THE GENERALIZED FINITE VOLUME SUSHI SCHEME FOR THE DISCRETIZATION OF THE PEACEMAN MODEL

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Abstract. We demonstrate some a priori estimates of a scheme using stabilization and hybrid interfaces applying to partial differential equations describing miscible displacement in porous media. This system is made of two coupled equations: an anisotropic diffusion equation on the pressure and a convection-diffusion-dispersion equation on the concentration of invading fluid. The anisotropic diffusion operators in both equations require special care while discretizing by a finite volume method SUSHI. Later, we present some numerical experiments.

Keywords: porous medium; nonconforming grid; finite volume scheme; a priori estimate; miscible fluid flow

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1. Introduction

In the literature, there exist several modelling approaches of the single-phase miscible displacement of one fluid by another in a porous medium. In [8], [11], [21] the authors introduced the Peaceman model, where the fluids are considered incompressible. This model is constituted of an elliptic parabolic coupled system. While if the fluids are compressible, the system becomes parabolic, see [9], [12]. We are interested in the study of the first model. It is constituted of an anisotropic diffusion equation on the pressure and a convection-diffusion-dispersion on the concentration of the invading fluid; see [18] for the theoretical analysis of this system of partial differential equations, see also [17], [3], [2].

Let us mention that the Peaceman model has been the object of several studies. The authors in [19], [10], [11] studied the finite element schemes for both equations. We refer to [22] and [23] for the Eulerian-Lagrangian localized adjoint method combined with the mixed finite element methods. See [1] for the convergence analysis for

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a discontinuous Galerkin finite element. The pressure equation was discretized by the finite element method and the concentration equation by the method of characteristics in [13], [14] and [20].

There are other works that treat both equations by a single method, for example Chainais-Hillairet, Krell and Mouton in [7] and [6] study the numerical and convergence analysis of a DDFV scheme for a system describing miscible fluid flows in porous media and in [5] Chainais-Hillairet and Droniou proposed the mixed finite volume scheme for both equations.

In this paper, we propose another method for both equations—the SUSHI (Scheme Using Stabilisation and Hybrid Interfaces) method.

1.1. The continuous problem and objectives. Let us consider that the unknowns of the problem are the pressure in the mixture p, its Darcy velocity U and the concentration of the invading fluid c. The porous medium is characterized by its porosity $\phi(x)$ and its permeability $A(x)$. We denote by $\mu(c)$ the viscosity of the fluid mixture, $\widehat{c}$ the injected concentration, q^+ and q^- are the injection and the production source terms, respectively. The model is defined on a time interval $(0,T)$ and a domain $\Omega\subset\mathbb{R}^2$ by:

$$\begin{cases} \operatorname{div}(U) = q^+ - q^- & \text{in } (0,T)\times\Omega, \\ U = -K(x,c)\nabla p & \text{in } (0,T)\times\Omega, \\ \displaystyle\int_\Omega p(\cdot,x)\,\mathrm{d}x = 0 & \text{on } (0,T), \end{cases} \tag{1.1}$$

$$\phi(x)\partial_t c - \operatorname{div}(D(x,U)\nabla c) + \operatorname{div}(Uc) + q^- c = \widehat{c}q^+ \quad \text{in } (0,T)\times\Omega, \tag{1.2}$$

where $K(x,c) = A(x)/\mu(c)$ and D are the diffusion-dispersion tensor including molecular diffusion and mechanical dispersion, respectively. The initial condition is

$$c(x,0) = c_0(x), \tag{1.3}$$

where

$$c_0 \in L^\infty(\Omega), \text{ and satisfies } 0 \leqslant c_0 \leqslant 1 \text{ almost everywhere (a.e.) in } \Omega, \tag{1.4}$$

and the boundary conditions are

$$\begin{cases} [K(x,c)\nabla p]\cdot n = 0 & \text{on } (0,T)\times\partial\Omega, \\ [D(x,U)\nabla c]\cdot n = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{1.5}$$

where n is the unit vector, Ω is an open, bounded connected subset of $\mathbb{R}^2$ which supported tube polygonal ($d=2$) and $\partial\Omega$ stands for its boundary.

The porous medium is characterized by the porosity $\phi(x)$ with

$$\phi \in L^\infty(\Omega) \text{ and there exists } \phi^* > 0 \text{ such that } \phi_* \leqslant \phi \leqslant \phi_*^{-1} \text{ a.e. in } \Omega, \tag{1.6}$$

and

$$\begin{cases} K\colon\ \Omega \times \mathbb{R} \to M_2(\mathbb{R}) \ \text{ is a Caratheodory matrix-valued function} \\ \text{satisfying: } \exists\, \alpha_K > 0 \text{ such that } K(x,s)\xi \cdot \xi \geqslant \alpha_K |\xi|^2 \\ \text{for a.e. } x \in \Omega,\ \text{ all } s \in \mathbb{R},\ \text{ and all } \xi \in \mathbb{R}^2, \\ \exists\, \Lambda_K > 0 \text{ such that } |K(x,s)| \leqslant \Lambda_K \text{ for a.e. } x \in \Omega \text{ and all } s \in \mathbb{R}. \end{cases} \tag{1.7}$$

Further, D is the diffusion-dispersion tensor including molecular diffusion and mechanical dispersion, satisfying the following

$$\begin{cases} D\colon\ \Omega \times \mathbb{R}^2 \to M_2(\mathbb{R}) \ \text{ is a Caratheodory matrix-valued function} \\ \text{such that: } \exists\, \alpha_D > 0 \text{ s.t. } D(x,V)\xi \cdot \xi \geqslant \alpha_D(1+|V|)|\xi|^2 \\ \text{for a.e. } x \in \Omega,\ \text{ all } (V,\xi) \in \mathbb{R}^2 \times \mathbb{R}^2,\ \exists\, \Lambda_D > 0 \\ \text{such that } |D(x,V)| \leqslant \Lambda_D(1+|V|) \text{ for a.e. } x \in \Omega \text{ and all } V \in \mathbb{R}^2, \end{cases} \tag{1.8}$$

where D is given by

$$D(x,U) = \phi(x)(d_m I + |U|(d_l E(U) + d_t(I - E(U)))). \tag{1.9}$$

Here I is the identity matrix, d_m is the molecular diffusion, d_l and d_t are the longitudinal and transverse dispersion coefficients, respectively, and

$$E(U) = \Big(\frac{U_i U_j}{|U|^2}\Big)_{1 \leqslant i,j \leqslant d}.$$

We denote by $\mu(c)$ the viscosity of the fluid mixture as

$$\mu(c) = \mu(0)(1 + (M^{1/4} - 1)c)^{-4} \text{ on } [0,1], \tag{1.10}$$

where $M = \mu(0)/\mu(1)$ is the mobility ratio (we extend μ to $\mathbb{R}$ by letting $\mu = \mu(0)$ on $(-\infty, 0)$ and $\mu = \mu(1)$ on $(1, \infty)$), $\widehat{c}$ is the injected concentration such that

$$\widehat{c} \in L^\infty((0,T) \times \Omega) \text{ satisfies } 0 \leqslant \widehat{c} \leqslant 1 \text{ a.e. in } (0,T) \times \Omega, \tag{1.11}$$

q^+ and q^- are the injection and the production source terms, respectively,

$$\begin{cases} (q^+, q^-) \in L^\infty(0,T; L^2(\Omega)) \text{ are non negative functions such that} \\ \int_\Omega q^+(\cdot,x)\,\mathrm{d}x = \int_\Omega q^-(\cdot,x)\,\mathrm{d}x \text{ a.e. on } (0,T). \end{cases} \tag{1.12}$$

Definition 1.1 (Weak solution). Under the hypotheses (1.3)–(1.12), a weak solution of (1.1) and (1.2) is a triple of functions $(\overline{p}, \overline{U}, \overline{c})$ satisfying: $\overline{p} \in L^\infty(0,T;H^1(\Omega))$, $\overline{U} \in L^\infty(0,T;L^2(\Omega))^2$ and $\overline{c} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$

$$
\begin{cases}
\int_\Omega \overline{p}(t,\cdot) = 0 & \text{for a.e. } t \in (0,T), \\
\overline{U} = -K(x,\overline{c})\nabla\overline{p} & \text{a.e. on } (0,T)\times\Omega, \\
a_1(\overline{p},\varphi_1) = \int_0^T \int_\Omega q^+\varphi_1 + \int_0^T \int_\Omega q^-\varphi_1 & \text{for all } \varphi_1 \in C^\infty([0,T]\times\overline{\Omega}), \\
a_2(\overline{c},\varphi_2) = \int_0^T \int_\Omega \widehat{c}q^+\varphi_2 & \text{for all } \varphi_2 \in C_c^\infty([0,T]\times\overline{\Omega}),
\end{cases}
\tag{1.13}
$$

with

$$
\begin{cases}
a_1(\overline{p},\varphi_1) = -\int_0^T \int_\Omega \overline{U}\cdot\nabla\varphi_1, \\
a_2(\overline{c},\varphi_2) = -\int_0^T \int_\Omega \phi(x)\overline{c}\partial_t\varphi_2 + \int_0^T \int_\Omega D(x,\overline{U})\nabla\overline{c}\cdot\nabla\varphi_2 \\
\qquad - \int_0^T \int_\Omega \overline{c}\overline{U}\cdot\nabla\varphi_2 + \int_0^T \int_\Omega q^+\overline{c}\varphi_2 - \int_\Omega \phi c_0(x)\varphi_2(0,\cdot).
\end{cases}
\tag{1.14}
$$

One of the disadvantages of the finite volume method is that it assumes the condition of orthogonality on the mesh in the sense of Eymard et al. [15]; this excludes other types of meshes that do not satisfy this condition. For example, in porous media, most geological layers are quite deformed, and therefore the mesh used to study these problems in general does not meet the orthogonality requirement. In this work, we want to apply one of the finite volume methods dedicated to anisotropic diffusion. We will examine the application of a finite volume scheme using stabilization and hybrid interfaces, which has been proposed by Eymard et al. [16], to the diffusion term in the pressure equation and in the concentration convection-diffusion-dispersion equation of the system describing miscible fluid flows in porous media (Peaceman model). This method is characterized by:

- ▹ using a single mesh that is very general, unstructured and does not take into account the condition of orthogonality (classical finite volume see [15]);
- ▹ avoiding to project the gradient on the edges of dual and primal mesh (method DDFV) by adding a term of stability which stabilizes the gradient obtained by the method of classical finite volume; then the number of variables of SUSHI method is less compared to the method (DDFV).

We present and study a numerical scheme for SUSHI method applied to this model, more precisely, we prove some a priori estimates on the pressure, the gradient of the pressure, the Darcy velocity and also a priori estimates on the concentration and the gradient of the concentration. Later, some numerical tests are also carried out to verify the validity of the proposed numerical scheme.

This article is organized as follows. In Section 2 we present meshes and the associated notations, then we introduce the different discrete operators (gradient and convection operators) and some proprieties. The main result of the paper is detailed in Section 3 as follows: Sections 3.1, 3.2, and 3.3 are devoted to the discretization of system (1.1)–(1.12), a priori estimates are proved in Section 3.4, and finally we present some numerical experiments in Section 3.5.

2. The finite volume schemes SUSHI

The SUSHI scheme is based on Hybrid Finite Volume (HFV) and cell-centric (SUCCES) schemes. They are based on two fundamental ideas: one, where unknowns on the edges are introduced only where they are needed, and second, where strangers on the edges are introduced on all edges of the mesh.

In this section, we will present different definitions, notations and conventions of writing that we will use later. Besides we follow the idea of Eymard et al. [16] to build flux using a stabilized discrete gradient. After we define the discretization of the convection term, we give some proprieties and definition of the schemes.

2.1. Notation and assumptions. Now let us define some notations of the discretization of Ω.

Definition 2.1.

