SCATTER HALFSPACE DEPTH: GEOMETRIC INSIGHTS

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Abstract. Scatter halfspace depth is a statistical tool that allows one to quantify the fitness of a candidate covariance matrix with respect to the scatter structure of a probability distribution. The depth enables simultaneous robust estimation of location and scatter, and nonparametric inference on these. A handful of remarks on the definition and the properties of the scatter halfspace depth are provided. It is argued that the currently used notion of this depth is well suited especially for symmetric random vectors. The scatter halfspace depth closely relates to an appropriate distance of matrix-generated ellipsoids from an upper level set of the (location) halfspace depth function. Several modifications and extensions to the scatter halfspace depth are considered, with their theoretical properties outlined.

Keywords: elliptical distributions; floating body; scatter halfspace depth; Tukey depth

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1. Introduction: The depth of points and matrices

In recent years, the idea of *depth functions* has attracted considerable attention in mathematical statistics. We consider the halfspace depth proposed by Tukey [12] and extensions of this function to scatter matrices.

1.1. Location halfspace depth. For $\mathcal{P}(\mathbb{R}^d)$, the collection of Borel probability measures on $\mathbb{R}^d$, and $X \sim P \in \mathcal{P}(\mathbb{R}^d)$, the halfspace depth of $x \in \mathbb{R}^d$ with respect to (w.r.t.) P is given as the minimum P-probability of a halfspace that contains x:

$$\mathrm{hD}(x; P) = \inf_{u \in \mathbf{S}^{d-1}} \mathsf{P}(u^\top X \leqslant u^\top x), \tag{1.1}$$

where $\mathbf{S}^{d-1}$ stands for the unit sphere in $\mathbb{R}^d$. For P given, the depth (1.1) acts as a centrality index of x. The higher $\mathrm{hD}(x; P)$ is, the more centrally located the

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point x is w.r.t. the mass of P. The maximum value $\Pi(P) = \sup_{x \in \mathbb{R}^d} \mathrm{hD}(x; P)$ can be seen as an indicator of the degree of symmetry of P ([8], Section 4), with more symmetric distributions attaining $\Pi(P)$ closer to $1/2$. It is, however, not true that only symmetric distributions attain depth $1/2$. For instance, $\Pi(P) \geqslant 1/2$ for all $P \in \mathcal{P}(\mathbb{R})$, where the depth is maximized at the usual (univariate) median of P. More generally, regions of the form

$$P_\delta = \{x \in \mathbb{R}^d \colon \mathrm{hD}(x; P) \geqslant \delta\} \quad \text{for } \delta \in (0, \Pi(P)] \tag{1.2}$$

are a collection of nested compact convex sets that naturally induce a P-dependent ordering on $\mathbb{R}^d$, and can be recognized as a generalization of quantiles to multivariate probability distributions. In particular, the *halfspace median*

$$x_P = \mathrm{Cen}(\{x \in \mathbb{R}^d \colon \mathrm{hD}(x; P) = \Pi(P)\}), \tag{1.3}$$

the centroid[1] of the set of maximizers of $\mathrm{hD}(\cdot; P)$, generalizes the median to $\mathbb{R}^d$.

In geometry, ideas closely related to hD are known at least since the early 1900's. A prominent concept equivalent to the depth (1.1) is that of *floating bodies*, pioneered by Dupin [2] in 1822. Take a convex body $K \subset \mathbb{R}^d$—a compact, convex set in $\mathbb{R}^d$ with nonempty interior. Since the uniform probability measure P_K on K is an element of $\mathcal{P}(\mathbb{R}^d)$, we identify K with P_K. Given $\delta > 0$, the floating body of K is defined as the set $K_{[\delta]} \subset K$ such that all the supporting hyperplanes to $K_{[\delta]}$ cut off a set of volume $\delta \, \mathrm{vol}_d(K)$ from K, for $\mathrm{vol}_d(K)$ the volume of K. If the floating body exists, it can be shown to coincide with the central region (1.2) of P_K. Thus, the central regions (1.2), and by extension the depth (1.1), generalize floating bodies to arbitrary probability measures. A comprehensive survey of these connections is [8]. We will draw from these relations throughout the paper, and obtain analogous insights into the geometric representation of the scatter halfspace depth.

