SOME CONSISTENT EXPONENTIALITY TESTS BASED ON PURI-RUBIN AND DESU CHARACTERIZATIONS

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Abstract. We present new goodness-of-fit tests for the exponential distribution based on equidistribution type characterizations. For the construction of the test statistics, we employ an L^2-distance between the corresponding V-empirical distribution functions. The resulting test statistics are V-statistics, free of the scale parameter.

The quality of the tests is assessed through local Bahadur efficiencies as well as the empirical power for small and moderate sample sizes. According to both criteria, for many common alternatives, our tests perform better than the integral and Kolmogorov-type tests based on the same characterizations.

Keywords: goodness-of-fit; asymptotic efficiency; V-statistics; characterization; test for exponentiality

MSC 2020: 62G10, 62G20

1. Introduction

The exponential distribution is one of the most frequently used distributions in probability and statistics. It appears in survival analysis, reliability theory, and many other domains. Therefore, numerous tests for testing the hypothesis of exponentiality based on various ideas have been proposed and studied.

The exponential distribution possesses a substantial amount of characterizations, see, e.g., [10], [8], [1], [16]. As a result, many tests for exponentiality have been developed using some of these characterizations, see, for instance, [19], [20], [9], [6], [14], [15], [13]. Most of these characterizations use the equidistribution

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of two statistics, namely ω_1 equals in distribution ω_2 if and only if the parent statistics has the exponential distribution with an unknown scale or location parameter.

A straightforward way to use an equidistribution-type characterization is to consider the difference of the U-empirical or V-empirical distribution functions (df's) generated by two statistics ω_1 and ω_2, which should be small under the null hypothesis of exponentiality.

There are several approaches for building a test statistic based on such a difference. One way is to integrate with respect to the usual empirical df. The resulting integral statistic can be reduced to some U-statistic which is asymptotically normal and usually has sufficiently high efficiency in the Bahadur sense against common alternatives. The disadvantage of such a test statistic is that it cannot be consistent against *any* alternative.

Another possibility is to take the maximum of modulus of the difference of U- or V-empirical df's. In this way we obtain statistics of the Kolmogorov type. These statistics are consistent with respect to any alternative but usually have relatively low Bahadur efficiency in comparison to the integral-type statistics. This is the common shortcoming of the Kolmogorov-type statistics, see [17].

The aim of the present paper is to introduce test statistics of ω^2-type, namely those based on the integrated squared difference of V-empirical df's, and to study their asymptotic properties. Such statistics are very natural and they are also consistent against any alternative. However, for a long time the quadratic statistics have not been used in the theory of tests based on characterizations. The reason for this is that such statistics lead to V-statistics with complicated *degenerate* kernels. The limiting distribution as well as the calculation of the Bahadur efficiency depend on the spectrum of a Fredholm integral operator with such a kernel. Due to the complexity of these kernels, the spectrum and even the first eigenvalue (which is only necessary for the calculation of Bahadur efficiency) can hardly be found by analytical methods.

However, there are many methods for solving Fredholm integral equations numerically, see, e.g., [21]. Due to the exponentiality of the underlying distribution, we may reduce the interval of integration to a bounded one. Using a standard quadrature method on the finite interval with a large number of equidistant nodes, we arrive at the evaluation of eigenvalues of some large matrix instead of the integral operator. The modern packages like Wolfram Mathematica allow to calculate the first eigenvalues of a matrix with sufficient accuracy. This method in the context of symmetry testing was already used in [4]. In this paper this idea is used for the calculation of the Bahadur efficiency of quadratic tests when testing exponentiality on the basis of its two well-known characterizations. We build corresponding quadratic tests and calculate their local Bahadur efficiency against common alternatives, which turns to be reasonably high.