- ▷ A discretization of Ω, denoted $\mathcal{D}$, is defined by a triplet $\mathcal{D} = (\mathcal{M}, \mathcal{E}, P)$.
- ▷ $\mathcal{M}$ is a family of connected nonempty open subspaces included in Ω (set of control volumes $\mathcal{K}$) such that $\Omega = \bigcup_{\mathcal{K}\in\mathcal{M}} \overline{\mathcal{K}}$. The boundary $\partial\mathcal{K} = \overline{\mathcal{K}} \setminus \mathcal{K}$ for any $\mathcal{K} \in \mathcal{M}$ is the boundary of $\mathcal{K}$; $m_{\mathcal{K}} > 0$ is the measure of $\mathcal{K}$, $x_{\mathcal{K}}$ is the barycentre of $\mathcal{K}$ and $d(\mathcal{K})$ is the diameter of $\mathcal{K}$.
- ▷ Let us define the set of interfaces of the mesh $\mathcal{D}$ by $\mathcal{E}$; this set is decomposed into two subdomains $\mathcal{E}_{\text{int}}$ and $\mathcal{E}_{\text{ext}}$, which respectively represent the set of internal faces and faces located on the edge $\partial\Omega$ of the domain.
- ▷ σ is a nonempty open subset of $\mathbb{R}$ $(\sigma \in \mathcal{E})$, x_σ is the center of σ and m_σ is the measure of interface σ. The symbol $\sigma_{\mathcal{K},\mathcal{L}}$ stands for the common interface between $\mathcal{K}$ and $\mathcal{L}$.
- ▷ For any $\sigma \in \mathcal{E}$, $M_\sigma = \{\mathcal{K} \in \mathcal{M}, \sigma \in \partial\mathcal{K}\}$. If M_σ contains one element, then $\sigma \in \mathcal{E}_{\text{ext}}$, else $\sigma \in \mathcal{E}_{\text{int}}$. Let $\mathcal{E}_{\mathcal{K}}$ be the set of the interfaces of $\mathcal{K}$.
- ▷ $n_{\mathcal{K},\sigma}$ is the unit vector normal to σ outward to $\mathcal{K}$ and $d_{\mathcal{K},\sigma} > 0$ is the Euclidean distance between x_σ and $x_{\mathcal{K}}$.

▷ Let P be the set of points of Ω.

▷ Let $\mathfrak{D}$ be the set of diamond $\mathcal{D}_{\mathcal{T}}$ such that $\bigcup_{\mathcal{D}_{\mathcal{T}}\in\mathfrak{D}} \overline{\mathcal{D}_{\mathcal{T}}} = \Omega$, and $C_{\mathcal{K},\sigma}$ be the cone with vertex $x_{\mathcal{K}}$ and basis σ (we note $\{\mathcal{K},\sigma\} = C_{\mathcal{K},\sigma}$).

▷ The size of the discretization $\mathcal{D}$ is defined by

$$h_{\mathcal{D}} = \sup_{\mathcal{K}\in\mathcal{M}} (d(\mathcal{K})). \tag{2.1}$$

Definition 2.2. We consider $X_{\mathcal{D}}$, $X_{\mathcal{D},0}$ and $X_{\mathcal{D},0,\mathcal{B}}$ three spaces defined as follows:

$$X_{\mathcal{D}} = \{v = ((v_{\mathcal{K}})_{\mathcal{K}\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{E}});\ v_{\mathcal{K}} \in \mathbb{R},\ v_\sigma \in \mathbb{R}\}, \tag{2.2}$$

$$X_{\mathcal{D},0} = \{v \in X_{\mathcal{D}};\ \Lambda_{\mathcal{K}} \nabla^n_{\mathcal{K},\sigma} v \cdot n_{\mathcal{K},\sigma} = 0 \text{ for all } \sigma \in \mathcal{E}_{\text{ext}}\}, \tag{2.3}$$

$$X_{\mathcal{D},0,\mathcal{B}} = \left\{v \in X_{\mathcal{D},0};\ \exists\, \beta^{\mathcal{K}}_\sigma \in \mathbb{R};\ v_\sigma = \sum_{\mathcal{K}\in\mathcal{M}} \beta^{\mathcal{K}}_\sigma v_{\mathcal{K}}\right\}, \tag{2.4}$$

where $\mathcal{B}$ is defined in the next definition and $\Lambda = K(x,c)$ if $v = p$ and $\Lambda = D(x,U)$ if $v = c$.

Definition 2.3. Let

$$u_\sigma = \sum_{\mathcal{K}\in\mathcal{M}} \beta^{\mathcal{K}}_\sigma u_{\mathcal{K}}, \tag{2.5}$$

where $(\beta^{\mathcal{K}}_\sigma)_{\mathcal{K}\in\mathcal{M},\sigma\in\mathcal{E}_{\text{int}}}$ is a family of real numbers with $\beta^{\mathcal{K}}_\sigma \neq 0$ only for some control volumes $\mathcal{K}$ close to σ, and such that

$$\sum_{\mathcal{K}\in\mathcal{M}} \beta^{\mathcal{K}}_\sigma = 1 \text{ and } x_\sigma = \sum_{\mathcal{K}\in\mathcal{M}} \beta^{\mathcal{K}}_\sigma x_{\mathcal{K}}. \tag{2.6}$$

Let $\mathcal{B}$ be the set of the eliminated unknowns using (2.5), and $\mathcal{H} = \mathcal{E}_{\text{int}}/\mathcal{B}$.

The projections in the spaces $X_{\mathcal{D}}, X_{\mathcal{D},0}$ and $X_{\mathcal{D},0,\mathcal{B}}$ are defined in the next definition.

Definition 2.4. $C_0(\overline{\Omega})$ is the set of continuous functions which are null in $\partial\Omega$. For all $\psi \in C_0(\overline{\Omega})$ we define:

(1) the projection in $X_{\mathcal{D}}$ by

$$\mathcal{P}_{\mathcal{D}}\colon C_0(\overline{\mathbb{R}}) \to X_{\mathcal{D}},$$
$$\psi \mapsto \mathcal{P}_{\mathcal{D}}\psi = ((\psi(x_{\mathcal{K}}))_{\mathcal{K}\in\mathcal{M}}, (\psi(x_\sigma))_{\sigma\in\mathcal{E}});$$

(2) $\mathcal{P}_{\mathcal{D},\mathcal{B}}\psi = v$ as an element of $X_{\mathcal{D},\mathcal{B}}$ such that $v_\mathcal{K} = \psi(x_\mathcal{K})$ for all $\mathcal{K} \in \mathcal{M}$; $v_\sigma = v_\mathcal{K}$ for all $\sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{K}$; $v_\sigma = \psi(x_\sigma)$ for all $\sigma \in \mathcal{H}$ and $v_\sigma = \sum_{\mathcal{K}\in\mathcal{M}} \beta_\sigma^\mathcal{K} \psi(x_\mathcal{K})$ for all $\sigma \in \mathcal{B}$;

(3) $H_\mathcal{M}(\Omega)$ as the set of the piece-wise functions on $\mathcal{M}$ and the operator $\mathcal{P}_\mathcal{M}$ such that for any $\psi\colon \Omega \to \mathbb{R}$, $\mathcal{P}_\mathcal{M}\psi$ is the piecewise function satisfying $\mathcal{P}_\mathcal{M}(\psi(x)) = \psi(x_\mathcal{K})$ for all $\mathcal{K} \in \mathcal{M}$.

The space $X_\mathcal{D}$ is equipped with the semi-norm $|\cdot|_{X_\mathcal{D}}$ defined by

$$|v|^2_{X_\mathcal{D}} = \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_\mathcal{K}} \frac{m_\sigma}{d_{\mathcal{K}\sigma}} (v_\sigma - v_\mathcal{K})^2 \quad \forall\, v \in X_\mathcal{D}. \tag{2.7}$$

Note that $|\cdot|_{X_\mathcal{D}}$ is a norm on the spaces $X_{\mathcal{D},0}$ and $X_{\mathcal{D},0,\mathcal{B}}$.

Definition 2.5. The time interval $(0,T)$ $(T > 0)$ is divided into N (N is an integer such that $N > 0$) small intervals with a time step $\delta t = t_{n+1} - t_n$ equal to T/N. We introduce the following spaces:

$$X_{\mathcal{D},\delta t} = \{(v^n)_{n\in\{0,\dots,N-1\}},\ v^n \in X_\mathcal{D}\}, \tag{2.8}$$

$$X_{\mathcal{D},\delta t,0} = \{(v^n)_{n\in\{0,\dots,N-1\}},\ v^n \in X_{\mathcal{D},0}\}, \tag{2.9}$$

$$X_{\mathcal{D},\delta t,\mathcal{B}} = \{(v^n)_{n\in\{0,\dots,N-1\}},\ v^n \in X_{\mathcal{D},0,\mathcal{B}}\}. \tag{2.10}$$

The semi-norm on $X_{\mathcal{D},\delta t}$ is defined by

$$|v|^2_{X_{\mathcal{D},\delta t}} = \sum_{n=0}^{N-1} \delta t |v^n|^2_{X_\mathcal{D}}. \tag{2.11}$$

2.2. The discrete gradient. It is always possible to deduce an expression for $\nabla_\mathcal{D} u(x)$ as a linear combination of $(u_\sigma - u_\mathcal{K})_{\sigma\in\mathcal{E}_\mathcal{K}}$.

Let us first define

$$\nabla_\mathcal{K}\colon\ X_\mathcal{D} \to H_\mathcal{M}(\Omega)^d,$$
$$u^{n+1} \mapsto \nabla_\mathcal{K} u^{n+1}$$

such that

$$u^{n+1} \in X_D, \nabla_\mathcal{K} u^{n+1} = \frac{1}{|\mathcal{K}|} \sum_{\sigma\in\mathcal{E}_\mathcal{K}} |\sigma| [u_\sigma^{n+1} - u_\mathcal{K}^{n+1}] n_{\mathcal{K},\sigma}.$$

However, we find that this discrete gradient is zero for any $u_\mathcal{K}^{n+1} \in \mathcal{K}$ if u_σ^{n+1} are zero, so it is not coercive. We thus seek a new discrete gradient coherent with the previous and coercive in the $C_{\mathcal{K},\sigma}$ (cone with the vertex $x_\mathcal{K}$ and basis σ). This corresponds to the previous step gradient to which we add a correction term. We define the discrete gradient as

$$\nabla_{\mathcal{K},\sigma} u^{n+1} = \nabla_{\mathcal{K}} u^{n+1} + \mathcal{R}_{\mathcal{K},\sigma}(u^{n+1}) n_{\mathcal{K},\sigma} \tag{2.12}$$

with

$$\mathcal{R}_{\mathcal{K},\sigma}(u^{n+1}) = \frac{\sqrt{d}}{d_{\mathcal{K},\sigma}} (u_\sigma^{n+1} - u_{\mathcal{K}}^{n+1} - \nabla_{\mathcal{K}} u^{n+1} \cdot [x_\sigma - x_{\mathcal{K}}]).$$

(Recall that d is the space dimension and $d_{\mathcal{K},\sigma}$ is the Euclidean distance between $x_{\mathcal{K}}$ and x_σ.) We obtain the following stable discrete gradient

$$\nabla_{\mathcal{K},\sigma} u^{n+1} = \nabla_{\mathcal{K}} u^{n+1} + \mathcal{R}_{\mathcal{K},\sigma} u^{n+1} \cdot n_{\mathcal{K},\sigma}. \tag{2.13}$$

We may then define $\nabla_{\mathcal{D}}$ as the piece-wise constant function equal to $\nabla_{\mathcal{K},\sigma}$ a.e. in the cone $C_{\mathcal{K},\sigma}$ with vertex $x_{\mathcal{K}}$ and basis σ:

$$\nabla_{\mathcal{D}} u^{n+1} = \nabla_{\mathcal{K},\sigma} u^{n+1} \quad \text{for a.e. } x \in C_{\mathcal{K},\sigma}. \tag{2.14}$$

Then we have

$$\nabla_{\mathcal{K},\sigma} u^{n+1} = \sum_{\sigma' \mathcal{E}_{\mathcal{K}}} Y^{\sigma,\sigma'} (u_{\sigma'}^{n+1} - u_{\mathcal{K}}^{n+1}) \tag{2.15}$$

with $Y^{\sigma,\sigma'}$ given by

$$Y^{\sigma,\sigma'} = \begin{cases} \dfrac{m_\sigma}{m_{\mathcal{K}}} n_{\mathcal{K}\sigma} + \dfrac{\sqrt{d}}{d_{\mathcal{K},\sigma}} \Big(1 - \dfrac{m_\sigma}{m_{\mathcal{K}}} n_{\mathcal{K}\sigma} \cdot [x_\sigma - x_{\mathcal{K}}]\Big) n_{\mathcal{K}\sigma} & \text{if } \sigma = \sigma', \\ \dfrac{m_{\sigma'}}{m_{\mathcal{K}}} n_{\mathcal{K}\sigma'} - \dfrac{\sqrt{d}}{d_{\mathcal{K},\sigma} m_{\mathcal{K}}} n_{\mathcal{K},\sigma'} \cdot [x_\sigma - x_{\mathcal{K}}] n_{\mathcal{K},\sigma} & \text{otherwise.} \end{cases} \tag{2.16}$$

2.3. The discrete convection term. To treat the convection term in the concentration equation, we define the following convection discrete operator as follows:

$$\begin{aligned} \int_\Omega \operatorname{div}(\xi, v) &\approx \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_\sigma \operatorname{div} c_\sigma(\xi_{\mathcal{D}}, v_{\mathcal{T}}) \\ &= \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_\sigma [(\xi_{\mathcal{D}} \cdot n_{\sigma,\mathcal{K}})^+ v_{\mathcal{K}} - (\xi_{\mathcal{D}} \cdot n_{\sigma,\mathcal{K}})^- v_{\mathcal{L}}], \end{aligned} \tag{2.17}$$

with $v_{\mathcal{T}} \in X_{\mathcal{D}}$ and $\xi_{\mathcal{D}} \in \mathbb{R}^2$.

