A CONVERGENCE RESULT AND NUMERICAL STUDY FOR A NONLINEAR PIEZOELECTRIC MATERIAL IN A FRICTIONAL CONTACT PROCESS WITH A CONDUCTIVE FOUNDATION

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Abstract. We consider two static problems which describe the contact between a piezoelectric body and an obstacle, the so-called foundation. The constitutive relation of the material is assumed to be electro-elastic and involves the nonlinear elastic constitutive Hencky's law. In the first problem, the contact is assumed to be frictionless, and the foundation is nonconductive, while in the second it is supposed to be frictional, and the foundation is electrically conductive. The contact is modeled with the normal compliance condition with finite penetration, the regularized Coulomb law, and the regularized electrical conductivity condition. The existence and uniqueness results are provided using the theory of variational inequalities and Schauder's fixed-point theorem. We also prove that the solution of the latter problem converges towards that of the former as the friction and electrical conductivity coefficients converge towards zero. The numerical solutions of the problems are achieved by using a successive iteration technique; their convergence is also established. The numerical treatment of the contact condition is realized using an Augmented Lagrangian type formulation that leads us to use Uzawa type algorithms. Numerical experiments are performed to show that the numerical results are consistent with the theoretical analysis.

Keywords: piezoelectric body; nonlinear elastic constitutive Hencky's law; normal compliance contact condition; Coulomb's friction law; iteration method; augmented Lagrangian; Uzawa block relaxation

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1. Introduction

A piezoelectric material is a material capable of converting mechanical energy into electrical energy (this is the direct piezoelectric effect) and vice versa. These

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properties are the source of a wide range of applications for these materials, and therefore they have been extensively studied and modernized. Two applications are well known for their involvement in daily life and their production at the industrial stage: electro-acoustic ultrasonic sensors/actuators at the heart of ultrasound sonographers (the same principle is also used in sonars), and quartz resonators for the manufacture of clocks.

In addition, most structural and mechanical systems admit situations in which a deformable piezoelectric body comes into contact with other bodies, and this is a very frequent and important phenomenon in our daily lives and has attracted attention of human beings since antiquity; this explains why scientists have tried to study and model it, for example in [6], [5] a linear unilateral contact problem with Tresca's and Coulomb's friction law between an electro-elastic structure and a conductive foundation has been formulated, analyzed, and then approximated numerically, [6] is concluded by numerical simulations. In [1], [2], [9] a mathematical model which describes the frictional contact problem between an electro-viscoelastic body and a conductive foundation has been considered, the contact is modeled with normal compliance, Coulomb's law of dry friction, and a regularized electrical conductivity condition. This article discusses a piezoelectric body in contact with an electrically conductive foundation. The constitutive relation of the material is assumed to be electro-elastic involving the nonlinear elastic constitutive Hencky's law. The contact is modeled with the normal compliance condition with finite penetration, Coulomb's nonlocal friction law, and the regularized electrical conductivity condition.

Our interest is to study the problem at the limit verified by the above problem as the friction and electrical conductivity coefficients converge towards zero, which is an interesting result from the physical and numerical point of view. In Theorem 4.1, we prove that the limit solution is the solution of the frictionless contact problem of a piezoelectric body with a nonconductive foundation. Besides that, an iteration technique to solve the problems numerically is considered and implemented in numerical codes, and numerical simulations are provided to show that the numerical results are in accordance with the theoretical analysis.

2. Physical problems and their variational formulations

The physical setting is as follows. An electro-elasto-plastic body occupies, in its reference configuration, an open and bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with a sufficiently regular boundary Γ, partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that $\text{meas}(\Gamma_1) > 0$, on one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, such that $\text{meas}(\Gamma_a) > 0$, on the other hand.

To facilitate the notation, we do not explicitly indicate the dependence of several functions on the spatial variable $\boldsymbol{x} \in \overline{\Omega}$. Moreover, in the sequel, the indices i, j run between 1 and d, the summation convention over a repeated index is used and the index that follows a comma indicates the partial derivative with respect to the corresponding component of the independent variable, e.g., $u_{i,j} = \partial u_i / \partial x_j$.

We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$. We define the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i; \qquad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij} \tau_{ij}; \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} : \boldsymbol{\tau})^{1/2} \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d.$$

Throughout the paper, we adopt the following notation: $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$ for the displacement field, $\boldsymbol{\sigma}\colon \Omega \to \mathbb{S}^d$, $\boldsymbol{\sigma} = (\sigma_{ij})$ for the stress tensor. Moreover, let $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ denote the linearized strain tensor given by $\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$, $\mathbf{D}\colon \Omega \to \mathbb{R}^d$ the electric displacements field, $\mathbf{E}(\varphi) = -\nabla\varphi$ the electric vector field, where $\varphi\colon \Omega \to \mathbb{R}$ is the electrical potential. We also use the notation for the normal and tangential components of the displacements vector and stress:

$$v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}, \quad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\nu = \boldsymbol{\sigma\nu} \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \boldsymbol{\sigma}_\nu \boldsymbol{\nu},$$

where $\boldsymbol{\nu}$ denotes the unit outward normal vector on Γ.

The governing equations consist of the equilibrium equations and the constitutive relation. Since here the process is assumed to be static, the equilibrium equations are given by

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = 0, \quad \operatorname{div} \mathbf{D} = q_0 \quad \text{in } \Omega, \tag{2.1}$$

where $\operatorname{Div} \boldsymbol{\sigma} = (\sigma_{ij,j})$ and $\operatorname{div} \mathbf{D} = (\mathrm{D}_{j,j})$, $\boldsymbol{f}_0$ and q_0 are the density of the volume forces and the volume electric charges, respectively. As a description, the constitutive laws of the material can be written as

$$\boldsymbol{\sigma} = \mathfrak{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{E}^*\mathbf{E}(\varphi), \quad \mathbf{D} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \beta\mathbf{E}(\varphi) \quad \text{in } \Omega, \tag{2.2}$$

in which $\mathfrak{F}\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is the nonlinear elasticity operator that describes the behavior of Hencky's materials, see for example [3], [8], [7], given by

$$\mathfrak{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) = k_0 \operatorname{tr}(\boldsymbol{\varepsilon}(\boldsymbol{u}))\mathrm{I} + 2g(\|\overline{\boldsymbol{\varepsilon}}(\boldsymbol{u})\|^2)\overline{\boldsymbol{\varepsilon}}(\boldsymbol{u}) \quad \text{in } \Omega; \tag{2.3}$$

here $k_0 > 0$ is a material coefficient, I is the identity tensor of second order, $\operatorname{tr}(\boldsymbol{\varepsilon}) = \varepsilon_{ii}$ is the trace of $\boldsymbol{\varepsilon}$ and $\overline{\boldsymbol{\varepsilon}}$ denotes its deviatoric part

$$\overline{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon} - \frac{1}{d}\operatorname{tr}(\boldsymbol{\varepsilon})\mathrm{I}.$$

Also, $\mathcal{E}\colon \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ is a linear piezoelectric operator, $\mathcal{E}^*$ is its transpose given by $\mathcal{E}\boldsymbol{\sigma}\boldsymbol{v} = \boldsymbol{\sigma}\mathcal{E}^*\boldsymbol{v}$ for all $\boldsymbol{\sigma} \in \mathbb{S}^d$, $\boldsymbol{v} \in \mathbb{R}^d$ and $\beta\colon \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a linear electric permittivity operator. To complete the model, we have to prescribe the mechanic and electric boundary conditions. According to the physical setting, we use

$$\boldsymbol{u} = 0 \quad \text{on } \Gamma_1, \quad \boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2, \quad \varphi = 0 \quad \text{on } \Gamma_a, \quad \mathbf{D}\cdot\boldsymbol{\nu} = q_2 \quad \text{on } \Gamma_b, \tag{2.4}$$

where $\boldsymbol{f}_2$ and q_2 are the density of tractions and surface electric charges, respectively. On the surface Γ_3, we model the contact with the normal compliance condition with finite penetration (see [1]), that is

$$\left.\begin{aligned} &u_\nu - \varrho \leqslant 0, \quad \boldsymbol{\sigma}_\nu(\boldsymbol{u},\varphi) + h_\nu(\varphi-\varphi_0)p_\nu(u_\nu-\varrho) \leqslant 0,\\ &(\boldsymbol{\sigma}_\nu(\boldsymbol{u},\varphi) + h_\nu(\varphi-\varphi_0)p_\nu(u_\nu-\varrho))(u_\nu-\varrho) = 0, \end{aligned}\right\} \quad \text{on } \Gamma_3. \tag{2.5}$$

In condition (2.5), p_ν is a prescribed nonnegative function which vanishes when its argument is negative, h_ν is a positive function which depends on the difference between the potential of the foundation and the body's surface and ϱ is the gap between the body and the foundation, measured along the outward normal $\boldsymbol{\nu}$. When the tangential stress on Γ_3 is supposed to be nil, and the foundation is nonconductive, i.e.,

$$\sigma_\tau = 0, \quad \mathbf{D}\cdot\boldsymbol{\nu} = 0 \quad \text{on } \Gamma_3, \tag{2.6}$$

the resulting physical problem is a frictionless contact problem of a nonlinear electro-elastic body with a nonconductive foundation, and it may be formulated classically as follows.

Problem (P_1)**.** Find a displacement field $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}\colon \Omega \to \mathbb{S}^d$, an electric potential $\varphi\colon \Omega \to \mathbb{R}$ and an electric displacement field $\mathbf{D}\colon \Omega \to \mathbb{R}^d$ that satisfies (2.1)–(2.6).

Otherwise, in the case where the tangential stress is not nil and the foundation is conductive, we can model the frictional contact with the regularized Coulomb's law and the condition of regularized electrical conductivity as follows:

$$\left.\begin{aligned} &\left\{\begin{aligned} &\|\boldsymbol{\sigma}_\tau\| \leqslant \mu|R\boldsymbol{\sigma}_\nu(\boldsymbol{u},\varphi)|,\\ &\|\boldsymbol{\sigma}_\tau\| < \mu|R\boldsymbol{\sigma}_\nu(\boldsymbol{u},\varphi)| \Rightarrow \boldsymbol{u}_\tau = 0,\\ &\|\boldsymbol{\sigma}_\tau\| = \mu|R\boldsymbol{\sigma}_\nu(\boldsymbol{u},\varphi)| \Rightarrow \exists\,\lambda \in \mathbb{R}^+ \text{ such that } \boldsymbol{\sigma}_\tau = -\lambda\boldsymbol{u}_\tau, \end{aligned}\right.\\ &\mathbf{D}\cdot\nu = \psi(u_\nu-\varrho)\phi_L(\varphi-\varphi_0), \end{aligned}\right\} \quad \text{on } \Gamma_3. \tag{2.7}$$

In (2.7), the tangential stress in norm cannot exceed the frictional resistance's maximum, and when it reaches the limit, the body slips on the foundation while the tangential stress opposes the movement. Furthermore, μ is the friction coefficient, R is a regularization operator and ϕ_L is the truncation function, used to control the boundedness of $\varphi-\varphi_0$, where φ_0 represents the electrical potential of the foundation and L is a large positive constant; in applications, we chose $\phi_L(\varphi-\varphi_0)=\varphi-\varphi_0$. A possible choice of the surface electrical conductivity and the truncation functions is (see [6], [5], [1], [2], [9])

$$\psi(s)=k_e\psi_0(s),\quad \phi_L(s)=\begin{cases}-L & \text{if } s<-L,\\ s & \text{if } -L\leqslant s\leqslant L,\\ L & \text{if } s>L,\end{cases} \tag{2.8}$$

in which

$$\psi_0(s)=\begin{cases}0 & \text{if } s<0,\\ \delta s & \text{if } 0\leqslant s\leqslant 1/\delta,\\ 1 & \text{if } s>1/\delta,\end{cases}$$

with $k_e>0$ being the electrical conductivity coefficient, $\delta>0$ a small parameter, and L a large positive constant. It is easy to verify that ψ satisfies (h_4) which will be introduced later.

The classical formulation of the physical problem of frictional contact of a nonlinear electro-elastic body with a conductive foundation is as follows:

Problem (P_2)**.** Find a displacement field $\boldsymbol{u}\colon\,\Omega\to\mathbb{R}^d$, a stress field $\boldsymbol{\sigma}\colon\,\Omega\to\mathbb{S}^d$, an electric potential $\varphi\colon\,\Omega\to\mathbb{R}$ and an electric displacement field $\mathbf{D}\colon\,\Omega\to\mathbb{R}^d$ satisfying (2.1)–(2.5) and (2.7).

