A CONTINUITY RESULT FOR A QUASILINEAR ELLIPTIC FREE BOUNDARY PROBLEM

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Received August 4, 2019. Published online January 31, 2020.

Abstract. We investigate a two dimensional quasilinear free boundary problem, and show that the free boundary is a union of graphs of continuous functions.

Keywords: quasilinear elliptic free boundary; continuity

MSC 2010: 35R35, 35J62

1. Introduction

In this paper we consider the quasilinear free boundary problem studied in [12]

$$\text{(P)}\qquad \begin{cases} \text{Find } (u,\chi)\in W^{1,A}(\Omega)\times L^\infty(\Omega) \text{ such that:}\\ \text{(i)}\quad 0\leqslant u\leqslant M,\quad 0\leqslant\chi\leqslant 1,\quad u(1-\chi)=0\quad \text{a.e. in } \Omega,\\ \text{(ii)}\quad \Delta_A u=-\operatorname{div}(\chi H(x))\quad \text{in } (W_0^{1,A}(\Omega))', \end{cases}$$

where Ω is an open bounded domain of $\mathbb{R}^2$, $x=(x_1,x_2)$, M is a positive constant,

$$A(t)=\int_0^t a(s)\,\mathrm{d}s,\quad \Delta_A u=\operatorname{div}\Big(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u\Big)$$

in the distributional sense is the A-Laplacian, a is a C^1 function from $[0,\infty)$ to $[0,\infty)$ such that $a(0)=0$, $a(t)>0$ for $t>0$, and for some positive constants a_0,a_1

$$a_0\leqslant\frac{ta'(t)}{a(t)}\leqslant a_1\quad \forall\,t>0. \tag{1.1}$$

As a consequence of (1.1), we have the following monotonicity inequality (see [8]):

$$\Big(\frac{a(|\xi|)}{|\xi|}\xi-\frac{a(|\zeta|)}{|\zeta|}\zeta\Big)\cdot(\xi-\zeta)>0\quad \forall\,\xi,\ \zeta\in\mathbb{R}^2\setminus\{0\},\ \xi\neq\zeta. \tag{1.2}$$

DOI: 10.21136/AM.2020.0190-19

For examples of functions $a(t)$, we refer to [13].

Let $H = (H_1, H_2)$ be a vector function that satisfies for some positive constants $\underline{h}$, $\bar{h}$

$$|H_1| \leqslant \bar{h}, \quad 0 < \underline{h} \leqslant H_2 \leqslant \bar{h} \quad \text{in } \Omega, \tag{1.3}$$

$$H \in C^{0,1}(\bar{\Omega}), \tag{1.4}$$

$$\operatorname{div}(H) \geqslant 0 \quad \text{a.e. in } \Omega, \tag{1.5}$$

$$\operatorname{div}(H) \leqslant \bar{h} \quad \text{a.e. in } \Omega. \tag{1.6}$$

We refer to [13] for the definition of the Orlicz-Sobolev space $W^{1,A}(\Omega)$ and its norm.

In [12], it was shown that the free boundary which is defined as the intersection of the sets $\{u = 0\}$ and $\overline{\{u > 0\}}$, is a union of graphs of lower semi-continuous functions depending only on the vector function H. In this paper, we will show that these functions are actually continuous and that χ is the characteristic function of the set $\{u > 0\}$.

Problem (P) describes a variety of free boundary problems including the lubrication problem [1] and the dam problem [16], [15], [2], [6], [3], [10], [18], and [19]. For a more general framework, we refer to [14], [4], [5], [9], [7], [11], [12] and [20].

Throughout this paper, we will denote by $B_r(x)$ or $\overline{B}_r(x)$ the open or closed ball, respectively, of center x and radius r in $\mathbb{R}^2$.

2. Preliminary results

When $H_1 = 0$ and H_2 is a constant function, it is easy to show as in [7] that $\chi_{x_2} \leqslant 0$ in $\mathcal{D}'(\Omega)$ and that the free boundary $\partial\{u > 0\} \cap \Omega$ is the graph of a continuous function $x_2 = \varphi(x_1)$. When H is not a constant vector, we can show as in [5] that

$$\operatorname{div}(\chi H) - \chi(\{u > 0\})\operatorname{div}(H) \leqslant 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{2.1}$$