2.4. The proprieties of the schemes. Let $\mathcal{D}$ be a discretization of Ω in the sense of Definition 2.1. The regularity of the mesh is defined by

$$\theta_{\mathcal{D}} = \max\Big(\max_{\sigma_{\mathcal{K},\mathcal{L}}\in\mathcal{E}_{\text{int}}} \Big(\frac{d_{\mathcal{K},\sigma}}{d_{\mathcal{L},\sigma}}\Big), \max_{\mathcal{K}\in\mathcal{M},\sigma\in\mathcal{E}_{\mathcal{K}}} \Big(\frac{d(\mathcal{K})}{d_{\mathcal{K},\sigma}}\Big)\Big). \tag{2.18}$$

For a given set $\mathcal{B} \in \mathcal{E}_{\text{int}}$ and for a given family $\beta_\sigma^{\mathcal{K}}$ satisfying property (2.5), we introduce a measure of the resulting regularity by

$$\theta_{\mathcal{D},\mathcal{B}} = \max\left(\theta_{\mathcal{D}}, \max_{\mathcal{K}\in\mathcal{M},\sigma\in\mathcal{E}_{\mathcal{K}}\cap\mathcal{B}} \frac{\sum_{\mathcal{L}\in\mathcal{M}} |\beta_\sigma^{\mathcal{L}}||x_{\mathcal{L}} - x_\sigma|^2}{h_{\mathcal{K}}^2}\right). \tag{2.19}$$

Definition 2.6. Let $\mathcal{D}$ be a discretization of Ω in the sense of Definition 2.1, and let δt be the time step defined in Definition 2.5. For $v \in H_{\mathcal{M}}(\Omega)$ we define the related norm

$$\|\mathcal{P}_{\mathcal{M}} v\|_{1,2,\mathcal{M}}^2 = \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} |\sigma| d_{\mathcal{K},\sigma} \Big(\frac{D_\sigma v}{d_\sigma}\Big)^2, \tag{2.20}$$

and for $v \in X_{\mathcal{D},\delta t}$, we define the related norm

$$\|\mathcal{P}_{\mathcal{M}} v\|_{1;1,2,\mathcal{M}}^2 = \sum_{n=0}^{N-1} \delta t \|\mathcal{P}_{\mathcal{M}} v^n\|_{1,2,\mathcal{M}}^2 \tag{2.21}$$

with $d_\sigma = |d_{\mathcal{K},\sigma} + d_{\mathcal{L},\sigma}|$, $D_\sigma v = |v_{\mathcal{K}} - v_{\mathcal{L}}|$ if $\mathcal{M}_\sigma = \{\mathcal{K},\mathcal{L}\}$, and $d_\sigma = d_{\mathcal{K},\sigma}$, $D_\sigma v = |v_{\mathcal{K}}|$ if $\mathcal{M}_\sigma = \{\mathcal{K}\}$.

A result stated in [16] gives the relation

$$\|\mathcal{P}_{\mathcal{M}} v\|_{1,2,\mathcal{M}}^2 \leqslant |v|_{X_{\mathcal{D}}}^2 \quad \forall\, v \in X_{\mathcal{D},0}. \tag{2.22}$$

Then we get

$$\|\mathcal{P}_{\mathcal{M}} v\|_{1;1,2,\mathcal{M}}^2 \leqslant |v|_{X_{\mathcal{D}},\delta t}^2 \quad \forall\, v \in X_{\delta t,\mathcal{D},0}. \tag{2.23}$$

A result stated in [16] gives the relation

$$\|\mathcal{P}_{\mathcal{M}} v\|_{1,2,\mathcal{M}}^2 \leqslant |v|_{X_{\mathcal{D}}}^2 \quad \forall\, v \in X_{\mathcal{D},0}. \tag{2.24}$$

Then we get

$$\|\mathcal{P}_{\mathcal{M}} v\|_{1;1,2,\mathcal{M}}^2 \leqslant |v|_{X_{\mathcal{D}},\delta t}^2 \quad \forall\, v \in X_{\delta t,\mathcal{D},0}. \tag{2.25}$$

We recall in this section some proprieties of SUSHI scheme. The proof of these proprieties can be found in [4].

Lemma 2.1. *Let $\mathcal{D}$ be a discretization of Ω in the sense of Definition 2.1. Let $\nu > 0$ be such that $\nu \leqslant d_{\mathcal{K},\sigma}/d_{\mathcal{L},\sigma} \leqslant 1/\nu$ for all $\sigma \in \mathcal{E}$, where $M_\sigma = \{\mathcal{K},\mathcal{L}\}$. Then there exists C_1 depending only on d, Ω and ν such that*

$$\|\mathcal{P}_{\mathcal{M}} v\|_{L^2(\Omega)} \leqslant C_1 \|\mathcal{P}_{\mathcal{M}} v\|_{1,2,\mathcal{M}} \quad \forall\, v \in X_{\mathcal{D}} \tag{2.26}$$

where $\|\mathcal{P}_{\mathcal{M}} v\|_{1,2,\mathcal{M}}$ is defined by (2.20).

Lemma 2.2. *Let $\mathcal{D}$ be a discretization of Ω in the sense of Definition 2.1, and and let δt be the time step defined in Definition 2.5. Let $\nu > 0$ be such that $\nu \leqslant d_{\mathcal{K},\sigma}/d_{\mathcal{L},\sigma} \leqslant 1/\nu$ for all $\sigma \in \mathcal{E}$, where $M_\sigma = \{\mathcal{K}, \mathcal{L}\}$. Then there exists $C_1 > 0$ depending only on δt and C_1 such that*

$$\|\mathcal{P}_{\mathcal{M}} v\|_{L^2(0,T;L^2(\Omega))} \leqslant C_2 \|\mathcal{P}_{\mathcal{M}} v\|_{1;1,2,\mathcal{M}} \quad \forall\, v \in X_{\mathcal{D},\delta t} \tag{2.27}$$

where $\|\mathcal{P}_{\mathcal{M}} v\|_{1;1,2,\mathcal{M}}$ is defined by (2.21).

P r o o f. The proof is a result of Lemma 2.1. □

Definition 2.7. Let $\mathcal{D}$ be a discretization of Ω in the sense of Definition 2.1, and let δt be the time step defined in Definition 2.5. We define the L^2-norm of the discrete gradient by

$$\|\nabla_{\mathcal{D}} v(x)\|^2_{L^2(\Omega)^d} = \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \frac{|\sigma| d_{\mathcal{K},\sigma}}{d} |\nabla_{\mathcal{K},\sigma} v|^2 \quad \forall\, v \in X_{\mathcal{D}},$$

and

$$\|\nabla_{\mathcal{D}} w(x,t)\|^2_{L^2(0,T;L^2(\Omega)^d)} = \sum_{n=1}^{N} \delta t \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \frac{|\sigma| d_{\mathcal{K},\sigma}}{d} |\nabla_{\mathcal{K},\sigma} w^n|^2 \quad \forall\, w \in X_{\mathcal{D},\delta t},$$

where $\nabla_{\mathcal{K},\sigma}$ and $\nabla_{\mathcal{D}}$ are defined by (2.13) and (2.14)

Lemma 2.3. *Let $\mathcal{D}$ be a discretization of Ω in the sense of Definition 2.1, let δt be the time step defined in Definition 2.5 and suppose that there exists a positive θ such that $\theta_{\mathcal{D}} \leqslant \theta$ for all $\mathcal{D}$.*

(1) *Then there exist positive constants C_3 and C_4 depending only on θ and d such that*

$$C_3 |v|^2_{X_{\mathcal{D}}} \leqslant \|\nabla_{\mathcal{D}} v\|^2_{L^2(\Omega)^d} \leqslant C_4 |v|^2_{X_{\mathcal{D}}} \quad \forall\, v \in X_{\mathcal{D}}. \tag{2.28}$$

(2) *Moreover, we have*

$$C_5 |w|^2_{X_{\mathcal{D},\delta t}} \leqslant \|\nabla_{\mathcal{D}} w\|^2_{L^2(0,T;L^2(\Omega)^d)} \leqslant C_6 |w|^2_{X_{\mathcal{D},\delta t}} \quad \forall\, v \in X_{\mathcal{D}\delta t}. \tag{2.29}$$

Definition 2.8. Let $\mathcal{D}$ be a discretization of Ω in the sense of Definition 2.1 and let δt be the time step defined in Definition 2.5. Let $u_{\mathcal{D},\delta t} \in X_{\mathcal{D},\delta t}$ be a solution of the problem. We say that $\mathcal{P}_{\mathcal{M}} u_{\mathcal{D},\delta t}(x,t)$ is an approximate solution of the problem.

3.1. Discrete weak formulation. In this section we present the discrete weak formulation for problem (1.1)–(1.2). We consider, for all $n = 0, \dots, N-1$ and all $\mathcal{K} \in \mathcal{M}$, the unknowns $c^{n+1} \in X_{\mathcal{D},\delta t}$, $U^n \in X_{\mathcal{D},\delta t}$ and $p^n \in X_{\mathcal{D},\delta t}$, which stand for approximate values of c, U and p on $[n; n+1]$.

3.1.1. Equation of the pressure. We begin with the discretization of equation

$$-\mathrm{div}(K(x,c)\nabla p) = q^+ - q^-. \tag{3.1}$$

We integer over $\mathcal{K}$ for any $\mathcal{K} \in \mathcal{M}$ and in the interval $(t^n, t^{n+1}) \subset (0,T)$ which yields

$$\int_{\mathcal{K}} \int_{t^n}^{t^{n+1}} -\mathrm{div}(K(x,c)\nabla p) = \int_{\mathcal{K}} \int_{t^n}^{t^{n+1}} (q^+ - q^-),$$

which gives

$$\delta t \int_{\mathcal{K}} -\mathrm{div}(K(x,c^n)\nabla p^{n+1}) = \delta t \int_{\mathcal{K}} (q^{+,n+1} - q^{-,n+1}),$$

then

$$\sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} \int_{\sigma} K(x,c^n)\nabla p^{n+1} \cdot n_{\mathcal{K},\sigma} = m_{\mathcal{K}}(q_{\mathcal{K}}^{+,n+1} - q_{\mathcal{K}}^{-,n+1}),$$

finally

$$\sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} \mathcal{F}^1_{\mathcal{K},\sigma}(p^{n+1}) = m_{\mathcal{K}} q_{\mathcal{K}}^{+,n+1} - \mathcal{K} q_{\mathcal{K}}^{-,n+1}. \tag{3.2}$$

For the border elements we obtain the equations by discretizing the second part of system (1.5), which gives that the flow is zero on the boundary as follows:

$$K(x,c^n_{\mathcal{K}})\nabla_{\mathcal{K},\sigma} p^{n+1} \cdot n_{\mathcal{K},\sigma} = 0 \quad \forall \sigma \in \mathcal{E}_{\mathrm{ext}} \text{ with } \sigma \in \mathcal{K}. \tag{3.3}$$

We use the fact that the numerical flow is locally conserve at the interface of the two elements, we then have

$$\mathcal{F}^1_{\mathcal{K},\sigma}(p^{n+1}) + \mathcal{F}^1_{\mathcal{L},\sigma}(p^{n+1}) = 0 \quad \forall \sigma \in \mathcal{E}_{\mathrm{int}} \text{ such that } \mathcal{M}_\sigma = \{\mathcal{K}, \mathcal{L}\}. \tag{3.4}$$

And now we have $\mathrm{card}(\mathcal{E}_{\mathrm{int}}) + \mathrm{card}(\mathcal{E}_{\mathrm{ext}}) + \mathrm{card}(\mathcal{M})$ unknowns and equations.

