{d}x$$

$$\equiv K_1 + K_2 + K_3.$$

For K_1,

(15)
$$K_1 \leqslant C\|u_3\|_{L^q}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^{2q/(q-2)}}\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}$$

$$\text{(by the Hölder inequality)}$$

$$\leqslant C\|u_3\|_{L^q}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{(q-3)/q}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^6}^{3/q}\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}$$

$$\text{(by the interpolation inequality)}$$

$$\leqslant C\|u_3\|_{L^q}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{(q-3)/q}\|\triangle(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{1/q}\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{(q+2)/q}$$

$$\text{(by (6) and classical elliptic estimates)}$$

$$\leqslant C\|u_3\|_{L^q}^{2q/(q-2)}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{2(q-3)/(q-2)}\|\triangle(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{2/(q-2)}$$

$$+ \frac{1}{6}\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2.$$

Similarly, K_2, K_3 can be bounded as

(16)
$$K_2 \leqslant C\|b_3\|_{L^{10}}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^{5/2}}\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}$$

$$\leqslant C\|b_3\|_{L^{10}}\cdot\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{7/10}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^6}^{3/10}\cdot\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}$$

$$\leqslant C\|b_3\|_{L^{10}}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{7/10}\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{1/5}\|\nabla^2(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{1/10}\cdot\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}$$

$$\leqslant C\|b_3\|_{L^{10}}^{5/2}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{7/4}\|\triangle(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{1/4} + \frac{1}{6}\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2.$$

(17)
$$K_3 \leqslant C\|\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^{2s/(s-1)}}^2\|j_3\|_{L^s}$$

$$\leqslant C\|\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{(2s-3)/s}\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{3/s}\cdot\|j_3\|_{L^s}$$

$$\leqslant C\|j_3\|_{L^s}^{2s/(2s-3)}\|\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 + \frac{1}{6}\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2.$$

Gathering (15), (16), and (17) into (14), and denoting by

(18)
$$\mathcal{K}^2(t) = \sup_{\Gamma\leqslant\tau\leqslant t}\|\nabla_h(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^2 + \int_\Gamma^t\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^2\,\mathrm{d}\tau,$$

$$\mathcal{M}^2(t) = \sup_{\Gamma\leqslant\tau\leqslant t}\|\nabla(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^2 + \int_\Gamma^t\|\triangle(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^2\,\mathrm{d}\tau,$$

we obtain

$$(19) \quad \mathcal{K}^2(t) \leqslant \|\nabla_h(\boldsymbol{u}, \boldsymbol{b})(\Gamma)\|_{L^2}^2$$

$$+ C \int_\Gamma^t \|u_3(\tau)\|_{L^q}^{2q/(q-2)} \|\nabla(\boldsymbol{u}, \boldsymbol{b})(\tau)\|_{L^2}^{2(q-3)/(q-2)} \|\triangle(\boldsymbol{u}, \boldsymbol{b})(\tau)\|_{L^2}^{2/(q-2)} \, d\tau$$

$$+ C \int_\Gamma^t \|b_3(\tau)\|_{L^{10}}^{5/2} \|\nabla(\boldsymbol{u}, \boldsymbol{b})(\tau)\|_{L^2}^{7/4} \|\triangle(\boldsymbol{u}, \boldsymbol{b})(\tau)\|_{L^2}^{1/4} \, d\tau$$

$$+ C \int_\Gamma^t \|j_3(\tau)\|_{L^s}^{2s/(2s-3)} \|\nabla_h(\boldsymbol{u}, \boldsymbol{b})(\tau)\|_{L^2}^2 \, d\tau$$

$$\leqslant C + \mathcal{K}_1(t) + \mathcal{K}_2(t) + \mathcal{K}_3(t).$$

We bound the terms one by one as

$$\mathcal{K}_1(t) \leqslant C \sup_{\Gamma \leqslant \tau \leqslant t} \|\nabla(\boldsymbol{u}, \boldsymbol{b})(\tau)\|_{L^2}^{(3q-10)/(2q-4)}$$

$$\times \int_\Gamma^t \|u_3(\tau)\|_{L^q}^{2q/(q-2)} \|\nabla(\boldsymbol{u}, \boldsymbol{b})(\tau)\|_{L^2}^{1/2} \|\triangle(\boldsymbol{u}, \boldsymbol{b})(\tau)\|_{L^2}^{2/(q-2)} \, d\tau$$