Next, we present the notation and recall some definitions needed in the sequel. The necessary functional spaces are

$$H=L^2(\Omega)^d,\quad H_1=H^1(\Omega)^d,\quad \mathcal{H}=\{\boldsymbol{\sigma}=(\sigma_{ij});\ \sigma_{ij}=\sigma_{ji}\in L^2(\Omega)\},$$
$$\mathcal{H}_1=\{\boldsymbol{\sigma}\in\mathcal{H};\ \sigma_{ij,j}\in H\},\quad \mathcal{W}=\{\mathbf{D}=(\mathrm{D}_i)\in L^2(\Omega)^d;\ \operatorname{div}\mathbf{D}\in L^2(\Omega)\}.$$

These are real Hilbert spaces endowed with the inner products

$$(\boldsymbol{u},\boldsymbol{v})_H=\int_\Omega u_iv_i\,\mathrm{d}x,\ (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}}=\int_\Omega\sigma_{ij}\tau_{ij}\,\mathrm{d}x,\ (\boldsymbol{u},\boldsymbol{v})_{H_1}=(\boldsymbol{u},\boldsymbol{v})_H+(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}},$$
$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1}=(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}}+(\operatorname{Div}\boldsymbol{\sigma},\operatorname{Div}\boldsymbol{\tau})_H,\quad (\mathbf{D},\mathbf{E})_{\mathcal{W}}=(\mathbf{D},\mathbf{E})_H+(\operatorname{div}\mathbf{D},\operatorname{div}\mathbf{E})_{L^2(\Omega)},$$

with the associated norms $\|\cdot\|_H$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{\mathcal{H}_1}$, and $\|\cdot\|_{\mathcal{W}}$, respectively.

Let $H_\Gamma=(H_\Gamma^{1/2})^d$ and let $\gamma\colon\,H_1\to H_\Gamma$ be the trace map. For every element $\boldsymbol{v}\in H_1$, we also use the notation $\boldsymbol{v}$ to note the trace $\gamma\boldsymbol{v}$ of $\boldsymbol{v}$ on Γ. Keeping in mind the boundary conditions, we introduce the spaces of the displacements and the electric potential

$$V = \{\boldsymbol{v} \in H_1;\ \boldsymbol{v} = 0 \text{ on } \Gamma_1\}, \quad W = \{\xi \in H^1(\Omega);\ \xi = 0 \text{ on } \Gamma_a\},$$

and let K be the set of admissible displacements

$$K = \{\boldsymbol{v} \in V;\ (v_\nu - \varrho) \leqslant 0 \text{ on } \Gamma_3\}.$$

Since $\text{meas}(\Gamma_1) > 0$ and $\text{meas}(\Gamma_a) > 0$, Korn's and the Friedrichs-Poincaré inequalities hold: There exist $c_K > 0$ and $c_F > 0$ which depend only on Ω, Γ_1, and Γ_a such that

$$\begin{aligned} \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} &\geqslant c_K \|\boldsymbol{v}\|_{H_1} \quad && \forall\, \boldsymbol{v} \in V, \\ \|\nabla \xi\|_H &\geqslant c_F \|\xi\|_{H^1(\Omega)} \quad && \forall\, \xi \in W. \end{aligned} \tag{2.9}$$

Therefore, the space V endowed with the inner product $(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}$ is a real Hilbert space, and its associated norm $\|\boldsymbol{v}\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}}$ is equivalent on V to the usual norm $\|\cdot\|_{H_1}$. On W we consider the inner product given by $(\varphi, \xi)_W = (\nabla\varphi, \nabla\xi)_H$. It follows from (2.9) that $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_W$ are equivalent norms on W and thus $(W, \|\cdot\|_W)$ is a real Hilbert space. By Sobolev's trace theorem, there exist two positive constants c_0 and $\tilde{c}_0$ which depend only on Ω, Γ_3, Γ_1, and Γ_a such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma)^d} \leqslant c_0 \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V, \tag{2.10}$$

$$\|\xi\|_{L^2(\Gamma_3)} \leqslant \tilde{c}_0 \|\xi\|_W \quad \forall\, \xi \in W. \tag{2.11}$$

To study the problems (P_1) and (P_2), we make the following assumptions on the data:

(h_1) The function g is continuously differentiable in $[0, \infty)$ and satisfies
- (a) $0 < g_0 \leqslant g(t) \leqslant \frac{1}{2} d k_0$,
- (b) $0 < \alpha_1 \leqslant g(t) + 2g'(t)t \leqslant \alpha_2$,

where g_0, α_1 and α_2 are given positive constants.

(h_2) The piezoelectric tensor $\mathcal{E} = (e_{ijk})$ satisfies $e_{ijk} = e_{ikj} \in L^\infty(\Omega)$.

(h_3) The electric permittivity tensor is satisfies
- (a) $\beta_{ij} = \beta_{ji} \in L^\infty(\Omega)$.
- (b) There exists $m_\beta > 0$ such that $\beta_{ij}\xi_i\xi_j \geqslant m_\beta \|\boldsymbol{\xi}\|^2$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $x \in \Omega$.

(h_4) The function $\psi\colon\ \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies
- (a) There exists $M_\psi > 0$ such that $|\psi(x, u)| \leqslant M_\psi$ for all $u \in \mathbb{R}$, a.e. $x \in \Gamma_3$.
- (b) The mapping $x \mapsto \psi(x, u)$ is measurable on Γ_3 for all $u \in \mathbb{R}$.

(c) The mapping $x \mapsto \psi(x, u) = 0$ for all $u \leqslant 0$.

(d) There exists $L_\psi > 0$ such that $|\psi(x, u_1) - \psi(x, u_2)| \leqslant L_\psi |u_1 - u_2|$ for all $u_1, u_2 \in \mathbb{R}$, a.e. $x \in \Gamma_3$.

(h_5) The functions $p_\nu \colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

(a) There exists $M_{p_\nu} > 0$ such that $|p_\nu(x, u)| \leqslant M_{p_\nu}$ for all $u \in \mathbb{R}$, a.e. $x \in \Gamma_3$.

(b) The mapping $x \mapsto p_\nu(x, u)$ is measurable on Γ_3 for all $u \in \mathbb{R}$.

(c) The mapping $x \mapsto p_\nu(x, u) = 0$ for all $u \leqslant 0$.

(d) There exists $L_{p_\nu} > 0$ such that $|p_\nu(x, u_1) - p_\nu(x, u_2)| \leqslant L_{p_\nu} |u_1 - u_2|$ for all $u_1, u_2 \in \mathbb{R}$, a.e. $x \in \Gamma_3$.

(h_6) The functions $h_\nu \colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfy

(a) There exists $M_{h_\nu} > 0$ such that $|h_\nu(x, u)| \leqslant M_{h_\nu}$ for all $u \in \mathbb{R}$, a.e. $x \in \Gamma_3$.

(b) The mapping $x \mapsto h_\nu(x, u)$ is measurable on Γ_3 for all $u \in \mathbb{R}$.

(c) There exists $L_{h_\nu} > 0$ such that $|h_\nu(x, u_1) - h_\nu(x, u_2)| \leqslant L_{h_\nu} |u_1 - u_2|$ for all $u_1, u_2 \in \mathbb{R}$, a.e. $x \in \Gamma_3$.

(h_7) The coefficient of friction μ satisfies

$$\mu \in L^\infty(\Gamma_3), \quad \mu \geqslant 0 \quad \text{and} \quad \|\mu\|_{L^\infty(\Gamma_3)} \leqslant \mu^*.$$

(h_8) The mapping $R \colon H_{\Gamma_3}^{-1/2} \to L^2(\Gamma_3)$ is linear and continuous with $\|R\| = c_R$.

(h_9) The densities of the body force, surface traction, volume electric charge, surface electric charge and the potential of the foundation have the regularity

$$\boldsymbol{f}_0 \in L^2(\Omega)^d, \quad \boldsymbol{f}_2 \in L^2(\Gamma_2)^d, \quad q_0 \in L^2(\Omega), \quad q_2 \in L^2(\Gamma_b), \quad \text{and} \quad \varphi_0 \in L^2(\Gamma_3).$$

Next, using Riesz's representation theorem, we define $\boldsymbol{f} \in V$ and $q \in W$ for all $\boldsymbol{v} \in V$ and $\xi \in W$ by

$$(\boldsymbol{f}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v} \, \mathrm{d}a, \quad (q, \xi)_W = \int_\Omega q_0 \xi \, \mathrm{d}x - \int_{\Gamma_2} q_2 \xi \, \mathrm{d}a, \tag{2.12}$$

and we define the mappings $l \colon V \times W \times W \to \mathbb{R}$, $j_1 \colon V \times W \times V \to \mathbb{R}$, $j_2 \colon V \times W \times V \to \mathbb{R}$ and $j \colon V \times W \times V \to \mathbb{R}$, respectively, by

$$l(\boldsymbol{u}, \varphi, \xi) = \int_{\Gamma_3} \psi(u_\nu - \varrho) \phi_L(\varphi - \varphi_0) \xi \, \mathrm{d}a, \tag{2.13}$$

$$j_1(\boldsymbol{u}, \varphi, \boldsymbol{v}) = \int_{\Gamma_3} h_\nu(\varphi - \varphi_0) p_\nu(u_\nu - \varrho) v_\nu \, \mathrm{d}a, \tag{2.14}$$

$$j_2(\boldsymbol{u}, \varphi, \boldsymbol{v}) = \int_{\Gamma_3} \mu |R\sigma_\nu(\boldsymbol{u}, \varphi)| \|\boldsymbol{v}_\tau\| \, \mathrm{d}a, \tag{2.15}$$

$$j(\boldsymbol{u}, \varphi, \boldsymbol{v}) = j_1(\boldsymbol{u}, \varphi, \boldsymbol{v}) + j_2(\boldsymbol{u}, \varphi, \boldsymbol{v}). \tag{2.16}$$

By virtue of the assumptions (h_4)(b), (h_5)(b), (h_6)(b), (h_7)–(h_9) it follows that the integrals in (2.13)–(2.16) are well-defined. Thus, with these notation and a standard procedure based on Green's formula, we can derive the following variational formulation of the physical problems (P_1) and (P_2).

Problem (PV_1)**.** Find a displacement field $\boldsymbol{u} \in K$ and an electric potential $\varphi \in W$ such that for all $\boldsymbol{v} \in K$ and $\xi \in W$ we have

$$(2.17)\qquad (\mathfrak{F}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_H + j_1(\boldsymbol{u}, \varphi, \boldsymbol{v}) - j_1(\boldsymbol{u}, \varphi, \boldsymbol{u}) \geqslant (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V,$$

$$(2.18)\qquad (\beta\nabla\varphi, \nabla\xi)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla\xi)_H = (q, \xi)_W.$$

Problem (PV_2)**.** Find a displacement field $\boldsymbol{u} \in K$ and an electric potential $\varphi \in W$ such that for all $\boldsymbol{v} \in K$ and $\xi \in W$ we have

$$(2.19)\qquad (\mathfrak{F}\boldsymbol{\varepsilon}(\boldsymbol{u}), \varepsilon(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_H + j(\boldsymbol{u}, \varphi, \boldsymbol{v}) - j(\boldsymbol{u}, \varphi, \boldsymbol{u}) \geqslant (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V,$$

$$(2.20)\qquad (\beta\nabla\varphi, \nabla\xi)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla\xi)_H + l(\boldsymbol{u}, \varphi, \xi) = (q, \xi)_W.$$

3. Existence and uniqueness results

The unique solvability of problems (PV_1) and (PV_2) follows from the following results.

Theorem 3.1. *Assume that* (h_1)–(h_3), (h_5)(a)–(c), (h_6)(a)–(b), *and* (h_9) *hold. Then*

(1) *Problem* (PV_1) *has at least one solution.*
(2) *Under the assumptions* (h_5)(d) *and* (h_6)(c) *there exists* $L_1^* > 0$*, such that if* $M_{h_\nu}L_{p_\nu} + L_{h_\nu}M_{p_\nu} < L_1^*$ *the solution is unique.*

Theorem 3.2. *Assume that* (h_1)–(h_4)(c), (h_5)(a)–(c), (h_6)(a)–(b) *and* (h_7)–(h_9) *hold. Then*

(1) *Problem* (PV_2) *has at least one solution.*
(2) *Under the assumptions* (h_4)(d), (h_5)(d) *and* (h_6)(c) *there exists* $L_2^* > 0$*, such that if* $M_{h_\nu}L_{p_\nu} + L_{h_\nu}M_{p_\nu} + \mu^* + L_\psi L + M_\psi < L_2^*$ *the solution is unique.*

P r o o f of Theorems 3.1 and 3.2. Let us consider the product space $X = V \times W$ endowed with the inner product

$$(\boldsymbol{x}, \boldsymbol{y})_X = (\boldsymbol{u}, \boldsymbol{v})_V + (\varphi, \xi)_W \quad \forall\, \boldsymbol{x} = (\boldsymbol{u}, \varphi),\ \boldsymbol{y} = (\boldsymbol{v}, \xi) \in X, \tag{3.1}$$

and the associated norms $\|\cdot\|_X$. Let $U = K \times W$ be a nonempty closed convex subset of X. We define the operator $A\colon X \to X$, the functions $\tilde{j}$ and $\tilde{l}$ on $X \times X$ and the element $\boldsymbol{f}_3 \in X$ by

$$(A\boldsymbol{x}, \boldsymbol{y})_X = (\mathfrak{F}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi, \boldsymbol{\varepsilon}(\boldsymbol{v}))_H + (\beta\nabla\varphi, \nabla\xi)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla\xi)_H, \tag{3.2}$$

$$\tilde{j}_1(\boldsymbol{x}, \boldsymbol{y}) = j_1(\boldsymbol{u}, \varphi, \boldsymbol{v}), \quad \tilde{j}_2(\boldsymbol{x}, \boldsymbol{y}) = j_2(\boldsymbol{u}, \varphi, \boldsymbol{v}), \quad \tilde{j}(\boldsymbol{x}, \boldsymbol{y}) = j(\boldsymbol{u}, \varphi, \boldsymbol{v}), \tag{3.3}$$

$$\tilde{l}(\boldsymbol{x}, \boldsymbol{y}) = l(\boldsymbol{u}, \varphi, \xi), \quad \boldsymbol{f}_3 = (\boldsymbol{f}, q) \in X \tag{3.4}$$

for all $\boldsymbol{x} = (\boldsymbol{u}, \varphi)$ and $\boldsymbol{y} = (\boldsymbol{v}, \xi)$ in X. With the above notation, we get the following equivalent problems.