1.2. Scatter halfspace depth. One way to look at the halfspace depth (1.1) is to consider it to be a gauge of appropriateness of x as a location parameter of P. In this context, hD is sometimes also called the location halfspace depth. The halfspace median (1.3) certainly is a reasonable location parameter. The next natural step is the extension of this idea to the scatter structure of multivariate distributions, recently pursued in [1] and [9]. For given $X \sim P \in \mathcal{P}(\mathbb{R}^d)$ and T (introduced below),

[1] We denote the centroid of a set $K \subset \mathbb{R}^d$ by $\mathrm{Cen}(K)$. For K of positive volume, the centroid is the expectation of the uniform probability distribution on K. We also write $\mathrm{Cen}(\{x\}) = x$ for $x \in \mathbb{R}^d$.

the depth of a symmetric, positive definite $d \times d$ matrix Σ (we write also $\Sigma \in \mathcal{P}_d$) is proposed to be evaluated by

$$\begin{aligned}(1.4)\quad & \operatorname{shD}(\Sigma; P) \\ & = \inf_{u \in \mathbf{S}^{d-1}} \min\{\mathsf{P}(|u^\top (X - T_P)| \leqslant \sqrt{u^\top \Sigma u}), \mathsf{P}(|u^\top (X - T_P)| \geqslant \sqrt{u^\top \Sigma u})\}.\end{aligned}$$

This depth is called the *scatter halfspace depth* of Σ w.r.t. P. In (1.4), $T\colon \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d\colon P \mapsto T_P$ is an affine equivariant location functional, that is a measurable map such that $T_{P_{A,b}} = AT_P + b$ for any $A \in \mathbb{R}^{d\times d}$ nonsingular and $b \in \mathbb{R}^d$, where $P_{A,b}$ is the distribution of $AX + b$. Examples of affine equivariant location functionals are the expected value, the centre of symmetry, or the halfspace median (1.3). However, in connection with the depth only the halfspace median has been considered in [9] since the other two functionals may be undefined for general distributions. Any matrix Σ that maximizes $\operatorname{shD}(\cdot; P)$ over $\mathcal{P}_d$ is called a *scatter median* of P.

It is instructive to see what form shD takes for univariate distributions. There, the scatter halfspace depth of $\sigma^2 > 0$ w.r.t. $X \sim P \in \mathcal{P}(\mathbb{R})$ is

$$\operatorname{shD}(\sigma^2; P) = \min\{\mathsf{P}(|X - T_P|^2 \leqslant \sigma^2), \mathsf{P}(|X - T_P|^2 \geqslant \sigma^2)\}.$$

If T is the (univariate) median and the support of P is connected, the scatter median of P is the squared median absolute deviation, (the square of) a well-known robust scale functional, with the maximum depth being at least $1/2$.

In this note we are concerned with the geometry behind the mapping shD, in the same way as the geometry of hD was studied in [8]. We discuss the general definition of shD, and interpret the depth (1.4) in terms of probabilities of halfspaces and floating bodies. We draw connections with the theory of approximation of convex bodies by ellipsoids. Numerous minor modifications of the depth (1.4) are suggested. These appear to aid the geometric intuition behind the concept, and possess more convenient theoretical properties, but at the same time do not lose any of the benefits of the original scatter halfspace depth. We conclude the present note with an important discussion about multivariate symmetry of measures in connection with shD.

Assumption. To avoid technical nuances, in the sequel we consider only distributions $P \in \mathcal{P}(\mathbb{R}^d)$ that are continuous in the sense that all hyperplanes attain null P-probability. This assumption is by no means necessary, but it will greatly facilitate our exposition.

2. Properties of the scatter halfspace depth

2.1. Centring: The choice of the location functional. Just as for the location halfspace depth, there are only few types of multivariate distributions whose scatter halfspace depth was computed exactly in the literature. The most important cases are the elliptically symmetric distributions[2] and K-symmetric distributions[3] [6]. All these measures are centrally symmetric[4] around a unique point $\theta \in \mathbb{R}^d$. Therefore, for each such P, its centre of symmetry $T_P = \theta$ is the unique affine invariant location functional [5], and there is no ambiguity in the choice of T in (1.4).

The situation is different for general probability measures. If P is not symmetric, it is natural to employ the halfspace median (1.3). Nevertheless, it appears that especially for asymmetric distributions, it may be beneficial to consider also other location functionals. One further reasonable functional that fits the framework of scatter halfspace depth will be introduced in Section 3.