$$\leqslant C \mathcal{M}^{(3q-10)/(2q-4)}(t) \cdot \left(\int_\Gamma^t \|u_3(\tau)\|_{L^q}^{8q/(3q-10)} \, d\tau \right)^{(3q-10)/(4q-8)}$$

$$\times \left(\int_\Gamma^t \|\nabla(\boldsymbol{u}, \boldsymbol{b})(\tau)\|_{L^2}^2 \, d\tau \right)^{1/4} \cdot \left(\int_\Gamma^t \|\triangle(\boldsymbol{u}, \boldsymbol{b})(\tau)\|_{L^2}^2 \, d\tau \right)^{1/(q-2)}$$

$$\leqslant C \mathcal{M}^{(3q-10)/(2q-4)}(t)$$

$$\times \left\{ \int_\Gamma^t \left[1 + \|u_3(\tau)\|_{L^q}^{30q/(7q-45)} \right] d\tau \right\}^{(3q-10)/(4q-8)} \cdot \varepsilon^{1/4} \mathcal{M}^{2/(q-2)}(t)$$

$$\leqslant C \varepsilon^{1/4} \mathcal{M}^{3/2}(t),$$

$$\mathcal{K}_2(t) \leqslant C \sup_{\Gamma \leqslant \tau \leqslant t} \|b_3(\tau)\|_{L^{10}}^{5/2} \cdot \left(\int_\Gamma^t \|\nabla(\boldsymbol{u}, \boldsymbol{b})(\tau)\|_{L^2}^2 \, d\tau \right)^{7/8}$$

$$\times \left(\int_\Gamma^t \|\triangle(\boldsymbol{u}, \boldsymbol{b})(\tau)\|_{L^2}^2 \, d\tau \right)^{1/8}$$

$$\leqslant C \sup_{\Gamma \leqslant \tau \leqslant t} \|b_3(\tau)\|_{L^{10}}^{5/2} \cdot \varepsilon^{7/8} \mathcal{M}^{1/4}(t),$$

$$\mathcal{K}_3(t) \leqslant C \sup_{\Gamma \leqslant \tau \leqslant t} \|\nabla_h(\boldsymbol{u}, \boldsymbol{b})(\tau)\|_{L^2}^2 \cdot \int_\Gamma^t \|j_3(\tau)\|_{L^s}^{2s/(2s-3)} \, d\tau$$

$$\leqslant C \delta \mathcal{K}^2(t).$$

Collecting the above estimates into (19), and taking δ to be sufficiently small, we deduce that

$$(20) \qquad \mathcal{K}^2(t) \leqslant C + C \varepsilon^{1/4} \mathcal{M}^{3/2}(t) + C \sup_{\Gamma \leqslant \tau \leqslant t} \|b_3(\tau)\|_{L^{10}}^{5/2} \cdot \varepsilon^{7/8} \mathcal{M}^{1/4}(t).$$

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To proceed further, we dominate b_3 in terms of u_3 via the following lemma, which be can tracked back to [16].

Lemma 1. *For $q \geqslant 5$, we have*

$$
(21) \qquad \|b_3(t)\|_{L^{10}} \leqslant \left[\|b_3(\Gamma)\|_{L^{10}}^{10q/(4q-15)} + C\int_\Gamma^t \|\nabla_h \boldsymbol{b}(\tau)\|_{L^2}^{20q/(12q-45)} \right.
$$
$$
\left. \times \|\nabla \boldsymbol{b}(\tau)\|_{L^2}^{10q/(12q-45)} \|u_3(\tau)\|_{L^q}^{10q/(4q-15)} \, \mathrm{d}\tau \right]^{(4q-15)/10q}
$$
$$
\text{for all } \Gamma \leqslant t < \Gamma^*.
$$

P r o o f.   Multiplying the equation of b_3:

$$
\partial_t b_3 + (\boldsymbol{u} \cdot \nabla)b_3 - (\boldsymbol{b} \cdot \nabla)u_3 - \triangle b_3 = 0
$$

with $10b_3^9$, integrating over $\mathbb{R}^3$, we deduce by suitable integration by parts that

$$
(22) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\|b_3^5\|_{L^2}^2 + \frac{18}{5}\|\nabla(b_3^5)\|_{L^2}^2 = 10\int [(\boldsymbol{b} \cdot \nabla)u_3] \cdot b_3^9 \, \mathrm{d}x \equiv L.
$$

