REGULARITY CRITERION FOR A NONHOMOGENEOUS INCOMPRESSIBLE GINZBURG-LANDAU-NAVIER-STOKES SYSTEM

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Received November 11, 2019. Published online December 14, 2020.

Abstract. We prove a regularity criterion for a nonhomogeneous incompressible Ginzburg-Landau-Navier-Stokes system with the Coulomb gauge in $\mathbb{R}^3$. It is proved that if the velocity field in the Besov space satisfies some integral property, then the solution keeps its smoothness.

Keywords: Ginzburg-Landau; Navier-Stokes; regularity criterion

MSC 2020: 35Q30, 35Q56, 76D03, 82D55

1. Introduction

In this work, we consider the following nonhomogeneous incompressible Ginzburg-Landau-Navier-Stokes system with the Coulomb gauge [6]:

$$\operatorname{div} u = 0, \tag{1.1}$$

$$\partial_t \varrho + u\cdot\nabla\varrho = 0, \tag{1.2}$$

$$\varrho\partial_t u + \varrho u\cdot\nabla u + \nabla\pi - \Delta u = |\psi|^2\nabla h, \tag{1.3}$$

$$\eta\partial_t\psi + \mathrm{i}\eta k\varphi\psi + u\cdot\nabla\psi + \Big(\frac{\mathrm{i}}{k}\nabla + A\Big)^2\psi + (|\psi|^2-1)\psi = 0, \tag{1.4}$$

$$\partial_t A + \nabla\varphi - \Delta A + \operatorname{Re}\Big\{\Big(\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big)\overline{\psi}\Big\} = 0, \tag{1.5}$$

$$\operatorname{div} A = 0 \quad \text{in } \mathbb{R}^3\times(0,\infty), \tag{1.6}$$

$$(u,\psi,A)(\cdot,0) = (u_0,\psi_0,A_0)(\cdot) \quad \text{in } \mathbb{R}^3, \tag{1.7}$$

The authors are indebted to the referees for careful reading of the manuscript and helpful suggestions. This work is partially supported by NSFC (No. 11971234) and University Natural Science Research Project in Anhui Province (No. KJ2017A622).

DOI: 10.21136/AM.2020.0298-19

where ϱ is the density, u is the velocity, π is the pressure, ψ is the complex order parameter, A is the vector potential and φ is the electric potential, respectively, η and k are the positive Ginzburg-Landau constants, $\overline{\psi}$ is the complex conjugate of ψ, $\operatorname{Re}\psi := (\psi + \overline{\psi})/2$ is the real part of ψ, $|\psi|^2 := \psi\overline{\psi}$ is the density of superconductivity carriers and i is the imaginary unit. The function $h := h(x)$ denotes a potential function. We will assume that h is a smooth function.

When h is a constant, system (1.1), (1.2), and (1.3) reduces to the nonhomogeneous incompressible Navier-Stokes equations. Choe and Kim [3] showed that if the data ϱ_0 and u_0 satisfy

$$0 \leqslant \varrho_0 \in L^{3/2} \cap H^2, \quad u_0 \in \dot{H}^1 \cap \dot{H}^2 \quad \text{and} \quad -\Delta u_0 + \nabla \pi_0 = \sqrt{\varrho_0} g \tag{1.8}$$

for some $(\pi_0, g) \in \dot{H}^1 \times L^2$, then there exists a positive time T_* and a unique strong solution (ϱ, u) to the problem such that

$$\begin{gathered} \varrho \in C([0, T_*]; L^{3/2} \cap H^2), \quad u \in C([0, T_*]; \dot{H}^1 \cap \dot{H}^2) \cap L^2(0, T_*; \dot{H}^3), \\ \partial_t u \in L^2(0, T_*; \dot{H}^1), \quad \text{and} \quad \sqrt{\varrho}\partial_t u \in L^\infty(0, T_*; L^2). \end{gathered} \tag{1.9}$$

