GLOBAL STRONG SOLUTIONS OF A 2-D NEW MAGNETOHYDRODYNAMIC SYSTEM

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Abstract. The main objective of this paper is to study the global strong solution of the parabolic-hyperbolic incompressible magnetohydrodynamic model in the two dimensional space. Based on Agmon, Douglis, and Nirenberg's estimates for the stationary Stokes equation and Solonnikov's theorem on L^p-L^q-estimates for the evolution Stokes equation, it is shown that this coupled magnetohydrodynamic equations possesses a global strong solution. In addition, the uniqueness of the global strong solution is obtained.

Keywords: global strong solution; magnetohydrodynamics; Stokes equation; L^p-L^q-estimates

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1. Introduction

We consider the following 2-D incompressible magnetohydrodynamic (MHD) model, which describes the interaction between moving conductive fluid flows and electromagnetic fields in [11],

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u = \nu\Delta u - \dfrac{1}{\varrho_0}\nabla p + \dfrac{\varrho_e}{\varrho_0}u\times \operatorname{curl} A + f(x) & \text{in } \Omega\times[0,T), \\ \dfrac{\partial^2 A}{\partial t^2} = \dfrac{1}{\varepsilon_0\mu_0}\Delta A + \dfrac{\varrho_e}{\varepsilon_0}u - \nabla\Phi, \quad \nabla\cdot u = 0, \quad \nabla\cdot A = 0 & \text{in } \Omega\times[0,T). \end{cases} \tag{1.1}$$

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Here $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain, $T > 0$ is any fixed time, $u(x,t) = (u_1(x,t), u_2(x,t))$, $A(x,t) = (A_1(x,t), A_2(x,t))$, $p(x,t)$ are the velocity field, the magnetic potential, and the pressure function, respectively, and $\Phi = \partial A_0/\partial t$ represents the magnetic pressure with the scalar electromagnetic potential A_0. The constants ν, ϱ_0, ϱ_e, ε_0, μ_0 denote the kinetic viscosity, mass density, equivalent charge density, electric permittivity, and magnetic permeability of free space, respectively.

In this paper, we focus on the system (1.1) with the initial-boundary conditions

$$u(0,x) = \varphi(x), \quad A(0,x) = \psi(x), \quad A_t(0,x) = \eta(x) \quad \text{in } \Omega, \tag{1.2}$$

$$u(t,x) = 0, \quad A(t,x) = 0 \quad \text{on } \partial\Omega \times [0,T). \tag{1.3}$$

Note that the authors have established the N-dimensional ($N \geqslant 2$) new MHD model (1.1) in [11] based on the fundamental physical principles – the Newton's second law and the Maxwell equations for the electromagnetic fields. Moreover, it is worth noticing that the new model (1.1) is established without any assumptions, which implies that (1.1) is a physical principle for the incompressible magnetohydrodynamics. Differing from the classical MHD equation, the MHD model (1.1) describes the motion of plasma under the standard Coulomb gauge and is also compatible with the Maxwell equations. In addition, the global weak solutions of the corresponding 3-D MHD model (1.1) with initial-boundary conditions has been obtained by using the Galerkin technique and standard energy estimates in [11]. In this paper, what we are mainly concerned with is the existence and regularity of a global strong solution of the 2-D MHD model (1.1) with the initial-boundary conditions (1.2)–(1.3).

It is known that there have been huge mathematical studies on the existence of solutions to the N-dimensional ($N \geqslant 2$) classical MHD model established by Chandrasekhar [4]. In particular, Duvaut and Lions [5] constructed a global weak solution and the local strong solution to the 3-D classical MHD equations with the initial boundary value problem, and the properties of such solutions have been investigated by Sermange and Temam in [18]. Furthermore, some sufficient conditions for smoothness were presented for the weak solution to the 3-D classical MHD equations in [7] and some sufficient conditions for local regularity of a suitable weak solution to the 3-D classical MHD system for the points belonging to a C^3-smooth part of the boundary were obtained in [21]. Also, the global strong solutions for heat conducting 3-D classical magnetohydrodynamic flows with non-negative density were proved in [26]. For other various results related to the classical MHD model, we refer to [10], [13], [17], [22], and the references therein.

Furthermore, let us recall some known results for the 2-D classical and generalized MHD equations. It is noticed in [5], [18] that the 2-D classical MHD equations admit a unique global strong solution. Then, Ren, Wu et al. [16] have proved the global existence and the decay estimates of small smooth solutions for the 2-D classical MHD equations without magnetic diffusion and Cao, Regmi, and Wu [3] have obtained the global regularity for the 2-D classical MHD equations with mixed partial dissipation and magnetic diffusion. Besides, Regmi [15] established the global weak solution for the 2-D classical MHD equations with partial dissipation and vertical diffusion. There are also very interesting investigations concerning the existence of strong solutions to the 2-D classical and generalized MHD equations, see [8], [9], [14], [18], [23], [24], [25], and the references therein.

However, it is worth pointing out that the incompressible MHD system (1.1) is a mixed-type model which is combined with the parabolic equation $(1.1)_1$ and the hyperbolic equation $(1.1)_2$. The main challenge in obtaining global strong solution of the 2-D MHD model (1.1) with (1.2)–(1.3) is the estimate for $\|u \times \operatorname{curl} A\|_{L^\infty(0,T;L^2)}$ and $\|(u\cdot\nabla)u\|_{L^\infty(0,T;L^2)}$. The difficulty is overcome by applying the Solonnikov's theorem [6], [12], [19] on L^p-L^q-estimates for the non-stationary Stokes equations and Agmon, Douglis, and Nirenberg's estimates [1], [2], [12] for the stationary Stokes equations. As we know, Solonnikov [19] first gave the proof of Maximal L^p-L^q-estimates for the Stokes equation (2.3) using potential theoretic arguments. Recently, Geissert, Hess, Hieber et al. [6] provided a short proof of the corresponding Solonnikov's theorem in [19].