To discretize the null average condition on the pressure $\int_\Omega p(\cdot,x)\,\mathrm{d}x = 0$, we consider $\Omega = \bigcup_{\mathcal{K}\in\mathcal{M}} \overline{\mathcal{K}}$. Then we have

$$\sum_{\mathcal{K}\in\mathcal{M}} m_{\mathcal{K}} p_{\mathcal{K}} = 0.$$

Multiplying equation (3.2) by $v_{\mathcal{K}}^{n+1}$ for all $\mathcal{K}\in\mathcal{M}$ and all $n = 0,\dots,N-1$, then summing over $\mathcal{K}$ and over $n = 0,\dots,N-1$, we get

$$\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \mathcal{F}^1_{\mathcal{K}}(p^{n+1}) v_{\mathcal{K}}^{n+1} = \sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}} v_{\mathcal{K}}^{n+1} m_{\mathcal{K}}(q_{\mathcal{K}}^{+,n+1} - q_{\mathcal{K}}^{-,n+1}), \tag{3.5}$$

which gives

$$\langle p, v\rangle_{F^1} = \sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}} v_{\mathcal{K}}^{n+1} m_{\mathcal{K}}(q_{\mathcal{K}}^{+,n+1} - q_{\mathcal{K}}^{-,n+1}) \tag{3.6}$$

with

$$\langle p, v\rangle_{F^1} = \sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \mathcal{F}^1_{\mathcal{K}}(p^{n+1})[v_{\mathcal{K}}^{n+1} - v_{\sigma}^{n+1}]. \tag{3.7}$$

We define also

$$[p^{n+1}, v^{n+1}]_{F^1} = \sum_{\mathcal{K}\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \mathcal{F}^1_{\mathcal{K}}(p^{n+1})[v_{\mathcal{K}}^{n+1} - v_{\sigma}^{n+1}]. \tag{3.8}$$

3.1.2. Via equation. For the second equation we have

$$U = -K(x,c)\nabla p, \quad \text{in } (0,T)\times\Omega. \tag{3.9}$$

We integer over $\mathcal{D}_{\mathcal{T}}$ and over the time interval $(t^n, t^{n+1}) \subset (0,T)$ and we obtain

$$\int_{t^n}^{t^{n+1}}\int_{\mathcal{D}_{\mathcal{T}}} U\,\mathrm{d}x\,\mathrm{d}t = \int_{t^n}^{t^{n+1}}\int_{\mathcal{D}_{\mathcal{T}}} -K(x,c)\nabla p\,\mathrm{d}x\,\mathrm{d}t, \tag{3.10}$$

after simplifications we obtain the formula

$$U_{\mathcal{D}_{\mathcal{T}}}^{n+1} = (-K(x_\sigma, c^n)\nabla p^{n+1})_{\mathcal{D}_{\mathcal{T}}}.$$

For any diamond $\mathcal{D}_{\mathcal{T}} \in \mathfrak{D}$ we have

$$\mathcal{D}_{\mathcal{T}} = \begin{cases} \{\mathcal{K},\sigma\}\cup\{\mathcal{L},\sigma\} & \text{if } \sigma\in\mathcal{E}_{\text{int}},\\ \{\mathcal{K},\sigma\} & \text{if } \sigma\in\mathcal{E}_{\text{ext}}.\end{cases}$$

Then

$$U_{\mathcal{D}_\mathcal{T}}^{n+1} = \begin{cases} U_{\mathcal{K},\sigma}^{n+1} + U_{\mathcal{L},\sigma}^{n+1} & \text{if } \sigma \in \mathcal{E}_{\text{int}}, \\ U_{\mathcal{K},\sigma}^{n+1} & \text{if } \sigma \in \mathcal{E}_{\text{ext}}, \end{cases}$$

and

$$\nabla^{\mathcal{D}_\mathcal{T}} p^{n+1} = \begin{cases} \nabla_{\mathcal{K},\sigma} p^{n+1} + \nabla_{\mathcal{L},\sigma} p^{n+1} & \text{if } \sigma \in \mathcal{E}_{\text{int}}, \\ \nabla_{\mathcal{K},\sigma} p^{n+1} & \text{if } \sigma \in \mathcal{E}_{\text{ext}}. \end{cases}$$

Finally, we get

$$\begin{cases} U_{\mathcal{K},\sigma}^{n+1} \cdot n_{\mathcal{K},\sigma} + U_{\mathcal{L},\sigma}^{n+1} \cdot n_{\mathcal{L},\sigma} & \\ \qquad = -K(x_\sigma, c_\mathcal{K}^n) \nabla_{\mathcal{K},\sigma} p^{n+1} - K(x_\sigma, c_\mathcal{L}^n) \nabla_{\mathcal{L},\sigma} p^{n+1} & \text{if } \sigma \in \mathcal{E}_{\text{int}}, \\ U_{\mathcal{K},\sigma}^{n+1} \cdot n_{\mathcal{K},\sigma} = -K(x_\sigma, c_\mathcal{K}^n) \nabla_{\mathcal{K},\sigma} p^{n+1} & \text{if } \sigma \in \mathcal{E}_{\text{ext}} \end{cases} \tag{3.11}$$

where $\nabla_{\mathcal{K},\sigma} p^{n+1}, \nabla_{\mathcal{L},\sigma} p^{n+1}$ are noted in (2.15).

We rewrite (3.11) as

$$\begin{cases} U_{\mathcal{K},\sigma}^{n+1} \cdot n_{\mathcal{K},\sigma} + U_{\mathcal{L},\sigma}^{n+1} \cdot n_{\mathcal{L},\sigma} = \mathcal{F}_{\mathcal{K},\sigma}^1(p^{n+1}) + \mathcal{F}_{\mathcal{L},\sigma}^1(p^{n+1}) & \text{if } \sigma \in \mathcal{E}_{\text{int}}, \\ U_{\mathcal{K},\sigma}^{n+1} \cdot n_{\mathcal{K},\sigma} = \mathcal{F}_{\mathcal{K},\sigma}^1(p^{n+1}) & \text{otherwise.} \end{cases} \tag{3.12}$$

3.1.3. Concentration equation. Now, we discretize the third equation

$$\phi(x)\partial_t c - \operatorname{div}(D(x,U)\nabla c) + \operatorname{div}(cU) + q^- c = q^+ \widehat{c}. \tag{3.13}$$

We integrate over the volume control $\mathcal{K} \in \mathcal{M}$ and over the time interval $(t^n, t^{n+1}) \subset [0, T]$ and we obtain

$$\int_{t^n}^{t^{n+1}} \int_\mathcal{K} \phi(x)\partial_t c - \int_{t^n}^{t^{n+1}} \int_\mathcal{K} \operatorname{div}(D(x,U)\nabla c) \\ + \int_{t^n}^{t^{n+1}} \int_\mathcal{K} \operatorname{div}(cU) + \int_{t^n}^{t^{n+1}} \int_\mathcal{K} q^- c = \int_{t^n}^{t^{n+1}} \int_\mathcal{K} q^+ \widehat{c}.$$

That gives

$$\int_\mathcal{K} \phi(x)(c^{n+1} - c^n) - \delta t \int_\mathcal{K} \operatorname{div}(D(x, U^{n+1})\nabla c^{n+1}) \\ + \delta t \int_\mathcal{K} \operatorname{div}(c^{n+1} U^{n+1}) + \delta t \int_\mathcal{K} q^{-,n+1} c^{n+1} = \delta t \int_\mathcal{K} q^{+,n+1} \widehat{c}^{n+1}.$$

Then

$$\int_\mathcal{K} \phi(x)(c^{n+1} - c^n) + \delta t \sum_{\sigma \in \mathcal{E}_\mathcal{K}} \int_\sigma D(x, U^{n+1})\nabla c^{n+1} \\ + \delta t \int_\mathcal{K} \operatorname{div}(c^{n+1} U^{n+1}) + \delta t \int_\mathcal{K} q^{-,n+1} c^{n+1} = \delta t \int_\mathcal{K} q^{+,n+1} \widehat{c}^{n+1}.$$

Finally,

$$m_{\mathcal{K}}\phi_{\mathcal{K}}(c_{\mathcal{K}}^{n+1}-c_{\mathcal{K}}^{n})+\delta t\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}\mathcal{F}^2_{\mathcal{K},\sigma}(c^{n+1})+\delta t\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}m_\sigma\operatorname{div}c_\sigma(c_{\mathcal{D}}^{n+1},U_{\mathcal{D}}^{n+1}) + \delta t m_{\mathcal{K}} q_{\mathcal{K}}^{-,n+1}c_{\mathcal{K}}^{n+1}=\delta t m_{\mathcal{K}} q_{\mathcal{K}}^{+,n+1}\widehat{c}_{\mathcal{K}}^{n+1}. \tag{3.14}$$

We have card($\mathcal{M}$) equations and card($\mathcal{E}$) + card($\mathcal{M}$) unknowns. For a reasonable system we need card($\mathcal{E}$) more equations; for that we use that the flow is null on the boundary:

$$D(x_{\mathcal{K}},U_{\mathcal{K}}^{n+1})\nabla_{\mathcal{K},\sigma}c^{n+1}\cdot n_{\mathcal{K},\sigma}=0 \quad \forall\,\sigma\in\mathcal{E}_{\mathrm{ext}}\cap\mathcal{K}. \tag{3.15}$$

And since the numerical flow is locally conserved, we have

$$\mathcal{F}^2_{\mathcal{K},\sigma}(c^{n+1})+\mathcal{F}^2_{\mathcal{L},\sigma}(c^{n+1})=0 \quad \forall\,\sigma\in\mathcal{E}_{\mathrm{int}} \text{ such that } \mathcal{M}_\sigma=\{\mathcal{K},\mathcal{L}\}. \tag{3.16}$$

We have now card($\mathcal{E}_{\mathrm{int}}$) + card($\mathcal{E}_{\mathrm{ext}}$) + card($\mathcal{M}$) unknowns and equations.

We multiply (3.14) by $w_{\mathcal{K}}^{n+1}$ for all $w_{\mathcal{K}}\in\mathcal{M}$ and all $n=0,\dots,N-1$. Then summing over $\mathcal{K}$ and over $n=0,\dots,N-1$, we get

$$\begin{aligned}\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}w_{\mathcal{K}}^{n+1}\phi_{\mathcal{K}}(c_{\mathcal{K}}^{n+1}-c_{\mathcal{K}}^{n})&+\delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}\mathcal{F}^2_{\mathcal{K},\sigma}(c^{n+1})w_{\mathcal{K}}^{n+1}\\ &+\delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}w_{\mathcal{K}}^{n+1}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}m_\sigma\operatorname{div}c_\sigma(c_{\mathcal{D}}^{n+1},U_{\mathcal{D}}^{n+1})\\ &+\delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}w_{\mathcal{K}}^{n+1}q_{\mathcal{K}}^{-,n+1}c_{\mathcal{K}}^{n+1}=\delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}w_{\mathcal{K}}^{n+1}q_{\mathcal{K}}^{+,n+1}\widehat{c}_{\mathcal{K}}^{n+1}.\end{aligned}$$

That gives

$$\begin{aligned}\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}w_{\mathcal{K}}^{n+1}\phi_{\mathcal{K}}(c_{\mathcal{K}}^{n+1}-c_{\mathcal{K}}^{n})&+\delta t\sum_{n=0}^{N-1}\sum_{\sigma\in\mathcal{E}_{\mathrm{int}}}[\mathcal{F}^2_{\mathcal{K},\sigma}(c^{n+1})w_{\mathcal{K}}^{n+1}+\mathcal{F}^2_{\mathcal{L},\sigma}(c^{n+1})w_{\mathcal{L}}^{n+1}]\\ &+\delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}w_{\mathcal{K}}^{n+1}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}m_\sigma\operatorname{div}c_\sigma(c_{\mathcal{D}}^{n+1},U_{\mathcal{D}}^{n+1})\\ &+\delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}w_{\mathcal{K}}^{n+1}q_{\mathcal{K}}^{-,n+1}c_{\mathcal{K}}^{n+1}=\delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}w_{\mathcal{K}}^{n+1}q_{\mathcal{K}}^{+,n+1}\widehat{c}_{\mathcal{K}}^{n+1}.\end{aligned}$$

Bearing in mind (3.16) and (3.15), we get

$$\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}} w_{\mathcal{K}}^{n+1}\phi_{\mathcal{K}}(c_{\mathcal{K}}^{n+1}-c_{\mathcal{K}}^{n}) + \delta t\sum_{n=0}^{N-1}\sum_{\sigma\in\mathcal{E}_{\text{int}}}[\mathcal{F}_{\mathcal{K},\sigma}^{2}(c^{n+1})w_{\mathcal{K}}^{n+1}-\mathcal{F}_{\mathcal{K},\sigma}^{2}(c^{n+1})w_{\mathcal{L}}^{n+1}] + \delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}} w_{\mathcal{K}}^{n+1}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_{\sigma}\operatorname{div} c_{\sigma}(c_{\mathcal{D}}^{n+1},U_{\mathcal{D}}^{n+1}) + \delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}} w_{\mathcal{K}}^{n+1}q_{\mathcal{K}}^{-,n+1}c_{\mathcal{K}}^{n+1} = \delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}} w_{\mathcal{K}}^{n+1}q_{\mathcal{K}}^{+,n+1}\widehat{c}_{\mathcal{K}}^{n+1}.$$