Kim [14] gave the following regularity criterion:

$$u \in L^{2p/(p-3)}(0, T; L^p_w) \text{ with } 3 < p \leqslant \infty. \tag{1.10}$$

Here L^p_w denotes the weak-L^p space and $L^\infty_w \equiv L^\infty$. Then Fan and Ozawa [7] refined it as

$$u \in L^{2/(1-\alpha)}(0, T; \dot{B}^{-\alpha}_{\infty,\infty}) \text{ with } 0 < \alpha < 1. \tag{1.11}$$

Very recently, Hou, Xu and Ye [13] improved (1.11) as

$$\int_0^T \frac{\|u(t)\|^{2/(1-\alpha)}_{\dot{B}^{-\alpha}_{\infty,\infty}}}{\log(\mathrm{e} + \|u(t)\|_{\dot{B}^{-\alpha}_{\infty,\infty}})} \, \mathrm{d}t < \infty \text{ with } 0 < \alpha < 1. \tag{1.12}$$

On the other hand, when $u = 0$, system (1.4), (1.5), and (1.6) reduces to the time-dependent Ginzburg-Landau, which has received many studies, e.g. [1], [2], [4], [5], [8], [9], [10], [11], [12], [16], [17]. Paper [5] showed the existence of global weak solutions. Paper [8], [4] proved the uniqueness of weak solutions.

The aim of this note is to prove (1.12) as a regularity criterion for (1.1)–(1.7) under the assumption that $|\psi_0| \leqslant 1$ on $\mathbb{R}^3$. We will prove:

Theorem 1.1. *Let* (1.8) *hold true. Let* $\psi_0, A_0 \in H^1$ *and* $h \in H^2$ *with* $\operatorname{div} u_0 = \operatorname{div} A_0 = 0$ *and* $|\psi_0| \leqslant 1$ *in* $\mathbb{R}^3$. *Let* $(\varrho, u, \pi, \psi, A, \varphi)$ *be a local strong solution to the problem* (1.1)–(1.7). *If* (1.12) *holds true with some* $0 < T < \infty$, *then the solution* $(\varrho, u, \pi, \psi, A, \varphi)$ *can be extended beyond* $T > 0$.

R e m a r k 1.1. We can prove similar results under the Lorentz gauge.

R e m a r k 1.2. Note that the system (1.1)–(1.5) holds its form under the scaling $(\varrho, u, \pi, \psi, A, \varphi, h) \to (\varrho_\lambda, u_\lambda, \pi_\lambda, \psi_\lambda, A_\lambda, \varphi_\lambda, h_\lambda) := (\varrho, \lambda u, \lambda^2\pi, \lambda\psi, \lambda A, \lambda^2\varphi, h)(\lambda^2 t, \lambda x)$ for any $\lambda > 0$ when neglecting the linear lower order term ψ in (1.4). Thus (1.10), (1.11), and (1.12) are optimal in this sense.

R e m a r k 1.3. The regularity criterion includes only one unknown velocity, which plays an important role. In fact, if $u = 0$, then the GL system has a unique global strong (smooth) solution.

Applying div to (1.5) and using (1.6), we have

$$-\Delta\varphi = \operatorname{div}\operatorname{Re}\Big\{\Big(\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big)\overline{\psi}\Big\}. \tag{1.13}$$

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Since it is easy to use the classical Banach fixed-point theorem to show the local well-posedness of strong solutions, we only need to establish a priori estimates.