The rest of this article is organized as follows. In Section 2, we introduce some elementary function spaces, a vital embedding theorem and some regularity results of both the non-stationary and stationary Stokes equations. Section 3 is mainly devoted to giving the main results and proofs.

2. Preliminaries

2.1. Notation and definitions. First, we introduce some notation and conventions used throughout this paper.

Let $\Omega \subset \mathbb{R}^2$ be a bounded sufficiently smooth domain. Let $H^\tau(\Omega)$ $(\tau = 1, 2)$ be the general Sobolev space on Ω with the norm $\|\cdot\|_{H^\tau}$ and $L^2(\Omega)$ the Hilbert space with the usual norm $\|\cdot\|$. By the space $H_0^1(\Omega)$ we mean the completion of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_{H^1}$. If $\mathfrak{F}$ is a Banach space, we denote by $L^p(0,T;\mathfrak{F})$ $(1 < p < \infty)$ the Banach space of the $\mathfrak{F}$-valued functions defined in the interval $(0,T)$ that are L^p-integrable.

We also consider the following spaces of divergence-free functions (see Temam [20]):

$$X = \{u \in C_0^\infty(\Omega, \mathbb{R}^2);\ \operatorname{div} u = 0 \quad \text{in } \Omega\},$$

$$Y = \text{the closure of } X \quad \text{in } L^2(\Omega,\mathbb{R}^2) = \{u \in L^2(\Omega,\mathbb{R}^2);\ \operatorname{div} u = 0 \quad \text{in } \Omega\},$$
$$Z = \text{the closure of } X \quad \text{in } H^1(\Omega,\mathbb{R}^2) = \{u \in H_0^1(\Omega,\mathbb{R}^2);\ \operatorname{div} u = 0 \quad \text{in } \Omega\}.$$

Definition 2.1. Suppose that $\varphi, \eta \in Y$, $\psi \in Z$. For any $T > 0$, a vector function (u, A) is called a global weak solution of problem (1.1)–(1.3) on $(0,T)\times\Omega$ if it satisfies the following conditions:

1. $u \in L^2(0,T;Z) \cap L^\infty(0,T;Y)$,
2. $A \in L^\infty(0,T;Z)$, $A_t \in L^\infty(0,T;Y)$,
3. for any function $v \in X$,

$$\int_\Omega u\cdot v \,\mathrm{d}x + \int_0^t\int_\Omega \Big((u\cdot\nabla)u\cdot v + \nu\nabla u\cdot\nabla v - \frac{\varrho_e}{\varrho_0}(u\times\operatorname{curl} A)\cdot v\Big)\,\mathrm{d}x\,\mathrm{d}t = \int_0^t\int_\Omega f\cdot v\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \varphi\cdot v\,\mathrm{d}x$$

and

$$\int_\Omega \frac{\partial A}{\partial t}\cdot v\,\mathrm{d}x + \int_0^t\int_\Omega\Big(\frac{1}{\varepsilon_0\mu_0}\nabla A\cdot\nabla v + \frac{\varrho_e}{\varepsilon_0\mu_0}u\cdot v\Big)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \eta v\,\mathrm{d}x.$$

Now, we define the strong solution of the problem (1.1)–(1.3).

Definition 2.2. Suppose that $\varphi, \psi \in H^2(\Omega,\mathbb{R}^2)\cap Z$, $\eta \in Z$. Then (u, A) is called a global strong solution to (1.1)–(1.3), if (u, A) satisfies

$$\begin{aligned}
&u \in L^\infty(0,T;H^2(\Omega,\mathbb{R}^2)\cap Z),\ u_t \in L^\infty(0,T;Y)\cap L^2(0,T;Z),\\
&p \in L^\infty(0,T;H^1(\Omega)),\\
&A \in L^\infty(0,T;H^2(\Omega,\mathbb{R}^2)\cap Z),\ A_t \in L^\infty(0,T;Z),\ A_{tt} \in L^\infty(0,T;Y),\\
&\Phi \in L^\infty(0,T;H^1(\Omega)),
\end{aligned}$$

where $0 < T < \infty$. Furthermore, (1.1) holds almost everywhere in $\Omega \times (0,T)$, and (1.2) holds pointwise in Ω.

2.2. Crucial lemmas. Some lemmas will be frequently used later. One is the following embedding result [12], so we omit the proof.

Lemma 2.1. *For any $k \geqslant 0$, the following inclusion holds:*

$$L^p(0,T,W^{k+1,p}(\Omega)) \cap L^\infty(0,T;L^r(\Omega)) \subset L^q(0,T;W^{k,q}(\Omega)), \tag{2.1}$$

where $q = (r(k+1)p + np)/(rk + n)$. In the special case of $k = 0$, (2.1) reduces to

$$L^p(0,T;W^{1,p}(\Omega)) \cap L^\infty(0,T;L^r(\Omega)) \subset L^q((\Omega)\times(0,T)), \tag{2.2}$$

provided that $q = (n + r)p/n$.

Another lemma is responsible for the estimates for u, p, u_t and follows from the L^p-L^q-estimates [6], [19] for non-stationary Stokes equations. For its proof, we refer to [6], [19].