Then we have

$$\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}} w_{\mathcal{K}}^{n+1}\phi_{\mathcal{K}}(c_{\mathcal{K}}^{n+1}-c_{\mathcal{K}}^{n}) +\delta t\sum_{n=0}^{N-1}\sum_{\sigma\in\mathcal{E}_{\text{int}}}[\mathcal{F}_{\mathcal{K},\sigma}^{2}(c^{n+1})(w_{\mathcal{K}}^{n+1}-w_{\sigma}^{n+1})+\mathcal{F}_{\mathcal{K},\sigma}^{2}(c^{n+1})(w_{\sigma}^{n+1}-w_{\mathcal{L}}^{n+1})] +\delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}} w_{\mathcal{K}}^{n+1}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_{\sigma}\operatorname{div} c_{\sigma}(c_{\mathcal{D}}^{n+1},U_{\mathcal{D}}^{n+1}) +\delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}} w_{\mathcal{K}}^{n+1}q_{\mathcal{K}}^{-,n+1}c_{\mathcal{K}}^{n+1} = \delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}} w_{\mathcal{K}}^{n+1}q_{\mathcal{K}}^{+,n+1}\widehat{c}_{\mathcal{K}}^{n+1},$$

$$\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}} w_{\mathcal{K}}\phi_{\mathcal{K}}(c_{\mathcal{K}}^{n+1}) - \delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}\mathcal{F}_{\mathcal{K},\sigma}^{2}(c^{n+1})[w_{\mathcal{K}}^{n+1}-w_{\sigma}^{n+1}] +\delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}} w_{\mathcal{K}}^{n+1}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_{\sigma}\operatorname{div} c_{\sigma}(c_{\mathcal{D}}^{n+1},U_{\mathcal{D}}^{n+1}) +\delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}} w_{\mathcal{K}}^{n+1}q_{\mathcal{K}}^{-,n+1}c_{\mathcal{K}}^{n+1} = \delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}} w_{\mathcal{K}}^{n+1}[q_{\mathcal{K}}^{+,n+1}\widehat{c}_{\mathcal{K}}^{n+1}+\phi_{\mathcal{K}}(c_{\mathcal{K}}^{n})],$$

thus, we give as a form of bilinear approximation the following formula

$$\langle c,w\rangle_{\mathcal{F}^2} = \sum_{n=0}^{N-1}[c^{n+1},w^{n+1}]_{\mathcal{F}^2}, \tag{3.17}$$

where

$$[c^{n+1}, w^{n+1}]_{\mathcal{F}^2} = \sum_{\mathcal{K}\in\mathcal{M}} \frac{w_{\mathcal{K}}^{n+1}\phi_{\mathcal{K}}}{\delta t}(c_{\mathcal{K}}^{n+1})$$
$$+ \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \mathcal{F}^2_{\mathcal{K},\sigma}(c^{n+1})[w_{\mathcal{K}}^{n+1} - w_{\sigma}^{n+1}]$$
$$+ \sum_{\mathcal{K}\in\mathcal{M}} w_{\mathcal{K}}^{n+1} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_\sigma \operatorname{div} c_\sigma(c_{\mathcal{D}}^{n+1}, U_{\mathcal{D}}^{n+1}) + \sum_{\mathcal{K}\in\mathcal{M}} w_{\mathcal{K}}^{n+1} q_{\mathcal{K}}^{-,n+1} c_{\mathcal{K}}^{n+1}. \tag{3.18}$$

3.2. The discrete flux. The discrete flux $\mathcal{F}^1_{\mathcal{K},\sigma}$ and $\mathcal{F}^2_{\mathcal{K},\sigma}$ are expressed in terms of the discrete unknowns. For this purpose we apply the SUSHI scheme proposed in [16]. The idea is based upon the identification of the numerical flux through the mesh dependent bilinear form, using the expressions of the discrete gradient

$$\sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \mathcal{F}^1_{\mathcal{K},\sigma}(p^{n+1})(u_{\mathcal{K}} - u_\sigma)$$
$$\approx \int_\Omega \nabla_{\mathcal{D}} p^{n+1} K(x, c^n) \nabla_{\mathcal{D}} u \quad \forall\, p^{n+1},\, u \in X_{0,\mathcal{D}}, \tag{3.19}$$

and

$$\sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \mathcal{F}^2_{\mathcal{K},\sigma}(c^{n+1})(v_{\mathcal{K}} - v_\sigma)$$
$$\approx \int_\Omega \nabla_{\mathcal{D}} c^{n+1} D(x, U^{n+1}) \nabla_{\mathcal{D}} v \quad \forall\, c^{n+1},\, v \in X_{0,\mathcal{D}}. \tag{3.20}$$

The identification of the numerical fluxes using relation (3.19) and (3.20) leads to the expression

$$\mathcal{F}^1_{\mathcal{K},\sigma}(p^n) = \sum_{\sigma'\in\mathcal{E}_{\mathcal{K}}} K_{\mathcal{K}}^{\sigma,\sigma'}(p_{\mathcal{K}}^{n+1} - p_{\sigma}^{n+1}), \tag{3.21}$$

$$\mathcal{F}^2_{\mathcal{K},\sigma}(c^{n+1}) = \sum_{\sigma'\in\mathcal{E}_{\mathcal{K}}} D_{\mathcal{K}}^{\sigma,\sigma'}(c_{\mathcal{K}}^{n+1} - c_{\sigma}^{n+1}). \tag{3.22}$$

Thus

$$\int_{\mathcal{K}} \nabla_{\mathcal{D}} p^{n+1} K(x, c^n) \nabla_{\mathcal{D}} u = \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \sum_{\sigma'\in\mathcal{E}_{\mathcal{K}}} K_{\mathcal{K}}^{\sigma,\sigma'}(p_{\mathcal{K}}^{n+1} - p_{\sigma'}^{n+1})(u_{\sigma'} - u_{\mathcal{K}}), \tag{3.23}$$

$$\int_{\mathcal{K}} \nabla_{\mathcal{D}} c^{n+1} D(x, U^{n+1}) \nabla_{\mathcal{D}} v = \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \sum_{\sigma'\in\mathcal{E}_{\mathcal{K}}} D_{\mathcal{K}}^{\sigma,\sigma'}(c_{\mathcal{K}}^{n+1} - c_{\sigma'}^{n+1})(v_{\sigma'} - v_{\mathcal{K}}), \tag{3.24}$$

with $\sigma, \sigma' \in \mathcal{E}_\mathcal{K}$ and

$$K_\mathcal{K}^{\sigma,\sigma'} = \sum_{\sigma'' \in \mathcal{E}_\mathcal{K}} Y^{\sigma'',\sigma} \Gamma_\mathcal{K}^{\sigma''} Y^{\sigma'',\sigma'} \quad \text{with } \Gamma_\mathcal{K}^{\sigma''} = \int_{\mathcal{K}, C_{\sigma''}} K(x, c^n)\,\mathrm{d}x,$$

$$D_\mathcal{K}^{\sigma,\sigma'} = \sum_{\sigma'' \in \mathcal{E}_\mathcal{K}} Y^{\sigma'',\sigma} \Theta_\mathcal{K}^{\sigma''} Y^{\sigma'',\sigma'} \quad \text{with } \Theta_\mathcal{K}^{\sigma''} = \int_{\mathcal{K}, C_{\sigma''}} D(x, U^{n+1})\,\mathrm{d}x.$$

The local matrices $K_\mathcal{K}^{\sigma,\sigma'}$ and $D_\mathcal{K}^{\sigma,\sigma'}$ are symmetric and positive.

3.3. Final scheme. Using (2.13), we have

$$\nabla_{\mathcal{K},\sigma} p^{n+1} = \nabla_\mathcal{K} p^{n+1} + \mathcal{R}_{\mathcal{K},\sigma} p^{n+1} \cdot n_{\mathcal{K},\sigma},$$
$$\nabla_{\mathcal{K},\sigma} c^{n+1} = \nabla_\mathcal{K} c^{n+1} + \mathcal{R}_{\mathcal{K},\sigma} c^{n+1} \cdot n_{\mathcal{K},\sigma},$$

and

$$\operatorname{div} c_\sigma(U_\mathcal{D}, v_\mathcal{T}) = (U_\mathcal{D} \cdot n_{\sigma,\mathcal{K}})^+ v_\mathcal{K} - (U_\mathcal{D} \cdot n_{\sigma,\mathcal{K}})^- v_\mathcal{L}.$$

The discretization of problems (1.1) and (1.2) is defined as follows:

$$(3.25)\quad \begin{cases} \text{Find for all } \mathcal{K} \in \mathcal{M} \text{ and for all } n, p^{n+1} \text{ and } c^{n+1} \\ \displaystyle\sum_{\sigma \in \mathcal{E}_\mathcal{K}} \sum_{\sigma' \in \mathcal{E}_\mathcal{K}} K_\mathcal{K}^{\sigma,\sigma'} [p_\sigma^{n+1} - p_\mathcal{K}^{n+1}][v_{\sigma'} - v_\mathcal{K}] = m_\mathcal{K} v_\mathcal{K} (q_\mathcal{K}^{+,n+1} - q_\mathcal{K}^{-,n+1}), \\ U_\mathcal{D}^{n+1} = K(x_\sigma, c_\mathcal{K}^n)) \nabla_\mathcal{D} p^{n+1}, \ \displaystyle\sum_{\mathcal{K} \in \mathcal{M}} m_\mathcal{K} p_\mathcal{K} = 0, \\ \text{with } K(x_\sigma, c_\mathcal{K}^n)) \nabla_\mathcal{D} p^{n+1} \cdot n_{\mathcal{K},\sigma} = 0 \text{ if } \sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{K}, \\ m_\mathcal{K} v_\mathcal{K} \phi_\mathcal{K}(c_\mathcal{K}^{n+1}) - \delta t \displaystyle\sum_{\sigma \in \mathcal{E}_\mathcal{K}} \sum_{\sigma' \in \mathcal{E}_\mathcal{K}} D_\mathcal{K}^{\sigma,\sigma'} [c_\sigma^{n+1} - c_\mathcal{K}^{n+1}][v_{\sigma'} - v_\mathcal{K}] \\ \qquad + \delta t v_\mathcal{K} \displaystyle\sum_{\sigma \in \mathcal{E}_\mathcal{K}} m_\sigma \operatorname{div} c_\sigma(c_\mathcal{D}^{n+1}, U_\mathcal{D}^{n+1}) + \delta t m_\mathcal{K} v_\mathcal{K} q_\mathcal{K}^{-,n+1} c_\mathcal{K}^{n+1} \\ \quad = \delta t m_\mathcal{K} v_\mathcal{K} [q_\mathcal{K}^{+,n+1} \widehat{c}_\mathcal{K}^{n+1} + \phi_\mathcal{K} c_\mathcal{K}^n], \\ c(x, 0) = \dfrac{1}{m_\mathcal{K}} \displaystyle\int_{\mathcal{K} \in \mathcal{M}} c_0(x)\,\mathrm{d}x, \\ \text{with } D(U_\mathcal{D}, c_\mathcal{K}^n)) \nabla_\mathcal{D} c^{n+1} \cdot n_{\mathcal{K},\sigma} = 0 \text{ if } \sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{K}, \end{cases}$$

where

$$(3.26)\quad \begin{cases} K_\mathcal{K}^{\sigma,\sigma'} = \displaystyle\sum_{\sigma'' \in \mathcal{E}_\mathcal{K}} Y^{\sigma'',\sigma} \cdot \Gamma_{\mathcal{K},\sigma''} Y^{\sigma'',\sigma'}, \\ D_\mathcal{K}^{\sigma,\sigma'} = \displaystyle\sum_{\sigma'' \in \mathcal{E}_\mathcal{K}} Y^{\sigma'',\sigma} \cdot \Theta_{\mathcal{K},\sigma''} Y^{\sigma'',\sigma'}, \end{cases}$$

and

$$
(3.27)\qquad \begin{cases} Y^{\sigma,\sigma'} \text{ is given by (2.16)}, \\ \Gamma_{\mathcal{K},\sigma''} = \displaystyle\int_{\mathcal{C}_{\mathcal{K},\sigma''}} K(x, c_{\mathcal{K}}^n))\,\mathrm{d}x, \\ \Theta_{\mathcal{K},\sigma''} = \displaystyle\int_{\mathcal{C}_{\mathcal{K},\sigma''}} D(x, U^{n+1})\,\mathrm{d}x \end{cases}
$$

and $\mathcal{C}_{\mathcal{K},\sigma''}$ is the cone with vertex $x_{\mathcal{K}}$ and basis σ''.