The structure of the paper is as follows. We formulate the so-called Puri-Rubin and Desu characterizations of exponentiality and construct the corresponding test statistics in Section 2. They turn to be V-statistics with complicated but bounded kernels. In the same section we study asymptotic properties of the new statistics. Section 3 is dedicated to calculations of the local Bahadur efficiency of the new tests. We also provide the description of the procedure for an approximate evaluation of the first eigenvalue of the integral operators related to the kernels of V-statistics built above. We present the values of simulated powers for our tests against a set of common alternatives in Section 4, while Section 5 contains some discussion of the results.

2. Test statistic

Consider the following characterizations from [22] and [7], respectively.

Characterization 2.1 (Puri-Rubin (1970)). *Let X_1 and X_2 be two independent copies of a random variable X with probability density function $f(x)$. Then X and $|X_1 - X_2|$ have the same distribution if and only if $f(x) = \lambda \mathrm{e}^{-\lambda x}$, $x \geqslant 0$, for some $\lambda > 0$.*

Characterization 2.2 (Desu (1971)). *Let $X_1, X_2, \ldots, X_m$ be m independent copies of a non-degenerate random variable X with distribution function $F(x)$. Then for each m, $m \min\{X_1, \ldots, X_m\}$ and X have the same distribution if and only if $F(x) = 1 - \mathrm{e}^{-\lambda x}$, $x \geqslant 0$, for some $\lambda > 0$.*

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d.) non-negative random variables with an unknown absolutely continuous distribution function F. In view of Characterization 2.1 and Characterization 2.2 for $m = 2$, we propose two novel, Cramér-von Mises-type test statistics

$$(2.1)\quad W_n^{\mathcal{P}} = \int_0^\infty \Big(\frac{1}{n} \sum_{i=1}^n \mathrm{I}\{X_i < t\} - \frac{1}{n^2} \sum_{i,j=1}^n \mathrm{I}\{|X_i - X_j| < t\} \Big)^2 \mathrm{d}F_n(t),$$

$$(2.2)\quad W_n^{\mathcal{D}} = \int_0^\infty \Big(\frac{1}{n} \sum_{i=1}^n \mathrm{I}\{X_i < t\} - \frac{1}{n^2} \sum_{i,j=1}^n \mathrm{I}\{2\min\{X_i, X_j\} < t\} \Big)^2 \mathrm{d}F_n(t)$$

for testing the composite hypothesis of exponentiality $H_0 : F(x) = 1 - \mathrm{e}^{-\lambda x}$, $x \geqslant 0$, $\lambda > 0$. After integration, these statistics turn out to be V-statistics of order 5, i.e., we have

$$W_n^{\mathcal{I}} = \frac{1}{n^5} \sum_{i_1,i_2,i_3,i_4,i_5} \Phi^{\mathcal{I}}(X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}, X_{i_5}), \quad \mathcal{I} \in \{\mathcal{P}, \mathcal{D}\},$$

where $\Phi^{\mathcal{I}}$ is a symmetric function of its arguments.

We consider large values of our statistics to be significant. Notice that under the null hypothesis both test statistics are scale free.

2.1. Asymptotic properties. The symmetric kernels of $W_n^{\mathcal{P}}$ and $W_n^{\mathcal{D}}$ are given by

$$\begin{aligned}
&\Phi^{\mathcal{P}}(X_1, \dots, X_5) \\
&\quad = \frac{1}{5!} \sum_{\pi(5)} (\mathrm{I}\{|X_{i_1} - X_{i_2}| < X_{i_5}\}\mathrm{I}\{|X_{i_3} - X_{i_4}| < X_{i_5}\} \\
&\qquad + \mathrm{I}\{X_{i_1} < X_{i_5}\}\mathrm{I}\{X_{i_2} < X_{i_5}\} - 2\mathrm{I}\{X_{i_1} < X_{i_5}\}\mathrm{I}\{|X_{i_2} - X_{i_3}| < X_{i_5}\}), \\
&\Phi^{\mathcal{D}}(X_1, \dots, X_5) \\
&\quad = \frac{1}{5!} \sum_{\pi(5)} (\mathrm{I}\{2\min(X_{i_1}, X_{i_2}) < X_{i_5}\}\mathrm{I}\{2\min(X_{i_3}, X_{i_4}) < X_{i_5}\} \\
&\qquad + \mathrm{I}\{X_{i_1} < X_{i_5}\}\mathrm{I}\{X_{i_2} < X_{i_5}\} - 2\mathrm{I}\{X_{i_1} < X_{i_5}\}\mathrm{I}\{2\min(X_{i_2}, X_{i_3}) < X_{i_5}\}),
\end{aligned}$$