Problem $(\widetilde{\mathrm{PV}}_1)$**.** Find $\boldsymbol{x} = (\boldsymbol{u}, \varphi) \in U$ such that

$$(A\boldsymbol{x}, \boldsymbol{y} - \boldsymbol{x})_X + \tilde{j}_1(\boldsymbol{x}, \boldsymbol{y}) - \tilde{j}_1(\boldsymbol{x}, \boldsymbol{x}) \geqslant (\boldsymbol{f}_3, \boldsymbol{y} - \boldsymbol{x})_X \quad \forall\, \boldsymbol{y} = (\boldsymbol{v}, \xi) \in U. \tag{3.5}$$

Problem $(\widetilde{\mathrm{PV}}_2)$**.** Find $\boldsymbol{x} = (\boldsymbol{u}, \varphi) \in U$ such that

$$(A\boldsymbol{x}, \boldsymbol{y} - \boldsymbol{x})_X + \tilde{j}(\boldsymbol{x}, \boldsymbol{y}) - \tilde{j}(\boldsymbol{x}, \boldsymbol{x}) + \tilde{l}(\boldsymbol{x}, \boldsymbol{y} - \boldsymbol{x}) \geqslant (\boldsymbol{f}_3, \boldsymbol{y} - \boldsymbol{x})_X \quad \forall\, \boldsymbol{y} = (\boldsymbol{v}, \xi) \in U. \tag{3.6}$$

Lemma 3.3. *The couple* $\boldsymbol{x} = (\boldsymbol{u}, \varphi) \in U$ *is a solution to Problem* (PV_1) *if and only if it is a solution of Problem* $(\widetilde{\mathrm{PV}}_1)$*.*

Lemma 3.4. *The couple* $\boldsymbol{x} = (\boldsymbol{u}, \varphi) \in U$ *is a solution to Problem* (PV_2) *if and only if it is a solution of Problem* $(\widetilde{\mathrm{PV}}_2)$*.*

The proof of Lemmas 3.3 and 3.4 can be obtained in a way similar to that in [6].

Next, we make sure that the operator A is strongly monotone and Lipschitz continuous. For this we can show, using an algebraic manipulation, that the nonlinear operator of elasticity $\mathfrak{F}$, defined in (2.3), is strongly monotone and Lipschitz continuous, i.e.,

$$(\mathfrak{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_1) - \mathfrak{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_2), \boldsymbol{\varepsilon}(\boldsymbol{u}_1) - \boldsymbol{\varepsilon}(\boldsymbol{u}_2))_{\mathcal{H}} \geqslant m_{\mathfrak{F}} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V^2 \quad \forall\, \boldsymbol{u}_1, \boldsymbol{u}_2 \in V,$$
$$\|\mathfrak{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_1) - \mathfrak{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_2)\|_{\mathcal{H}} \leqslant M_{\mathfrak{F}} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \quad \forall\, \boldsymbol{u}_1, \boldsymbol{u}_2 \in V,$$

with $m_{\mathfrak{F}} = 2\alpha_1$ and $M_{\mathfrak{F}} = 2d^2 k_0$ (see [3]). Then, we study the properties of the operator A. By algebraic manipulations similar to those used in [5], we can easily demonstrate that there exists $m_A > 0$ depending only on $\mathfrak{F}$, β, Ω, Γ_a and there exists $M_A > 0$ depending only on $\mathfrak{F}$, β and $\mathcal{E}$ such that

$$(Ax_1 - Ax_2, x_1 - x_2)_X \geqslant m_A \|x_1 - x_2\|_X^2 \quad \forall\, x_1, x_2 \in X, \tag{3.7}$$

$$\|Ax_1 - Ax_2\|_X \leqslant M_A \|x_1 - x_2\|_X \quad \forall\, x_1, x_2 \in X. \tag{3.8}$$

The continuation of the proof is obtained by using Schauder's fixed-point theorem combined with arguments of abstract variational inequalities [4], in a way similar to that in [5]. □

4. A convergence result

We are now interested in the problem of the limit verified by

$$\boldsymbol{x}_{(\mu,k_e)} = (\boldsymbol{u}_{(\mu,k_e)}, \varphi_{(\mu,k_e)})$$

(solution to Problem $(\widetilde{\mathrm{PV}}_2)$ with μ and k_e being, respectively, the friction and electrical conductivity coefficients). When we take $\mu = 0$ and $k_e = 0$ in the conditions for the limits given by (2.7), we obtain $\boldsymbol{\sigma}_\tau = 0$ and $\mathbf{D} \cdot \boldsymbol{\nu} = 0$ on Γ_3. We are then in the presence of a frictionless contact problem with a nonconductive foundation. The next theorem shows that $\boldsymbol{x}_{(\mu,k_e)} = (\boldsymbol{u}_{(\mu,k_e)}, \varphi_{(\mu,k_e)})$ converges towards the solution to Problem $(\widetilde{\mathrm{PV}}_1)$.

Theorem 4.1. *Assume that the assumptions of Theorems* 3.1 *and* 3.2 *hold. Let us denote by $\boldsymbol{x} = (\boldsymbol{u}, \varphi)$ and $\boldsymbol{x}_{(\mu,k_e)} = (\boldsymbol{u}_{(\mu,k_e)}, \varphi_{(\mu,k_e)})$ the respective solutions to problems $(\widetilde{\mathrm{PV}}_1)$ and $(\widetilde{\mathrm{PV}}_2)$. Then we have*

$$\boldsymbol{x}_{(\mu,k_e)} = (\boldsymbol{u}_{(\mu,k_e)}, \varphi_{(\mu,k_e)}) \to \boldsymbol{x} = (\boldsymbol{u}, \varphi) \quad \text{as } (\|\mu\|_{L^\infty(\Gamma_3)}, k_e) \to (0,0).$$

Proof. Let $\psi\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ be the application given in (2.8).

Taking $\boldsymbol{y} = (0,0)$ in (3.6), by using the facts that $\tilde{j}_2(\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{x}_{(\mu,k_e)}) > 0$ and $\phi_L(\varphi_{(\mu,k_e)} - \varphi_0)(\varphi_{(\mu,k_e)} - \varphi_0) > 0$, we get

$$(A\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{x}_{(\mu,k_e)})_X \leqslant (\boldsymbol{f}_3, \boldsymbol{x}_{(\mu,k_e)})_X - \tilde{j}_1(\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{x}_{(\mu,k_e)}) - l(\boldsymbol{u}_{(\mu,k_e)}, \varphi_{(\mu,k_e)}, \varphi_0).$$

Taking into account the boundedness of h_ν, p_ν and ϕ_L, (h_1) and (h_3)–(h_6), we get

$$\begin{aligned} m_{\mathfrak{F}} \|\boldsymbol{u}_{(\mu,k_e)}\|_V^2 &+ m_\beta \|\varphi_{(\mu,k_e)}\|_W^2 \\ &\leqslant \|\boldsymbol{f}\|_V \|\boldsymbol{u}_{(\mu,k_e)}\|_V + \|q\|_W \|\varphi_{(\mu,k_e)}\|_W \\ &\quad + M_{h_\nu} M_{p_\nu} \operatorname{meas}(\Gamma_3)^{1/2} c_0 \|\boldsymbol{u}_{(\mu,k_e)}\|_V + k_e c_0 L_{\psi_0} L \|\varphi_0\|_{L^2(\Gamma_3)} \|\boldsymbol{u}_{(\mu,k_e)}\|_V. \end{aligned}$$

Thus,

$$\|\boldsymbol{x}_{(\mu,k_e)}\|_X \leqslant c_1 c\big(2\|\boldsymbol{f}_3\|_X + M_{h_\nu} M_{p_\nu} \operatorname{meas}(\Gamma_3)^{1/2} c_0 + \underbrace{k_e c_0 M L_{\psi_0} L \|\varphi_0\|_{L^2(\Gamma_3)}}_{\to 0 \text{ as } k_e \to 0}\big)$$

with $c_1 = 1/m_A$ and c being a constant independent of μ and k_e. This shows that the sequence $(\boldsymbol{x}_{(\mu,k_e)}) = (\boldsymbol{u}_{(\mu,k_e)}, \varphi_{(\mu,k_e)})$ is bounded in X, hence there exist $\tilde{\boldsymbol{x}} = (\tilde{\boldsymbol{u}}, \tilde{\varphi}) \in X$ and a subsequence, denoted again $(\boldsymbol{x}_{(\mu,k_e)}) = (\boldsymbol{u}_{(\mu,k_e)}, \varphi_{(\mu,k_e)})$, such that $(\boldsymbol{x}_{(\mu,k_e)})$ converge weakly to $\tilde{\boldsymbol{x}} \in X$ as $(\|\mu\|_{L^\infty(\Gamma_3)}, k_e) \to (0,0)$. Since U is a closed convex set in a real Hilbert space X, therefore U is weakly closed, hence $\tilde{\boldsymbol{x}} \in U$. Moreover, using the compactness of the trace map $\gamma\colon X \to L^2(\Gamma_3)^d \times L^2(\Gamma_3)$, we conclude from the weak convergence of $(\boldsymbol{x}_{(\mu,k_e)})$ that $\boldsymbol{x}_{(\mu,k_e)} \to \tilde{\boldsymbol{x}}$ strongly in $L^2(\Gamma_3)^d \times L^2(\Gamma_3)$ as $(\|\mu\|_{L^\infty(\Gamma_3)}, k_e) \to (0,0)$.

Next, let us prove that $\tilde{\boldsymbol{x}} = (\tilde{\boldsymbol{u}}, \tilde{\varphi})$ is the solution to Problem $(\widetilde{\mathrm{PV}}_1)$. Using (3.3) and keeping in mind the proprieties of μ, R, ψ, ϕ_L, h_ν and p_ν, we get

$$\begin{aligned} &|\tilde{j}_1(\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{x}_{(\mu,k_e)}) - \tilde{j}_1(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{x}})| \\ &\qquad \leqslant c_0 (L_{h_\nu} M_{p_\nu} + M_{h_\nu} L_{p_\nu}) \|\boldsymbol{x}_{(\mu,k_e)}\|_X \|\boldsymbol{u}_{(\mu,k_e)} - \tilde{\boldsymbol{u}}\|_{L^2(\Gamma_3)^d} \\ &\qquad\quad + M_{h_\nu} M_{p_\nu} \operatorname{meas}(\Gamma_3)^{1/2} \|\boldsymbol{u}_{(\mu,k_e)} - \tilde{\boldsymbol{u}}\|_{L^2(\Gamma_3)}. \end{aligned} \tag{4.1}$$

$$|\tilde{j}_2(\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{x}_{(\mu,k_e)})| \leqslant \|\mu\|_{L^\infty(\Gamma_3)} c_R c_0^2 \|\boldsymbol{x}_{(\mu,k_e)}\|_X^2. \tag{4.2}$$

Moreover,

$$|\tilde{l}(\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{y} - \boldsymbol{x}_{(\mu,k_e)})| \leqslant k_e M_{\psi_0} L \operatorname{meas}(\Gamma_3)^{1/2} \|\xi - \varphi_{(\mu,k_e)}\|_{L^2(\Gamma_3)}. \tag{4.3}$$

Since $\boldsymbol{x}_{(\mu,k_e)} \to \tilde{\boldsymbol{x}}$ strongly in $L^2(\Gamma_3)^d \times L^2(\Gamma_3)$, it follows from the boundedness of $(\boldsymbol{x}_{(\mu,k_e)})$ in X, (4.1)–(4.3) that

$$\left.\begin{aligned} &\tilde{j}_1(\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{x}_{(\mu,k_e)}) \to \tilde{j}_1(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{x}}), \\ &\tilde{j}_2(\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{x}_{(\mu,k_e)}) \to 0, \\ &\tilde{l}(\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{y} - \boldsymbol{x}_{(\mu,k_e)}) \to 0, \end{aligned}\right\} \quad \text{as } (\|\mu\|_{L^\infty(\Gamma_3)}, k_e) \to (0,0).$$

We deduce from (3.6) that for all $\boldsymbol{y} = (\boldsymbol{v}, \xi) \in U$

$$\limsup_{(\|\mu\|_{L^\infty(\Gamma_3)}, k_e) \to (0,0)} (A\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{x}_{(\mu,k_e)} - \boldsymbol{y})_X \leqslant (\boldsymbol{f}_3, \tilde{\boldsymbol{x}} - \boldsymbol{y})_X + \tilde{j}_1(\tilde{\boldsymbol{x}}, \boldsymbol{y}) - \tilde{j}_1(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{x}}).$$