2.2. Symmetrization. To get some intuition for the geometric meaning of shD, we explore the sets whose probabilities are compared in (1.4). First, note that these sets are complementary in the sense that, up to the common boundary hyperplanes $\{x \in \mathbb{R}^d\colon\ u^\top(x - T_P) = \pm\sqrt{u^\top \Sigma u}\}$, whose probabilities are supposed to be zero, they decompose $\mathbb{R}^d$. Thus, it is enough to analyse only the first term in the minimum in (1.4). For any $c > 0$ the region

$$S_{u,c} = \{x \in \mathbb{R}^d\colon\ |u^\top(x - T_P)| \leqslant c\}$$

is the closed slab between two parallel hyperplanes whose unit normal is u, in the same distance c from the central point T_P. Thus, $\min\{P(S_{u,c}), 1 - P(S_{u,c})\} = 1/2 - |1/2 - P(S_{u,c})|$ is a measure of deviation of the probability of the slab $S_{u,c}$ from $1/2$. The distance $c = \sqrt{u^\top \Sigma u} = \|\Sigma^{1/2} u\|$, of course, depends on both Σ and u. This will be discussed in detail in Section 2.3.

Geometrically, in (1.4) we therefore

- ▷ consider the collection of all slabs $\{S_{u,\sqrt{u^\top \Sigma u}}\}_{u \in \mathbb{S}^{d-1}}$ centred at T_P whose width is controlled by a given function of u and Σ, and

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[2] We say that $P \in \mathcal{P}(\mathbb{R}^d)$ is spherically symmetric if it is invariant with respect to orthogonal transformations. P is said to be elliptically symmetric if it is a nonsingular affine image of a spherically symmetric distribution. For details see [3].

[3] For an origin-symmetric convex body K, the measure $P \in \mathcal{P}(\mathbb{R}^d)$ is said to be K-symmetric if its characteristic function $\psi(t)$ depends on $t \in \mathbb{R}^d$ only through its Minkowski functional $\|t\|_K = \inf\{\lambda > 0\colon\ t \in \lambda K\}$.

[4] A distribution $X \sim P \in \mathcal{P}(\mathbb{R}^d)$ is said to be centrally symmetric around $\theta \in \mathbb{R}^d$ if $X - \theta \overset{d}{=} -(X - \theta)$, with $\overset{d}{=}$ standing for "is equal in distribution".

▷ search for a direction u, where the probability of a slab deviates the most from $1/2$.

The absolute value in (1.4) means that any potential asymmetries of P around T_P are disregarded. Consider the two halfspaces in $\mathbb{R}^d$ complementary to $S_{u,c}$

$$\begin{aligned} H^-_{u,c} &= \{x \in \mathbb{R}^d\colon\ u^\top(x - T_P) \leqslant -c\},\\ H^+_{u,c} &= \{x \in \mathbb{R}^d\colon\ u^\top(x - T_P) \geqslant c\}. \end{aligned} \tag{2.1}$$

The fact that $P(H^-_{u,c}) \neq P(H^+_{u,c})$ does not affect the depth of Σ at all, as in (1.4) only $P(S_{u,c}) = 1 - (P(H^-_{u,c}) + P(H^+_{u,c}))$ is considered. These remarks are formalized in the following theorem.

Theorem 2.1. *For $X \sim P \in \mathcal{P}(\mathbb{R}^d)$, denote by $P^1 \in \mathcal{P}(\mathbb{R}^d)$ the distribution of $2T_P - X$, and by $P^2 \in \mathcal{P}(\mathbb{R}^d)$ the distribution of the mixture of X and $2T_P - X$ with equal mixing proportions. Then*

$$\mathrm{shD}(\Sigma; P) = \mathrm{shD}(\Sigma; P^1) = \mathrm{shD}(\Sigma; P^2) \quad \forall\, \Sigma \in \mathcal{P}_d.$$

P r o o f. We may assume that $T_P = 0$. Directly from (1.4) we have

$$\begin{aligned} \mathrm{shD}(\Sigma; P) &= \inf_{u\in\mathbf{S}^{d-1}} \min\bigl\{\mathsf{P}\bigl(|u^\top X| \leqslant \sqrt{u^\top\Sigma u}\bigr), \mathsf{P}\bigl(|u^\top X| \geqslant \sqrt{u^\top\Sigma u}\bigr)\bigr\}\\ &= \inf_{u\in\mathbf{S}^{d-1}} \min\bigl\{\mathsf{P}\bigl(|-u^\top X| \leqslant \sqrt{u^\top\Sigma u}\bigr), \mathsf{P}\bigl(|-u^\top X| \geqslant \sqrt{u^\top\Sigma u}\bigr)\bigr\}\\ &= \mathrm{shD}(\Sigma; P^1) = \inf_{u\in\mathbf{S}^{d-1}} \tfrac{1}{2}\bigl(\min\bigl\{\mathsf{P}\bigl(|u^\top X| \leqslant \sqrt{u^\top\Sigma u}\bigr), \mathsf{P}\bigl(|u^\top X| \geqslant \sqrt{u^\top\Sigma u}\bigr)\bigr\}\\ &\quad + \min\bigl\{\mathsf{P}\bigl(|-u^\top X| \leqslant \sqrt{u^\top\Sigma u}\bigr), \mathsf{P}\bigl(|-u^\top X| \geqslant \sqrt{u^\top\Sigma u}\bigr)\bigr\}\bigr) = \mathrm{shD}(\Sigma; P^2). \end{aligned}$$