where $\pi(5)$ is the set of all permutations $\{i_1, i_2, i_3, i_4, i_5\}$ of the set $\{1, 2, 3, 4, 5\}$.

It can be shown that the first projections of the kernels of our two test statistics are equal to zero. The second projection is defined as

$$\varphi_2^{\mathcal{I}}(s,t) = E(\Phi^{\mathcal{I}}(X_1, X_2, X_3, X_4, X_5)|X_1 = s, X_2 = t), \quad \mathcal{I} \in \{\mathcal{P}, \mathcal{D}\}.$$

After some calculations, one can obtain that the second projections of the kernels $\Phi^{\mathcal{P}}$ and $\Phi^{\mathcal{D}}$ are given by

$$\begin{aligned}
\varphi_2^{\mathcal{P}}(s,t) &= \frac{1}{30} + \frac{3}{10}(\mathrm{e}^{-2s-t} + \mathrm{e}^{-s-2t}) - \frac{3}{20}(\mathrm{e}^{-2t} + \mathrm{e}^{-2s}) - \frac{16}{15}\mathrm{e}^{-s-t} \\
&\quad + \frac{1}{15}\mathrm{e}^{-\min(s,t)}(2 - 3\min(s,t)) + \frac{1}{30}\mathrm{e}^{-\max(s,t)}(19 - 6\min(s,t)), \\
\varphi_2^{\mathcal{D}}(s,t) &= \frac{8}{75} + \frac{2}{25}(\mathrm{e}^{-5s} + \mathrm{e}^{-5t}) + \frac{2}{15}(\mathrm{e}^{-3s} + \mathrm{e}^{-3t}) \\
&\quad - \frac{1}{20}(\mathrm{e}^{-2s} + \mathrm{e}^{-2t}) - \frac{1}{10}(\mathrm{e}^{-s} + \mathrm{e}^{-t}) + \frac{2}{15}(\mathrm{e}^{-\frac{3}{2}t} - \mathrm{e}^{-3s})\mathrm{I}\{t \leqslant 2s\} \\
&\quad - \frac{1}{10}((1 - \mathrm{e}^{-s})\mathrm{I}\{t \leqslant s\} + (1 - \mathrm{e}^{-t})\mathrm{I}\{s \leqslant t\}) \\
&\quad + \frac{2}{15}(\mathrm{e}^{-\frac{3}{2}s} - \mathrm{e}^{-3t})\mathrm{I}\{s \leqslant 2t\} - \frac{1}{5}\mathrm{e}^{-4\min(s,t)}.
\end{aligned}$$

The following theorem presents the asymptotic distribution of the considered tests statistics under the null hypothesis.

Theorem 2.1. *Let $X_1, \dots, X_n$ be an i.i.d. sample with the distribution function $F(x) = 1 - \mathrm{e}^{-\lambda x}$, $x \geqslant 0$ for some $\lambda > 0$. Then*

$$nW_n^{\mathcal{I}} \xrightarrow{d} 10 \sum_{k=1}^{\infty} \delta_k^{\mathcal{I}} Z_k^2, \tag{2.3}$$

$\mathcal{I} \in \{\mathcal{P}, \mathcal{D}\}$, where $\xrightarrow{d}$ denotes convergence in distribution, $\{\delta_k\}$, $k = 1, 2, \dots$, is the sequence of eigenvalues of the integral operator $A^{\mathcal{I}}$ defined by $A^{\mathcal{I}} q(x) = \int_0^{\infty} \varphi_2^{\mathcal{I}}(x, y) q(y) \mathrm{e}^{-y} \, \mathrm{d}y$, and Z_k, $k = 1, 2, \dots$, are independent standard normal variables.