Let us consider the Stokes equations

$$
\begin{cases} \dfrac{\partial u}{\partial t} = \nu \Delta u - \nabla p + f(x,t), \\ \nabla \cdot u = 0, \\ u|_{\partial\Omega} = 0, \\ u(0) = u_0, \end{cases} \tag{2.3}
$$

where $\nu > 0$ is a constant.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a domain with compact C^3-boundary, $1 < r,\ p < \infty$, $0 < T < \infty$. Then for any $f \in L^r(0,T; L^q(\Omega, \mathbb{R}^n))$ and $u_0 \in W^{2,q}(\Omega, \mathbb{R}^n)$, there exists a unique solution (u, p) of (2.3) satisfying*

$$
\begin{aligned} &u \in L^r(0,T; W^{2,q}(\Omega, \mathbb{R}^n)),\ u_t \in L^r(0,T; L^q(\Omega, \mathbb{R}^n)), \\ &p \in L^r(0,T; W^{1,q}(\Omega)), \end{aligned}
$$

such that

$$
\|u\|_{L^r(0,T;W^{2,q})} + \|u_t\|_{L^r(0,T;L^q)} + \|p\|_{L^r(0,T;W^{1,q})} \leqslant C(\|f\|_{L^r(0,T;L^q)} + \|u_0\|_{W^{2,q}}),
$$

where $C > 0$ is a constant.

Finally, we give some regularity results for the stationary Stokes system. For their proof, we refer to [1], [2], [12].

Lemma 2.3. *Assume that $(v, p) \in W^{2,p}(\Omega, \mathbb{R}^n) \times W^{1,p}(\Omega)$ $(1 < p < \infty)$ is a solution of the stationary Stokes equations*

$$
\begin{cases} -\nu \Delta v - \nabla p = F(x) & \text{in } \Omega, \\ \nabla \cdot v = 0 & \text{in } \Omega, \\ v|_{\partial\Omega} = 0 & \text{on } \partial\Omega, \end{cases}
$$

and $F \in W^{k,q}(\Omega, \mathbb{R}^n)$ $(k \geqslant 0,\ 1 < q < \infty)$. Then

$$
(v, p) \in W^{k+2,q}(\Omega, \mathbb{R}^n) \times W^{k+1,q}(\Omega)
$$

and

$$
\|v\|_{W^{k+2,q}} + \|p\|_{W^{k+1,q}} \leqslant C(\|F\|_{W^{k,q}} + \|(u, p)\|_{L^q})
$$

with some constant C depending on n, Ω, and q.

3. Main results and proofs

In this section, we state the existence results of the global weak solution and the global strong solution for the 2-D problem (1.1)–(1.3), and the partial regularity for the 3-D problem (1.1)–(1.3), and also prove them.

3.1. Strong solutions of the 2-D new MHD model.

Theorem 3.1. *Let the initial values be $\varphi, \eta \in Y$, $\psi \in Z$. If $f \in Y$, then there exists a global weak solution for the problem* (1.1)–(1.3).

Proof. By the standard Galerkin method and estimates similar to those in [11], the existence of global weak solution of (1.1)–(1.3) is also valid, so we omit the proof. □

Theorem 3.2. *Let Ω be a bounded domain with compact C^3-boundary. If φ, $\psi \in H^2(\Omega, \mathbb{R}^2) \cap Z$, $\eta \in Z$ for any $f \in Y$, then there exists a unique global strong solution to the problem* (1.1)–(1.3), *i.e., for any $0 < T < \infty$,*

$$\begin{aligned}
&u \in L^\infty(0,T; H^2(\Omega,\mathbb{R}^2) \cap Z),\ u_t \in L^\infty(0,T;Y) \cap L^2(0,T;Z),\\
&p \in L^\infty(0,T;H^1(\Omega)),\\
&A \in L^\infty(0,T;H^2(\Omega,\mathbb{R}^2)\cap Z),\ A_t \in L^\infty(0,T;Z),\ A_{tt} \in L^\infty(0,T;Y),\\
&\Phi \in L^\infty(0,T;H^1(\Omega)).
\end{aligned}$$

Proof. The proof will be divided into 3 steps. We will use the same generic constant C to denote various constants that depend on $\mu_0, \varrho_0, \varrho_e, \varepsilon_0$, and T only.

Step 1. The estimates and regularity for A.

From Theorem 3.1, for any $0 < T < \infty$ we get the global weak solution

$$\begin{aligned}
&u \in L^2(0,T;Z) \cap L^\infty(0,T;Y),\\
&A \in L^\infty(0,T;Z), \quad A_t \in L^\infty(0,T;Y).
\end{aligned} \tag{3.1}$$

Multiplying both sides of $(1.1)_2$ by $-\Delta A_t$ and integrating over Ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \Big(|\nabla A_t|^2 + \frac{1}{\varepsilon_0\mu_0}|\Delta A|^2\Big)\,\mathrm{d}x\Big) = \frac{\varrho_e}{\varepsilon_0}\int_\Omega \nabla u \nabla A_t\,\mathrm{d}x \tag{3.2}$$

since $\operatorname{div} A = 0$ and (1.3).