□

The scatter depth (1.4) can therefore be seen to act on the symmetrized measures P^2 from Theorem 2.1, instead of the original ones. This, of course, does not affect the analysis of scatter patterns of centrally symmetric distributions ([1], [9]), but it does limit the overall resolution of the depth in terms of its discrimination between different measures. A related concern connects with the *halfspace depth characterization problem* [7]. If a (location) depth is intended to serve as a generalization of quantiles, the knowledge of the depth of all points of the sample space w.r.t. P should fully determine P. Theorem 2.1 shows that the scatter halfspace depth does not recognize asymmetric distributions. Therefore, it may be worthwhile to relax the implicit symmetry in (1.4) by considering the probabilities of the halfspaces (2.1) separately. This will be done in Section 3.

2.3. The special role of ellipticity. Elliptically symmetric distributions play a crucial role in the theory of depth. They form an invariance class for the location halfspace depth (1.1), as the depth contours (and floating bodies) of any such distribution P are ellipsoids of the same centre and shape as the scatter ellipsoid that governs the geometry of P.

To any central point $\mu \in \mathbb{R}^d$ and a shape matrix $\Sigma \in \mathcal{P}_d$ it is possible to assign a nondegenerate ellipsoid

$$K(\mu, \Sigma) = \left\{x \in \mathbb{R}^d \colon \sqrt{(x-\mu)^\top \Sigma^{-1}(x-\mu)} \leqslant 1\right\},$$

sometimes called the Mahalanobis ellipsoid, with centre μ and shape Σ. For μ and Σ taken to be a location and a scatter parameter, respectively, of an elliptically symmetric $P \in \mathcal{P}(\mathbb{R}^d)$, $K(\mu, \Sigma)$ may be used to visualise P.

The expression $c = \sqrt{u^\top \Sigma u}$ from (1.4) that appears in (2.1) naturally relates with the support function of the set $K(\mu, \Sigma)$. Recall that for a convex body $K \subset \mathbb{R}^d$, the support function of K is defined as

$$h_K \colon \mathbf{S}^{d-1} \to \mathbb{R} \colon u \mapsto \sup\{u^\top x \colon x \in K\}.$$

It has a nice geometric interpretation—$h_K(u)$ is the signed distance from the origin to a hyperplane with normal u that supports K, see Fig. 1. For ellipsoids we get

$$h_{K(\mu,\Sigma)}(u) = u^\top \mu + \sqrt{u^\top \Sigma u}.$$

The orthogonal distance c from the centre T_P to a boundary hyperplane of a slab $S_{u,c}$ is given by $u^\top (X - T_P) \leqslant \sqrt{u^\top \Sigma u}$, or equivalently by $u^\top X \leqslant h_{K(T_P,\Sigma)}(u)$. This offers an attractive alternative formulation to shD. One computes the depth of $\Sigma \in \mathcal{P}_d$ as $1/2$ minus the largest deviation of the probability of a slab between the two parallel hyperplanes supporting $K(T_P, \Sigma)$ from $1/2$ (see Fig. 1). Now we recognize the special role played by ellipsoids in (1.4). Likewise, this makes it easy to accommodate also geometric shapes and convex bodies different from ellipsoids into shD, the only modification being the replacement of the support function of an ellipsoid $\sqrt{u^\top \Sigma u}$ by a support function of another body. Such replacements are quite natural in the context of K-symmetric distributions, such as the multivariate Cauchy distribution with independent marginals studied in [9]. There, it is convenient to consider the support function of the convex body

$$K^\circ = \{x \in \mathbb{R}^d \colon x^\top y \leqslant 1 \quad \forall\, y \in K\}$$

that is polar to K to detect deviations from the K-symmetry of P. For more details we refer to [6] and Sections 1.6.1 and 1.7.1 from [11]. Note that a polar body to