First, it follows from (1.1) and (1.2) that

$$\|\varrho(t)\|_{L^p} = \|\varrho_0\|_{L^p} \text{ with } \frac{3}{2} \leqslant p \leqslant \infty. \tag{2.1}$$

Similarly to the method in [2], [5], it is standard to prove that

$$|\psi| \leqslant 1. \tag{2.2}$$

Testing (1.4) by $\overline{\psi}$, taking the real parts and using (1.1), we get

$$\frac{\eta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\psi|^2\,\mathrm{d}x + \int\Big|\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big|^2\,\mathrm{d}x + \int|\psi|^4\,\mathrm{d}x = \int|\psi|^2\,\mathrm{d}x,$$

which leads to

$$\int|\psi|^2\,\mathrm{d}x + \int_0^T\int\Big|\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C. \tag{2.3}$$

Testing (1.5) by A, using (1.6), (2.2), and (2.3), we find that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|A|^2\,\mathrm{d}x + \int|\nabla A|^2\,\mathrm{d}x &= -\operatorname{Re}\int\Big(\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big)\overline{\psi}A\,\mathrm{d}x\\ &\leqslant \Big\|\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big\|_{L^2}\|\psi\|_{L^\infty}\|A\|_{L^2} \leqslant \Big\|\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big\|_{L^2}\|A\|_{L^2},\end{aligned}$$

which implies

$$\|A\|_{L^\infty(0,T;L^2)} + \|A\|_{L^2(0,T;H^1)} \leqslant C. \tag{2.4}$$

It follows from (2.2), (2.3), and (2.4) that

$$\int_0^T \int |\psi A|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \|\psi\|_{L^\infty(0,T;L^\infty)} \int_0^T \int |A|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant C,$$

whence

$$\|\psi\|_{L^2(0,T;H^1)} \leqslant C. \tag{2.5}$$

Testing (1.13) by φ, and using (2.2) and (2.3), we compute

$$\begin{aligned} \|\varphi\|_{L^2(0,T;L^2)} &\leqslant C\|\nabla\varphi\|_{L^2(0,T;L^{6/5})} \\ &\leqslant C\Big\|\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big\|_{L^2(0,T;L^2)} \|\psi\|_{L^\infty(0,T;L^3)} \leqslant C \end{aligned} \tag{2.6}$$

and

$$\|\nabla\varphi\|_{L^2(0,T;L^2)} \leqslant C\Big\|\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big\|_{L^2(0,T;L^2)} \|\psi\|_{L^\infty(0,T;L^\infty)} \leqslant C. \tag{2.7}$$

Testing (1.3) by u and using (1.1), (1.2), (2.1), and (2.2), we find that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \varrho |u|^2 \,\mathrm{d}x + \int |\nabla u|^2 \,\mathrm{d}x = \int |\psi|^2 \nabla h \cdot u \,\mathrm{d}x &\leqslant \|\psi\|_{L^6}^2 \|\nabla h\|_{L^2} \|u\|_{L^6} \\ &\leqslant C\|\nabla u\|_{L^2} \leqslant \frac{1}{2}\int |\nabla u|^2 \,\mathrm{d}x + C, \end{aligned}$$

which gives

$$\|\sqrt{\varrho} u\|_{L^\infty(0,T;L^2)} + \|\nabla u\|_{L^2(0,T;L^2)} \leqslant C. \tag{2.8}$$

Testing (1.5) by $-\Delta A$, using (1.6), (2.3), and (2.2), we derive

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla A|^2 \,\mathrm{d}x + \int |\Delta A|^2 \,\mathrm{d}x &= \mathrm{Re}\int \Big(\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big)\overline{\psi} \cdot \Delta A \,\mathrm{d}x \\ &\leqslant \Big\|\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big\|_{L^2} \|\psi\|_{L^\infty} \|\Delta A\|_{L^2} \leqslant \frac{1}{2}\|\Delta A\|_{L^2}^2 + C\Big\|\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big\|_{L^2}^2, \end{aligned}$$

which yields

$$\|A\|_{L^\infty(0,T;H^1)} + \|A\|_{L^2(0,T;H^2)} \leqslant C. \tag{2.9}$$

Equation (1.4) can be written as

$$\eta\partial_t\psi - \frac{1}{k^2}\Delta\psi = -\mathrm{i}\eta k\varphi\psi - u\cdot\nabla\psi - \frac{2\mathrm{i}}{k}A\cdot\nabla\psi - |A|^2\psi - (|\psi|^2 - 1)\psi. \tag{2.10}$$