On the other hand, for all $\boldsymbol{y} = (\boldsymbol{v}, \xi) \in U$ we have

$$\begin{aligned}
&\limsup_{(\|\mu\|_{L^\infty(\Gamma_3)}, k_e)\to(0,0)} (A\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{x}_{(\mu,k_e)} - \tilde{\boldsymbol{x}})_X \\
&= \limsup_{(\|\mu\|_{L^\infty(\Gamma_3)}, k_e)\to(0,0)} [(A\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{x}_{(\mu,k_e)} - \boldsymbol{y})_X + (A\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{y} - \tilde{\boldsymbol{x}})_X] \\
&\leqslant \limsup_{(\|\mu\|_{L^\infty(\Gamma_3)}, k)\to(0,0)} [(A\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{x}_{(\mu,k_e)} - \boldsymbol{y})_X + \|A\boldsymbol{x}_{(\mu,k_e)}\|_X \|\boldsymbol{y} - \tilde{\boldsymbol{x}}\|_X] \\
&\leqslant (\boldsymbol{f}_3, \tilde{\boldsymbol{x}} - \boldsymbol{y})_X + \tilde{j}_1(\tilde{\boldsymbol{x}}, \boldsymbol{y}) - \tilde{j}_1(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{x}}) + \limsup_{(\|\mu\|_{L^\infty(\Gamma_3)}, k_e)\to(0,0)} \|A\boldsymbol{x}_{(\mu,k_e)}\|_X \|\boldsymbol{y} - \tilde{\boldsymbol{x}}\|_X.
\end{aligned}$$

Note that $\|A\boldsymbol{x}_{(\mu,k_e)}\|_X$ is bounded on X, hence we may substitute $\boldsymbol{y} = \tilde{\boldsymbol{x}}$ into the last inequality to obtain

$$\limsup_{(\|\mu\|_{L^\infty(\Gamma_3)}, k_e)\to(0,0)} (A\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{x}_{(\mu,k_e)} - \tilde{\boldsymbol{x}})_X \leqslant 0.$$

Therefore, by pseudo-monotonicity of A, we get

$$(A\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{x}} - \boldsymbol{y})_X \leqslant \liminf_{(\|\mu\|_{L^\infty(\Gamma_3)}, k_e)\to(0,0)} (A\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{x}_{(\mu,k_e)} - \boldsymbol{y})_X. \tag{4.4}$$

Combining (3.6) and (4.3), we deduce

$$(A\tilde{\boldsymbol{x}}, \boldsymbol{y} - \tilde{\boldsymbol{x}})_X + \tilde{j}_1(\tilde{\boldsymbol{x}}, \boldsymbol{y}) - \tilde{j}_1(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{x}}) \geqslant (\boldsymbol{f}_3, \boldsymbol{y} - \tilde{\boldsymbol{x}})_X,$$

which means that $\tilde{\boldsymbol{x}} \in U$ is a solution to Problem $(\widetilde{\mathrm{PV}}_1)$, and from the uniqueness of the solution for this variational inequality we obtain $\tilde{\boldsymbol{x}} = \boldsymbol{x}$. Since $\boldsymbol{x}$ is the unique weak limit of any subsequence of $(\boldsymbol{x}_{(\mu,k_e)})$, we deduce that the whole sequence $(\boldsymbol{x}_{(\mu,k_e)})$ is weakly convergent in X to $\boldsymbol{x}$.

Let us now prove that

$$\|\boldsymbol{x}_{(\mu,k_e)} - \boldsymbol{x}\|_X \to 0 \quad \text{as } (\|\mu\|_{L^\infty(\Gamma_3)}, k_e) \to (0,0).$$

To this end, let $\boldsymbol{x}_{(\mu,k_e)} = (\boldsymbol{u}_{(\mu,k_e)}, \varphi_{(\mu,k_e)}) \in U$ be a solution to Problem $(\widetilde{\mathrm{PV}}_2)$ and $\boldsymbol{x} = (\boldsymbol{u}, \varphi) \in U$ a solution to Problem $(\widetilde{\mathrm{PV}}_1)$, thus we have

$$\begin{aligned}
(A\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{x}_{(\mu,k_e)} - \boldsymbol{y})_X &\leqslant (\boldsymbol{f}_3, \boldsymbol{x}_{(\mu,k_e)} - \boldsymbol{y})_X \\
&\quad + \tilde{j}(\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{y}) - \tilde{j}(\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{x}_{(\mu,k_e)}) + \tilde{l}(\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{y} - \boldsymbol{x}_{(\mu,k_e)}), \\
(A\boldsymbol{x}, \boldsymbol{x} - \boldsymbol{y})_X &\leqslant (\boldsymbol{f}_3, \boldsymbol{x} - \boldsymbol{y})_X + \tilde{j}_1(\boldsymbol{x}, \boldsymbol{y}) - \tilde{j}_1(\boldsymbol{x}, \boldsymbol{x}).
\end{aligned}$$

Taking $\boldsymbol{y} = \boldsymbol{x}$ in the former inequality, $\boldsymbol{y} = \boldsymbol{x}_{(\mu,k_e)}$ in the latter and adding the two resulting inequalities, we get

$$(A\boldsymbol{x}_{(\mu,k_e)} - A\boldsymbol{x}, \boldsymbol{x}_{(\mu,k_e)} - \boldsymbol{x})_X \leqslant G + \tilde{l}(\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{x} - \boldsymbol{x}_{(\mu,k_e)}) \tag{4.5}$$

with

$$G = \tilde{j}(\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{x}) - \tilde{j}(\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{x}_{(\mu,k_e)}) + \tilde{j}_1(\boldsymbol{x}, \boldsymbol{x}_{(\mu,k_e)}) - \tilde{j}_1(\boldsymbol{x}, \boldsymbol{x}).$$

From (2.14)–(2.16), it is straightforward to show that

$$\begin{aligned} G &= \tilde{j}_1(\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{x}) - \tilde{j}_1(\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{x}_{(\mu,k_e)}) + \tilde{j}_1(\boldsymbol{x}, \boldsymbol{x}_{(\mu,k_e)}) - \tilde{j}_1(\boldsymbol{x}, \boldsymbol{x}) \\ &\quad + \tilde{j}_2(\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{x}) - \tilde{j}_2(\boldsymbol{x}_{(\mu,k_e)}, \boldsymbol{x}_{(\mu,k_e)}) \\ &\leqslant M_{h_\nu} L_{p_\nu} c_0^2 \|\boldsymbol{u}_{(\mu,k_e)} - \boldsymbol{u}\|_V^2 + L_{h_\nu} M_{p_\nu} \tilde{c}_0 c_0 \|\varphi_{(\mu,k_e)} - \varphi\|_W \|\boldsymbol{u}_{(\mu,k_e)} - \boldsymbol{u}\|_V \\ &\quad + \|\mu\|_{L^\infty(\Gamma_3)} c_R c_0^2 \|\boldsymbol{u}_{(\mu,k_e)}\|_V \|\boldsymbol{u}_{(\mu,k_e)} - \boldsymbol{u}\|_V \\ &\leqslant (M_{h_\nu} L_{p_\nu} c_0^2 + L_{h_\nu} M_{p_\nu} \tilde{c}_0 c_0) \|\boldsymbol{x}_{(\mu,k_e)} - \boldsymbol{x}\|_X^2 \\ &\quad + \|\mu\|_{L^\infty(\Gamma_3)} c_R c_0^2 \|\boldsymbol{x}_{(\mu,k_e)}\|_X \|\boldsymbol{x}_{(\mu,k_e)} - \boldsymbol{x}\|_X. \end{aligned} \tag{4.6}$$

So, we combine (4.5), (4.6) and the strong monotonicity of the operator A to deduce that

$$\begin{aligned} m_A \|\boldsymbol{x}_{(\mu,k_e)} - \boldsymbol{x}\|_X^2 &\leqslant (M_{h_\nu} L_{p_\nu} c_0^2 + L_{h_\nu} M_{p_\nu} \tilde{c}_0 c_0) \|\boldsymbol{x}_{(\mu,k_e)} - \boldsymbol{x}\|_X^2 \\ &\quad + \|\mu\|_{L^\infty(\Gamma_3)} c_R c_0^2 \|\boldsymbol{x}_{(\mu,k_e)}\|_X \|\boldsymbol{x}_{(\mu,k_e)} - \boldsymbol{x}\|_X \\ &\quad + k_e M_{\psi_0} L \tilde{c}_0 \operatorname{meas}(\Gamma_3)^{1/2} \|\boldsymbol{x}_{(\mu,k_e)} - \boldsymbol{x}\|_X. \end{aligned}$$

Thus,

$$\|\boldsymbol{x}_{(\mu,k_e)} - \boldsymbol{x}\|_X \leqslant c(\|\mu\|_{L^\infty(\Gamma_3)} + k_e)$$

with c being a constant independent of μ and k_e. This proves that $(\boldsymbol{x}_{(\mu,k_e)})$ converges strongly to $\boldsymbol{x}$ in X as μ and k_e converge towards zero. □

5. Iteration method

The iteration method for problems (PV_1) and (PV_2) consists of the following procedures, respectively,

$$\begin{cases} \text{Given an initial guess } \boldsymbol{x}_0 = (\boldsymbol{u}_0, \varphi_0) \in U, \\ \text{find } \boldsymbol{x}_{n+1} = (\boldsymbol{u}_{n+1}, \varphi_{n+1}) \in U \text{ such that} \\ B(\boldsymbol{x}_n; \boldsymbol{x}_{n+1}, \boldsymbol{y} - \boldsymbol{x}_{n+1}) + \tilde{j}_1(\boldsymbol{x}_n, \boldsymbol{y}) - \tilde{j}_1(\boldsymbol{x}_n, \boldsymbol{x}_{n+1}) \geqslant (\boldsymbol{f}_3, \boldsymbol{y} - \boldsymbol{x}_{n+1})_X. \end{cases} \tag{5.1}$$

$$\begin{cases} \text{Given an initial guess } \boldsymbol{x}_0 = (\boldsymbol{u}_0, \varphi_0) \in U, \\ \text{find } \boldsymbol{x}_{n+1} = (\boldsymbol{u}_{n+1}, \varphi_{n+1}) \in U \text{ such that} \\ B(\boldsymbol{x}_n; \boldsymbol{x}_{n+1}, \boldsymbol{y} - \boldsymbol{x}_{n+1}) + \tilde{j}(\boldsymbol{x}_n, \boldsymbol{y}) - \tilde{j}(\boldsymbol{x}_n, \boldsymbol{x}_{n+1}) + \tilde{l}(\boldsymbol{x}_n, \boldsymbol{y} - \boldsymbol{x}_{n+1}) \\ \qquad\qquad \geqslant (\boldsymbol{f}_3, \boldsymbol{y} - \boldsymbol{x}_{n+1})_X \end{cases} \tag{5.2}$$

for all $\boldsymbol{y} = (\boldsymbol{v}, \xi)$ in U, where the operator $B\colon\ U \times X \times X \to \mathbb{R}$ is defined by

$$\begin{aligned} B(\boldsymbol{x}; \boldsymbol{y}, \boldsymbol{z}) &= (k_0 \operatorname{tr}(\boldsymbol{\varepsilon}(\boldsymbol{v}))\mathrm{I} + 2g(\|\overline{\boldsymbol{\varepsilon}}(\boldsymbol{u})\|^2)\overline{\boldsymbol{\varepsilon}}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} \\ &\quad + (\mathcal{E}^* \nabla \eta, \boldsymbol{\varepsilon}(\boldsymbol{w}))_H + (\beta \nabla \eta, \nabla \xi)_H - (\mathcal{E} \boldsymbol{\varepsilon}(\boldsymbol{v}), \nabla \xi)_H \end{aligned}$$

for all $\boldsymbol{x} = (\boldsymbol{u}, \varphi)$, $\boldsymbol{y} = (\boldsymbol{v}, \eta)$ and $\boldsymbol{z} = (\boldsymbol{w}, \xi) \in X$. For a fixed $\boldsymbol{x} = (\boldsymbol{u}, \varphi) \in X$, it is clear that $(\boldsymbol{y}, \boldsymbol{z}) \mapsto B(\boldsymbol{x}; \boldsymbol{y}, \boldsymbol{z})$ is a bilinear form and arguments similar to those used in Section 3 show that

$$\begin{cases} B(\boldsymbol{x}; \boldsymbol{y}, \boldsymbol{y}) \geqslant m_A \|\boldsymbol{y}\|_X^2 & \forall\, \boldsymbol{y} = (\boldsymbol{v}, \eta) \in X, \\ |B(\boldsymbol{x}; \boldsymbol{y}, \boldsymbol{z})| \leqslant M_A \|\boldsymbol{y}\|_X \|\boldsymbol{z}\|_X & \forall\, \boldsymbol{y} = (\boldsymbol{v}, \eta),\ \boldsymbol{z} = (\boldsymbol{w}, \xi) \in X. \end{cases}$$

We have the following convergence results.

Theorem 5.1. *Under the assumptions of Theorem* 3.1, *the iteration method* (5.1) *converges to the solution of Problem* $(\widetilde{\mathrm{PV}}_1)$, *i.e.,*

$$\|\boldsymbol{x}_n - \boldsymbol{x}\|_X \to 0 \quad \text{as } n \to \infty,$$

where $\boldsymbol{x}$ *is the unique solution to Problem* $(\widetilde{\mathrm{PV}}_1)$.