3.4. A priori estimates. In this part we will show some a priori estimates following the same process of a priori estimates demonstrated in [6].

Lemma 3.1. *Let Ω be an open bounded connected polygonal domain of $\mathbb{R}^2$ and let $\mathcal{D}$ be a SUSHI mesh of Ω in the sense of Definition* 2.1. *Assume* (1.4), (1.6)–(1.7) *and* (1.12) *hold and that the scheme* (3.25) *has a solution $(p_{\mathcal{D},\delta t}, U_{\mathcal{D},\delta t}, c_{\mathcal{D},\delta t})$. Then there exists $C_7 > 0$ depending only on Ω, α, C_1, C_2, C_5 and $\Lambda_{\mathcal{K}}$, such that we have for all $n \in \{0, \dots, N-1\}$:*

$$
(3.28)\qquad \|\mathcal{P}_{\mathcal{M}} p_{\mathcal{D},\delta t}\|^2_{1;1,2,\mathcal{M}} + \|\nabla p_{\mathcal{D},\delta t}\|^2_{L^2(0,T;L^2(\Omega))} + \|U_{\mathcal{D},\delta t}\|^2_{L^2(0,T;L^2(\Omega))} \leqslant C_7 \|q^+ - q^-\|_{L^\infty(0,T;L^2(\Omega))}.
$$

P r o o f. In order to obtain the estimate for $\mathcal{P}_{\mathcal{M}} p_{\mathcal{D},\delta t}$ in (3.28), we use $p_{\mathcal{D},\delta t}$ as a test element:

$$
\langle -\mathrm{div}(K(x,c)\nabla p_{\mathcal{D},\delta t}), p_{\mathcal{D},\delta t}\rangle_{F^1} = \left\langle q^+ - q^-, p_{\mathcal{D},\delta t}\right\rangle_{F^1}.
$$

That gives

$$
\langle K(x,c)\nabla p_{\mathcal{D},\delta t}, \nabla p_{\mathcal{D},\delta t}\rangle_{F^1} = \left\langle q^+ - q^-, p_{\mathcal{D},\delta t}\right\rangle_{F^1}.
$$

Then, using hypothesis (1.7) and the Cauchy-Schwartz inequality, we have

$$
(3.29)\qquad \alpha \|\nabla p_{\mathcal{D},\delta t}\|^2_{L^2(0,T;L^2(\Omega))} \leqslant \|q^+ - q^-\|_{L^\infty(0,T;L^2(\Omega))} \|p_{\mathcal{D},\delta t}\|_{L^2(0,T;L^2(\Omega))}.
$$

Applying now the discrete Poincaré Inequality (2.29), we get

$$
C_5 \alpha |p_{\mathcal{D},\delta t}|^2_{X_{\mathcal{D},\delta t}} \leqslant \|q^+ - q^-\|_{L^\infty(0,T;L^2(\Omega))} \|p_{\mathcal{D},\delta t}\|_{L^2(0,T;L^2(\Omega))}.
$$

The formula (2.25) gives

$$
C_5 \alpha \|\mathcal{P}_{\mathcal{M}} p_{\mathcal{D},\delta t}\|^2_{1;1,2,\mathcal{M}} \leqslant \|q^+ - q^-\|_{L^\infty(0,T;L^2(\Omega)} \|p_{\mathcal{D},\delta t}\|_{L^2(0,T;L^2(\Omega))}.
$$

Using (2.27), we have

$$\|\mathcal{P}_{\mathcal{M}} p_{\mathcal{D},\delta t}\|_{1;1,2,\mathcal{M}} \leqslant \frac{C_2}{C_5 \alpha} \|q^+ - q^-\|_{L^\infty(0,T;L^2(\Omega))}. \tag{3.30}$$

Now, formulas (3.29) and (2.27) give

$$\alpha \|\nabla p_{\mathcal{D},\delta t}\|^2_{L^2(0,T;L^2(\Omega))} \leqslant \|q^+ - q^-\|_{L^\infty(0,T;L^2(\Omega))} C_2 \|p_{\mathcal{D},\delta t}\|_{1;1,2,\mathcal{M}}.$$

Thanks to formula (2.25)

$$\alpha \|\nabla p_{\mathcal{D},\delta t}\|^2_{L^2(0,T;L^2(\Omega))} \leqslant \|q^+ - q^-\|_{L^\infty(0,T;L^2(\Omega))} C_2 |p_{\mathcal{D},\delta t}|_{X_{\mathcal{D}}},$$

(2.28) gives

$$\|\nabla p_{\mathcal{D},\delta t}\|^2_{L^2(0,T;L^2(\Omega))} \leqslant \frac{C_2}{\alpha \sqrt{C_3}} \|q^+ - q^-\|_{L^\infty(0,T;L^2(\Omega))} \|\nabla p_{\mathcal{D},\delta t}\|_{L^2(0,T;L^2(\Omega))}.$$

Then

$$\|\nabla p_{\mathcal{D},\delta t}\|_{L^2(0,T;L^2(\Omega))} \leqslant \frac{C_2}{\alpha \sqrt{C_3}} \|q^+ - q^-\|_{L^\infty(0,T;L^2(\Omega))}. \tag{3.31}$$

The estimation of the third term U^n is deduced by (3.31) and (1.7):

$$\begin{aligned} \|U_{\mathcal{D},\delta t}\|_{L^2(0,T;L^2(\Omega))} &\leqslant \Lambda_{\mathcal{K}} \|\nabla p_{\mathcal{D},\delta t}\|_{L^2(0,T;L^2(\Omega))} \\ &\leqslant \frac{C_2 \Lambda_{\mathcal{K}}}{\alpha \sqrt{C_3}} \|q^+ - q^-\|_{L^\infty(0,T;L^2(\Omega)}. \end{aligned} \tag{3.32}$$

Finally, using (3.30), (3.31) and (3.32), we have the proof with

$$C_7 = \frac{C_2}{\alpha C_3} \Big(\sqrt{C_3} \Lambda_{\mathcal{K}} + \sqrt{C_3} + \frac{C_3}{C_5} \Big).$$

□

Lemma 3.2. *Let Ω be an open bounded connected polygonal domain of $\mathbb{R}^2$ and let $\mathcal{D}$ be a SUSHI mesh of Ω in the sense of Definition* 2.1. *Assume* (1.4), (1.6)–(1.8), (1.11) *and* (1.12) *hold and that the scheme* (3.25) *has a solution* $(p_{\mathcal{D},\delta t}, U_{\mathcal{D},\delta t}, c_{\mathcal{D},\delta t})$. *Then there exists $C_8 > 0$ depending only on Ω, $\alpha_{\mathcal{D}}$, ϕ_*, c_0, C_2, C_6 and q^+ such that we have*

$$\begin{aligned} \frac{\phi_*}{2} \|c^N_{\mathcal{D}}\|^2_{L^2(\Omega)} &+ \alpha_D \||U^n_{\mathcal{K}}|^{1/2} \nabla_{\mathcal{D}} c_{\mathcal{D},\delta t}\|^2_{L^2(0,T;L^2(\Omega))} \\ &+ (1 + \alpha_D) \|\nabla c_{\mathcal{D},\delta t}\|_{L^2(0,T;L^2(\Omega))} \leqslant C_8. \end{aligned} \tag{3.33}$$

P r o o f. Multiplying (3.14) by $c_{\mathcal{K}}^{n+1}$, we get

$$T_1 + T_2 + T_3 + T_4 = T_5 \tag{3.34}$$

with

$$\begin{aligned}
T_1 &= m_{\mathcal{K}}\phi_{\mathcal{K}}(c_{\mathcal{K}}^{n+1} - c_{\mathcal{K}}^{n})c_{\mathcal{K}}^{n+1},\\
T_2 &= c_{\mathcal{K}}^{n+1} \sum_{\sigma\in\mathcal{E}_{\rm int}} \mathcal{F}_{\mathcal{K},\sigma}^{2}(c^{n+1}),\\
T_3 &= c_{\mathcal{K}}^{n+1} \sum_{\sigma\in\mathcal{E}_{\rm int}} m_\sigma((U_{\mathcal{K},\sigma}^{n+1})^{+}c_{\mathcal{K}}^{n+1} - (U_{\mathcal{K},\sigma}^{n+1})^{-}c_{\mathcal{L}}^{n+1}),\\
T_4 &= c_{\mathcal{K}}^{n+1} m_{\mathcal{K}} q_{\mathcal{K}}^{-,n+1} c_{\mathcal{K}}^{n+1},\\
T_5 &= m_{\mathcal{K}} c_{\mathcal{K}}^{n+1} q^{+,n+1}.
\end{aligned}$$

The relation

$$(a-b)a \geqslant \frac{1}{2}(a^2 - b^2) \tag{3.35}$$

ensures

$$T_1 \geqslant \phi_{\mathcal{K}} m_{\mathcal{K}} \frac{1}{2}((c_{\mathcal{K}}^{n+1})^2 - (c_{\mathcal{K}}^{n})^2). \tag{3.36}$$

Summing formula (3.36), over $n = 0, \ldots, N-1$ with $N \geqslant 0$, we get

$$\sum_{n=0}^{n=N-1} T_1 \geqslant \phi_{\mathcal{K}} m_{\mathcal{K}}((c_{\mathcal{K}}^{N})^2 - (c_{\mathcal{K}}^{0})^2).$$

Applying the hypothesis (1.6), we have

$$\begin{aligned}
\sum_{\mathcal{K}\in\mathcal{M}} \sum_{n=0}^{n=N-1} T_1 &\geqslant \frac{1}{2} \sum_{\mathcal{K}\in\mathcal{M}} \phi_{\mathcal{K}} m_{\mathcal{K}}((c_{\mathcal{K}}^{N})^2 - (c_{\mathcal{K}}^{0})^2),\\
&\geqslant \frac{\phi_*}{2} \sum_{\mathcal{K}\in\mathcal{M}} m_{\mathcal{K}}(c_{\mathcal{K}}^{N})^2 - \frac{\phi_*^{-1}}{2} \sum_{\mathcal{K}\in\mathcal{M}} m_{\mathcal{K}}(c_{\mathcal{K}}^{0})^2,
\end{aligned}$$

which gives

$$\sum_{\mathcal{K}\in\mathcal{M}} \sum_{n=0}^{N-1} T_1 \geqslant \phi_* \|c_{\mathcal{D}}^{N}\|_{L^2(\Omega)} - \phi_*^{-1} \|c_{\mathcal{D}}^{0}\|_{L^2(\Omega)}. \tag{3.37}$$

Using (3.12) and noting that $\mathcal{F}_{\mathcal{K},\sigma}^{1}(p^{n+1}) = \mathcal{F}_{\mathcal{K},\sigma}^{1}$, we have

$$\sum_{\mathcal{K}\in\mathcal{M}} T_3 = \sum_{\mathcal{K}\in\mathcal{M}} c_{\mathcal{K}}^{n+1} \sum_{\sigma\in\mathcal{E}_{\rm int}} ((\mathcal{F}_{\mathcal{K},\sigma}^{1})^{+} c_{\mathcal{K}}^{n+1} - (\mathcal{F}_{\mathcal{K},\sigma}^{1})^{-} c_{\mathcal{L}}^{n+1}).$$

$\mathcal{F}^1_{\mathcal{K},\sigma}$ is continuous, then we have $(\mathcal{F}^1_{\mathcal{K},\sigma})^+ = -(\mathcal{F}^1_{\mathcal{K},\sigma})^-$, so

$$\begin{aligned}\sum_{\mathcal{K}\in\mathcal{M}} T_3 &= \sum_{\mathcal{K}\in\mathcal{M}} c^{n+1}_{\mathcal{K}} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} (\mathcal{F}^1_{\mathcal{K},\sigma})^+ c^{n+1}_{\mathcal{K}} - (\mathcal{F}^1_{\mathcal{K},\sigma})^- c^{n+1}_{\mathcal{L}}) \\ &= \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} ((\mathcal{F}^1_{\mathcal{K},\sigma})^+ (c^{n+1}_{\mathcal{K}} - c^{n+1}_{\mathcal{L}}) c^{n+1}_{\mathcal{K}} - (\mathcal{F}^1_{\mathcal{K},\sigma})^- (c^{n+1}_{\mathcal{L}} - c^{n+1}_{\mathcal{K}}) c^{n+1}_{\mathcal{L}}).\end{aligned}$$