Using the Hölder inequality, it is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\nabla A_t\|_{L^2}^2 + \frac{1}{\varepsilon_0\mu_0}\|\Delta A\|_{L^2}^2\Big) \leqslant 2\Big(\|\nabla A_t\|_{L^2}^2 + \frac{1}{\varepsilon_0\mu_0}\|\Delta A\|_{L^2}^2 + \frac{\varrho_e^2}{\varepsilon_0^2}\|\nabla u\|_{L^2}^2\Big). \tag{3.3}$$

Then, by the Gronwall inequality, (3.3) implies

$$\|\nabla A_t\|_{L^2}^2 + \|\Delta A\|_{L^2}^2 \leqslant \mathrm{e}^{2T} C\Big(\|\Delta\psi\|_{L^2}^2 + \|\nabla\eta\|_{L^2}^2 + 2\frac{\varrho_e^2}{\varepsilon_0^2}\int_0^T \|\nabla u\|_{L^2}^2\,\mathrm{d}s\Big) \tag{3.4}$$

for all $0 < T < \infty$.

Therefore, we conclude that

$$\nabla A_t \in L^\infty(0,T;Y), \quad \Delta A \in L^\infty(0,T;Y). \tag{3.5}$$

Next, we need to derive an estimate for $\|A_{tt}\|_{L^\infty(0,T;Y)}$.

Multiplying both sides of $(1.1)_2$ by A_{tt} and integrating over Ω leads to

$$\int_\Omega |A_{tt}|^2\,\mathrm{d}x = \frac{1}{\varepsilon_0\mu_0}\int_\Omega \Delta A A_{tt}\,\mathrm{d}x + \frac{\varrho_e}{\varepsilon_0}\int_\Omega u A_{tt}\,\mathrm{d}x, \tag{3.6}$$

since $-\int_\Omega \nabla\Phi A_{tt}\,\mathrm{d}x = \int_\Omega \Phi\,\mathrm{div}\,A_{tt}\,\mathrm{d}x = 0$.

Using the Hölder inequality and the Young inequality, we deduce from (3.6) that

$$\int_\Omega |A_{tt}|^2\,\mathrm{d}x \leqslant \frac{1}{\varepsilon_0^2\mu_0^2}\int_\Omega |\Delta A|^2\,\mathrm{d}x + \frac{\varrho_e^2}{\varepsilon_0^2}\int_\Omega |u|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |A_{tt}|^2\,\mathrm{d}x. \tag{3.7}$$

It is easy to see that

$$\operatorname*{ess\,sup}_{0\leqslant t\leqslant T}\int_\Omega |A_{tt}|^2\,\mathrm{d}x \leqslant \sup_{0\leqslant t\leqslant T}\frac{2}{\varepsilon_0^2\mu_0^2}\int_\Omega |\Delta A|^2\,\mathrm{d}x + \sup_{0\leqslant t\leqslant T}\frac{2\varrho_e^2}{\varepsilon_0^2}\int_\Omega |u|^2\,\mathrm{d}x. \tag{3.8}$$

Putting the estimates (3.1), (3.5), and (3.8) together, we have

$$A_{tt} \in L^\infty(0,T;Y). \tag{3.9}$$

Hence, (3.5) and (3.9) imply the regularity for A.

Step 2. The $L^{4/3}$-$L^{4/3}$-estimates for $u\cdot\nabla u$ and $u\times A$.

From (3.1) and Lemma 2.3 (the case that $k=0$), it is easy to check that

$$u \in L^4((0,T)\times\Omega). \tag{3.10}$$

Note that

$$\int_0^T\!\!\int_\Omega |Du|^{4/3}|u|^{4/3}\,\mathrm{d}x\,\mathrm{d}t \leqslant \Big(\int_0^T\!\!\int_\Omega |Du|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{2/3}\Big(\int_0^T\!\!\int_\Omega |u|^4\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/3}, \tag{3.11}$$

which implies that

$$u\cdot\nabla u\in L^{4/3}(0,T;L^{4/3}(\Omega,\mathbb{R}^2)). \tag{3.12}$$

Combining (3.1) and (3.10), we get

$$\begin{aligned}\int_0^T\int_\Omega |u\times \operatorname{curl} A|^{4/3}\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \Big(\int_0^T\int_\Omega |u|^4\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/3}\Big(\int_0^T\int_\Omega |\operatorname{curl} A|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{2/3}\\ &\leqslant C\Big(\int_0^T\int_\Omega |u|^4\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/3}\Big(\int_0^T\int_\Omega |\nabla A|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{2/3}\\ &< \infty,\end{aligned} \tag{3.13}$$

which in turn implies that

$$u\times \operatorname{curl} A\in L^{4/3}(0,T;L^{4/3}(\Omega,\mathbb{R}^2)). \tag{3.14}$$

Recall that (u,p) satisfies the Stokes system

$$\begin{cases}\dfrac{\partial u}{\partial t}=\nu\Delta u-\dfrac{1}{\varrho_0}\nabla p+F(x,t),\\ \nabla\cdot u=0,\\ u|_{\partial\Omega}=0,\\ u(0)=\varphi,\end{cases} \tag{3.15}$$

where $F(x,t)=f-(u\cdot\nabla)u+(\varrho_e/\varrho_0)(u\times\operatorname{curl} A)$.

By (3.12) and (3.14), we get $F\in L^{4/3}(0,T;L^{4/3}(\Omega,\mathbb{R}^2))$. Applying this to Lemma 2.4, we obtain that

$$\begin{aligned}&u\in L^{4/3}(0,T;W^{2,4/3}(\Omega,\mathbb{R}^2)),\quad u_t\in L^{4/3}(0,T;L^{4/3}(\Omega,\mathbb{R}^2)),\\ &p\in L^{4/3}(0,T;W^{1,4/3}(\Omega)).\end{aligned} \tag{3.16}$$

In the next step, Lemma 2.5 will be used, since (3.15) can be rewritten as the Stokes equations

$$\begin{cases}-\nu\Delta u+\dfrac{1}{\varrho_0}\nabla p=\widetilde{F}(x,t),\\ \nabla\cdot u=0,\\ u|_{\partial\Omega}=0,\\ u(0)=\varphi,\end{cases} \tag{3.17}$$

where $\widetilde{F}(x,t)=f-(u\cdot\nabla)u+(\varrho_e/\varrho_0)(u\times\operatorname{curl} A)-u_t$.