Actually (2.1) can be obtained from (P) (ii) by adapting the proof of Lemma 2.4. As a consequence of this property, the function χ is decreasing along the orbit $\gamma(w)$ (see Figure 1) of the following differential equation (see [12]):

$$(E(w,h)) \begin{cases} X'(t,w,h) = H(X(t,w,h)), \\ X(0,w,h) = (w,h), \end{cases}$$

where $h \in \pi_{x_2}(\Omega)$, $w \in \pi_{x_1}(\Omega \cap \{x_2 = h\})$, and where π_{x_1} and π_{x_2} are respectively the orthogonal projections on the x_1 and x_2 axes. We will denote by $X(\cdot, w)$ the maximal solution of $E(w,h)$ defined on the interval $(\alpha_-(w), \alpha_+(w))$. We know [5] that the limits $\lim_{t\to\alpha_-(w)^+} X(t,w) = X(\alpha_-(w),w) \in \partial\Omega \cap \{x_2 < h\}$ and $\lim_{t\to\alpha_+(w)^-} X(t,w) = X(\alpha_+(w),w) \in \partial\Omega \cap \{x_2 > h\}$ both exist.

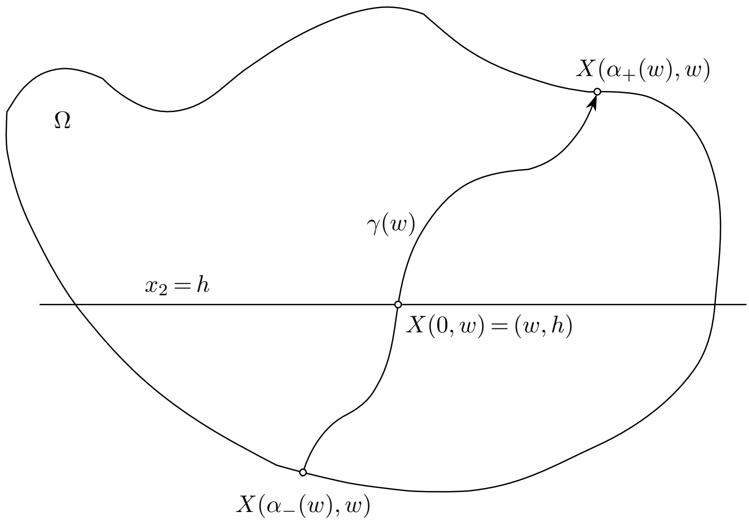

Figure 1.

Now, we recall for the reader's convenience a few technical properties and definitions established in [5] and [12]:

▷ α_+ and α_- are uniformly bounded.

▷ For each $h \in \pi_{x_2}(\Omega)$, the following mapping is one to one

$$T_h\colon\ D_h \to T_h(D_h),$$
$$(t,w) \mapsto T_h(t,w) = (T_h^1, T_h^2)(t,w) = X(t,w),$$

where $D_h = \{(t,w)/w \in \pi_{x_1}(\Omega \cap \{x_2 = h\}),\, t \in (\alpha_-(w), \alpha_+(w))\}$.

▷ $\Omega = \bigcup_{h\in\pi_{x_2}(\Omega)} T_h(D_h)$.

▷ T_h and T_h^{-1} are $C^{0,1}$.

▷ The determinant $Y_h(t,w)$ of the Jacobian matrix of T_h satisfies:

(i) $Y_h(t,w) = -H_2(w,h)\exp(\int_0^t (\mathrm{div} H)(X(s,w))\,\mathrm{d}s)$ a.e. in D_h.

(ii) $\underline{h} \leqslant |Y_h(t,w)| \leqslant C\bar{h}$, $C > 0$, a.e. in D_h.

The following interior regularity, established in [12], will be useful in Section 3.

Theorem 2.1. *For any solution* (u,χ) *of* (P) *we have* $u \in C^{0,1}_{\mathrm{loc}}(\Omega)$.

The following monotonicity of χ based on (2.1) (see [5], [12]) is the key point in parameterizing the free boundary:

$$\frac{\partial}{\partial t}(\chi \circ T_h) \leqslant 0 \quad \text{in } \mathcal{D}'(D_h). \tag{2.2}$$

Property (2.2) means that χ decreases along the orbits of the differential equation $(E(w,h))$. The consequence of this monotonicity is materialized in the next theorem (see Figure 2).