The term L may be dominated as

$$
(23) \quad L = -10\int_{\mathbb{R}^3} \boldsymbol{b}u_3 \cdot \nabla(b_3^9) \, \mathrm{d}x = -18\int_{\mathbb{R}^3} \boldsymbol{b}u_3(b_3^5)^{4/5} \cdot \nabla(b_3^5) \, \mathrm{d}x
$$
$$
\leqslant C\|\boldsymbol{b}\|_{L^6}\|u_3\|_{L^q}\|b_3^5\|_{L^{12q/(5q-15)}}^{4/5}\|\nabla(b_3^5)\|_{L^2}
$$
$$
\leqslant C\|\nabla_h\boldsymbol{b}\|_{L^2}^{2/3}\|\nabla\boldsymbol{b}\|_{L^2}^{1/3} \cdot \|u_3\|_{L^q} \cdot (\|b_3^5\|_{L^2}^{3(q-5)/4q}\|\nabla(b_3^5)\|_{L^2}^{(q+15)/4q})^{4/5}\|\nabla(b_3^5)\|_{L^2}
$$
$$
\leqslant C\|\nabla_h\boldsymbol{b}\|_{L^2}^{20q/(12q-45)}\|\nabla\boldsymbol{b}\|_{L^2}^{10q/(12q-45)}\|u_3\|_{L^q}^{10q/(4q-15)}\|b_3^5\|_{L^2}^{6(q-5)/(4q-15)}
$$
$$
+ \frac{9}{5}\|\nabla(b_3^5)\|_{L^2}^2.
$$

Putting (23) into (22), we find that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|b_3\|_{L^{10}}^{10} \leqslant C\|\nabla_h\boldsymbol{b}\|_{L^2}^{20q/(12q-45)}\|\nabla\boldsymbol{b}\|_{L^2}^{10q/(12q-45)}\|u_3\|_{L^q}^{10q/(4q-15)}\|b_3\|_{L^{10}}^{30(q-5)/(4q-15)}.
$$

Dividing by $\|b_3\|_{L^{10}}^{30(q-5)/(4q-15)}$, we get

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|b_3\|_{L^{10}}^{10q/(4q-15)} \leqslant C\|\nabla_h\boldsymbol{b}\|_{L^2}^{20q/(12q-45)}\|\nabla\boldsymbol{b}\|_{L^2}^{10q/(12q-45)}\|u_3\|_{L^q}^{10q/(4q-15)}.
$$

Integrating in time, we obtain (21) as desired. $\qquad\qquad\qquad\qquad\square$

Now, substituting (21) into (20), we obtain

$$
\mathcal{K}^2(t) \leqslant C + C\varepsilon^{1/4}\mathcal{M}^{3/2}(t) + C\left[C + C\int_\Gamma^t \|\nabla_h \boldsymbol{b}(\tau)\|_{L^2}^{20q/(12q-45)}\right.
$$

$$
\left.\times \|\nabla \boldsymbol{b}(\tau)\|_{L^2}^{10q/(12q-45)} \|u_3(\tau)\|_{L^q}^{10q/(4q-15)}\, \mathrm{d}\tau\right]^{(4q-15)/4q} \cdot \varepsilon^{7/8}\mathcal{M}^{1/4}(t)
$$

$$
\leqslant C + C\varepsilon^{1/4}\mathcal{M}^{3/2}(t) + C\varepsilon^{7/8}\mathcal{M}^{1/4}(t) + C\left[\sup_{\Gamma\leqslant\tau\leqslant t} \|\nabla_h \boldsymbol{b}(\tau)\|_{L^2}^{20q/(12q-45)}\right.
$$

$$
\left.\times \int_\Gamma^t \|\nabla \boldsymbol{b}(\tau)\|_{L^2}^{10q/(12q-45)} \|u_3(\tau)\|_{L^q}^{10q/(4q-15)}\, \mathrm{d}\tau\right]^{(4q-15)/4q} \cdot \varepsilon^{7/8}\mathcal{M}^{1/4}(t)
$$

$$
\leqslant C + C\varepsilon^{1/4}\mathcal{M}^{3/2}(t) + C\varepsilon^{7/8}\mathcal{M}^{1/4}(t) + C\mathcal{K}^{5/3}(t)
$$

$$
\times \left(\int_\Gamma^t \|\nabla \boldsymbol{b}(\tau)\|_{L^2}^{10q/(12q-45)} \|u_3(\tau)\|_{L^q}^{10q/(4q-15)}\, \mathrm{d}\tau\right)^{(4q-15)/4q} \cdot \varepsilon^{7/8}\mathcal{M}^{1/4}(t).
$$