Step 3. The estimate for $\|\widetilde{F}\|_{L^\infty(0,T;L^2(\Omega,\mathbb{R}^2))}$.

(i) The estimate for $\|\nabla u\|_{L^\infty(0,T;L^2(\Omega,\mathbb{R}^2))}$.

Multiplying $(1.1)_1$ by u_t and integrating over Ω, we have

$$\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^2\,\mathrm{d}x+\int_\Omega |u_t|^2\,\mathrm{d}x=\int_\Omega\Big(-(u\cdot\nabla)u\cdot u_t+\frac{\varrho_e}{\varrho_0}(u\times\operatorname{curl}A)u_t+fu_t\Big)\,\mathrm{d}x. \tag{3.18}$$

Note that the following continuous embeddings hold:

$$W^{2,\frac{4}{3}}(\Omega,\mathbb{R}^2)\hookrightarrow W^{1,4}(\Omega,\mathbb{R}^2)\hookrightarrow C^{1/2}(\Omega,\mathbb{R}^2)\hookrightarrow C^0(\Omega,\mathbb{R}^2). \tag{3.19}$$

Combining (3.19), the Hölder inequality and the ε-Young inequality, we derive that

$$\int_\Omega |(u\cdot\nabla)u\cdot u_t|\,\mathrm{d}x\leqslant C\|u_t\|_{L^2}\|u\|_{C^0}\|\nabla u\|_{L^2}\leqslant\frac{1}{4}\|u_t\|_{L^2}^2+C^2\|u\|_{C^0}^2\|\nabla u\|_{L^2}^2 \tag{3.20}$$

and

$$\begin{aligned}\frac{\varrho_e}{\varrho_0}\int_\Omega |(u\times\operatorname{curl}A)u_t|\,\mathrm{d}x&\leqslant C\|u\|_{C^0}\|\nabla A\|_{L^2}\|u_t\|_{L^2}\\&\leqslant\frac{1}{4}\|u_t\|_{L^2}^2+C^2\|u\|_{C^0}^2\|\nabla A\|_{L^2}^2,\end{aligned} \tag{3.21}$$

which together with Gronwall's inequality implies that

$$\operatorname*{ess\,sup}_{0<t<T}\|\nabla u\|_{L^2}^2<\infty. \tag{3.22}$$

(ii) The estimate for $\|u_t\|_{L^\infty(0,T;L^2(\Omega,\mathbb{R}^2))}$.

Taking the t-derivative of $(1.1)_1$, one gets that

$$u_{tt}-\nu\Delta u_t=-(u_t\cdot\nabla)u-(u\cdot\nabla)u_t-\frac{1}{\varrho_0}\nabla p_t+\frac{\varrho_e}{\varrho_0}u_t\times\operatorname{curl}A+\frac{\varrho_e}{\varrho_0}u\times\operatorname{curl}A_t. \tag{3.23}$$

Multiplying (3.23) by u_t and integrating over Ω, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_t|^2\,\mathrm{d}x+\nu\int_\Omega|\nabla u_t|^2\,\mathrm{d}x=\int_\Omega\Big(-(u_t\cdot\nabla)u\cdot u_t+\frac{\varrho_e}{\varrho_0}(u\times\operatorname{curl}A_t)u_t\Big)\,\mathrm{d}x \tag{3.24}$$

since

$$(u_t\times\operatorname{curl}A)\cdot u_t=0,\ \int_\Omega(u\cdot\nabla)u_t\cdot u_t\,\mathrm{d}x=-\int_\Omega\frac{1}{2}u_t^2\operatorname{div}u\,\mathrm{d}x=0.$$

Next, we estimate the two terms on the right-hand side of (3.24). By (3.19) and integrating by parts, we have

$$\begin{aligned}-\int_\Omega(u_t\cdot\nabla)u\cdot u_t\,\mathrm{d}x&=\int_\Omega(u_t^iu^j\partial_iu_t^j-u_t^i\partial_i(u^ju_t^j))\,\mathrm{d}x\\&=\int_\Omega u_t^iu^j\partial_iu_t^j\,\mathrm{d}x\leqslant C\|u\|_{C^0}(\|u_t\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2).\end{aligned} \tag{3.25}$$

Similarly,

$$\frac{\varrho_e}{\varrho_0}\int_\Omega |(u\times \operatorname{curl} A_t)u_t|\,\mathrm{d}x \leqslant \frac{C\varrho_e}{\varrho_0}\int_\Omega |uu_t\nabla A_t|\,\mathrm{d}x \leqslant \frac{C\varrho_e}{\varrho_0}\|u\|_{C^0}(\|u_t\|_{L^2}^2+\|\nabla A_t\|_{L^2}^2). \tag{3.26}$$