Since $(\mathcal{F}^1_{\mathcal{K},\sigma})^+ + (\mathcal{F}^1_{\mathcal{K},\sigma})^- = \mathcal{F}^1_{\mathcal{K},\sigma}$,

$$\sum_{\mathcal{K}\in\mathcal{M}} T_3 = \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} \mathcal{F}^1_{\mathcal{K},\sigma} (c^{n+1}_{\mathcal{K}} - c^{n+1}_{\mathcal{L}}) c^{n+1}_{\mathcal{K}}.$$

Using (3.35), we have

$$\sum_{\mathcal{K}\in\mathcal{M}} T_3 \geqslant -\frac{1}{2} \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} \mathcal{F}^1_{\mathcal{K},\sigma} ((c^{n+1}_{\mathcal{K}})^2 - (c^{n+1}_{\mathcal{L}})^2). \tag{3.38}$$

Applying (3.4), we get

$$\begin{aligned}\sum_{\mathcal{K}\in\mathcal{M}} T_3 &\geqslant \frac{1}{2} \sum_{\mathcal{K}\in\mathcal{M}} (c^{n+1}_{\mathcal{K}})^2 \Big(- \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} \mathcal{F}^1_{\mathcal{K},\sigma} \Big) \\ &\geqslant \frac{1}{2} \sum_{\mathcal{K}\in\mathcal{M}} m_{\mathcal{K}} (c^{n+1}_{\mathcal{K}})^2 (q^{+,n+1}_{\mathcal{K}} - q^{-,n+1}_{\mathcal{K}}).\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{2} \sum_{\mathcal{K}\in\mathcal{M}} m_{\mathcal{K}} (c^{n+1}_{\mathcal{K}})^2 (q^{+,n+1}_{\mathcal{K}} - q^{-,n+1}_{\mathcal{K}}) + \sum_{\mathcal{K}\in\mathcal{M}} T_4 \\ = \frac{1}{2} \sum_{\mathcal{K}\in\mathcal{M}} m_{\mathcal{K}} (c^{n+1}_{\mathcal{K}})^2 (q^{+,n+1}_{\mathcal{K}} + q^{-,n+1}_{\mathcal{K}}) \geqslant 0,\end{aligned} \tag{3.39}$$

we deduce

$$\sum_{n=0}^{N-1} \sum_{\mathcal{K}\in\sigma} T_3 + T_4 \geqslant 0. \tag{3.40}$$

Relations (3.24) and (1.8) give

$$\begin{aligned}\delta t \sum_{n=0}^{N-1} \sum_{\mathcal{K}\in\mathcal{M}} T_2 \geqslant \alpha_{\mathcal{D}} \|\nabla_{\mathcal{D}} c_{\mathcal{D},\delta t}\|^2_{L^2(0,T;L^2(\Omega))} \\ + \alpha_D \| |U^n_{\mathcal{K}}|^{1/2} \nabla_{\mathcal{D}} c_{\mathcal{D},\delta t}\|^2_{L^2(0,T;L^2(\Omega))}.\end{aligned} \tag{3.41}$$

Using Young's inequality, we have

$$\delta t \sum_{n=0}^{N-1} \sum_{\mathcal{K}\in\mathcal{M}} T_5 \leqslant \frac{\varepsilon}{2}\|q^+\widehat{c}\|^2_{L^2(0,T;L^2(\Omega))} + \frac{1}{2\varepsilon}\|c_{\mathcal{D},\delta t}\|^2_{L^2(0,T;L^2(\Omega))}. \tag{3.42}$$

From (3.34), (3.37), (3.40), (3.41) and (3.42) we deduce

$$\begin{aligned}\frac{\phi_*}{2}\|c^N_{\mathcal{D}}\|^2_{L^2(\Omega)} - \frac{\phi_*^{-1}}{2}\|c^0_{\mathcal{D}}\|^2_{L^2(\Omega)} + \alpha_{\mathcal{D}}\|\nabla_{\mathcal{D}} c_{\mathcal{D},\delta t}\|^2_{L^2(0,T;L^2(\Omega))}& \\ + \alpha_D\||U^n_{\mathcal{K}}|^{1/2}\nabla_{\mathcal{D}} c_{\mathcal{D},\delta t}\|^2_{L^2(0,T;L^2(\Omega))}& \\ \leqslant \frac{\varepsilon}{2}\|q^+\widehat{c}\|^2_{L^2(0,T;L^2(\Omega))} + \frac{1}{2\varepsilon}\|c_{\mathcal{D},\delta t}\|^2_{L^2(0,T;L^2(\Omega))}.&\end{aligned}$$

Using (2.25), (2.27) and (2.29), we get

$$\begin{aligned}\frac{\phi_*}{2}\|c^N_{\mathcal{D}}\|^2_{L^2(\Omega)} - \frac{\phi_*^{-1}}{2}\|c^0_{\mathcal{D}}\|^2_{L^2(\Omega)} + \alpha_{\mathcal{D}}\|\nabla_{\mathcal{D}} c_{\mathcal{D},\delta t}\|^2_{L^2(0,T;L^2(\Omega))}& \\ + \alpha_D\||U^n_{\mathcal{K}}|^{1/2}\nabla_{\mathcal{D}} c_{\mathcal{D},\delta t}\|^2_{L^2(0,T;L^2(\Omega))} + \frac{C_2}{2\varepsilon C_6}\|\nabla c_{\mathcal{D},\delta t}\|_{L^2(0,T;L^2(\Omega))}& \\ \leqslant \frac{\varepsilon}{2}\|q^+\widehat{c}\|^2_{L^2(0,T;L^2(\Omega))}.&\end{aligned}$$

With $\varepsilon = C_2/(2C_6)$ we have

$$\begin{aligned}\frac{\phi_*}{2}\|c^N_{\mathcal{D}}\|^2_{L^2(\Omega)} + \alpha_D\||U^n_{\mathcal{K}}|^{1/2}\nabla_{\mathcal{D}} c_{\mathcal{D},\delta t}\|^2_{L^2(0,T;L^2(\Omega))}& \\ + (1+\alpha_D)\|\nabla c_{\mathcal{D},\delta t}\|_{L^2(0,T;L^2(\Omega))}& \\ \leqslant \frac{\phi_*^{-1}}{2}\|c^0_{\mathcal{D}}\|^2_{L^2(\Omega)} + \frac{C_2}{4C_6}\|q^+\widehat{c}\|^2_{L^2(0,T;L^2(\Omega))}.&\end{aligned}$$

□

3.5. Existence and uniqueness of $(c^n_{\mathcal{D}}; U^n_{\mathcal{D}}; p^n_{\mathcal{D}})$.

Lemma 3.3. *Let $\mathcal{D}$ be a SUSHI mesh of Ω (Ω is an open bounded connected polygonal domain of $\mathbb{R}^2$). Let $T > 0$ and δt be a time step such that $N = T/\delta t$ is an integer. Assume that* (1.3)–(1.12) *hold. Then the scheme* (3.25)–(3.27) *admits a unique solution* $(c^n_{\mathcal{D}}; U^n_{\mathcal{D}}; p^n_{\mathcal{D}})_{1\leqslant n\leqslant N}$.

P r o o f. To demonstrate this lemma we adapt the demonstration of Theorem 3.4 in [6]. □

3.6. Numerical results. In this section, we consider four numerical tests to validate the effectiveness of our algorithm. First, we apply the SUSHI-2D scheme to a diffusion equation under Neumann-like and Dirichlet boundary conditions in a nonstructured mesh (see Figure 1). Then we present the numerical results obtained by application of SUSHI-2D scheme to problem (1.1)–(1.5) on three different examples, in a nonstrictured mesh (see Figure 2).

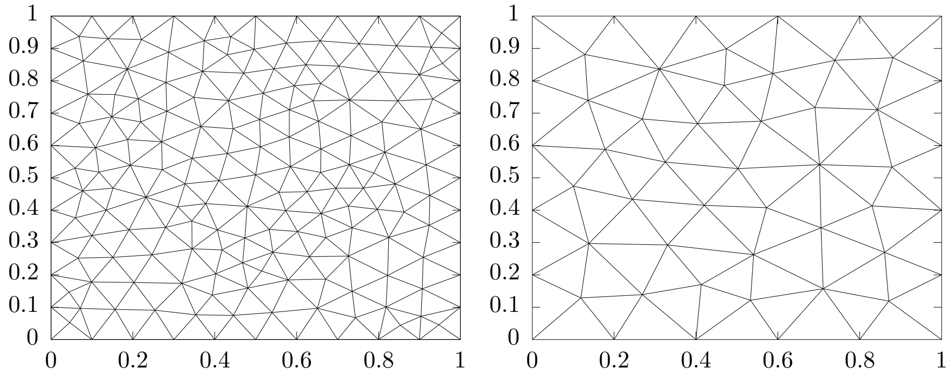

Figure 1. Unstructured mesh in the case where $\Omega = (0,1)^2$ with $h = 0.1$ (right) and $h = 0.2$ (left).

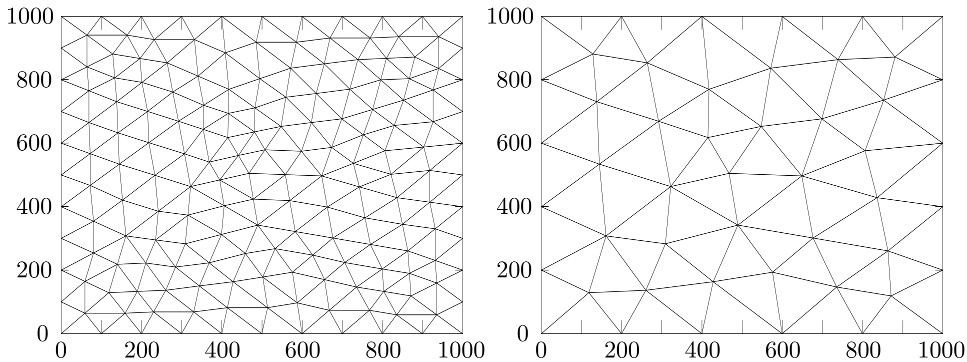

Figure 2. Unstructured mesh in the case where $\Omega = (0,1000)^2$ with $h = 236.0117$ (right) and $h = 119.6810$ (left).

3.6.1. Test 1 (Convergence of the pressure equation). In this numerical test, we are interested in demonstrating the convergence of the pressure equation with Neumann and Dirichlet boundary.

First, we consider the Neumann boundary with the following exact solution $p_1(x,y) = \cos(\pi x)\cos(\pi y)$ and the permeability

$$K_1(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Second, we study the case with Dirichlet boundary. Let $p_2(x, y) = x^2y^2(x-1)^2(y-1)^2$ and the permeability

$$K(x, y) = K_2(x, y) = \frac{1}{x^2 + y^2} \begin{bmatrix} 10^{-3}x^2 + y^2 & (10^{-3} - 1)xy \\ (10^{-3} - 1)xy & 10^{-3}y^2 + x^2 \end{bmatrix}.$$

In both cases $\Omega = (0, 1)^2$ and for the level number 5 of the mesh, the number of unknowns equals to 12880, the number of triangles equals to 5120 and the size of the mesh equals to 0.0294. Then we get the convergence tables (Table 1 and 2) in norm L^2, L^1, and L^∞.

Refinement level	$\|p - p_{\text{ext}}\|_{L^2(\Omega)}$	$\|p - p_{\text{ext}}\|_{L^1(\Omega)}$	$\|p - p_{\text{ext}}\|_{L^\infty(\Omega)}$
1	0.0379	0.0268	0.3814
2	0.0090	0.0068	0.1776
3	0.0023	0.0017	0.0878
4	6.1482e − 04	4.8368e − 04	0.0444

Table 1. Convergence results of the SUSHI on the pressure p, with $p_{\text{ext}} = p_1$ and $K = K_1$.

Refinement level	$\|p - p_{\text{ext}}\|_{L^2(\Omega)}$	$\|p - p_{\text{ext}}\|_{L^1(\Omega)}$	$\|p - p_{\text{ext}}\|_{L^\infty(\Omega)}$
1	0.0159	0.0017	0.3076
2	0.0050	4.7231e − 04	0.1941
3	0.0015	1.2424e − 04	0.1124
4	4.2783e − 04	3.1108e − 05	0.0623

Table 2. Convergence results of the SUSHI on the pressure p, with $p_{\text{ext}} = p_2$ and $K = K_2$.