Theorem 5.2. *Under the assumptions of Theorem* 3.2, *the iteration method* (5.2) *converges to the solution of Problem* $(\widetilde{\mathrm{PV}}_2)$, *i.e.,*

$$\|\boldsymbol{x}_n - \boldsymbol{x}\|_X \to 0 \quad \text{as } n \to \infty,$$

where $\boldsymbol{x}$ *is the unique solution to Problem* $(\widetilde{\mathrm{PV}}_2)$.

Proof of Theorem 5.1. Let $\boldsymbol{x}_n = (\boldsymbol{u}_n, \varphi_n) \in U$ the solution of the problem (5.1), thus we have for all $\boldsymbol{y} = (\boldsymbol{v}, \xi) \in U$

$$B(\boldsymbol{x}_{n-1}; \boldsymbol{x}_n, \boldsymbol{y} - \boldsymbol{x}_n) + \tilde{j}_1(\boldsymbol{x}_{n-1}, \boldsymbol{y}) - \tilde{j}_1(\boldsymbol{x}_{n-1}, \boldsymbol{x}_n) \geqslant (\boldsymbol{f}_3, \boldsymbol{y} - \boldsymbol{x}_n)_X. \tag{5.3}$$

Taking $\boldsymbol{y} = (0, 0)$ in (5.3), we get

$$B(\boldsymbol{x}_{n-1}; \boldsymbol{x}_n, \boldsymbol{x}_n) \leqslant (\boldsymbol{f}_3, \boldsymbol{x}_n)_X - \tilde{j}_1(\boldsymbol{x}_{n-1}, \boldsymbol{x}_n).$$

It follows from the properties of B, the boundedness of h_ν and p_ν and (2.10) that

$$\|\boldsymbol{x}_n\|_X \leqslant c_1(\|\boldsymbol{f}_3\|_X + c_0 M_{h_\nu} M_{p_\nu} \operatorname{meas}(\Gamma_3)^{1/2}).$$

Thereafter, as the sequence $(\boldsymbol{x}_n)_{n \geqslant 1}$ is bounded in X, there exist $\tilde{\boldsymbol{x}} = (\tilde{\boldsymbol{u}}, \tilde{\varphi}) \in X$ and a subsequence, denoted again $(\boldsymbol{x}_n)_{n \geqslant 1}$, such that $(\boldsymbol{x}_n)_{n \geqslant 1}$ converges weakly to $\tilde{\boldsymbol{x}} \in X$. Since U is a closed convex set in a real Hilbert space X, therefore U is weakly closed, and hence $\tilde{\boldsymbol{x}} \in U$. Moreover, due to the compactness of the trace map $\gamma\colon X \to L^2(\Gamma_3)^d \times L^2(\Gamma_3)$, it follows from the weak convergence of $(\boldsymbol{x}_n)_{n \geqslant 1}$ that $\boldsymbol{x}_n \to \tilde{\boldsymbol{x}}$ strongly in $L^2(\Gamma_3)^d \times L^2(\Gamma_3)$.

Next, let us prove that $\tilde{\boldsymbol{x}}$ is the solution of Problem $(\widetilde{\mathrm{PV}}_1)$. Using (3.3)–(3.4) and keeping in mind the properties of h_ν and p_ν, we get

$$
(5.4) \qquad |\tilde{j}_1(\boldsymbol{x}_{n-1}, \boldsymbol{x}_n) - \tilde{j}_1(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{x}})| \leqslant M_{h_\nu} L_{p_\nu} \|\boldsymbol{u}_{n-1} - \tilde{\boldsymbol{u}}\|_{L^2(\Gamma_3)^d} \|\boldsymbol{u}_n\|_{L^2(\Gamma_3)^d} + M_{h_\nu} M_{p_\nu} \operatorname{meas}(\Gamma_3)^{1/2} \|\boldsymbol{u}_n - \tilde{\boldsymbol{u}}\|_{L^2(\Gamma_3)^d} + L_{h_\nu} M_{p_\nu} \|\varphi_{n-1} - \tilde{\varphi}\|_{L^2(\Gamma_3)} \|\tilde{\boldsymbol{u}}\|_{L^2(\Gamma_3)^d}.
$$

Since $\boldsymbol{x}_n \to \tilde{\boldsymbol{x}}$ strongly in $L^2(\Gamma_3)^d \times L^2(\Gamma_3)$, it follows from (5.4) that

$$
\tilde{j}_1(\boldsymbol{x}_{n-1}, \boldsymbol{x}_n) \to \tilde{j}_1(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{x}}), \quad \text{as } n \to \infty.
$$

We deduce from (5.1) that for all $\boldsymbol{y} = (\boldsymbol{v}, \xi) \in U$

$$
\limsup_{n\to\infty} B(\boldsymbol{x}_{n-1}; \boldsymbol{x}_n, \boldsymbol{x}_n - \boldsymbol{y}) \leqslant (\boldsymbol{f}_3, \boldsymbol{x} - \boldsymbol{y})_X + \tilde{j}_1(\tilde{\boldsymbol{x}}, \boldsymbol{y}) - \tilde{j}_1(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{x}}).
$$

On the other hand, we have for all $\boldsymbol{y} = (\boldsymbol{v}, \xi) \in U$

$$
\begin{aligned}
&\limsup_{n\to\infty} B(\boldsymbol{x}_{n-1}; \boldsymbol{x}_n, \boldsymbol{x}_n - \tilde{\boldsymbol{x}}) \\
&\qquad = \limsup_{n\to\infty} [B(\boldsymbol{x}_{n-1}; \boldsymbol{x}_n, \boldsymbol{x}_n - \boldsymbol{y}) + B(\boldsymbol{x}_{n-1}; \boldsymbol{x}_n, \boldsymbol{y} - \tilde{\boldsymbol{x}})] \\
&\qquad \leqslant \limsup_{n\to\infty} [B(\boldsymbol{x}_{n-1}; \boldsymbol{u}_n, \boldsymbol{x}_n - \boldsymbol{y}) + M_A \|\boldsymbol{x}_n\|_X \|y - \tilde{\boldsymbol{x}}\|_X] \\
&\qquad \leqslant (\boldsymbol{f}_3, \tilde{\boldsymbol{x}} - \boldsymbol{y})_X + \tilde{j}_1(\tilde{\boldsymbol{x}}, \boldsymbol{y}) - \tilde{j}_1(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{x}}) + \limsup_{n\to\infty} M_A \|\boldsymbol{x}_n\|_X \|\boldsymbol{y} - \tilde{\boldsymbol{x}}\|_X.
\end{aligned}
$$

Note that $\|\boldsymbol{x}_n\|_X$ is bounded on X, so we may then substitute $\boldsymbol{y} = \tilde{\boldsymbol{x}}$ into the last inequality to obtain

$$
\limsup_{n\to\infty} B(\boldsymbol{x}_{n-1}; \boldsymbol{x}_n, \boldsymbol{x}_n - \tilde{\boldsymbol{x}}) \leqslant 0.
$$

Therefore, by pseudo-monotonicity of B, we get

$$
(5.5) \qquad B(\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{x}} - \boldsymbol{y}) \leqslant \liminf_{n\to\infty} B(\boldsymbol{x}_{n-1}; \boldsymbol{x}_n, \boldsymbol{x}_n - \boldsymbol{y}).
$$

Combining (5.1) and (5.5), we deduce

$$
B(\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{x}}, \boldsymbol{y} - \tilde{\boldsymbol{x}}) + \tilde{j}_1(\tilde{\boldsymbol{x}}, \boldsymbol{y}) - \tilde{j}_1(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{x}}) \geqslant (\boldsymbol{f}_3, \boldsymbol{y} - \tilde{\boldsymbol{x}})_X.
$$

This means that $\tilde{\boldsymbol{x}} \in U$ is a solution of Problem $(\widetilde{\mathrm{PV}}_1)$, and from the uniqueness of the solution for this variational inequality we obtain $\tilde{\boldsymbol{x}} = \boldsymbol{x}$. Since $\boldsymbol{x}$ is the unique weak limit of any subsequence of $(\boldsymbol{x}_n)_{n\geqslant 1}$, we deduce that the whole sequence $(\boldsymbol{x}_n)_{n\geqslant 1}$ is weakly convergent in X to $\boldsymbol{x}$.

Let us now prove that

$$\|\boldsymbol{x}_n - \boldsymbol{x}\|_X \to 0 \quad \text{as } n \to \infty.$$

To this end, let $\boldsymbol{x}_n \in U$ be a solution of (5.1), and $\boldsymbol{x} \in U$ a solution of the problem (PV_1). By using the strong monotonicity of A, we get

$$m_A\|\boldsymbol{x}_n - \boldsymbol{x}\|_X^2 \leqslant (A\boldsymbol{x}_n - A\boldsymbol{x}, \boldsymbol{x}_n - \boldsymbol{x})_X = (A\boldsymbol{x}_n, \boldsymbol{x}_n - \boldsymbol{x})_X - (A\boldsymbol{x}, \boldsymbol{x}_n - \boldsymbol{x})_X.$$

Using (3.5) with $\boldsymbol{y} = \boldsymbol{x}_n$ and the fact that $(A\boldsymbol{x}, \boldsymbol{y})_X = B(\boldsymbol{x}; \boldsymbol{x}, \boldsymbol{y})$ for all $\boldsymbol{x}$, $\boldsymbol{y}$ in X, we obtain

$$m_A\|\boldsymbol{x}_n - \boldsymbol{x}\|_X^2 \leqslant B(\boldsymbol{x}_n; \boldsymbol{x}_n, \boldsymbol{x}_n - \boldsymbol{x}) - (\boldsymbol{f}_3, \boldsymbol{x}_n - \boldsymbol{x})_X + \tilde{j}_1(\boldsymbol{x}, \boldsymbol{x}_n) - \tilde{j}_1(\boldsymbol{x}, \boldsymbol{x}).$$

We conclude by using the boundedness of h_ν and p_ν, the fact that $(\boldsymbol{x}_n)_{n\geqslant 1}$ is bounded, weakly convergent to $\boldsymbol{x}$ in X, and the continuity properties of B, $\tilde{j}_1$ and $(\boldsymbol{f}_3, \cdot)_X$, that $\boldsymbol{x}_n \to \boldsymbol{x}$ strongly in X. □

P r o o f of Theorem 5.2. To show that the solution of (5.2) converges strongly towards that of Problem $(\widetilde{\mathrm{PV}}_2)$, we will follow the same steps as above. We start by the weak convergence, for this reason, we show that the solution is bounded.

By taking $\boldsymbol{y} = (0, 0)$ in (5.2), and using the fact that $\tilde{j}_2(\boldsymbol{x}_{n-1}, \boldsymbol{x}_n) > 0$, the properties of B, the boundedness of h_ν, p_ν, ψ and ϕ_L, (2.10)–(2.11), we get

$$\|\boldsymbol{x}_n\|_X \leqslant c_1(\|\boldsymbol{f}_3\|_X + c_0 M_{h_\nu} M_{p_\nu} \operatorname{meas}(\Gamma_3)^{1/2} + M_\psi L \tilde{c}_0 \operatorname{meas}(\Gamma_3)^{1/2}).$$

Subsequently, there exists a subsequence $(\boldsymbol{x}_n)_{n\geqslant 1}$ such that $(\boldsymbol{x}_n)_{n\geqslant 1}$ converges weakly to $\tilde{\boldsymbol{x}} = (\tilde{\boldsymbol{u}}, \tilde{\varphi}) \in X$. By taking advantage of the properties of U (weakly closed), we get $\tilde{\boldsymbol{x}} \in U$. Moreover, from the compactness of the trace map $\gamma\colon X \to L^2(\Gamma_3)^d \times L^2(\Gamma_3)$, we obtain $\boldsymbol{x}_n \to \tilde{\boldsymbol{x}}$ strongly in $L^2(\Gamma_3)^d \times L^2(\Gamma_3)$. Next, using (3.3)–(3.4) and keeping in mind the properties of μ, R, h_ν, p_ν, ψ, and ϕ_L, we get

$$\begin{aligned}
(5.6)\quad |\tilde{j}_1(\boldsymbol{x}_{n-1}, \boldsymbol{x}_n) - \tilde{j}_1(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{x}})| &\leqslant M_{h_\nu} L_{p_\nu} \|\boldsymbol{u}_{n-1} - \tilde{\boldsymbol{u}}\|_{L^2(\Gamma_3)^d} \|\boldsymbol{u}_n\|_{L^2(\Gamma_3)^d} \\
&\quad + M_{h_\nu} M_{p_\nu} \operatorname{meas}(\Gamma_3)^{1/2} \|\boldsymbol{u}_n - \tilde{\boldsymbol{u}}\|_{L^2(\Gamma_3)^d} \\
&\quad + L_{h_\nu} M_{p_\nu} \|\varphi_{n-1} - \tilde{\varphi}\|_{L^2(\Gamma_3)} \|\tilde{\boldsymbol{u}}\|_{L^2(\Gamma_3)^d}, \\
(5.7)\quad |\tilde{l}(\boldsymbol{x}_{n-1}, \boldsymbol{y} - \boldsymbol{x}_n) - \tilde{l}(\tilde{\boldsymbol{x}}, \boldsymbol{y} - \tilde{\boldsymbol{x}})| &\leqslant M_\psi \|\varphi_{n-1} - \tilde{\varphi}\|_{L^2(\Gamma_3)} \|\xi - \varphi_n\|_{L^2(\Gamma_3)} \\
&\quad + L_\psi L \|\boldsymbol{u}_{n-1} - \tilde{\boldsymbol{u}}\|_{L^2(\Gamma_3)^d} \|\xi - \varphi_n\|_{L^2(\Gamma_3)} \\
&\quad + M_\psi L \operatorname{meas}(\Gamma_3)^{1/2} \|\varphi_n - \tilde{\varphi}\|_{L^2(\Gamma_3)}.
\end{aligned}$$