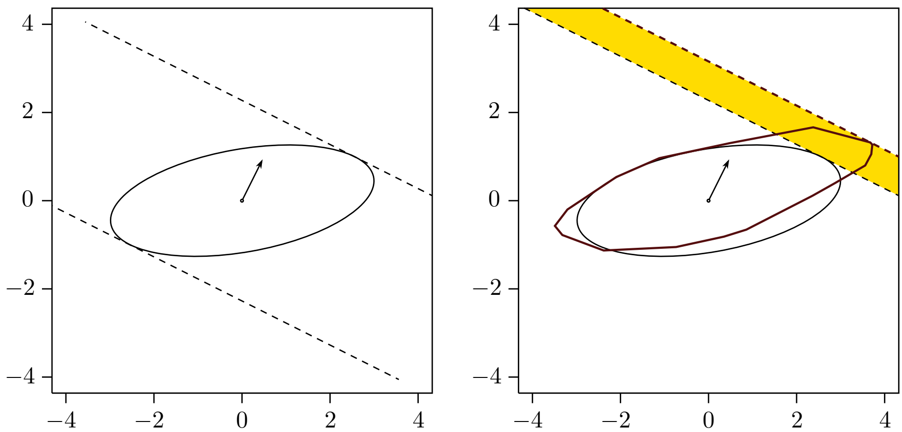

Figure 1. Left: A slab $S_{u,c}$ (region between dashed lines) in the direction u (arrow) from $T_P = 0$ (cross) that corresponds to $\Sigma \in \mathcal{P}_d$ (ellipse). The P-probability of all such slabs is compared with 1/2 in the computation of (the symmetrized) shD in (1.4). Right: A floating body $P_{1/4}$ (brown) and the slab between two parallel supporting hyperplanes of the ellipse and $P_{1/4}$, respectively, (coloured region); see Theorem 3.1.

an ellipsoid is an ellipsoid. Therefore, our theory is consistent with the well-known fact that the K-symmetric probability measures for K an ellipsoid are exactly the elliptically symmetric distributions.

3. Generalized scatter halfspace depth

Let T be an affine equivariant location functional. Typically, we mean the halfspace median (1.3); another option will be considered below. In accordance with the discussion from Section 2, consider the asymmetric version of the scatter depth of $\Sigma \in \mathcal{P}_d$ w.r.t. $X \sim P \in \mathcal{P}(\mathbb{R}^d)$

$$\widetilde{\mathrm{shD}}(\Sigma; P) = 2\Big(\tfrac{1}{4} - \sup_{u \in \mathbf{S}^{d-1}} \big|\tfrac{1}{4} - \mathsf{P}(u^\top(X - T_P) \geqslant \sqrt{u^\top \Sigma u})\big|\Big)_+, \tag{3.1}$$

where x_+ is the positive part of $x \in \mathbb{R}$. If $X \sim P$ is centrally symmetric about T_P, we have $2\mathsf{P}(u^\top(X - T_P) \geqslant \sqrt{u^\top \Sigma u}) = \mathsf{P}(|u^\top(X - T_P)| \geqslant \sqrt{u^\top \Sigma u})$ and

$$\widetilde{\mathrm{shD}}(\Sigma; P) = \inf_{u \in \mathbf{S}^{d-1}} \Big(\tfrac{1}{2} - \big|\tfrac{1}{2} - \mathsf{P}(|u^\top(X - T_P)| \geqslant \sqrt{u^\top \Sigma u})\big|\Big) = \mathrm{shD}(\Sigma; P).$$

For asymmetric distributions, $\widetilde{\mathrm{shD}}$ is a depth different from shD. Take, for instance, $d = 1$. In that case, the depth reduces for $\sigma > 0$ to

$$\widetilde{\mathrm{shD}}(\sigma^2; P) = 2\big(\tfrac{1}{4} - \max\big\{\big|\tfrac{1}{4} - \mathsf{P}(X \geqslant T_P + \sigma)\big|, \big|\tfrac{1}{4} - \mathsf{P}(X \leqslant T_P - \sigma)\big|\big\}\big)_+.$$

Here, we see why the positive part in (3.1) is necessary. For very asymmetric distributions, the probability $\mathsf{P}(X \geqslant T_P + \sigma)$ may exceed[5] $1/2$.

Our motivation for writing the supremum in (3.1) comes from an interesting observation connecting the depth shD with the floating body of P.