Testing (2.10) by $-\Delta\overline{\psi}$, taking the real parts, using (1.1), (2.6), (2.9), and (2.2), we compute

$$
\begin{aligned}
&\frac{\eta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla\psi|^2\,\mathrm{d}x+\frac{1}{k^2}\int|\Delta\psi|^2\,\mathrm{d}x\\
&\quad=\mathrm{Re}\int\mathrm{i}\eta k\varphi\psi\Delta\overline{\psi}\,\mathrm{d}x-\sum_j\mathrm{Re}\int\nabla u_j\partial_j\psi\nabla\overline{\psi}\,\mathrm{d}x+\mathrm{Re}\int\frac{2\mathrm{i}}{k}A\cdot\nabla\psi\cdot\Delta\overline{\psi}\,\mathrm{d}x\\
&\qquad+\mathrm{Re}\int|A|^2\psi\Delta\overline{\psi}\,\mathrm{d}x+\mathrm{Re}\int(|\psi|^2-1)\psi\Delta\overline{\psi}\,\mathrm{d}x\\
&\quad\leqslant C\|\varphi\|_{L^2}\|\Delta\psi\|_{L^2}+C\|\nabla u\|_{L^2}\|\nabla\psi\|_{L^4}^2+C\|A\|_{L^4}\|\nabla\psi\|_{L^4}\|\Delta\psi\|_{L^2}\\
&\qquad+C\|A\|_{L^4}^2\|\Delta\psi\|_{L^2}+C\|\psi\|_{L^2}\|\Delta\psi\|_{L^2}\\
&\quad\leqslant C\|\varphi\|_{L^2}\|\Delta\psi\|_{L^2}+C\|\nabla u\|_{L^2}\cdot\|\psi\|_{L^\infty}\|\Delta\psi\|_{L^2}\\
&\qquad+C\|\psi\|_{L^\infty}^{1/2}\|\Delta\psi\|_{L^2}^{3/2}+C\|\Delta\psi\|_{L^2}\\
&\quad\leqslant\frac{1}{2k^2}\|\Delta\psi\|_{L^2}^2+C\|\varphi\|_{L^2}^2+C\|\nabla u\|_{L^2}^2+C,
\end{aligned}
$$

which implies

$$\|\psi\|_{L^\infty(0,T;H^1)}+\|\psi\|_{L^2(0,T;H^2)}\leqslant C. \tag{2.11}$$

Here we have used the Gagliardo-Nirenberg inequality

$$\|\nabla\psi\|_{L^4}^2\leqslant C\|\psi\|_{L^\infty}\|\Delta\psi\|_{L^2}. \tag{2.12}$$