Here, we push all $\|\nabla_h \boldsymbol{b}(\tau)\|_{L^2}^{20q/(12q-45)}$ out of the integral, rather than $\|\nabla_h \boldsymbol{b}(\tau)\|_{L^2}^s$ for suitable smaller s as in [15], [18], [21], [22]. This ensures the optimality of the present method. Estimation further by the Young inequality yields

$$
\mathcal{K}^2(t) \leqslant C + C\varepsilon^{1/4}\mathcal{M}^{3/2}(t) + C\varepsilon^{7/8}\mathcal{M}^{1/4}(t) + C\mathcal{K}^{5/3}(t)
$$

$$
\times \left(\int_\Gamma^t \|\nabla \boldsymbol{b}(\tau)\|_{L^2}^2 + \|u_3(\tau)\|_{L^q}^{30q/(7q-45)}\, \mathrm{d}\tau\right)^{(4q-15)/4q} \cdot \varepsilon^{7/8}\mathcal{M}^{1/4}(t)
$$

$$
\leqslant C + C\varepsilon^{1/4}\mathcal{M}^{3/2}(t) + C\varepsilon^{7/8}\mathcal{M}^{1/4}(t) + C\mathcal{K}^{5/3}(t)\cdot\varepsilon^{7/8}\mathcal{M}^{1/4}(t)
$$

$$
\leqslant C + C\varepsilon^{1/4}\mathcal{M}^{3/2}(t) + C\varepsilon^{7/8}\mathcal{M}^{1/4}(t) + \frac{5}{6}\mathcal{K}^2(t) + \frac{1}{6}C^6\varepsilon^{21/4}\mathcal{M}^{3/2}(t).
$$

Consequently,

$$
(24) \qquad \mathcal{K}^2(t) \leqslant C + C\varepsilon^{1/4}\mathcal{M}^{3/2}(t) + C\varepsilon^{7/8}\mathcal{M}^{1/4}(t) + \frac{1}{6}C^6\varepsilon^{21/4}\mathcal{M}^{3/2}(t).
$$

Now, we can argue as in [21], Step 3, in particular, equation (22) to obtain

$$
\mathcal{M}^2(t) \leqslant C + C\varepsilon\mathcal{K}^2(t)\cdot\mathcal{M}^{1/2}(t).
$$

Putting (24) into this above inequality, and choosing ε sufficiently small, we see $\mathcal{M}(t)$ is uniformly bounded for $t \in [\Gamma, \Gamma^*)$, as desired.

The proof of Theorem 1 under condition (10) is completed. $\qquad\square$

References

[1] *C. Cao, J. Wu*: Two regularity criteria for the 3D MHD equations. J. Differ. Equations *248* (2010), 2263–2274. `zbl` `MR` `doi`

[2] *J.-Y. Chemin, P. Zhang*: On the critical one component regularity for 3-D Navier-Stokes systems. Ann. Sci. Éc. Norm. Supér. *49* (2016), 131–167. `zbl` `MR` `doi`

[3] *Q. Chen, C. Miao, Z. Zhang*: On the regularity criterion of weak solutions for the 3D viscous magneto-hydrodynamics equations. Commun. Math. Phys. *284* (2008), 919–930. `zbl` `MR` `doi`

[4] *H. Duan*: On regularity criteria in terms of pressure for the 3D viscous MHD equations. Appl. Anal. *91* (2012), 947–952. `zbl` `MR` `doi`

[5] *G. Duvaut, J. L. Lions*: Inéquations en thermoélasticité et magnétohydrodynamique. Arch. Ration. Mech. Anal. *46* (1972), 241–279. (In French.) `zbl` `MR` `doi`