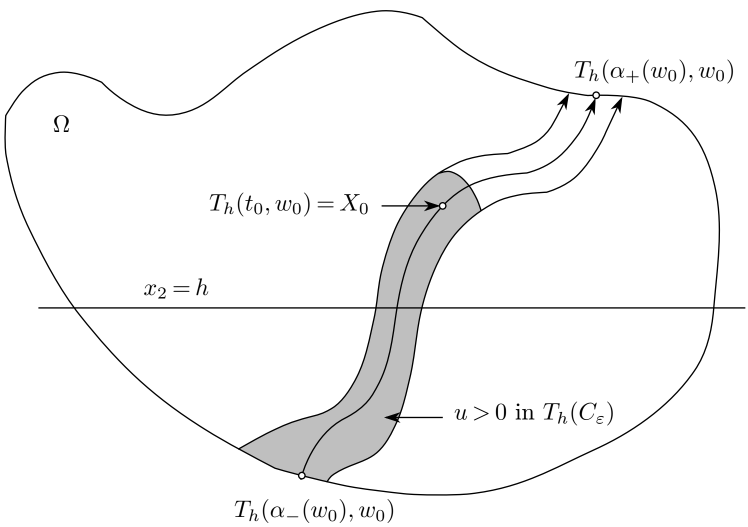

Figure 2.

Theorem 2.2. *Let* (u, χ) *be a solution of* (P) *and* $x_0 = T_h(t_0, w_0) \in T_h(D_h)$.

(i) *If* $u(x_0) = u \circ T_h(t_0, w_0) > 0$, *then there exists* $\varepsilon > 0$ *such that*

$$u \circ T_h(t, w) > 0 \quad \forall\, (t, w) \in C_\varepsilon = \{(t, w) \in D_h / |w - w_0| < \varepsilon,\ t < t_0 + \varepsilon\}.$$

(ii) *If* $u(x_0) = u \circ T_h(t_0, w_0) = 0$, *then* $u \circ T_h(t, w_0) = 0$ *for all* $t \geqslant t_0$.

The proof of Theorem 2.2 is based on the following strong maximum principle (see [12]):

Lemma 2.1. *If* $u \in W^{1,A}(U) \cap C^1(U) \cap C^0(\overline{U})$ *satisfies* $u \geqslant 0$ *and* $\Delta_A u \leqslant 0$ *in* U, *then* $u \equiv 0$ *in* U *or* $u > 0$ *in* U.

Thanks to Theorem 2.2, we can define for each $h \in \pi_{x_2}(\Omega)$, the following function φ_h on $\pi_{x_1}(\Omega \cap \{x_2 = h\})$ (see [12]):

$$\varphi_h(w) = \begin{cases} \sup\{t\colon\ (t,w) \in D_h,\ u \circ T_h(t,w) > 0\} & \text{if this set is not empty,} \\ \alpha_-(w) & \text{otherwise.} \end{cases}$$

Then we have (see [12]):

Proposition 2.1. *For each $h \in \pi_{x_2}(\Omega)$, the function φ_h is lower semi-continuous at each $w \in \pi_{x_1}(\Omega \cap \{x_2 = h\})$ such that $T_h(\varphi_h(w), w) \in \Omega$. Moreover,*

$$\{u \circ T_h(t,w) > 0\} \cap D_h = \{t < \varphi_h(w)\}. \tag{2.3}$$

The following lemma will be of interest in Section 3.

Lemma 2.2. *Let $h \in \pi_{x_2}(\Omega)$. For each $k \in \pi_{x_2}(\Omega)$ and $w \in \pi_{x_1}(\Omega \cap \{x_2 = h\})$, let $t_k(w)$ be the unique value of t at which the orbit $\gamma(w)$ of $X(\cdot, w)$ intersects the line $\{x_2 = k\}$ if it exists. Then the function $S(k,w) = t_k(w)$ is Lipschitz continuous in its domain. More precisely, we have for some positive constant C:*

$$|S(k,w) - S(k_0,w_0)| \leqslant C(|k-k_0| + |w-w_0|) \quad \forall\, (k,w), (k_0,w_0) \in \operatorname{domain}(S).$$

Proof. Let $(k,w), (k_0,w_0) \in \operatorname{domain}(S)$. First we have from the differential equation $(E(w,h))$

$$k = h + \int_0^{t_k(w)} H_2(X(s,w))\,\mathrm{d}s \quad \text{and} \quad k_0 = h + \int_0^{t_{k_0}(w_0)} H_2(X(s,w_0))\,\mathrm{d}s.$$

If we subtract these two equalities, we obtain

$$k - k_0 = \int_0^{t_k(w)} H_2(X(s,w))\,\mathrm{d}s - \int_0^{t_{k_0}(w_0)} H_2(X(s,w_0))\,\mathrm{d}s. \tag{2.4}$$