P r o o f. Since the kernel Φ is bounded and degenerate, the result follows from the theorem for the asymptotic distribution of U-statistics with degenerate kernels [12], Corollary 4.4.2 and the Bönner and Kirschner [3] formula connecting U- and V-statistics, see also [12], §1.3. □

3. Local Bahadur efficiency

Let $\mathcal{G} = \{G(x; \theta), \theta > 0\}$ be a family of alternative distribution functions such that $G(x; 0)$ is exponential and the regularity conditions for V-statistics with weakly degenerate kernels from [18], Assumptions WD are satisfied. For close alternatives from $\mathcal{G}$, the *absolute local Bahadur efficiency* for any sequence $\{T_n\}$ of test statistics is defined as

$$\mathrm{eff}(T) = \lim_{\theta \to 0} \frac{c_T(\theta)}{2K(\theta)},$$

where $c_T(\theta)$ is the Bahadur exact slope, a function proportional to the exponential rate of decrease of the test size when the sample size increases, and $K(\theta)$ is the minimal Kullback-Leibler distance from the alternative to the class of null hypotheses.

The *Bahadur exact slopes* are defined as follows. Suppose that under an alternative the sequence $\{T_n\}$ of test statistics converges in probability to some finite function $b(\theta)$. Suppose also that the large deviations limit

$$\lim_{n \to \infty} n^{-1} \ln P_{H_0}(T_n \geqslant t) = -f(t)$$

exists for any t in an open interval I, on which f is continuous and $\{b(\theta), \theta > 0\} \subset I$. Then the Bahadur exact slope is

$$c_T(\theta) = 2f(b(\theta)). \tag{3.1}$$

For more details on the Bahadur theory we refer to [2] and [17].

Theorem 3.1. *For the statistic $W_n^{\mathcal{I}}$, $\mathcal{I} \in \{\mathcal{P}, \mathcal{D}\}$, and a given alternative density $g(x;\theta)$ from $\mathcal{G}$, the local Bahadur exact slope is given by*

$$c_W^{\mathcal{I}}(\theta) = \frac{1}{\delta_1^{\mathcal{I}}} \int_0^\infty \int_0^\infty \varphi_2^{\mathcal{I}}(x,y) g'_\theta(x;0) g'_\theta(y;0)\,\mathrm{d}x\,\mathrm{d}y \cdot \theta^2 + o(\theta^2), \quad \theta \to 0,$$

where $\delta_1^{\mathcal{I}}$ is the largest eigenvalue of the integral operator $A^{\mathcal{I}}$ defined by $A^{\mathcal{I}} q(x) = \int_0^\infty \varphi_2^{\mathcal{I}}(x,y) q(y) \mathrm{e}^{-y}\,\mathrm{d}y$.

The proof of this theorem follows from [18], Theorem 4.

To calculate the efficiency one needs to find $\delta_1^{\mathcal{I}}$, the largest eigenvalue. Since we cannot obtain this eigenvalue analytically, we use the following approximation (see also [4]).

First, we consider the "symmetrized" operator

$$\overline{A}^{\mathcal{I}} q(x) = \int_0^\infty \varphi_2^{\mathcal{I}}(x,y) q(y) \mathrm{e}^{-x/2} \cdot \mathrm{e}^{-y/2}\,\mathrm{d}y, \tag{3.2}$$

which has the same spectrum as the operator $A^{\mathcal{I}}$.

Next, let us consider the "truncated" $T^{\mathcal{I}}$ operator on the set of real functions with support $[0,B]$ defined by

$$T^{\mathcal{I}} q(x) = \int_0^\infty \varphi_2^{\mathcal{I}}(x,y) q(y) \mathrm{e}^{-x/2} \cdot \mathrm{e}^{-y/2} \mathrm{I}(y \leqslant B)(1-\mathrm{e}^{-B})^{-1}\,\mathrm{d}y.$$

For a sufficiently large B, the operators $T^{\mathcal{I}}$ and $\overline{A}^{\mathcal{I}}$ differ on a set of a negligible measure.