Hence, by (3.24), (3.25) and (3.26), we get that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_t|^2\,\mathrm{d}x + \nu\int_\Omega |\nabla u_t|^2\,\mathrm{d}x \leqslant C\|u\|_{C^0}((1+\varrho_e/\varrho_0)\|u_t\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2) + \frac{C\varrho_e}{\varrho_0}\|u\|_{C^0}\|\nabla A_t\|_{L^2}^2, \tag{3.27}$$

which together with Gronwall's inequality completes the estimate

$$\operatorname*{ess\,sup}_{0<t<T}\|u_t(t)\|_{L^2}^2<\infty. \tag{3.28}$$

(iii) The estimates for $\|(u\cdot\nabla u)\|_{L^\infty(0,T;L^2(\Omega,\mathbb{R}^2))}$ and $\|u\times A\|_{L^\infty(0,T;L^2(\Omega,\mathbb{R}^2))}$. From (3.22) it is easy to see that

$$\nabla u\in L^\infty(0,T;Y). \tag{3.29}$$

Therefore,

$$u\in L^\infty(0,T;H^1).$$

It is known that $H^1\hookrightarrow L^q$ $(1<q<\infty)$ when $n=2$. Note that

$$\Big(\int_\Omega |(u\cdot\nabla)u|^r\,\mathrm{d}x\Big)^{1/r}\leqslant\Big(\int_\Omega|\nabla u|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega |u|^{2r/(2-r)}\,\mathrm{d}x\Big)^{(2-r)/2r}<\infty \tag{3.30}$$

provided that $1<r<2$. Hence,

$$(u\cdot\nabla)u\in L^\infty(0,T;L^r(\Omega,\mathbb{R}^2)). \tag{3.31}$$

By using the Hölder inequality and the Sobolev embedding theorem, it follows that

$$\int_\Omega |u\times\operatorname{curl} A|^2\,\mathrm{d}x\leqslant C\int_\Omega |u|^2|\nabla A|^2\,\mathrm{d}x \leqslant C\Big(\int_\Omega |u|^4\,\mathrm{d}x+\int_\Omega|\nabla A|^4\,\mathrm{d}x\Big) \leqslant C\Big(\int_\Omega|\nabla u|^2\,\mathrm{d}x+\int_\Omega|\Delta A|^2\,\mathrm{d}x\Big). \tag{3.32}$$

Combining (3.5) with (3.32), we have

$$u \times \operatorname{curl} A \in L^\infty(0,T; L^2(\Omega, \mathbb{R}^2)). \tag{3.33}$$

According to (3.28), (3.31), (3.33), and the assumption, $\widetilde{F}$ in (3.17) satisfies

$$\widetilde{F} \in L^r(\Omega, \mathbb{R}^2)\ (1 < r < 2) \quad \text{for any } 0 < T < \infty. \tag{3.34}$$

Applying (3.34) to Lemma 2.5, we get

$$u \in L^\infty(0,T; W^{2,r}(\Omega, \mathbb{R}^2)), \quad p \in L^\infty(0,T; W^{1,r}(\Omega)). \tag{3.35}$$

Using the Sobolev embedding theorem $W^{2,r} \hookrightarrow C^\alpha \hookrightarrow C^0$ $(0 < \alpha < 1,\ n = 2)$, we deduce from (3.29) and (3.35) that

$$(u \cdot \nabla)u \in L^2(\Omega, \mathbb{R}^2) \quad \text{for any } 0 < T < \infty. \tag{3.36}$$

By (3.28), (3.33), and (3.36), we get that

$$\widetilde{F} = f - (u \cdot \nabla)u + \frac{\varrho_e}{\varrho_0}(u \times \operatorname{curl} A) - u_t \in L^\infty(\Omega, L^2(\Omega, \mathbb{R}^2)). \tag{3.37}$$

Applying (3.37) to Lemma 2.5, we obtain that for any $T > 0$

$$u \in L^\infty(0,T; W^{2,2}(\Omega, \mathbb{R}^2)), \quad p \in L^\infty(0,T; W^{1,2}(\Omega)). \tag{3.38}$$

Therefore, (3.1), (3.5), (3.9), (3.28), and (3.38) yield the existence of the global strong solution of (1.1)–(1.3).

Now, we prove the global strong solution of (1.1)–(1.3) is unique. Without loss of generality, we suppose that (u^i, p^i, A^i, Φ^i) $(i = 1, 2)$ are two different strong solutions of (1.1)–(1.3).

Let $\overline{u} = u^1 - u^2$, $\overline{p} = p^1 - p^2$, $\overline{A} = A^1 - A^2$, $\overline{\Phi} = \Phi^1 - \Phi^2$. Then we have

$$\begin{cases} \dfrac{\partial \overline{u}}{\partial t} + (\overline{u} \cdot \nabla)\overline{u} = \nu \Delta \overline{u} - \dfrac{1}{\varrho_0}\nabla \overline{p} + \dfrac{\varrho_e}{\varrho_0}\overline{u} \times \operatorname{curl} A^1 + u^2 \times \operatorname{curl} \overline{A} \\ \qquad\qquad -(\overline{u} \cdot \nabla)u^2 - (u^2 \cdot \nabla)\overline{u}, \\ \dfrac{\partial \overline{A}}{\partial t} = \overline{A}_t, \\ \dfrac{\partial \overline{A}}{\partial t} = \dfrac{1}{\varepsilon_0 \mu_0}\Delta \overline{A} + \dfrac{\varrho_e}{\varepsilon_0}\overline{u} - \nabla \overline{\Phi}, \\ \nabla \cdot \overline{u} = 0, \\ \nabla \cdot \overline{A} = 0, \end{cases} \tag{3.39}$$

and the initial-boundary conditions

$$\overline{u}(0,x)=0,\quad \overline{A}(0,x)=0,\quad \overline{A}_t(0,x)=0 \quad \text{in } \Omega, \tag{3.40}$$

$$\overline{u}(t,x)=0,\quad \overline{A}(t,x)=0 \quad \text{on } \partial\Omega\times[0,T). \tag{3.41}$$