[6] *C. He, Z. Xin*: On the regularity of weak solutions to the magnetohydrodynamic equations. J. Differ. Equations *213* (2005), 235–254. `zbl` `MR` `doi`

[7] *E. Ji, J. Lee*: Some regularity criteria for the 3D incompressible magnetohydrodynamics. J. Math. Anal. Appl. *369* (2010), 317–322. `zbl` `MR` `doi`

[8] *X. Jia*: A new scaling invariant regularity criterion for the 3D MHD equations in terms of horizontal gradient of horizontal components. Appl. Math. Lett. *50* (2015), 1–4. `zbl` `MR` `doi`

[9] *X. Jia, Y. Zhou*: Regularity criteria for the 3D MHD equations involving partial components. Nonlinear Anal., Real World Appl. *13* (2012), 410–418. `zbl` `MR` `doi`

[10] *X. Jia, Y. Zhou*: Ladyzhenskaya-Prodi-Serrin type regularity criteria for the 3D incompressible MHD equations in terms of 3×3 mixture matrices. Nonlinearity *28* (2015), 3289–3307. `zbl` `MR` `doi`

[11] *X. Jia, Y. Zhou*: On regularity criteria for the 3D incompressible MHD equations involving one velocity component. J. Math. Fluid Mech. *18* (2016), 187–206. `zbl` `MR` `doi`

[12] *L. Ni, Z. Guo, Y. Zhou*: Some new regularity criteria for the 3D MHD equations. J. Math. Anal. Appl. *396* (2012), 108–118. `zbl` `MR` `doi`

[13] *P. Penel, M. Pokorný*: On anisotropic regularity criteria for the solutions to 3D Navier-Stokes equations. J. Math. Fluid Mech. *13* (2011), 341–353. `zbl` `MR` `doi`

[14] *M. Sermange, R. Temam*: Some mathematical questions related to the MHD equations. Commun. Pure Appl. Math. *36* (1983), 635–664. `zbl` `MR` `doi`

[15] *K. Yamazaki*: Regularity criteria of MHD system involving one velocity and one current density component. J. Math. Fluid Mech. *16* (2014), 551–570. `zbl` `MR` `doi`

[16] *K. Yamazaki*: Remarks on the regularity criteria of three-dimensional magnetohydrodynamics system in terms of two velocity field components. J. Math. Phys. *55* (2014), Article ID 031505, 16 pages. `zbl` `MR` `doi`

[17] *K. Yamazaki*: On the three-dimensional magnetohydrodynamics system in scaling-invariant spaces. Bull. Sci. Math. *140* (2016), 575–614. `zbl` `MR` `doi`

[18] *K. Yamazaki*: Regularity criteria of the three-dimensional MHD system involving one velocity and one vorticity component. Nonlinear Anal., Theory Methods Appl., Ser. A *135* (2016), 73–83. `zbl` `MR` `doi`

[19] *K. Yamazaki*: Horizontal Biot-Savart law in general dimension and an application to the 4D magneto-hydrodynamics. Differ. Integral Equ. *31* (2018), 301–328. `zbl` `MR`

[20] *Z. Zhang*: Regularity criteria for the 3D MHD equations involving one current density and the gradient of one velocity component. Nonlinear Anal., Theory Methods Appl., Ser. A *115* (2015), 41–49. `zbl` `MR` `doi`

[21] *Z. Zhang*: Remarks on the global regularity criteria for the 3D MHD equations via two components. Z. Angew. Math. Phys. *66* (2015), 977–987. `zbl` `MR` `doi`

[22] *Z. Zhang*: Refined regularity criteria for the MHD system involving only two components of the solution. Appl. Anal. *96* (2017), 2130–2139. `zbl` `MR` `doi`

[23] *Z. Zhang, Z.-a. Yao, M. Lu, L. Ni*: Some Serrin-type regularity criteria for weak solutions to the Navier-Stokes equations. J. Math. Phys. *52* (2011), Article ID 053103, 7 pages. `zbl` `MR` `doi`

[24] *Y. Zhou*: Remarks on regularities for the 3D MHD equations. Discrete Contin. Dyn. Syst. *12* (2005), 881–886. `zbl` `MR` `doi`

[25] *Y. Zhou, J. Fan*: Logarithmically improved regularity criteria for the 3D viscous MHD equations. Forum Math. *24* (2012), 691–708. `zbl` `MR` `doi`

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