The sequence of symmetric linear operators defined by $(m+1)\times(m+1)$ matrices $M^{(m)} = \|m_{i,j}^{(m)}\|$, $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant m$, where

$$m_{i,j}^{(m)} = \varphi_2^{\mathcal{I}}\Big(\frac{Bi}{m}, \frac{Bj}{m}\Big) \sqrt{\mathrm{e}^{Bi/m} - \mathrm{e}^{B(i+1)/m}} \cdot \sqrt{\mathrm{e}^{Bj/m} - \mathrm{e}^{B(j+1)/m}} \cdot \frac{1}{1-\mathrm{e}^{-B}}, \tag{3.3}$$

converges in norm to $T^{\mathcal{I}}$ when $m \to \infty$. The operators $T^{\mathcal{I}}$ and $M^{(m)}$ are symmetric and self-adjoint, and the norm of their difference tends to zero as m tends to infinity. Using the perturbation theory, see [11], Theorem 4.10, page 291, we obtain that the spectra of these two operators are at the distance that tends to zero. Hence, $\delta_1^{(m)}$—the sequence of the largest eigenvalues of $M^{(m)}$—must converge to $\delta_1^{\mathcal{I}}(B)$, the largest eigenvalue of $T^{\mathcal{I}}$. When B goes to ∞, the eigenvalue $\delta_1^{\mathcal{I}}(B)$ approaches $\delta_1^{\mathcal{I}}$. Hence, for B large enough, $\delta_1^{\mathcal{I}}(B)$ and $\delta_1^{\mathcal{I}}$ coincide up to a desired number of digits.

We apply this approximation procedure for $B = 10$. The values for different m are presented in Table 1. The calculations were performed using the Wolfram Mathematica 11.0 software.

m	1000	2000	3000	4000	5000	6000
$\delta_1^{\mathcal{P}}$	$6.11 \cdot 10^{-3}$	$6.05 \cdot 10^{-3}$	$6.03 \cdot 10^{-3}$	$6.02 \cdot 10^{-3}$	$6.01 \cdot 10^{-3}$	$6.01 \cdot 10^{-3}$
$\delta_1^{\mathcal{D}}$	$4.37 \cdot 10^{-3}$	$4.42 \cdot 10^{-3}$	$4.44 \cdot 10^{-3}$	$4.45 \cdot 10^{-3}$	$4.46 \cdot 10^{-3}$	$4.46 \cdot 10^{-3}$

Table 1. The largest eigenvalue of $M^{(m)}$.

We consider the following four classes of alternatives

▷ a Weibull distribution with density

$$g(x;\theta) = \mathrm{e}^{-x^{1+\theta}}(1+\theta)x^{\theta}, \quad \theta > 0,\ x \geqslant 0; \tag{3.4}$$

▷ a gamma distribution with density

$$g(x;\theta) = \frac{x^{\theta}\mathrm{e}^{-x}}{\Gamma(\theta+1)}, \quad \theta > 0,\ x \geqslant 0; \tag{3.5}$$

▷ a linear failure rate (LFR) distribution with density

$$g(x;\theta) = \mathrm{e}^{-x-\theta x^2/2}(1+\theta x), \quad \theta > 0,\ x \geqslant 0; \tag{3.6}$$

▷ a mixture of exponential distributions with negative weights (EMNW(β)) with density

$$g(x;\theta) = (1+\theta)\mathrm{e}^{-x} - \theta\beta\mathrm{e}^{-\beta x}, \quad \theta \in \Big(0, \frac{1}{\beta-1}\Big], \ x \geqslant 0.$$