Multiplying both sides of (3.39) by $(\overline{u},\overline{A},\overline{A}_t)$ and integrating over Ω, it follows from (3.40) and (3.41) that

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_\Omega |\overline{u}|^2\,\mathrm{d}x + 2\nu\int_\Omega |\nabla\overline{u}|^2\,\mathrm{d}x \\ \qquad\qquad = 2\displaystyle\int_\Omega (u^2\times \operatorname{curl}\overline{A})\cdot\overline{u}\,\mathrm{d}x - 2\int_\Omega (\overline{u}\cdot\nabla)u^2\cdot\overline{u}\,\mathrm{d}x, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_\Omega |A|^2\,\mathrm{d}x = 2\int_\Omega \overline{A}_t\cdot\overline{A}\,\mathrm{d}x, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_\Omega |\overline{A}_t|^2\,\mathrm{d}x = \frac{2}{\varepsilon_0\mu_0}\int_\Omega \Delta\overline{A}\cdot\overline{A}_t\,\mathrm{d}x + \frac{2\varrho_e}{\varepsilon_0}\int \overline{u}\cdot\overline{A}_t\,\mathrm{d}x. \end{cases} \tag{3.42}$$

By using the Sobolev embedding theorem and the Hölder inequality, we can obtain that

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\|\overline{u}\|_{L^2}^2 + \nu\|\nabla\overline{u}\|_{L^2}^2 \leqslant C(\|u^2\|_{C^0}\|\nabla\overline{A}\|_{L^2}\|\overline{u}\|_{L^2} + \|\nabla u^2\|_{L^4}\|\overline{u}\|_{L^4}\|\overline{u}\|_{L^2}), \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\|\overline{A}\|_{L^2}^2 \leqslant 2\|\overline{A}_t\|_{L^2}\|\overline{A}\|_{L^2}, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\|\overline{A}_t\|_{L^2}^2 + \dfrac{2}{\varepsilon_0\mu_0}\dfrac{\mathrm{d}}{\mathrm{d}t}\|\nabla\overline{A}\|_{L^2}^2 \leqslant \dfrac{2\varrho_e}{\varepsilon_0}\|\overline{u}\|_{L^2}\|\overline{A}_t\|_{L^2}, \end{cases} \tag{3.43}$$

where C is a constant depending on Ω, N and p.

By applying the Sobolev embedding theorem, it is easy to check that

$$\|\nabla u_2\|_{L^4}\|\overline{u}\|_{L^4}\|\overline{u}\|_{L^2} \leqslant \|\nabla u_2\|_{L^4}\|\nabla\overline{u}\|_{L^2}\|\overline{u}\|_{L^2}. \tag{3.44}$$

Combining (3.43)with (3.44), we can get the following inequalities by using the Young inequality and the Hölder inequality:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{u}\|_{L^2}^2 + 2\nu\|\nabla\overline{u}\|_{L^2}^2 \\ &\qquad \leqslant C(\|u^2\|_{C^0}^2\|\nabla\overline{A}\|_{L^2}^2 + \|\overline{u}\|_{L^2}^2 + \varepsilon^{-1}\|\nabla u^2\|_{L^4}^2\|\overline{u}\|_{L^2}^2) + \varepsilon\|\nabla\overline{u}\|_{L^2}^2, \end{aligned} \tag{3.45}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{A}\|_{L^2}^2 \leqslant \|\overline{A}_t\|_{L^2}^2 + \|\overline{A}\|_{L^2}^2, \tag{3.46}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{A}_t\|_{L^2}^2 + \frac{2}{\varepsilon_0\mu_0}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\overline{A}\|_{L^2}^2 \leqslant \frac{\varrho_e^2}{\varepsilon_0^2}\|\overline{u}\|_{L^2}^2 + \|\overline{A}_t\|_{L^2}^2. \tag{3.47}$$

If we take $\varepsilon=\nu$, then (3.45) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{u}\|_{L^2}^2+\nu\|\nabla\overline{u}\|_{L^2}^2\leqslant C(\|u^2\|_{C^0}^2\|\nabla\overline{A}\|_{L^2}^2+\|\overline{u}\|_{L^2}^2+\frac{1}{\nu}\|\nabla u^2\|_{L^4}^2\|\overline{u}\|_{L^2}^2). \tag{3.48}$$

Moreover, we can infer from (3.46)–(3.48) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\overline{u}\|_{L^2}^2+\|\overline{A}\|_{L^2}^2+\|\overline{A}_t\|_{L^2}^2+\frac{2}{\varepsilon_0\mu_0}\|\nabla\overline{A}\|_{L^2}^2\Big)+\nu\|\nabla\overline{u}\|_{L^2}^2\\&\leqslant C(\|\overline{u}\|_{L^2}^2+\|\overline{A}\|_{L^2}^2+\|\overline{A}_t\|_{L^2}^2+\|\nabla\overline{A}\|_{L^2}^2).\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|\overline{u}\|_{L^2}^2+\|\overline{A}\|_{L^2}^2+\|\overline{A}_t\|_{L^2}^2+\|\nabla\overline{A}\|_{L^2}^2)\\&\leqslant C(\|\overline{u}\|_{L^2}^2+\|\overline{A}\|_{L^2}^2+\|\overline{A}_t\|_{L^2}^2+\|\nabla\overline{A}\|_{L^2}^2),\end{aligned} \tag{3.49}$$

where $C=C(\Omega,\varepsilon_0,\|u_2\|_{H^2},\varrho_e,\nu)$.