Testing (1.3) by $\partial_t u$, using (1.1), (1.2), (2.1), and (2.2), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla u|^2\,\mathrm{d}x+\int\varrho|\partial_t u|^2\,\mathrm{d}x=-\int\varrho u\cdot\nabla u\cdot\partial_t u\,\mathrm{d}x+\int|\psi|^2\nabla h\cdot\partial_t u\,\mathrm{d}x \qquad (2.13)\\
&\quad\leqslant\|\sqrt{\varrho}\|_{L^\infty}\|\sqrt{\varrho}\partial_t u\|_{L^2}\|u\cdot\nabla u\|_{L^2}+\frac{\mathrm{d}}{\mathrm{d}t}\int|\psi|^2\nabla h\cdot u\,\mathrm{d}x-\int\nabla h\cdot u\partial_t|\psi|^2\,\mathrm{d}x\\
&\quad\leqslant C\|\sqrt{\varrho}\partial_t u\|_{L^2}\|u\|_{\dot{B}^{-\alpha}_{\infty,\infty}}\|u\|_{\dot{H}^{1+\alpha}}+\frac{\mathrm{d}}{\mathrm{d}t}\int|\psi|^2\nabla h\cdot u\,\mathrm{d}x\\
&\qquad+C\|u\|_{L^6}\|\nabla h\|_{L^3}(\|\Delta\psi\|_{L^2}+\|\varphi\|_{L^2}+\|A\|_{L^3}\|\nabla\psi\|_{L^6}+\|A\|_{L^4}^2+1)\\
&\qquad+C\|u\|_{L^6}^2\|\nabla\psi\|_{L^2}\|\nabla h\|_{L^6}\\
&\quad\leqslant C\|\sqrt{\varrho}\partial_t u\|_{L^2}\|u\|_{\dot{B}^{-\alpha}_{\infty,\infty}}\|\nabla u\|_{L^2}^{1-\alpha}\|\Delta u\|_{L^2}^{\alpha}\\
&\qquad+\frac{\mathrm{d}}{\mathrm{d}t}\int|\psi|^2\nabla h\cdot u\,\mathrm{d}x+C\|\nabla u\|_{L^2}^2+C\|\Delta\psi\|_{L^2}^2+C\|\varphi\|_{L^2}^2+C\\
&\quad\leqslant\delta\|\sqrt{\varrho}\partial_t u\|_{L^2}^2+\delta\|\Delta u\|_{L^2}^2+C\|u\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{2/(1-\alpha)}\|\nabla u\|_{L^2}^2\\
&\qquad+\frac{\mathrm{d}}{\mathrm{d}t}\int|\psi|^2\nabla h\cdot u\,\mathrm{d}x+C\|\nabla u\|_{L^2}^2+C\|\Delta\psi\|_{L^2}^2+C\|\varphi\|_{L^2}^2+C
\end{aligned}
$$

for any $0<\delta<1$.

Here we have used the inequality [15]:

$$\|u\cdot\nabla u\|_{L^2}\leqslant C\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}\|u\|_{\dot H^{1+\alpha}} \text{ with } 0<\alpha<1. \tag{2.14}$$

On the other hand, thanks to the H^2-theory of the Stokes system, it follows from (1.3), (2.1), (2.2), and (2.14) that

$$\begin{aligned}\|\Delta u\|_{L^2}&\leqslant C\|\nabla\pi-\Delta u\|_{L^2}\leqslant C\|\varrho\partial_t u+\varrho u\cdot\nabla u-|\psi|^2\nabla h\|_{L^2}\\&\leqslant C\|\sqrt{\varrho}\partial_t u\|_{L^2}+C\|u\cdot\nabla u\|_{L^2}+C\\&\leqslant C\|\sqrt{\varrho}\partial_t u\|_{L^2}+C\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}\|\nabla u\|^{1-\alpha}_{L^2}\|\Delta u\|^{\alpha}_{L^2}+C,\end{aligned}$$

which leads to

$$\|\Delta u\|_{L^2}\leqslant C\|\sqrt{\varrho}\partial_t u\|_{L^2}+C\|u\|^{1/(1-\alpha)}_{\dot B^{-\alpha}_{\infty,\infty}}\|\nabla u\|_{L^2}+C. \tag{2.15}$$

Inserting (2.15) into (2.13), taking δ small enough and summing with (2.15), we have