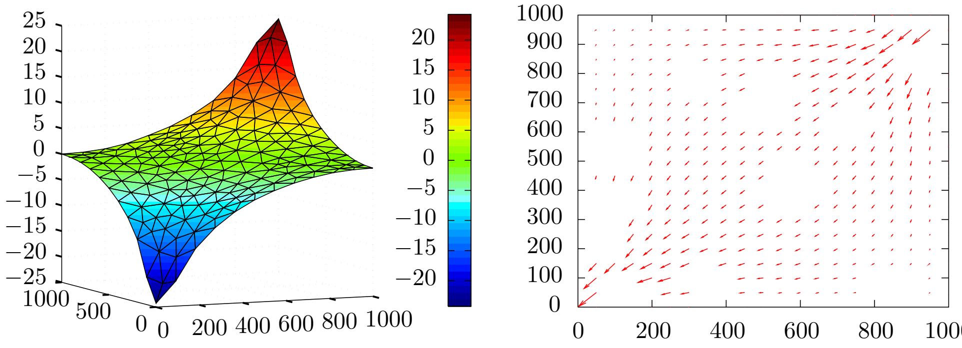

Figure 3. The pressure (left) and the gradient of the pressure (right) at $t = 3600$.

3.6.2. Test 2 (Peaceman model with continuous permeability). In the numerical tests 2 and 3, the spatial domain is $\Omega = (0, 1000) \times (0, 1000) ft^2$, and the time period is $[0, 3600]$ days. The injection and the production well are respectively located at the upper-right corner $(1000, 1000)$ and the lower-left corner $(0, 0)$ with an injection rate $q^+ = 30ft^2/\text{day}$ and a production rate $q^- = 30ft^2/\text{day}$. The viscosity of the oil is $\mu(0) = 1.0\,\text{cp}$, the injection concentration is $\widehat{c} = 1.0$.

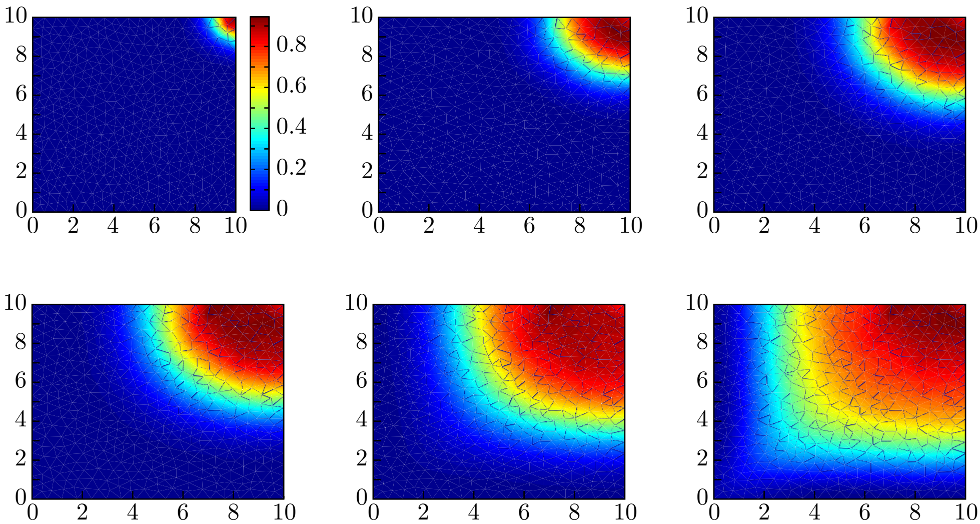

Figure 4. Surfaces plot of concentration at $t = 36$ days, $t = 108$ days, $t = 216$ days, $t \simeq 1$ year, $t \simeq 3$ years and $t \simeq 10$ years, with $\delta t = 36$ days and the mesh of the domain made of 928 triangles of maximal edge length 50 ft. Units on x, y axes in hundreds.

For the numerical test 2 (see Figures 3 and 4), the initial concentration is $c_0(x) = 0$ and the porosity of the medium is specified as $\phi(x) = 0.1$. We consider that the porous medium is homogeneous and isotropic and the permeability tensor is given by $K = 80I$. Let $M = 1$ and $\mu(c) = 1.0\,\text{cp}$. We assume that $\phi d_m = 1.0ft^2/\text{day}$, $\phi d_l = 5.0ft$ and $\phi d_t = 0.5ft$.

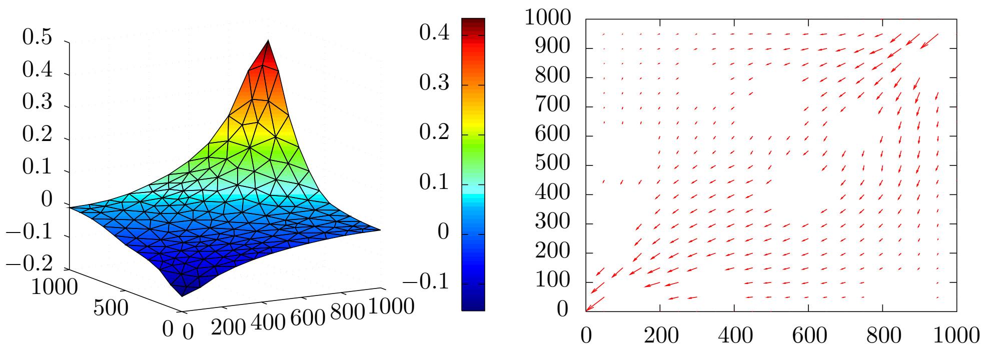

Figure 5. The pressure (left) and the gradient of the pressure (right) at $t = 3600$ days ≈ 10 years.

3.6.3. Test 3 (Peaceman model with discontinuous permeability). Now, we consider that the porous medium is homogeneous and isotropic and the permeability tensor is given by $K = (80 1_{y<500} + 20 1_{y>500})I$. Let $c_0(x) = 0$ and the porosity of the medium be specified as $\phi(x) = 0.1$, $M = 1$, and $\mu(c) = 1.0\,\text{cp}$. We assume that $\phi d_m = 1.0 ft^2/\text{day}$, $\phi d_l = 5.0 ft$, and $\phi d_t = 0.5 ft$. The results are illustrated in Figures 5, 6, and 7.

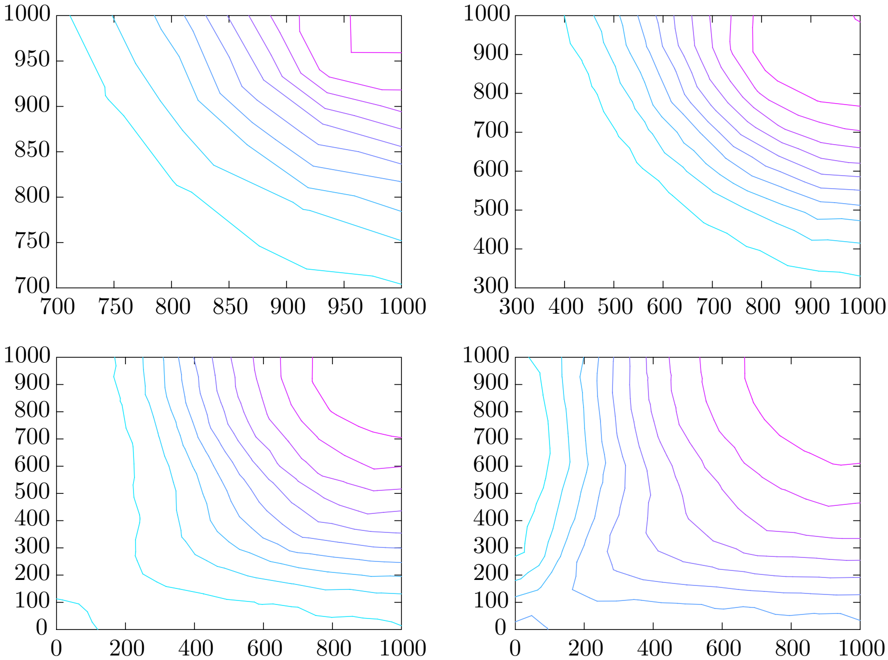

Figure 6. Surfaces plot of concentration at $t = 36$ days, $t \simeq 1$ year, $t \simeq 3$ years, and $t \simeq 10$ years, with $\delta t = 36$ days.

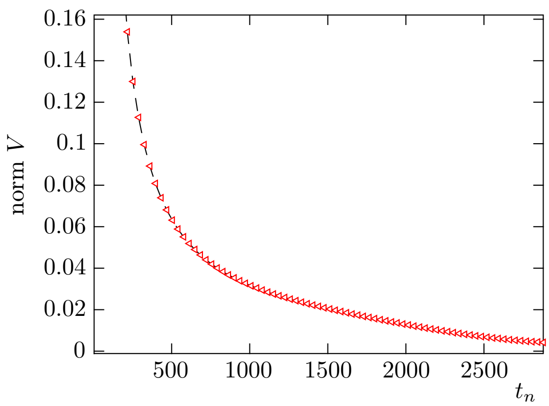

Figure 7. The norm $V = \|c^{t_{k+1}} - c^{t_k}\|_{L^2(\Omega)} / \|c^{t_{k+1}}\|_{L^2(\Omega)}$ computed from $c_0 = 0$.

3.6.4. Test 4 (The coupling between the diffusion problem in p and the Peaceman equation in c is strong). In this test case, we are interested in the case where the relation between the equation of the pressure and that of the concentration is strong with a discontinuity of the permeability $K(x, c)$, i.e. $\mu(c) = (1 + (M^{1/4} - 1)c)^{-4}$ with $M = 41$ and if $(x, y) \in [200, 400] \times [200, 400] \cup [600, 800] \times [200, 400] \cup [200, 400] \times [600, 800] \cup [600, 800] \times [600, 800]$, $K(x, y) = 80$ and else $K(x, y) = 20$. Let $c_0(x) = 0$ and the porosity of the medium be specified as $\phi(x) = 0.1$, and we assume that $\phi d_m = 0 ft^2/\text{day}$, $\phi d_l = 5.0 ft$, and $\phi d_t = 0.5 ft$. For the results see Figures 8 and 9.

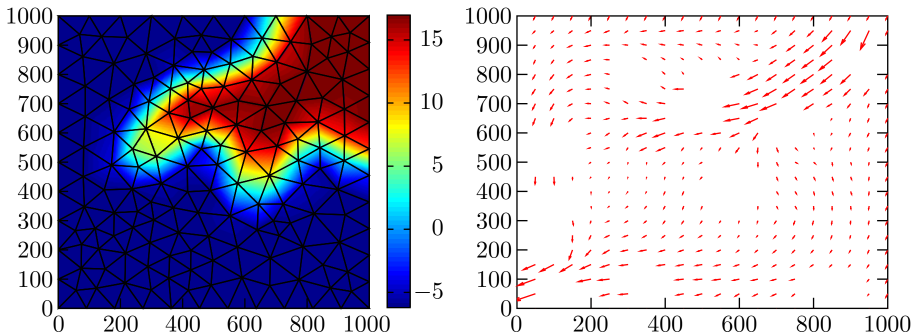

Figure 8. The pressure (left) and the gradient of the pressure (right) at $t = 3600$ days ≈ 10 years.

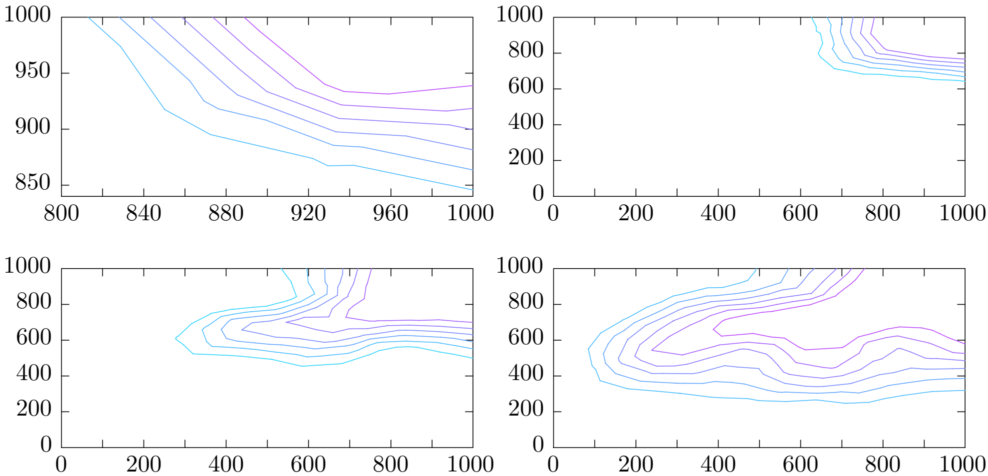

Figure 9. Surfaces plot of concentration at $t = 36$ days, $t \simeq 1$ year, $t \simeq 3$ years and $t \simeq 10$ years, with $\delta t = 36$ days.

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