Combining the Gronwall inequality and (3.40), we get

$$\overline{u}=0,\quad \overline{A}=0,\quad \overline{A}_t=0. \tag{3.50}$$

Therefore, the global strong solution of (1.1)–(1.3) is unique, which completes the proof. □

3.2. Partial regularity of the 3-D new MHD model. Analogously, we establish the following partial regularity of the 3-D new MHD model (1.1) with (1.2)–(1.3).

For convenience, we set

$$\begin{aligned}&\widetilde{X}=\{u\in C_0^\infty(\Omega,\mathbb{R}^3);\ \operatorname{div}u=0\ \text{ in }\Omega\},\\&\widetilde{Y}=\text{the closure of }X\text{ in }L^2(\Omega,\mathbb{R}^3)=\{u\in L^2(\Omega,\mathbb{R}^3);\ \operatorname{div}u=0\ \text{ in }\Omega\},\\&\widetilde{Z}=\text{the closure of }X\text{ in }H^1(\Omega,\mathbb{R}^3)=\{u\in H_0^1(\Omega,\mathbb{R}^3);\ \operatorname{div}u=0\ \text{ in }\Omega\}.\end{aligned}$$

Theorem 3.3. *Let Ω be a bounded domain with compact C^3-boundary. If φ, $\psi\in H^2(\Omega,\mathbb{R}^3)\cap\widetilde{Z}$, $\eta\in\widetilde{Z}$ for any $f\in\widetilde{Y}$, then there exists a partial regularity solution for the 3D problem* (1.1)–(1.3), *i.e., for any $0<T<\infty$*

$$\begin{aligned}&u\in L^{5/4}((0,T),W^{2,5/4}(\Omega,\mathbb{R}^3)),\quad u_t\in L^{5/4}(0,T;L^{5/4}(\Omega,\mathbb{R}^3)),\\&p\in L^{5/4}(0,T;W^{1,5/4}(\Omega)),\\&A\in L^\infty(0,T;\widetilde{Z}),\quad A_t\in L^\infty(0,T;\widetilde{Y}).\end{aligned}$$

P r o o f. For any $0 < T < \infty$ we can get the following results from Theorem 3.2 in [11]:

$$u \in L^2((0,T),\widetilde{Z}) \cap L^\infty((0,T),\widetilde{Y}),$$
$$A \in L^\infty((0,T),\widetilde{Z}), \quad A_t \in L^\infty((0,T),\widetilde{Y}). \tag{3.51}$$

Combining (3.51) and Lemma 2.1 ($k = 0$), we can infer that

$$u \in L^{10/3}((0,T)\times\Omega). \tag{3.52}$$

Notice that

$$\begin{aligned}\int_0^T\int_\Omega |(u\cdot\nabla)u|^{5/4}\,\mathrm{d}x\,\mathrm{d}t &\leqslant \int_0^T\int_\Omega |Du|^{5/4}|u|^{5/4}\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant \Big(\int_0^T\int_\Omega |Du|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{5/8}\Big(\int_0^T\int_\Omega |u|^{10/3}\,\mathrm{d}x\,\mathrm{d}t\Big)^{3/8}.\end{aligned} \tag{3.53}$$

It follows from (3.53) that

$$(u\cdot\nabla)u \in L^{5/4}((0,T),L^{5/4}(\Omega,\mathbb{R}^3)). \tag{3.54}$$

Combining (3.51) with (3.52), we derive that

$$\begin{aligned}&\int_0^T\int_\Omega |u\times\operatorname{curl}A|^{5/4}\,\mathrm{d}x\,\mathrm{d}t\\ &\quad\leqslant \int_0^T\int_\Omega |u|^{5/4}|\operatorname{curl}A|^{5/4}\,\mathrm{d}x\,\mathrm{d}t\\ &\quad\leqslant \Big(\int_0^T\int_\Omega |u|^{10/3}\,\mathrm{d}x\,\mathrm{d}t\Big)^{3/8}\Big(\int_0^T\int_\Omega |\operatorname{curl}A|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{5/8}\\ &\quad\leqslant C\Big(\int_0^T\int_\Omega |u|^{10/3}\,\mathrm{d}x\,\mathrm{d}t\Big)^{3/8}\Big(\int_0^T\int_\Omega |DA|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{5/8} < \infty,\end{aligned} \tag{3.55}$$

which implies that

$$u\times\operatorname{curl}A \in L^{5/4}((0,T),L^{5/4}(\Omega,\mathbb{R}^3)). \tag{3.56}$$

Now, we consider (u,p) satisfying the Stokes equations

$$\begin{cases}\dfrac{\partial u}{\partial t} = \nu\Delta u - \dfrac{1}{\varrho_0}\nabla p + F(x,t),\\ \operatorname{div} u = 0,\\ u|_{\partial\Omega} = 0,\\ u(0) = \varphi,\end{cases} \tag{3.57}$$

where $F(x,t) = f - (u\cdot\nabla)u + (\varrho_e/\varrho_0)(u\times\operatorname{curl}A)$.

By (3.54) and (3.56), it is clear that $F \in L^{5/4}((0,T), L^{5/4}(\Omega,\mathbb{R}^3))$. Moreover, by Lemma 2.2 we have

$$\begin{aligned} &u \in L^{5/4}((0,T), W^{2,5/4}(\Omega,\mathbb{R}^3)),\ u_t \in L^{5/4}((0,T), L^{5/4}(\Omega,\mathbb{R}^3)),\\ &p \in L^{5/4}((0,T), W^{1,5/4}(\Omega)). \end{aligned} \tag{3.58}$$

Therefore, (3.58) and Theorem 3.2 in [11] complete the proof. □

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