$$\begin{aligned}&\frac{\mathrm d}{\mathrm dt}\int|\nabla u|^2\,\mathrm dx+\int\varrho|\partial_t u|^2\,\mathrm dx+\int|\Delta u|^2\,\mathrm dx\\&\leqslant C\|u\|^{2/(1-\alpha)}_{\dot B^{-\alpha}_{\infty,\infty}}\|\nabla u\|^2_{L^2}+C\|\nabla u\|^2_{L^2}+C\|\Delta\psi\|^2_{L^2}\\&\quad+C\|\varphi\|^2_{L^2}+C+\frac{\mathrm d}{\mathrm dt}\int|\psi|^2\nabla h\cdot u\,\mathrm dx\\&=\frac{C\|u\|^{2/(1-\alpha)}_{\dot B^{-\alpha}_{\infty,\infty}}}{\log(\mathrm e+\|u\|_{\dot B^{-\alpha}_{\infty,\infty}})}\|\nabla u\|^2_{L^2}\log(\mathrm e+\|u\|_{\dot B^{-\alpha}_{\infty,\infty}})\\&\quad+C\|\nabla u\|^2_{L^2}+C\|\Delta\psi\|^2_{L^2}+C\|\varphi\|^2_{L^2}+C+\frac{\mathrm d}{\mathrm dt}\int|\psi|^2\nabla h\cdot u\,\mathrm dx\\&\leqslant\frac{C\|u\|^{2/(1-\alpha)}_{\dot B^{-\alpha}_{\infty,\infty}}}{\log(\mathrm e+\|u\|_{\dot B^{-\alpha}_{\infty,\infty}})}\|\nabla u\|^2_{L^2}\log(\mathrm e+y)\\&\quad+C\|\nabla u\|^2_{L^2}+C\|\Delta\psi\|^2_{L^2}+C\|\varphi\|^2_{L^2}+C+\frac{\mathrm d}{\mathrm dt}\int|\psi|^2\nabla h\cdot u\,\mathrm dx,\end{aligned}$$

which implies

$$\|\nabla u\|^2_{L^\infty(t_0,t;L^2)}+\int_{t_0}^t\int(\varrho|\partial_t u|^2+|\Delta u|^2)\,\mathrm dx\,\mathrm ds\leqslant C(\mathrm e+y)^{C_0\varepsilon} \tag{2.16}$$

with

$$y(t):=\sup_{[t_0,t]}\|D^{3/2-\alpha}u(s)\|_{L^2}$$

for any $0<t_0\leqslant t\leqslant T$ where C_0 is an absolute constant, provided that

$$\int_{t_0}^T\frac{\|u\|^{2/(1-\alpha)}_{\dot B^{-\alpha}_{\infty,\infty}}}{\log(\mathrm e+\|u\|_{\dot B^{-\alpha}_{\infty,\infty}})}\,\mathrm dt\leqslant\varepsilon<<1. \tag{2.17}$$

Applying ∂_t to (1.3), testing by $\partial_t u$, and using (1.1) and (1.2), we get

$$
\begin{aligned}
(2.18)\qquad & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \varrho|\partial_t u|^2\,\mathrm{d}x + \int |\nabla\partial_t u|^2\,\mathrm{d}x \\
&= -\int \partial_t\varrho|\partial_t u|^2\,\mathrm{d}x - \int \partial_t\varrho u\cdot\nabla u\cdot\partial_t u\,\mathrm{d}x \\
&\quad - \int \varrho\partial_t u\cdot\nabla u\cdot\partial_t u\,\mathrm{d}x - \int \partial_t|\psi|^2\nabla h\cdot\partial_t u\,\mathrm{d}x \\
&= -\int \varrho u\cdot\nabla|\partial_t u|^2\,\mathrm{d}x - \int \varrho u\cdot\nabla(u\cdot\nabla u\cdot\partial_t u)\,\mathrm{d}x \\
&\quad - \int \varrho\partial_t u\cdot\nabla u\cdot\partial_t u\,\mathrm{d}x - \int \partial_t|\psi|^2\nabla h\cdot\partial_t u\,\mathrm{d}x =: \sum_{j=1}^{4} I_j.
\end{aligned}
$$

We use the Hölder inequality, (2.1), and (2.2) to bound I_j $(j = 1,\ldots,4)$ as follows.