In Table 2 we present the absolute local Bahadur efficiencies of our tests accompanied with the local Bahadur efficiencies of integral and Kolmogorov-type tests based on the same two characterizations whose statistics are given below:

$$I_n^{\mathcal{P}} = \int_0^{\infty}\Big(\frac{1}{n}\sum_{i=1}^{n}\mathrm{I}\{X_i < t\} - \frac{1}{n^2}\sum_{i,j=1}^{n}\mathrm{I}\{|X_i - X_j| < t\}\Big)\,\mathrm{d}F_n(t);$$

$$D_n^{\mathcal{P}} = \sup_{t\geqslant 0}\Big|\frac{1}{n}\sum_{i=1}^{n}\mathrm{I}\{X_i \leqslant t\} - \frac{1}{n^2}\sum_{i,j=1}^{n}\mathrm{I}\{|X_i - X_j| \leqslant t\}\Big|;$$

$$I_n^{\mathcal{D}} = \int_0^{\infty}\Big(\frac{1}{n}\sum_{i=1}^{n}\mathrm{I}\{X_i < t\} - \frac{1}{n^2}\sum_{i,j=1}^{n}\mathrm{I}\{\min(X_i, X_j) < t\}\Big)\,\mathrm{d}F_n(t);$$

$$D_n^{\mathcal{D}} = \sup_{t\geqslant 0}\Big|\frac{1}{n}\sum_{i=1}^{n}\mathrm{I}\{X_i \leqslant t\} - \frac{1}{n^2}\sum_{i,j=1}^{n}\mathrm{I}\{2\min(X_i, X_j) \leqslant t\}\Big|.$$

Bahadur efficiencies of these tests can be found in [5].

	Weibull	Gamma	LFR	EMNW(3)
$I_n^{\mathcal{P}}$	0.821	0.788	0.337	0.949
$D_n^{\mathcal{P}}$	0.437	0.448	0.192	0.591
$W_n^{\mathcal{P}}$	0.733	0.719	0.308	0.891
$I_n^{\mathcal{D}}$	0.697	0.790	0.149	0.746
$D_n^{\mathcal{D}}$	0.158	0.174	0.073	0.247
$W_n^{\mathcal{D}}$	0.485	0.544	0.138	0.600

Table 2. Absolute local Bahadur efficiency.

We can notice that the new tests have reasonable to high efficiencies and are always more efficient than the corresponding Kolmogorov-type tests, while slightly less efficient but comparable to the integral-type tests. In addition, the tests based on the Puri-Rubin characterization are more efficient than those based on the Desu characterization.

4. Power study

The empirical sizes and the powers of our tests are estimated by the Monte Carlo method with 10 000 replicates at the level of significance 0.05. The set of alternatives—commonly used in the literature—is given as

▷ a Weibull W(θ) distribution with density

$$g(x;\theta) = \mathrm{e}^{-x^{\theta}}\theta x^{\theta-1}, \quad \theta > 0,\ x \geqslant 0;$$

▷ a gamma $\Gamma(\theta)$ distribution with density

$$g(x;\theta) = \frac{x^{\theta-1}\mathrm{e}^{-x}}{\Gamma(\theta)}, \quad \theta > 0,\ x \geqslant 0;$$

▷ a half-normal HN distribution with density

$$g(x) = \sqrt{\frac{2}{\pi}}\mathrm{e}^{-x^2/2}, \quad x \geqslant 0;$$

▷ a uniform U distribution with density

$$g(x) = 1, \quad 0 \leqslant x \leqslant 1;$$

▷ a Chen's CH(θ) distribution with density

$$g(x;\theta) = 2\theta x^{\theta-1}\mathrm{e}^{x^\theta - 2(1-\mathrm{e}^{x^\theta})}, \quad x \geqslant 0;$$

▷ a linear failure rate LF(θ) distribution with density

$$g(x;\theta) = \mathrm{e}^{-x-\theta x^2/2}(1+\theta x), \quad \theta > 0,\ x \geqslant 0;$$