We deduce from (5.2), (5.4), (5.6), and (5.7) that for all $\boldsymbol{y} = (\boldsymbol{v}, \xi) \in U$

$$\limsup_{n\to\infty} B(\boldsymbol{x}_{n-1}; \boldsymbol{x}_n, \boldsymbol{x}_n - \boldsymbol{y}) \leqslant (\boldsymbol{f}_3, \boldsymbol{x} - \boldsymbol{y})_X + \tilde{j}(\tilde{\boldsymbol{x}}, \boldsymbol{y}) - \tilde{j}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{x}}) + \tilde{l}(\tilde{\boldsymbol{x}}, \boldsymbol{y} - \tilde{\boldsymbol{x}}).$$

On the other hand, we have for all $\boldsymbol{y} = (\boldsymbol{v}, \xi) \in U$

$$\limsup_{n\to\infty} B(\boldsymbol{x}_{n-1}; \boldsymbol{x}_n, \boldsymbol{x}_n - \tilde{\boldsymbol{x}}) \leqslant (\boldsymbol{f}_3, \tilde{\boldsymbol{x}} - \boldsymbol{y})_X + \tilde{j}(\tilde{\boldsymbol{x}}, \boldsymbol{y}) - \tilde{j}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{x}}) + \tilde{l}(\tilde{\boldsymbol{x}}, \boldsymbol{y} - \tilde{\boldsymbol{x}}) + \limsup_{n\to\infty} M_A \|\boldsymbol{x}_n\|_X \|\boldsymbol{y} - \tilde{\boldsymbol{x}}\|_X.$$

Note that $\|\boldsymbol{x}_n\|_X$ is bounded on X, we may then substitute $\boldsymbol{y} = \tilde{\boldsymbol{x}}$ into the last inequality to obtain

$$\limsup_{n\to\infty} B(\boldsymbol{x}_{n-1}; \boldsymbol{x}_n, \boldsymbol{x}_n - \tilde{\boldsymbol{x}}) \leqslant 0.$$

The pseudo-monotonicity of B combined with (5.2) leads to

$$B(\tilde{\boldsymbol{x}}; \tilde{\boldsymbol{x}}, \boldsymbol{y} - \tilde{\boldsymbol{x}}) + \tilde{j}(\tilde{\boldsymbol{x}}, \boldsymbol{y}) - \tilde{j}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{x}}) + \tilde{l}(\tilde{\boldsymbol{x}}, \boldsymbol{y} - \tilde{\boldsymbol{x}}) \geqslant (\boldsymbol{f}_3, \boldsymbol{y} - \tilde{\boldsymbol{x}})_X.$$

From the uniqueness of the solution for this variational inequality we obtain $\tilde{\boldsymbol{x}} = \boldsymbol{x}$. Hence, the whole sequence $(\boldsymbol{x}_n)_{n\geqslant 1}$ is weakly convergent in X to $\boldsymbol{x}$. To obtain the strong convergence, we take advantage of the strong monotonicity of the operator A, so, for $\boldsymbol{x}_n \in U$ a solution of (5.2) and $\boldsymbol{x} \in U$ a solution of the problem $(\widetilde{\mathrm{PV}}_2)$ we get

$$m_A \|\boldsymbol{x}_n - \boldsymbol{x}\|_X^2 \leqslant (A\boldsymbol{x}_n - A\boldsymbol{x}, \boldsymbol{x}_n - \boldsymbol{x})_X = (A\boldsymbol{x}_n, \boldsymbol{x}_n - \boldsymbol{x})_X - (A\boldsymbol{x}, \boldsymbol{x}_n - \boldsymbol{x})_X.$$

Using (3.6) with $\boldsymbol{y} = \boldsymbol{x}_n$ and the fact that $(A\boldsymbol{x}, \boldsymbol{y})_X = B(\boldsymbol{x}; \boldsymbol{x}, \boldsymbol{y})$ for all $\boldsymbol{x}$, $\boldsymbol{y}$ in X, we obtain

$$m_A \|\boldsymbol{x}_n - \boldsymbol{x}\|_X^2 \leqslant B(\boldsymbol{x}_n; \boldsymbol{x}_n, \boldsymbol{x}_n - \boldsymbol{x}) - (\boldsymbol{f}_3, \boldsymbol{x}_n - \boldsymbol{x})_X + \tilde{j}(\boldsymbol{x}, \boldsymbol{x}_n) - \tilde{j}(\boldsymbol{x}, \boldsymbol{x}) + \tilde{l}(\boldsymbol{x}, \boldsymbol{x}_n - \boldsymbol{x}).$$

We conclude by using the boundedness of h_ν, p_ν, ψ and ϕ_L, the fact that $(\boldsymbol{x}_n)_{n\geqslant 1}$ is bounded, weakly convergent to $\boldsymbol{x}$ in X, and the continuity properties of B, $\tilde{j}$, $\tilde{l}$, and $(\boldsymbol{f}_3, \cdot)_X$, to get $\boldsymbol{x}_n \to \boldsymbol{x}$ strongly in X. □

The bilinear form B is positive definite, but not symmetric since the global matrix of piezoelectricity is antisymmetric. However, it is possible to use another equivalent variational formulation characterized by a symmetric bilinear form $\check{B}$. To this end, we will need the following additional step. By subtracting the equation (2.18) from the inequality (2.17), we obtain

$$(6.1)\quad (\check{A}\boldsymbol{x}, \boldsymbol{y}-\boldsymbol{x})_X + \tilde{j}(\boldsymbol{x}, \boldsymbol{y}) - \tilde{j}(\boldsymbol{x}, \boldsymbol{x}) + \check{l}(\boldsymbol{x}, \boldsymbol{y}-\boldsymbol{x}) \geqslant (\boldsymbol{f}_3, \boldsymbol{y}-\boldsymbol{x})_X \quad \forall\, \boldsymbol{y} = (\boldsymbol{v}, \xi) \in U,$$

with

$$(6.2)\quad (\check{A}\boldsymbol{x}, \boldsymbol{y})_X = (\mathfrak{F}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi, \boldsymbol{\varepsilon}(\boldsymbol{v}))_H - (\beta\nabla\varphi, \nabla\xi)_H + (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla\xi)_H,$$

$$(6.3)\quad (\check{\boldsymbol{f}}_3, \boldsymbol{y})_X = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v}\,\mathrm{d}a - \int_\Omega q_0\xi\,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_2} q_2\xi\,\mathrm{d}a,$$

$$(6.4)\quad \check{l}(\boldsymbol{x}, \boldsymbol{y}) = -\int_{\Gamma_3} \psi(u_\nu - \varrho)\phi_L(\varphi - \varphi_0)\xi\,\mathrm{d}a$$

for all $\boldsymbol{y} = (\boldsymbol{v}, \xi) \in X$.

Lemma 6.1. *The variational formulations* (6.1) *and* (3.6) *are equivalent.*

P r o o f. Let $\boldsymbol{x} = (\boldsymbol{u}, \varphi) \in U$ be the solution to the variational inequality (3.6). Since for all $\xi \in W$ we have $-\xi \in W$, we get for all $(\boldsymbol{v}, \xi) \in U$

$$(A\boldsymbol{x}, (\boldsymbol{v}, -\xi) - \boldsymbol{x})_X + \tilde{j}(\boldsymbol{x}, (\boldsymbol{v}, -\xi)) - \tilde{j}(\boldsymbol{x}, \boldsymbol{x}) + \tilde{l}(\boldsymbol{x}, (\boldsymbol{v}, -\xi) - \boldsymbol{x}) \geqslant (\boldsymbol{f}_3, (\boldsymbol{v}, -\xi) - \boldsymbol{x})_X.$$

For all $(\boldsymbol{v}, \xi) \in X$ we have

$$(\check{A}\boldsymbol{x}, \boldsymbol{y})_X = (A\boldsymbol{x}, (\boldsymbol{v}, -\xi))_X,\ \check{l}(\boldsymbol{x}, \boldsymbol{y}) = \tilde{l}(\boldsymbol{x}, (\boldsymbol{v}, -\xi)) \text{ and } (\check{\boldsymbol{f}}_3, \boldsymbol{y})_X = (f_3, (\boldsymbol{v}, -\xi))_X.$$

So $\boldsymbol{x} = (\boldsymbol{u}, \varphi)$ is also a solution of (6.1). Similarly, we show that the solution of (3.6) is also a solution of (6.1). □

Next, by applying the iteration method (5.2) presented in the preceding section to the variational inequality (6.1), we get the iterative problem

$$(6.5)\quad \begin{cases} \text{Given an initial guess } \boldsymbol{x}_0 = (\boldsymbol{u}_0, \varphi_0) \in U, \\ \text{find } \boldsymbol{x}_{n+1} = (\boldsymbol{u}_{n+1}, \varphi_{n+1}) \in U \text{ such that} \\ \check{B}(\boldsymbol{x}_n; \boldsymbol{x}_{n+1}, \boldsymbol{y} - \boldsymbol{x}_{n+1}) + \tilde{j}(\boldsymbol{x}_n, \boldsymbol{y}) - \tilde{j}(\boldsymbol{x}_n, \boldsymbol{x}_{n+1}) + \check{l}(\boldsymbol{x}_n, \boldsymbol{y} - \boldsymbol{x}_{n+1}) \\ \qquad\qquad\qquad \geqslant (\check{\boldsymbol{f}}_3, \boldsymbol{y} - \boldsymbol{x}_{n+1})_X \end{cases}$$

for all $\boldsymbol{y} = (\boldsymbol{v}, \xi)$ in U, where the bilinear symmetric form $\check{B}\colon U \times X \times X \to \mathbb{R}$ is given by

$$\check{B}(\boldsymbol{x}; \boldsymbol{y}, \boldsymbol{z}) = B(\boldsymbol{x}; \boldsymbol{y}, (\boldsymbol{w}, -\xi)) \quad \forall\, \boldsymbol{x} = (\boldsymbol{u}, \varphi),\ \boldsymbol{y} = (\boldsymbol{v}, \eta) \text{ and } \boldsymbol{z} = (\boldsymbol{w}, \xi) \in X.$$

Hence, a constrained minimization problem equivalent to (6.5) can be formulated. The proposed minimization problem is

$$\text{(6.6)} \qquad \begin{cases} \text{Find } \boldsymbol{x} = (\boldsymbol{u}, \varphi) \in U \text{ such that} \\ J_n(\boldsymbol{x}) + \tilde{j}_{2,n}(\boldsymbol{x}) \leqslant J_n(\boldsymbol{y}) + \tilde{j}_{2,n}(\boldsymbol{y}) \quad \forall\, \boldsymbol{y} = (\boldsymbol{v}, \xi) \in U, \end{cases}$$

J_n is the piezoelectric deformation energy functional due to nonfrictional effects given by

$$J_n(\boldsymbol{y}) = \frac{1}{2}\check{B}(\boldsymbol{x}_n; \boldsymbol{y}, \boldsymbol{y}) - (\boldsymbol{f}_{1,n}, \boldsymbol{y})_X \quad \forall\, \boldsymbol{y} = (\boldsymbol{v}, \xi) \in X,$$

where

$$(\boldsymbol{f}_{1,n}, \boldsymbol{y})_X = (\check{\boldsymbol{f}}_3, \boldsymbol{y})_X - \tilde{j}_1(\boldsymbol{x}_n, \boldsymbol{y}) - \check{l}(\boldsymbol{x}_n, \boldsymbol{y}) \quad \forall\, \boldsymbol{y} = (\boldsymbol{v}, \xi) \in X,$$
$$\tilde{j}_{1,n}(\boldsymbol{y}) = \tilde{j}_1(\boldsymbol{x}_n, \boldsymbol{y}), \quad \tilde{j}_{2,n}(\boldsymbol{y}) = \tilde{j}_2(\boldsymbol{x}_n, \boldsymbol{y}) \quad \forall\, \boldsymbol{y} = (\boldsymbol{v}, \xi) \in X.$$

The quadratic functional J_n is strictly convex and Gâteaux differentiable on X. Moreover, the friction functional $\tilde{j}_{2,n}$ is convex and lower semi-continuous on X, thus there exists a unique solution to (6.6).