Theorem 3.1. *Let $P \in \mathcal{P}(\mathbb{R}^d)$ be such that the central region $P_{1/4}$ is a floating body, i.e. each supporting hyperplane of $P_{1/4}$ cuts off a halfspace of P-probability $1/4$. Then*

$$\sup_{u \in \mathbf{S}^{d-1}} \left| \tfrac{1}{4} - \mathsf{P}\big(u^\top (X - T_P) \geqslant \sqrt{u^\top \Sigma u}\big) \right| = \mathrm{d}_P(K(T_P, \Sigma), P_{1/4}), \tag{3.2}$$

where d_P is a pseudometric in the space of convex bodies given by

$$\mathrm{d}_P(K, L) = \sup_{u \in \mathbf{S}^{d-1}} \mathsf{P}(u^\top X \in [h_K(u), h_L(u)]).$$

In the last formula, by $[a, b]$ we mean the interval between a and b even if $b < a$. If, in addition, the support of P is $\mathbb{R}^d$, d_P is a metric.

P r o o f. If $P_{1/4}$ is a floating body, $\mathsf{P}(u^\top X \geqslant h_{P_{1/4}}(u)) = 1/4$ for each $u \in \mathbf{S}^{d-1}$ since the hyperplane $\{x \in \mathbb{R}^d \colon u^\top x = h_{P_{1/4}}(u)\}$ supports $P_{1/4}$. This gives the first part of the statement.

To verify that d_P is a (pseudo)metric, we need to show its nonnegativity, symmetry and verify the triangle inequality. The first two conditions are obviously satisfied. For the triangle inequality, take convex bodies $K, L, M \subset \mathbb{R}^d$. Take $\varepsilon > 0$ small and find a direction $v \in \mathbf{S}^{d-1}$ such that $\mathsf{P}(v^\top X \in [h_K(v), h_L(v)]) \geqslant \mathrm{d}_P(K, L) - \varepsilon$. We may assume that $h_K(v) \leqslant h_L(v)$. If M satisfies $h_M(v) \leqslant h_K(v)$, necessarily

$$\begin{aligned} \mathrm{d}_P(K, M) + \mathrm{d}_P(M, L) &\geqslant \mathrm{d}_P(M, L) = \sup_{u \in \mathbf{S}^{d-1}} \mathsf{P}(u^\top X \in [h_M(u), h_L(u)]) \\ &\geqslant \mathsf{P}(v^\top X \in [h_M(v), h_L(v)]) \geqslant \mathsf{P}(v^\top X \in [h_K(v), h_L(v)]) \geqslant \mathrm{d}_P(K, L) - \varepsilon. \end{aligned}$$

A similar inequality can be shown if $h_M(v) \geqslant h_L(v)$. In the remaining situation when $h_K(v) < h_M(v) < h_L(v)$, we write

$$\begin{aligned} \mathrm{d}_P(K, M) + \mathrm{d}_P(M, L) &\geqslant \mathsf{P}(v^\top X \in [h_K(v), h_M(v)]) + \mathsf{P}(v^\top X \in [h_M(v), h_L(v)]) \\ &= \mathsf{P}(v^\top X \in [h_K(v), h_L(v)]) \geqslant \mathrm{d}_P(K, L) - \varepsilon. \end{aligned}$$

In all three cases, these inequalities are true for any $\varepsilon > 0$, and taking a limit as $\varepsilon \to 0$ allows us to conclude.

[5] Take, for example, T_P the expected value and P extremely skewed.

The equality of the support functions of convex bodies is equivalent with the equality of the bodies ([11], Section 1.7.1). This means that for any pair of distinct $K, L \subset \mathbb{R}^d$ we can find $u \in \mathbf{S}^{d-1}$ where the support functions differ, and the condition on the support of P guarantees that the P-mass of the slab between the supporting hyperplanes of K and L in direction u is nonnull. Thus, necessarily $\mathrm{d}_P(K, L) > 0$. □

Theorem 3.1 gives a valuable insight into the geometry of shD—in regular situations when the 1/4-floating body of P exists, the depth (3.1) is just a monotone transform of the distance of the Mahalanobis ellipsoid $K(T_P, \Sigma)$ from $P_{1/4}$. In particular, the asymmetric scatter halfspace depth $\widetilde{\mathrm{shD}}$ attains its maximum value of 1/2 if and only if the 1/4-floating body of P is an ellipsoid centred at T_P. By Dupin's theorem ([8], Sections 4.3.1 and 6.0.1), if the boundary of the central region $P_{1/4}$ from (1.2) is smooth[6], $P_{1/4}$ must be a floating body of P. Therefore, for P such that $P_{1/4}$ is an ellipsoid, Theorem 3.1 is valid.