$$
\begin{aligned}
|I_1| &\leqslant C\|u\|_{L^6}\|\sqrt{\varrho}\partial_t u\|_{L^3}\|\nabla\partial_t u\|_{L^2} \\
&\leqslant C\|\nabla u\|_{L^2}\|\sqrt{\varrho}\partial_t u\|_{L^2}^{1/2}\|\sqrt{\varrho}\partial_t u\|_{L^6}^{1/2}\|\nabla\partial_t u\|_{L^2} \\
&\leqslant C\|\nabla u\|_{L^2}\|\sqrt{\varrho}\partial_t u\|_{L^2}^{1/2}\|\nabla\partial_t u\|_{L^2}^{3/2} \\
&\leqslant \delta\|\nabla\partial_t u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\sqrt{\varrho}\partial_t u\|_{L^2}^2
\end{aligned}
$$

for any $0 < \delta < 1$;

$$
\begin{aligned}
|I_2| &\leqslant C\|u\|_{L^6}\|\nabla u\|_{L^6}\|\nabla u\|_{L^2}\|\partial_t u\|_{L^6} + C\|u\|_{L^6}^2\|\Delta u\|_{L^2}\|\partial_t u\|_{L^6} \\
&\quad + C\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla\partial_t u\|_{L^2} \\
&\leqslant C\|\nabla u\|_{L^2}^2\|\Delta u\|_{L^2}\|\nabla\partial_t u\|_{L^2} \\
&\leqslant \delta\|\nabla\partial_t u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\Delta u\|_{L^2}^2
\end{aligned}
$$

for any $0 < \delta < 1$;

$$
\begin{aligned}
|I_3| &\leqslant \|\nabla u\|_{L^2}\|\sqrt{\varrho}\partial_t u\|_{L^4}^2 \leqslant \|\nabla u\|_{L^2}\|\sqrt{\varrho}\partial_t u\|_{L^2}^{1/2}\|\sqrt{\varrho}\partial_t u\|_{L^6}^{3/2} \\
&\leqslant C\|\nabla u\|_{L^2}\|\sqrt{\varrho}\partial_t u\|_{L^2}^{1/2}\|\nabla\partial_t u\|_{L^2}^{3/2} \\
&\leqslant \delta\|\nabla\partial_t u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\sqrt{\varrho}\partial_t u\|_{L^2}^2
\end{aligned}
$$

for any $0 < \delta < 1$;

$$
|I_4| \leqslant C\|\partial_t\psi\|_{L^2}\|\nabla h\|_{L^3}\|\partial_t u\|_{L^6} \leqslant C\|\partial_t\psi\|_{L^2}^2 + \delta\|\nabla\partial_t u\|_{L^2}^2
$$

for any $0 < \delta < 1$.

Inserting the above estimates into (2.18) and taking δ small enough, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \varrho|\partial_t u|^2\,\mathrm{d}x + \frac{1}{2}\int |\nabla\partial_t u|^2\,\mathrm{d}x \leqslant C\|\nabla u\|_{L^2}^4\|\sqrt{\varrho}\partial_t u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\Delta u\|_{L^2}^2 + C\|\partial_t\psi\|_{L^2}^2. \tag{2.19}$$

Testing (2.10) by $\partial_t\overline{\psi}$, taking the real parts, using (2.2), (2.9), and (2.16), we have

$$\begin{aligned}\frac{1}{2k^2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla\psi|^2\,\mathrm{d}x + \int \eta|\partial_t\psi|^2\,\mathrm{d}x &\leqslant C(\|\varphi\|_{L^2} + \|u\|_{L^6}\|\nabla\psi\|_{L^3} + \|A\|_{L^4}\|\nabla\psi\|_{L^4} + \|A\|_{L^4}^2 + \|\psi\|_{L^2})\|\partial_t\psi\|_{L^2}\\ &\leqslant \frac{1}{2}\eta\|\partial_t\psi\|_{L^2}^2 + C\|\varphi\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla\psi\|_{L^3}^2 + C\|\nabla\psi\|_{L^4}^2 + C,\end{aligned}$$

which gives

$$\int_{t_0}^t\int |\partial_t\psi|^2\,\mathrm{d}x\,\mathrm{d}s \leqslant C(\mathrm{e}+y)^{C_0\varepsilon}. \tag{2.20}$$