Next, if we assume that $t_k(w) > t_{k_0}(w_0)$, then we get by (1.3)

$$\underline{h}(t_k(w) - t_{k_0}(w_0)) \leqslant \int_{t_{k_0}(w_0)}^{t_k(w)} H_2(X(s,w))\,\mathrm{d}s. \tag{2.5}$$

Now, observe that

$$(2.6)\quad \int_{t_{k_0}(w_0)}^{t_k(w)} H_2(X(s,w))\,\mathrm{d}s) = \int_0^{t_k(w)} H_2(X(s,w))\,\mathrm{d}s - \int_0^{t_{k_0}(w_0)} H_2(X(s,w))\,\mathrm{d}s$$
$$= \int_0^{t_k(w)} H_2(X(s,w))\,\mathrm{d}s - \int_0^{t_{k_0}(w_0)} H_2(X(s,w_0))\,\mathrm{d}s$$
$$+ \int_0^{t_{k_0}(w_0)} (H_2(X(s,w_0)) - H_2(X(s,w)))\,\mathrm{d}s.$$

Using (2.4), (2.6) and the fact that $H_2 \circ X$ is Lipschitz continuous in $\overline{D}_h$, and since $t_{k_0}(w_0)$ is bounded independently of k_0 and w_0, we obtain from (2.5) for some positive constant C_0

$$\underline{h}(t_k(w) - t_{k_0}(w_0)) \leqslant k - k_0 + C_0|w - w_0|,$$

which leads for $C = \max(1, C_0)/\underline{h}$, to

$$(2.7)\qquad t_k(w) - t_{k_0}(w_0) \leqslant C(|k - k_0| + |w - w_0|).$$

If $t_k(w) < t_{k_0}(w_0)$, we get in a similar fashion

$$(2.8)\qquad t_{k_0}(w_0) - t_k(w) \leqslant C(|k - k_0| + |w - w_0|).$$

Combining (2.7) and (2.8), the lemma follows. □

Remark 2.1. (i) Our main goal is to prove that for each $h \in \pi_{x_2}(\Omega)$, the function φ_h is actually continuous. Due to the local character of this result, we will confine ourselves to the following situation:

We assume that $u = 0$ on an open and connected subset Γ of $\partial\Omega$ and consider an open subset $U = T_h(D_h^+ \cap \{w_* < w < w^*\})$ of $T_h(D_h)$ (see Figure 3), where $D_h^+ = \{(t,w)/w \in \pi_{x_1}(\Omega \cap \{x_2 = h\}),\ t \in (0, \alpha_+(w))\}$ so that $T_h(\{(\alpha_+(w), w),\ w \in (w_*, w^*)\}) \subset\subset \Gamma$. Hence, we are led to the following problem:

$$(\mathrm{P})\qquad \begin{cases} \text{Find } (u,\chi) \in W^{1,A}(U) \times L^\infty(U) \text{ such that: } u = 0 \text{ on } \Gamma, \\ \quad 0 \leqslant u \leqslant M,\ 0 \leqslant \chi \leqslant 1,\ u(1-\chi) = 0 \quad \text{a.e. in } U, \\ \displaystyle\int_U \Big(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u + \chi H(x)\Big) \cdot \nabla\zeta\,\mathrm{d}x \leqslant 0, \\ \quad \forall \zeta \in W^{1,A}(U), \quad \zeta \geqslant 0 \quad \text{on } \Gamma, \quad \zeta = 0 \quad \text{on } \partial U \setminus \Gamma. \end{cases}$$

(ii) We observe that the free boundary $(\partial\{u > 0\}) \cap U$ is the graph of the lower semi-continuous function φ_h in (w_*, w^*). Our objective is to prove the continuity of the function φ_h, which we will do in Section 3 by showing that it is also upper semi-continuous. To this end, we need to generalize a few lemmas previously established for a linear operator in [5]. In the sequel and without notice, we will denote by (u,χ) a solution of the problem (P).