Let $\mathfrak{p} = (\mathfrak{p}_c, \mathfrak{p}_f)$, where $\mathfrak{p}_c$ (contact) and $\mathfrak{p}_f$ (friction) are auxiliary variables. We introduce the set

$$C = \{\mathfrak{p}_c \in L^2(\Gamma_3);\ (\mathfrak{p}_c - \varrho) \leqslant 0 \text{ on } \Gamma_3\},$$

and the characteristic functional $I_C\colon\ L^2(\Gamma_3) \to \mathbb{R} \cup \{\infty\}$ of the set C is defined by

$$I_C(\mathfrak{p}_c) = \begin{cases} 0 & \text{if } \mathfrak{p}_c \in C, \\ \infty & \text{if } \mathfrak{p}_c \notin C. \end{cases}$$

It is easy to see that the problem given in (6.6) is equivalent to the following constrained minimization problem:

Find $\boldsymbol{x} = (\boldsymbol{u}, \varphi) \in X$ and $\mathfrak{p} = (\mathfrak{p}_f, \mathfrak{p}_c) \in L^2(\Gamma_3)^2$ such that for all $\boldsymbol{y} = (\boldsymbol{v}, \xi) \in X$ and $\mathfrak{q} = (\mathfrak{q}_f, \mathfrak{q}_c) \in L^2(\Gamma_3)^2$

$$\text{(6.7)} \qquad J_n(\boldsymbol{x}) + \tilde{j}_{2,n}(\mathfrak{p}_f) + I_C(\mathfrak{p}_c) \leqslant J_n(\boldsymbol{y}) + \tilde{j}_{2,n}(\mathfrak{q}_f) + I_C(\mathfrak{q}_c),$$

$$\text{(6.8)} \qquad \left.\begin{aligned} u_\nu - \mathfrak{p}_c = 0, \\ \boldsymbol{u}_\tau - \mathfrak{p}_f = 0, \end{aligned}\right\} \quad \text{on } \Gamma_3.$$

Due to (6.7)–(6.8) the Augmented Lagrangian functional $\mathcal{L}_{1,r}$ is defined over $X \times L^2(\Gamma_3)^2 \times L^2(\Gamma_3)^2$ by

$$\mathcal{L}_{1,r}(\boldsymbol{y}, \mathfrak{q}; \boldsymbol{\theta}) = J_n(\boldsymbol{y}) + \tilde{j}_{2,n}(\mathfrak{q}_f) + I_C(\mathfrak{q}_c) + (\theta_c, v_\nu - \mathfrak{q}_c)_{L^2(\Gamma_3)} + (\boldsymbol{\theta}_f, \boldsymbol{v}_\tau - \mathfrak{q}_f)_{L^2(\Gamma_3)} + \frac{r}{2}\|v_\nu - \mathfrak{q}_c\|^2_{L^2(\Gamma_3)} + \frac{r}{2}\|\boldsymbol{v}_\tau - \mathfrak{q}_f\|^2_{L^2(\Gamma_3)},$$

where the constant $r > 0$ is the penalty parameter and $\boldsymbol{\theta} = (\theta_c, \boldsymbol{\theta}_f)$. The Uzawa block relaxation method is obtained as follows, starting with $\mathfrak{p}^0$ and $\boldsymbol{\lambda}^0$:

$$\mathcal{L}_{1,r}(\boldsymbol{x}^{k+1}, \mathfrak{p}^k; \boldsymbol{\lambda}^k) = \min_{\boldsymbol{y}} \mathcal{L}_{1,r}(\boldsymbol{y}, \mathfrak{p}^k; \boldsymbol{\lambda}^k), \tag{6.9}$$

$$\mathcal{L}_{1,r}(\boldsymbol{x}^{k+1}, \mathfrak{p}^{k+1}; \boldsymbol{\lambda}^k) = \min_{\mathfrak{p}} \mathcal{L}_{1,r}(\boldsymbol{x}^{k+1}, \mathfrak{q}; \boldsymbol{\lambda}^k), \tag{6.10}$$

$$\boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k + r(\boldsymbol{u}^{k+1} - \mathfrak{p}^{k+1}). \tag{6.11}$$

The solution of (6.9) can be characterized by the Euler-Lagrange equation [6], since $\boldsymbol{y} \mapsto \mathcal{L}_{1,r}(\boldsymbol{y}, \mathfrak{p}; \boldsymbol{\theta})$ is convex and differentiable:

$$\check{B}(\boldsymbol{x}_n; \boldsymbol{x}^{k+1}, \boldsymbol{y}) + r(u_\nu^{k+1}, v_\nu)_{L^2(\Gamma_3)} + r(\boldsymbol{u}_\tau^{k+1}, \boldsymbol{v}_\tau)_{L^2(\Gamma_3)} = (\boldsymbol{f}_{1,n}, \boldsymbol{y})_X + (r\mathfrak{p}_c^k - \lambda_c^k, v_\nu)_{L^2(\Gamma_3)} + (r\mathfrak{p}_f^k - \boldsymbol{\lambda}_f^k, \boldsymbol{v}_\tau)_{L^2(\Gamma_3)}.$$

In (6.10) the subproblems in $\mathfrak{p}_c$ and $\mathfrak{p}_f$ are uncoupled. Consequently, we can minimize the functional $\mathfrak{p} \to \mathcal{L}_{1,r}(\boldsymbol{x}^{k+1}, \mathfrak{p}; \boldsymbol{\lambda}^k)$ separately in $\mathfrak{p}_c$ and $\mathfrak{p}_f$. For the contact subproblem, straightforward calculations using Karush-Kuhn-Tucker optimality conditions yield (see [6])

$$\mathfrak{p}_c^{k+1} = u_\nu^{k+1} + \frac{1}{r}[\lambda_c^k - (\lambda_c^k + r(u_\nu^{k+1} - \varrho))^+].$$

For the friction subproblem, using the Fenchel duality theory we get (see [6])

$$\mathfrak{p}_f^{k+1} = \begin{cases} \dfrac{|\boldsymbol{\lambda}_f^k + r\boldsymbol{u}_\tau^{k+1}| - s_n}{r|\boldsymbol{\lambda}_f^k + r\boldsymbol{u}_\tau^{k+1}|}(\boldsymbol{\lambda}_f^k + r\boldsymbol{u}_\tau^{k+1}) & \text{if } |\boldsymbol{\lambda}_f^k + r\boldsymbol{u}_\tau^{k+1}| > s_n, \\ 0 & \text{if } |\boldsymbol{\lambda}_f^k + r\boldsymbol{u}_\tau^{k+1}| \leqslant s_n, \end{cases}$$

where

$$s_n = \mu|R\sigma_\nu(\boldsymbol{u}_n, \varphi_n)|.$$

With the previous results, we can now present our Uzawa block relaxation method Algorithm 2. We iterate until the relative error on $\boldsymbol{x}^k$, $\mathfrak{p}_c^k$ and $\mathfrak{p}_f^k$ is sufficiently "small", i.e.,

$$\frac{\|\boldsymbol{x}^{k+1} - \boldsymbol{x}^k\|^2_{L^2(\Omega)} + \|\mathfrak{p}_c^{k+1} - \mathfrak{p}_c^k\|^2_{L^2(\Gamma_3)} + \|\mathfrak{p}_f^{k+1} - \mathfrak{p}_f^k\|^2_{L^2(\Gamma_3)}}{\|\boldsymbol{x}^{k+1}\|^2_{L^2(\Omega)} + \|\mathfrak{p}_c^{k+1}\|^2_{L^2(\Gamma_3)} + \|\mathfrak{p}_f^{k+1}\|^2_{L^2(\Gamma_3)}} < \varepsilon^2. \tag{6.12}$$

Algorithm 2. Uzawa block relaxation for (6.6).

Initialization. $r > 0$, $\mathfrak{p}^0 = (\mathfrak{p}_c^0, \mathfrak{p}_f^0)$ and $\boldsymbol{\lambda}^0 = (\lambda_c^0, \boldsymbol{\lambda}_f^0)$ are given.

Iteration $k > 0$. Compute successively $\boldsymbol{x}^{k+1} = (\boldsymbol{u}^{k+1}, \varphi^{k+1})$, $\mathfrak{p}^{k+1} = (\mathfrak{p}_c^{k+1}, \mathfrak{p}_f^{k+1})$ and $\boldsymbol{\lambda}^{k+1} = (\lambda_c^{k+1}, \boldsymbol{\lambda}_f^{k+1})$ as follows

Step 1. Find $\boldsymbol{x}^{k+1} = (\boldsymbol{u}^{k+1}, \varphi^{k+1}) \in X$ such that

$$\begin{aligned}\check{B}(\boldsymbol{x}_n; \boldsymbol{x}^{k+1}, \boldsymbol{y}) &+ r(u_\nu^{k+1}, v_\nu)_{L^2(\Gamma_3)} + r(\boldsymbol{u}_\tau^{k+1}, \boldsymbol{v}_\tau)_{L^2(\Gamma_3)} \\ &= (\boldsymbol{f}_{1,n}, \boldsymbol{y})_X + (r\mathfrak{p}_c^k - \lambda_c^k, v_\nu)_{L^2(\Gamma_3)} + (r\mathfrak{p}_f^k - \boldsymbol{\lambda}_f^k, \boldsymbol{v}_\tau)_{L^2(\Gamma_3)}.\end{aligned}$$

Step 2. Compute the auxiliary contact and friction variables

$$\mathfrak{p}_c^{k+1} = u_\nu^{k+1} + \frac{1}{r}[\lambda_c^k - (\lambda_c^k + r(u_\nu^{k+1} - \varrho))^+],$$

$$\mathfrak{p}_f^{k+1} = \begin{cases} \dfrac{|\boldsymbol{\lambda}_f^k + r\boldsymbol{u}_\tau^{k+1}| - s_n}{r|\boldsymbol{\lambda}_f^k + r\boldsymbol{u}_\tau^{k+1}|}(\boldsymbol{\lambda}_f^k + r\boldsymbol{u}_\tau^{k+1}) & \text{if } |\boldsymbol{\lambda}_f^k + r\boldsymbol{u}_\tau^{k+1}| > s_n, \\ 0 & \text{if } |\boldsymbol{\lambda}_f^k + r\boldsymbol{u}_\tau^{k+1}| \leqslant s_n. \end{cases}$$

Step 3. Update the Lagrange multipliers

$$\begin{aligned}\lambda_c^{k+1} &= \lambda_c^k + r(u_\nu^{k+1} - \mathfrak{p}_c^{k+1}), \\ \boldsymbol{\lambda}_f^{k+1} &= \boldsymbol{\lambda}_f^k + r(\boldsymbol{u}_\tau^{k+1} - \mathfrak{p}_f^{k+1}).\end{aligned}$$

With the above results, the solution method for (6.1) is presented in Algorithm 1.

Algorithm 1. Solution for (6.1).

Initialization. s_0 and $\boldsymbol{x}_0 = (\boldsymbol{u}_0, \varphi_0) \in X$ are given.

Iteration $n \geqslant 0$. Compute $\boldsymbol{x}_{n+1}$ and s_{n+1} successively as follows

- ▷ Compute $\boldsymbol{x}_{n+1} = (\boldsymbol{u}_{n+1}, \varphi_{n+1}) \in X$ using Algorithm 2.
- ▷ Update $s_{n+1} = \mu|R\sigma_\nu(\boldsymbol{u}_{n+1}, \varphi_{n+1})|$ and $(\boldsymbol{f}_{1,n+1}, \cdot)_X = (\check{\boldsymbol{f}}_3, \cdot)_X - \tilde{j}_1(x_{n+1}, \cdot) - \check{l}(x_{n+1}, \cdot)$.

The fixed-point iteration terminates if the relative error on s_n becomes sufficiently "small", i.e.,

$$\frac{\|s_{n+1} - s_n\|^2_{L^2(\Gamma_3)}}{\|s_{n+1}\|^2_{L^2(\Gamma_3)}} < \varepsilon_{fp}^2. \tag{6.13}$$

An equivalent variational formulation to (3.5) is given by

$$(\check{A}\boldsymbol{x}, \boldsymbol{y} - \boldsymbol{x})_X + \tilde{j}_1(\boldsymbol{x}, \boldsymbol{y}) - \tilde{j}_1(\boldsymbol{x}, \boldsymbol{x}) \geqslant (\check{\boldsymbol{f}}_3, \boldsymbol{y} - \boldsymbol{x})_X \quad \forall\, \boldsymbol{y} = (\boldsymbol{v}, \xi) \in U, \tag{6.14}$$

In order to define the solution method for (6.14), we follow the same steps used to define that of (6.1). The result is

Algorithm 3. Solution for (6.14).

Initialization. $\boldsymbol{x}_0 = (\boldsymbol{u}_0, \varphi_0) \in X$ is given.

Iteration $n \geqslant 0$. Compute $\boldsymbol{x}_{n+1} \in X$ using Algorithm 4. Then, update

$$(\boldsymbol{f}_{2,n+1}, \cdot)_X = (\check{\boldsymbol{f}}_3, \cdot)_X - \tilde{j}_1(\boldsymbol{x}_{n+1}, \cdot).$$

The fixed-point iteration terminates if the relative error on $\boldsymbol{x}_n$ becomes sufficiently "small", i.e.,

$$\frac{\|\boldsymbol{x}_{n+1} - \boldsymbol{x}_n\|^2_{L^2(\Omega}}{\|\boldsymbol{x}_{n+1}\|^2_{L^2(\Omega)}} < \varepsilon^2_{fp} \tag{6.15}$$

where

Algorithm 4. Uzawa block relaxation.