Integrating (2.19) over (t_0,t) and using (2.20), we arrive at

$$\int \varrho|\partial_t u|^2\,\mathrm{d}x + \int_{t_0}^t\int |\nabla\partial_t u|^2\,\mathrm{d}x\,\mathrm{d}s \leqslant C(\mathrm{e}+y)^{C_0\varepsilon}. \tag{2.21}$$

Similarly to (2.15), we find that

$$\begin{aligned}\|\Delta u\|_{L^2} &\leqslant C\|\sqrt{\varrho}\partial_t u\|_{L^2} + C\|u\cdot\nabla u\|_{L^2} + C \leqslant C\|\sqrt{\varrho}\partial_t u\|_{L^2} + C\|u\|_{L^6}\|\nabla u\|_{L^3} + C\\ &\leqslant C\|\sqrt{\varrho}\partial_t u\|_{L^2} + C\|\nabla u\|_{L^2}\cdot\|\nabla u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{1/2} + C,\end{aligned}$$

which yields

$$\|\Delta u\|_{L^2} \leqslant C\|\sqrt{\varrho}\partial_t u\|_{L^2} + C\|\nabla u\|_{L^2}^3 + C. \tag{2.22}$$

Here we have used the Gagliardo-Nirenberg inequality

$$\|\nabla u\|_{L^3}^2 \leqslant C\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}. \tag{2.23}$$

Similarly to (2.15) again, we have

$$\begin{aligned}\|\Delta u\|_{L^{3/(2+\alpha)}} &\leqslant C\|\nabla\pi - \Delta u\|_{L^{3/(2+\alpha)}} \leqslant C\|\varrho\partial_t u + \varrho u\cdot\nabla u - |\psi|^2\nabla h\|_{L^{3/(2+\alpha)}}\\ &\leqslant C(\|\sqrt{\varrho}\|_{L^{6/(1+2\alpha)}}\|\sqrt{\varrho}\partial_t u\|_{L^2}\\ &\quad + \|\varrho\|_{L^{3/\alpha}}\|u\|_{L^6}\|\nabla u\|_{L^2} + \|\psi\|_{L^{12/(3+2\alpha)}}^2\|\nabla h\|_{L^6})\\ &\leqslant C(\|\sqrt{\varrho}\partial_t u\|_{L^2} + \|\nabla u\|_{L^2}^2 + 1).\end{aligned} \tag{2.24}$$

Noting the imbedding inequality

$$\|D^{3/2-\alpha}u\|_{L^2} \leqslant C\|\Delta u\|_{L^{3/(2+\alpha)}}, \tag{2.25}$$

we have

$$y \leqslant C(\mathrm{e}+y)^{C_0\varepsilon}, \tag{2.26}$$

which gives

$$\|\Delta u\|_{L^2} + \|\Delta u\|_{L^{3/(2+\alpha)}} \leqslant C. \tag{2.27}$$

Thus

$$\|\partial_t u\|_{L^2(0,T;H^1)} \leqslant C. \tag{2.28}$$

Equation (1.3) can be rewritten as

$$-\Delta u + \nabla\pi = f := |\psi|^2\nabla h - \varrho\partial_t u - \varrho u\cdot\nabla u \in L^2(0,T;L^2\cap L^6). \tag{2.29}$$

We have

$$\|\nabla u\|_{W^{1,q}} \leqslant C\|f\|_{L^q} + C\|\nabla u\|_{L^2} \text{ with } 3 < q \leqslant 6$$

and therefore,

$$\|\nabla u\|_{L^2(0,T;W^{1,q})} \leqslant C. \tag{2.30}$$

Now it is standard to deduce that

$$\|\varrho\|_{C([0,T];L^{3/2}\cap H^2)} \leqslant C. \tag{2.31}$$

This completes the proof. □

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