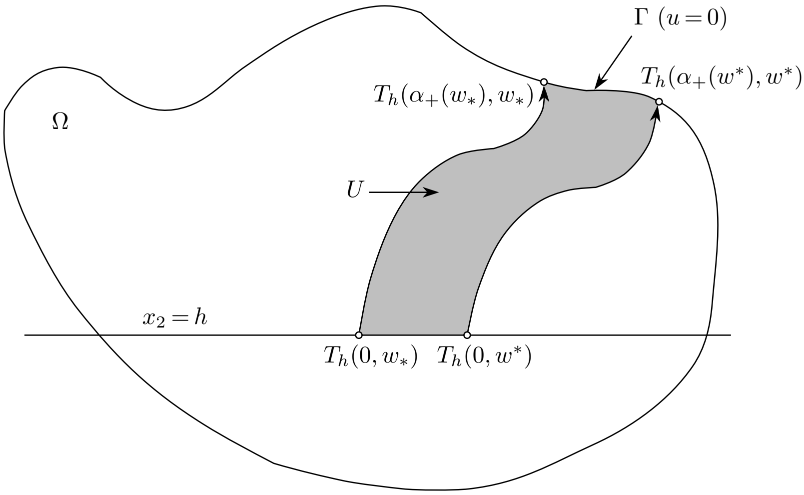

Figure 3.

Lemma 2.3. *Let $w_1, w_2 \in (w_*, w^*)$, $k \in \pi_{x_2}(U)$ be such that $w_1 < w_2$ and $\{x_2 = k\} \cap \gamma(w_i) \neq \emptyset$, $i = 1, 2$. If (see Figure 4)*

$$Z_k = T_h(\{(t, w) \in D_h,\ w \in (w_1, w_2),\ t > t_k(w)\}) = T_h(\{w_1 < w < w_2\}) \cap \{x_2 > k\},$$

and $u \circ T_h(t_k(w_i), w_i) = 0$ for $i = 1, 2$, then we have

$$\int_{Z_k} \Big(\frac{a(|\nabla u|)}{|\nabla u|} \nabla u + \chi H(x) \Big) \cdot \nabla \zeta \,\mathrm{d}x \leqslant 0 \quad \forall \zeta \in W^{1,A}(Z_k) \cap C^0(\overline{Z}_k),$$
$$\zeta \geqslant 0 \quad \text{on} \quad \overline{Z}_k \setminus \{x_2 = k\}, \quad \zeta = 0 \quad \text{on } \overline{Z}_k \cap \{x_2 = k\}.$$

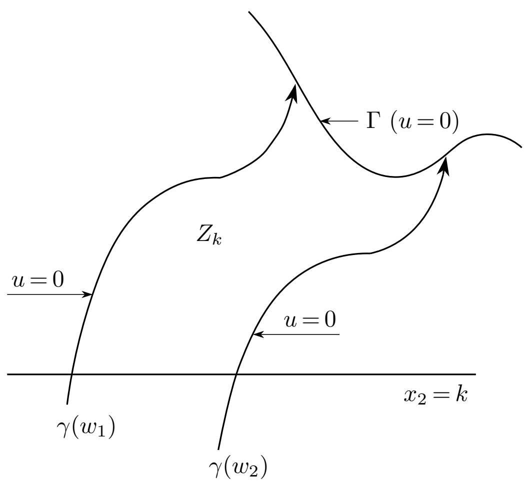

Figure 4.

The proof of Lemma 2.3 is inspired by the one of a similar lemma in [14] for the case $H(x)=(h(x),0)$. Our proof is based on the next lemma.

Lemma 2.4. *Under the assumptions of Lemma* 2.3*, we have*

$$\int_{Z_k}\frac{a(|\nabla u|)}{|\nabla u|}\nabla u\cdot\nabla\zeta\,\mathrm{d}x-\int_{Z_k}\chi_{\{u>0\}}\operatorname{div}(H)\zeta\,\mathrm{d}x\leqslant 0$$
$$\forall\zeta\in W^{1,A}(Z_k)\cap C^0(\overline{Z}_k),\ \zeta\geqslant 0\text{ on }\overline{Z}_k\setminus\{x_2=k\},\ \zeta=0\text{ on }\overline{Z}_k\cap\{x_2=k\}.$$

Proof. Let ζ be as in the lemma, $\varepsilon>0$, and $F_\varepsilon(u)=\min\{u^+/\varepsilon,1\}$. Using $\chi(Z_k)F_\varepsilon(u)\zeta$ as a test function for (P), we get

$$\int_{Z_k}F_\varepsilon(u)\frac{a(|\nabla u|)}{|\nabla u|}\nabla u\cdot\nabla\zeta\,\mathrm{d}x+\int_{Z_k}H(x)\cdot\nabla(F_\varepsilon(u)\zeta)\,\mathrm{d}x$$
$$\leqslant-\int_{Z_k}F_\varepsilon'(u)\zeta|\nabla u|a(|\nabla u|)\,\mathrm{d}x\leqslant 0.$$