▷ a modified extreme value EV(θ) distribution with density

$$g(x;\theta) = \frac{1}{\theta}\mathrm{e}^{(1-\mathrm{e}^x)/\theta + x}, \quad x \geqslant 0;$$

▷ a log-normal LN(θ) distribution with density

$$g(x;\theta) = \frac{1}{x\sqrt{2\pi\theta^2}}\mathrm{e}^{-(\log x)^2/(2\theta^2)}, \quad x \geqslant 0;$$

▷ a Dhillon DL(θ) distribution with density

$$g(x;\theta) = \frac{\theta+1}{x+1}(\log(x+1))^\theta \mathrm{e}^{-(\log(x+1))^{\theta+1}}, \quad x \geqslant 0.$$

The results are shown in Table 3 (sample size of 20) and in Table 4 (sample size of 50). In most cases the tests based on the Puri-Rubin characterization are more powerful. The noticeable exceptions are heavy-tailed alternatives ($\Gamma(0.4)$ and W(0.7)), in which case the tests based on the Desu characterization are superior. It is important to notice that the integral-type and Cramér-von Mises-type tests have similar power, however we recommend the latter, taking into account their consistency.

Alt.	Exp(1)	W(1.4)	$\Gamma(2)$	HN	U	CH(0.5)	CH(1)	CH(1.5)	LF(2)	LF(4)	EV(1.5)	LN(0.8)	LN(1.5)	DL(1)	DL(1.5)	W(0.7)	$\Gamma(0.4)$
$I_n^{\mathcal{P}}$	5	46	60	28	77	0	21	88	38	52	54	43	0	33	77	0	0
$D_n^{\mathcal{P}}$	5	38	52	23	65	0	16	76	30	42	44	47	1	32	69	0	0
$W_n^{\mathcal{P}}$	5	47	60	27	75	2	20	61	35	50	55	48	0	34	76	2	0
$I_n^{\mathcal{D}}$	5	39	56	19	49	46	15	74	27	38	37	57	5	37	77	20	65
$D_n^{\mathcal{D}}$	5	23	34	13	42	0	8	49	15	23	23	39	3	24	50	1	0
$W_n^{\mathcal{D}}$	5	27	42	15	45	0	11	86	20	30	30	44	3	27	59	1	5

Table 3. Percentage of rejected hypotheses for $n = 20$.

Alt.	Exp(1)	W(1.4)	Γ(2)	HN	U	CH(0.5)	CH(1)	CH(1.5)	LF(2)	LF(4)	EV(1.5)	LN(0.8)	LN(1.5)	DL(1)	DL(1.5)	W(0.7)	Γ(0.4)
$I_n^{\mathcal{P}}$	5	85	96	56	99	14	41	100	72	89	90	86	0	69	99	1	37
$D_n^{\mathcal{P}}$	5	75	91	46	98	23	29	99	60	78	83	90	9	67	98	7	51
$W_n^{\mathcal{P}}$	5	85	95	54	99	16	38	100	69	86	90	89	1	70	99	2	44
$I_n^{\mathcal{D}}$	5	74	94	37	83	94	24	99	51	72	68	95	18	74	99	64	99
$D_n^{\mathcal{D}}$	5	47	69	23	87	9	15	89	33	49	50	75	3	50	88	5	20
$W_n^{\mathcal{D}}$	5	59	82	29	86	69	19	96	41	60	61	90	15	60	96	24	88

Table 4. Percentage of rejected hypotheses for $n = 50$.

5. Conclusions

In this paper we proposed a new class of characterization based exponentiality tests of Cramér-von Mises-type. We derived their asymptotic distributions, and calculated, for the first time for such a class of tests, their local Bahadur efficiencies against some common alternatives. It turned out that our new tests have much higher local Bahadur efficiencies than the Kolmogorov type tests based on the same characterization and the tests comparable to the corresponding integral type statistics. Taking into account their consistency against all fixed alternatives, they are good candidates to be considered in a battery of exponentiality tests. This conclusion is backed up by the power study in small sample size cases.

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