The assumption of P having a floating body is not crucial in Theorem 3.1. For $X \sim P$ not satisfying this condition, an analogous result is readily available in terms of a function $q\colon \mathbf{S}^{d-1} \to \mathbb{R}$ that assigns to $u \in \mathbf{S}^{d-1}$ the (univariate) 3/4-quantile of $u^\top X$. The scatter halfspace depth (3.1) is in this situation a (pseudo)metric between q and $h_{K(T_P,\Sigma)}$ in the space of real-valued functions defined on $\mathbf{S}^{d-1}$. Furthermore, the conclusion of the previous paragraph is still true — for P with full support, the maximum depth (3.1) equals 1/2 if and only if $P_{1/4}$ is an ellipsoid centered at T_P.

In the definition (3.1) it is also possible to avoid the implicit dependence on the halfspace median having to be the centre of the region $P_{1/4}$. In connection with floating bodies, the literature in convex geometry offers a rich collection of plausible alternative location functionals in the form of *affine invariant points* [5]. A particularly well suited example appears to be $x_{P,\delta} = \mathrm{Cen}(P_\delta)$. Obviously, for $\delta = \Pi(P)$ we recover the halfspace median. Nevertheless, this more general setting directly allows to consider in (3.1) the point $T_P = x_{P,1/4}$. If $P_{1/4}$ is an ellipsoid, this choice assures that the maximum scatter halfspace depth $\widetilde{\mathrm{shD}}$ is 1/2. In view of the preceding discussion and Theorem 3.1, we obtain the following corollary.

Theorem 3.2. *For any $P \in \mathcal{P}(\mathbb{R}^d)$ such that $\Pi(P) > 1/4$ and T the centroid of $P_{1/4}$, the following statements are equivalent:*

(1) *The central region $P_{1/4}$ is an ellipsoid.*
(2) $\sup_{\Sigma \in \mathcal{P}_d} \widetilde{\mathrm{shD}}(\Sigma; P) = 1/2$.

[6] Through each point x on the boundary of $P_{1/4}$ passes a single hyperplane supporting $P_{1/4}$.

Note that for $d = 1$, the affine invariant point $x_{P,1/4}$ is the mid-quartile of $P \in \mathcal{P}(\mathbb{R})$, each distribution satisfies the conditions from Theorem 3.2, and a maximizer of $\widetilde{\mathrm{shD}}(\cdot; P)$ is the square of the semi-interquartile range of P, another well studied scale functional.

3.1. Additional theoretical properties. The scatter depth (3.1) has many good properties. Because for centrally symmetric distributions it coincides with the depth (1.4), it inherits the minimax optimality under Huber's contamination model [1]. Due to the affine equivariance of the central regions (1.2), the depth $\widetilde{\mathrm{shD}}$ is affine invariant in the sense that $\widetilde{\mathrm{shD}}(A\Sigma A^\top; P_{A,b}) = \widetilde{\mathrm{shD}}(\Sigma; P)$ for any $A \in \mathbb{R}^{d\times d}$ nonsingular, $b \in \mathbb{R}^d$, and $P_{A,b}$ the distribution of $AX + b$ with $X \sim P$. The depth also satisfies the property of monotonicity relative to the deepest scatter, as requested by [9], using a proof analogous to that from [9], Theorem 3.3. By Theorem 3.2, the depth $\widetilde{\mathrm{shD}}$ is Fisher consistent not only for elliptically symmetric distributions, but for any distribution whose 1/4-floating body is an ellipsoid (centred at T_P). Finally, for any Σ such that $\mathrm{d}_P(P_{1/4}, \Sigma) \geqslant 1/4$ we have $\widetilde{\mathrm{shD}}(\Sigma; P) = 0$. In particular, if for a sequence $\{\Sigma_n\}_{n=1}^\infty \subset \mathcal{P}_d$ the collection of ellipsoids $K(0, \Sigma_n)$ is unbounded, or approaches (in the Hausdorff distance) a set with empty interior, $\widetilde{\mathrm{shD}}(\Sigma_n; P) \to 0$. We have verified conditions (Q1)–(Q4) requested from a scatter depth in [9], and much more.

3.2. Some issues with the scatter depth. It is important to note that there are also difficulties with the definition of the scatter depth. Just as the original shD, the depth $\widetilde{\mathrm{shD}}$ is neither quasi-concave, nor monotone, with respect to the geodesic topology of positive definite matrices ([9], Fig. 2). This is caused by the quadratic form $u^\top \Sigma u$, a function not directly compatible with the topology of $\mathcal{P}_d$, in the definition of the depth (3.1); see also [6], Section 5. Another shortcoming appears to be the arbitrariness of the floating body $P_{1/4}$ used in formula (3.2). The constant 1/4 is well justified for (centrally) symmetric distributions, or more generally if $\Pi(P) = 1/2$. For asymmetric distributions in higher dimensions, it may however happen that the set $P_{1/4}$ is empty. In that case, it is natural to consider a depth based on the distance $\mathrm{d}_P(\cdot, P_\delta)$ for a different choice of $\delta < \Pi(P)$. Sensible alternatives appear to be $\delta = \Pi(P)/2$, or the largest δ such that the P-probability of P_δ is 1/2. In these situations, $\Pi(P)$ and δ are typically unknown, and must be estimated from a random sample from P.