Integrating by parts, we obtain

$$\int_{Z_k}F_\varepsilon(u)\frac{a(|\nabla u|)}{|\nabla u|}\nabla u\cdot\nabla\zeta\,\mathrm{d}x-\int_{Z_k}\operatorname{div}(H)F_\varepsilon(u)\zeta\,\mathrm{d}x\leqslant 0. \tag{2.9}$$

The lemma follows by letting ε go to 0 in (2.9). □

Proof of Lemma 2.3. For $\varepsilon>0$ small enough, let

$$\alpha_\varepsilon(w)=\min\Big\{1,\frac{(w-w_1)^+}{\varepsilon},\frac{(w_2-w)^+}{\varepsilon}\Big\},$$

and observe that

$$\begin{aligned}\int_{Z_k}\Big(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u+\chi H(x)\Big)\cdot\nabla\zeta\,\mathrm{d}x &\\ =\int_{Z_k}\Big(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u+\chi H(x)\Big)\cdot\nabla[(\alpha_\varepsilon\circ T_h^{-1})\zeta]\,\mathrm{d}x&\\ +\int_{Z_k}\Big(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u+\chi H(x)\Big)\cdot\nabla[(1-\alpha_\varepsilon\circ T_h^{-1})\zeta]\,\mathrm{d}x.&\end{aligned}\tag{2.10}$$

Since $\chi(Z_k)(\alpha_\varepsilon\circ T_h^{-1})\zeta$ is a test function for (P), we have:

$$\int_{Z_k}\Big(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u+\chi H(x)\Big)\cdot\nabla[(\alpha_\varepsilon\circ T_h^{-1})\zeta]\,\mathrm{d}x\leqslant 0. \tag{2.11}$$

Applying Lemma 2.4 to the function $(1-\alpha_\varepsilon \circ T_h^{-1})\zeta$, we get

$$\int_{Z_k} \frac{a(|\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla[(1-\alpha_\varepsilon \circ T_h^{-1})\zeta]\,\mathrm{d}x \leqslant \int_{Z_k} \chi_{\{u>0\}} \operatorname{div}(H)(1-\alpha_\varepsilon \circ T_h^{-1})\zeta\,\mathrm{d}x. \tag{2.12}$$

Taking into account (2.11)–(2.12), we obtain from (2.10)

$$\begin{aligned} \int_{Z_k} \Big(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u + \chi H(x)\Big) \cdot \nabla\zeta\,\mathrm{d}x \qquad & \\ \leqslant \int_{Z_k} \chi_{\{u>0\}} \operatorname{div}(H)(1-\alpha_\varepsilon \circ T_h^{-1})\zeta\,\mathrm{d}x & \\ + \int_{Z_k} \chi H(x) \cdot \nabla[(1-\alpha_\varepsilon \circ T_h^{-1})\zeta]\,\mathrm{d}x. & \end{aligned} \tag{2.13}$$

Using the change of variables $x = T_h(t,w)$ and arguing as in the proof of Theorem 2.1 in [5], we obtain

$$\begin{aligned} \int_{Z_k} \chi H(x) \cdot \nabla[(1-\alpha_\varepsilon \circ T_h^{-1})\zeta]\,\mathrm{d}x & \\ = \int_{T_h^{-1}(Z_k)} -Y_h \chi \circ T_h \frac{\partial}{\partial t}[1-\alpha_\varepsilon \zeta \circ T_h]\,\mathrm{d}t\,\mathrm{d}w & \\ = -\int_{T_h^{-1}(Z_k)} (1-\alpha_\varepsilon) Y_h \chi \circ T_h \frac{\partial}{\partial t}[\zeta \circ T_h]\,\mathrm{d}t\,\mathrm{d}w. & \end{aligned} \tag{2.14}$$

Then we derive from (2.13) and (2.14)

$$\begin{aligned} \int_{Z_k} \Big(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u + \chi H(x)\Big) \cdot \nabla\zeta\,\mathrm{d}x & \\ \leqslant \int_{Z_k} \chi_{\{u>0\}} \operatorname{div}(H)(1-\alpha_\varepsilon \circ T_h^{-1})\zeta\,\mathrm{d}x & \\ - \int_{T_h^{-1}(Z_k)} (1-\alpha_\varepsilon) Y_h \chi \circ T_h \frac{\partial}{\partial t}[\zeta \circ T_h]\,\mathrm{d}t\,\mathrm{d}w. & \end{aligned} \tag{2.15}$$

Hence, the lemma follows by letting ε go to 0 in (2.15). □

Lemma 2.5. *Let $x_0 = T_h(t_0, w_0) \in U$. If $u \circ T_h = 0$ in $B_r(t_0, w_0)$, then*

$$u \circ T_h = 0 \quad \text{in } C_r \quad \text{and} \quad \chi \circ T_h = 0 \quad \text{a.e. in } C_r,$$

where $C_r = \{(t,w) \in D_h,\ |w-w_0| < r,\ t > t_0\} \cup B_r(t_0, w_0)$.