Initialization. $r > 0$, $\mathfrak{p}_c^0$ and λ_c^0 are given.

Iteration $k > 0$. Compute successively $\boldsymbol{x}^{k+1} = (\boldsymbol{u}^{k+1}, \varphi^{k+1})$, $\mathfrak{p}_c^{k+1}$, and λ_c^{k+1} as follows

Step 1. Find $\boldsymbol{x}^{k+1} = (\boldsymbol{u}^{k+1}, \varphi^{k+1}) \in X$ such that

$$\check{B}(\boldsymbol{x}_n; \boldsymbol{x}^{k+1}, \boldsymbol{y}) + r(u_\nu^{k+1}, v_\nu)_{L^2(\Gamma_3)} = (\boldsymbol{f}_{2,n}, \boldsymbol{y})_X + (r\mathfrak{p}_c^k - \lambda_c^k, v_\nu)_{L^2(\Gamma_3)}.$$

Step 2. Compute the auxiliary contact variable

$$\mathfrak{p}_c^{k+1} = u_\nu^{k+1} + \frac{1}{r}[\lambda_c^k - (\lambda_c^k + r(u_\nu^{k+1} - \varrho))^+].$$

Step 3. Update the Lagrange multipliers

$$\lambda_c^{k+1} = \lambda_c^k + r(u_\nu^{k+1} - \mathfrak{p}_c^{k+1}).$$

We iterate until the relative error on $\boldsymbol{x}^k$ and $\mathfrak{p}_c^k$ is sufficiently "small", i.e.,

$$\frac{\|\boldsymbol{x}^{k+1} - \boldsymbol{x}^k\|^2_{L^2(\Omega)} + \|\mathfrak{p}_c^{k+1} - \mathfrak{p}_c^k\|^2_{L^2(\Gamma_3)}}{\|\boldsymbol{x}^{k+1}\|_{L^2(\Omega)} + \|\mathfrak{p}_c^{k+1}\|^2_{L^2(\Gamma_3)}} < \varepsilon^2, \tag{6.16}$$

7. Numerical experiments

We describe in this section numerical results for problems (PV_1) and (PV_2) in two dimensions to verify the performance of the iterative schemes presented in the previous section.

We implemented the algorithms described in Section 6 in MATLAB using the $\mathbb{P}_1$ triangular finite element method. A possible choice of the functions h_ν and p_ν is (see [1])

$$h_\nu(s) = c_\nu \times \begin{cases} \alpha_\nu & \text{if } |s| > 128, \\ 1 + (\alpha_\nu - 1) \times \dfrac{|s|}{128} & \text{if } 0 \leqslant |s| \leqslant 128, \end{cases}$$

$$p_\nu(s) = \begin{cases} 0 & \text{if } s < 0, \\ s & \text{if } 0 \leqslant s \leqslant n_\nu, \\ n_\nu & \text{if } s > n_\nu, \end{cases}$$

where c_ν, α_ν and n_ν are positive constants, $\alpha_\nu > 1$. The tolerances in the stopping criteria (6.12) and (6.13) are

$$\varepsilon = 10^{-4}, \quad \varepsilon_{fp} = 10^{-4}. \tag{7.1}$$

We assume that the nonlinear function g in (2.3), has the form:

$$g(t) = \begin{cases} \mu_1 & \text{if } t \leqslant t_0, \\ \mu_1 \dfrac{t_0}{t}\Big(1 + \ln \dfrac{t}{t_0}\Big) & \text{if } t > t_0. \end{cases}$$

i.e., The material behaves linearly for sufficiently small strains (see [8]).

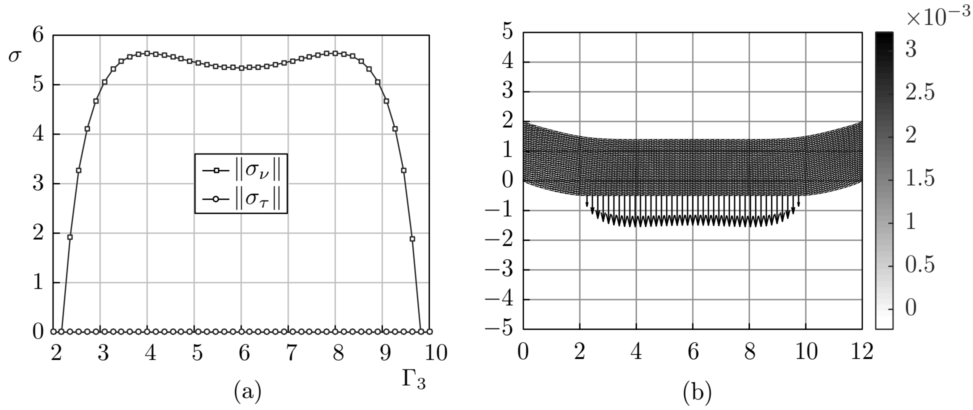
We choose $\mu_1 = E/(2 + 2\nu)$, $k_0 = E/(3 - 6\nu)$ and an elasticity limit $t_0 = 1.8$.

Example 7.1. In this example, we study the academic example of a parallelepiped bar which has the following dimensions: $\Omega = ([0, 12] \times [0, 2])$, with $\Gamma_1 = \Gamma_b = (\{0\} \times [0, 2] \cup \{12\} \times [0, 2])$, $\Gamma_2 = \Gamma_a = ([0, 12] \times \{2\} \cup [0, 2] \times \{0\} \cup [10, 12] \times \{0\})$ and $\Gamma_3 = ([2, 10] \times \{0\})$. The body force $\boldsymbol{f}_0$ and the volume electric charge q_0 are assumed to be zero. The body is clamped on Γ_1 and thus $\boldsymbol{u} = 0$ there. A surface traction and electric charge of densities $\boldsymbol{f}_2(x) = (0, -5)\,\mathrm{N/m^2}$, $q_2(x) = 0\,\mathrm{C/m^2}$ act, respectively, on Γ_2, Γ_b. The gap between the body and the conductive foundation is $\varrho = 0.5\,\mathrm{m}$. The penalty parameter is $r = 0.25 \times E$ for all mesh sizes. The characteristics of the material are given in Table 1 (see [10]):

Elasticity (GPa)				
Young's modulo (GPa)			Poisson's ratio	
58.7102			0.3912	
Piezoelectricity (C/m^2)			Permittivity (C^2/Nm^2)	
e_{32}	e_{33}	e_{24}	β_{22}/ε_0	β_{33}/ε_0
-5.4	15.8	12.3	916	830

Table 1. The material PZT-5A coefficient values with $\varepsilon_0 = 8.885e^{-12}\,C^2/Nm^2$.

Algorithm 3 stops after 2 iterations. The normal and tangential stress distributions on Γ_3 are shown in Figure 1(a) while Figure 1(b) shows the deformed configuration with electrical potential distribution.



(a) (b)

Figure 1. (a) Normal and tangential stress distributions on Γ_3. (b) Deformed configuration and electrical potential distribution with contact forces (arrows).

E x a m p l e 7.2. We consider here the same data as in the previous example. For $\mu = 0.6$, $k_e = 1$ and $\varphi_0 = 32\,V$, the normal and tangential stress distributions on Γ_3 are shown in Figure 2(a) while Figure 2(b) shows the deformed configuration with electrical potential distribution. The sticking zone $\|\sigma_\tau\| < s_n$ and sliding zone $\|\sigma_\tau\| = s_n$ are clearly identified.

To show that the numerical results are in good agreement with the theoretical analysis given in Section 4, we will study the evolution of the number of iterations, the distribution of the electrical potential, normal and tangential stresses on Γ_3 as a function of the friction coefficient μ and the electrical conduction coefficient k_e. The results are given in Table 2, Figure 3(a)–(b) and Figure 4.

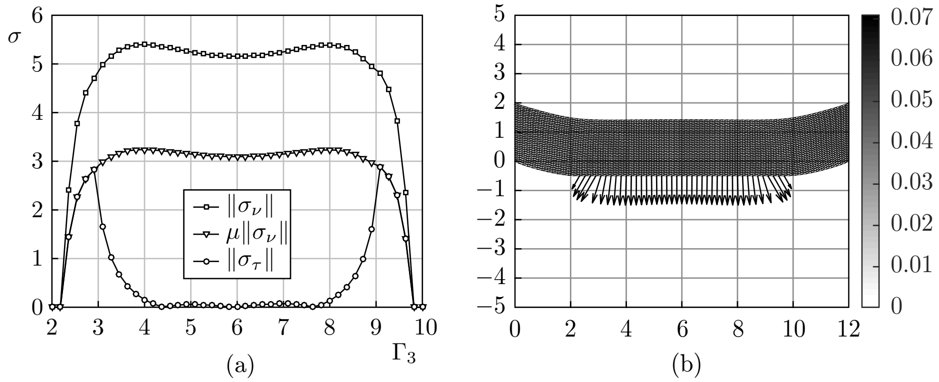


Figure 2. For $\mu = 0.6$, $k_e = 1$ and $\varphi_0 = 32\,\mathrm{V}$. (a) Normal and tangential stress distributions on Γ_3. (b) Deformed configuration and electrical potential distribution with contact forces (arrows).

$\mu = k_e$	Number of iterations
6.0×10^{-1}	7
2.0×10^{-1}	5
6.8×10^{-2}	4
5.1×10^{-3}	3
6.2×10^{-4}	2
1.4×10^{-5}	2
3.5×10^{-6}	2
8.5×10^{-7}	2

Table 2. Number of iterations for different values of (μ, k_e).

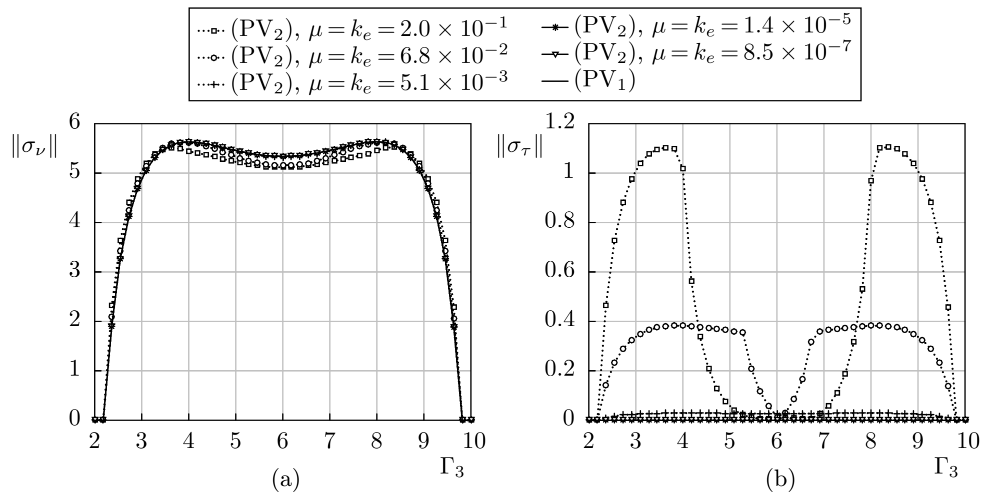


Figure 3. For $\varphi_0 = 32\,\mathrm{V}$ and different values of $\mu = k_e$ (a) Normal stress distribution on Γ_3. (b) Tangential stress distribution on Γ_3.

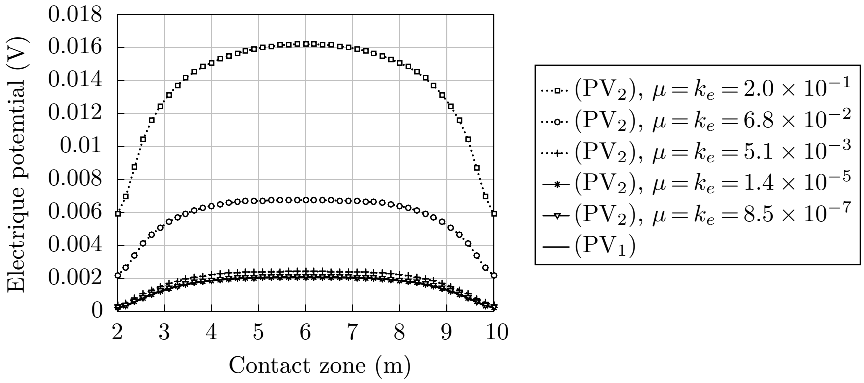


Figure 4. Electrical potential distribution on Γ_3 for $\varphi_0 = 32\,\mathrm{V}$ and different values of $\mu = k_e$.

For $\varphi_0 = 32\,\mathrm{V}$ and different values of $\mu = k_e$, Figure 3(a)–(b) show the normal and tangential stress distribution on Γ_3, while Figure 4 shows the electrical potential distribution on Γ_3, where we can clearly see that the contact zone's electrical potential, normal and tangential stress of Problem (PV$_2$) approaches those of Problem (PV$_1$) when $\mu = k_e$ approaches zero.

Conclusion

In this paper we have studied an interesting result from a physical and numerical point of view, where we can observe that the solution of the problem which describes the nonlocal frictional contact between a nonlinear piezoelectric body and an electrically conductive foundation, approached so closely to that of the frictionless contact problem between a nonlinear piezoelectric body and nonconductive foundation, as the friction and electrical conductivity coefficients have approached so close to zero.

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