4. Scatter halfspace depth and the exterior Radon transform

In this final section of our note, we are concerned with a question closely related to Theorem 3.2: Do there exist nonelliptical probability distributions $P \in \mathcal{P}(\mathbb{R}^d)$ whose floating bodies P_δ are ellipsoids for some $\delta > 0$? A trivial positive answer to this general question can be found in [6], Section 4—P_δ of any distribution P whose restriction to $C = \mathbb{R}^d \setminus P_\delta$ is elliptically symmetric is an ellipsoid. Here we resolve a more interesting problem. We construct an example of a distribution whose restriction to $C = \{x \in \mathbb{R}^d \colon \|x\| > 1\}$ is not spherically symmetric, yet $P_\delta = B^d$, the unit ball in $\mathbb{R}^d$. This gives a nontrivial nonelliptical distribution whose maximum scatter halfspace depth (3.1) is 1/2. This example is also interesting in its own right, in connection with the still open *homothety conjecture* ([7], Section 4), a major problem from convex geometry. The proof of the following theorem is based on an advanced result from the theory of Radon transforms [4], [10].

Theorem 4.1. *For any $d = 1, 2, \ldots$, there exists $P \in \mathcal{P}(\mathbb{R}^d)$ whose restriction to $C = \{x \in \mathbb{R}^d \colon \|x\| > 1\}$ is not spherically symmetric, but $P_\delta = B^d$ for some $\delta > 0$.*

P r o o f. Consider the exterior Radon transform $\mathcal{R}$ on C that to an integrable function $f \colon \mathbb{R}^d \to \mathbb{R}$ assigns the collection of all integrals of f over hyperplanes that do not intersect B^d. By the singular value decomposition for the exterior Radon

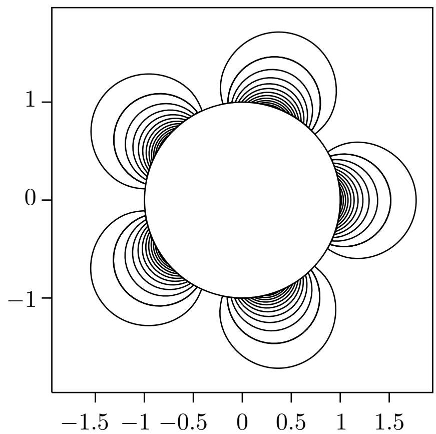

Figure 2. Several contours of the density f from the proof of Theorem 4.1. This gives an example of a distribution that is not spherically symmetric, but the assumptions of Theorem 3.2 are satisfied.

transform ([10], Theorem 3.2), the null space of $\mathcal{R}$ is nontrivial, and surprisingly large. In what follows we take a single bounded, nonelliptically contoured function from the null space of $\mathcal{R}$, and add a small multiple of this function to a spherically symmetric density on $\mathbb{R}^d$. If the resulting function is nonnegative, we may multiply it by a constant to obtain a density of $P \in \mathcal{P}(\mathbb{R}^d)$ with the desired property, as follows from Fubini's theorem.

By the decomposition from Theorem 3.2 from [10], such a function from the null space of $\mathcal{R}$ can be found in any dimension d. For $d = 2$, for instance, we may consider

$$f(x) = \frac{1}{\pi}\Bigl(\frac{1}{\|x\|^4} + \frac{x_1(x_1^4 - 10x_1^2x_2^2 + 5x_2^4)}{\|x\|^{10}}\Bigr)\mathbb{I}[\|x\| > 1], \quad \text{where } x = (x_1, x_2).$$

For $d > 2$, one employs spherical harmonics to obtain analogous examples. It is straightforward to verify that f is a density of a distribution $X \sim P \in \mathcal{P}(\mathbb{R}^2)$, and at the same time $\mathsf{P}(u^\top X \geqslant 1) = 1/4$ for any $u \in \mathbf{S}^1$. Therefore, the floating body $P_{1/4}$ exists, and equals B^2, even though $f(x)$ is (on C) not a function of $\|x\|$ only. $\square$

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