Proof. By Theorem 2.2 (ii), we have $u \circ T_h = 0$ in C_r. Applying Lemma 2.3 with domains $Z_k = T_h(\{w_1 < w < w_2\}) \cap \{x_2 > k\} \subset T_h(C_r)$, $(k \in \pi_{x_2}(U))$, and taking $\zeta = x_2 - k$, we obtain $\int_{Z_k} \chi H_2 \, \mathrm{d}x \leqslant 0$. Then we deduce from (1.3) that $\chi = 0$ a.e. in Z_k. This holds for all domains Z_k in $T_h(C_r)$. Hence, $\chi = 0$ a.e. in $T_h(C_r)$. $\square$

Lemma 2.6. *Let $x_0 = T_h(t_0, w_0) \in U$ such that $B_r = B_r(t_0, w_0) \subset D_h$. Then the following three situations are impossible:*

$$\text{(i)} \quad \begin{cases} u \circ T_h(t, w_0) = 0 & \forall\, t \in (t_0 - r, t_0 + r), \\ u \circ T_h(t, w) > 0 & \forall\, t \in (t_0 - r, t_0 + r), \quad \forall\, w \neq w_0, \end{cases}$$

$$\text{(ii)} \quad \begin{cases} u \circ T_h(t, w) = 0 & \forall\, (t, w) \in B_r \cap \{w \leqslant w_0\}, \\ u \circ T_h(t, w) > 0 & \forall\, (t, w) \in B_r \cap \{w > w_0\}, \end{cases}$$

$$\text{(iii)} \quad \begin{cases} u \circ T_h(t, w) = 0 & \forall\, (t, w) \in B_r \cap \{w \geqslant w_0\}, \\ u \circ T_h(t, w) > 0 & \forall\, (t, w) \in B_r \cap \{w < w_0\}. \end{cases}$$

Proof. Assume that (ii) holds. The proofs of (i) and (iii) are based on similar arguments. Let $\zeta \in \mathcal{D}(T_h(B_r))$, $\zeta \geqslant 0$. Using the fact that, by Lemma 2.5, $\chi \circ T_h = 0$ a.e. in $B_r \cap \{w \leqslant w_0\}$, we obtain after using the change of variable T_h, and taking into account (1.3) and (1.5),

$$\begin{aligned} \int_{T_h(B_r)} \frac{a(|\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla \zeta \, \mathrm{d}x &= \int_{B_r \cap \{w > w_0\}} \frac{\partial}{\partial t}(-Y_h(t, w)) \zeta \circ T_h \, \mathrm{d}t \, \mathrm{d}w \\ &= \int_{B_r \cap \{w > w_0\}} H_2(w, h)(\mathrm{div} H)(X(t, w)) \zeta \circ T_h \, \mathrm{d}t \, \mathrm{d}w \geqslant 0. \end{aligned}$$

This means that $\triangle_A u \leqslant 0$ in $\mathcal{D}'(T_h(B_r))$. By Lemma 2.1, either $u > 0$ or $u = 0$ in $T_h(B_r)$, which contradicts the assumption. $\square$

3. Continuity of the free boundary

As pointed out in Section 2, in order to prove the continuity of the function φ_h, it is enough to show that it is upper semi-continuous. The main idea to do that is to compare u with a suitable barrier function near a free boundary point. In the following step, we construct such a function. For this purpose, let $\varepsilon > 0$, $w_1, w_2 \in (w_*, w^*)$ such that $w_1 < w_2$, $k \in \pi_{x_2}(U)$, and assume that ε is small enough to guarantee that $Z_k^{k+\varepsilon}(w_1, w_2) = T_h(\{w_1 < w < w_2\}) \cap \{k < x_2 < k + \varepsilon\} \subset\subset U$ and $\varepsilon < \underline{h}/2\bar{h}$.

The proof of the main result requires a number of lemmas. First, observe that since $a(t) > 0$ for $t > 0$, we deduce from (1.1) that a is one-to-one. Then we consider the function

$$\overline{v}_\varepsilon(x_1,x_2) = \vartheta_\varepsilon(k+\varepsilon-x_2) \text{ with } \vartheta_\varepsilon(t) = \int_0^t a^{-1}(2\overline{h}\varepsilon - \overline{h}s)\,\mathrm{d}s \quad \text{for } t\in[0,\varepsilon],$$

which satisfies

$$\Delta_A \overline{v}_\varepsilon = -\overline{h} \quad \text{in } Z_k^{k+\varepsilon}(w_1,w_2). \tag{3.1}$$

Next, let v_ε be the unique solution in $W^{1,A}(Z_k^{k+\varepsilon}(w_1,w_2))$ of the problem

$$\begin{cases} \Delta_A v_\varepsilon = -\operatorname{div}(H) & \text{in } Z_k^{k+\varepsilon}(w_1,w_2),\\ v_\varepsilon = \overline{v}_\varepsilon & \text{on } \partial Z_k^{k+\varepsilon}(w_1,w_2). \end{cases} \tag{3.2}$$

Then we obtain:

Lemma 3.1. *We have*

$$0 \leqslant v_\varepsilon \leqslant \overline{v}_\varepsilon \quad \textit{in } Z_k^{k+\varepsilon}(w_1,w_2). \tag{3.3}$$

Proof. To simplify the notation, we drop the dependence of $Z_k^{k+\varepsilon}(w_1,w_2)$ on (w_1,w_2).

(i) Note that $v_\varepsilon^- \in W^{1,A}(Z_k^{k+\varepsilon})$ and $v_\varepsilon^- = 0$ on $\partial Z_k^{k+\varepsilon}(w_1,w_2)$. Therefore, we obtain from (3.2) and (1.5)

$$\begin{aligned} \int_{Z_k^{k+\varepsilon}} \frac{a(|\nabla v_\varepsilon|)}{|\nabla v_\varepsilon|}\nabla v_\varepsilon\cdot\nabla v_\varepsilon^-\,\mathrm{d}x &= \int_{Z_k^{k+\varepsilon}} \operatorname{div}(H)v_\varepsilon^-\,\mathrm{d}x,\\ \int_{Z_k^{k+\varepsilon}} \frac{a(|\nabla v_\varepsilon^-|)}{|\nabla v_\varepsilon^-|}\nabla v_\varepsilon^-\cdot\nabla v_\varepsilon^-\,\mathrm{d}x &= \int_{Z_k^{k+\varepsilon}} -\operatorname{div}(H)v_\varepsilon^-\,\mathrm{d}x,\\ \int_{Z_k^{k+\varepsilon}} |\nabla v_\varepsilon^-|a(|\nabla v_\varepsilon^-|)\,\mathrm{d}x &= \int_{Z_k^{k+\varepsilon}} -\operatorname{div}(H)v_\varepsilon^-\,\mathrm{d}x \leqslant 0. \end{aligned} \tag{3.4}$$

Taking into account (3.4) and the fact that $ta(t)$ is an increasing function, we deduce that $\nabla v_\varepsilon^- = 0$ a.e. in $Z_k^{k+\varepsilon}$. Since $v_\varepsilon^- = 0$ on $\partial Z_k^{k+\varepsilon}$, we must have $v_\varepsilon^- = 0$ in $Z_k^{k+\varepsilon}$. Hence, $v_\varepsilon \geqslant 0$ in $Z_k^{k+\varepsilon}$.

(ii) Similarly, we observe that $(v_\varepsilon - \overline{v}_\varepsilon)^+ \in W^{1,A}(Z_k^{k+\varepsilon})$ and $(v_\varepsilon - \overline{v}_\varepsilon)^+ = 0$ on $\partial Z_k^{k+\varepsilon}$. Therefore, we obtain from (3.1) and (3.2)

$$\int_{Z_k^{k+\varepsilon}} \frac{a(|\nabla v_\varepsilon|)}{|\nabla v_\varepsilon|}\nabla v_\varepsilon\cdot\nabla(v_\varepsilon-\overline{v}_\varepsilon)^+\,\mathrm{d}x = \int_{Z_k^{k+\varepsilon}} \operatorname{div}(H)(v_\varepsilon-\overline{v}_\varepsilon)^+\,\mathrm{d}x, \tag{3.5}$$

$$\int_{Z_k^{k+\varepsilon}} \frac{a(|\nabla \overline{v}_\varepsilon|)}{|\nabla \overline{v}_\varepsilon|}\nabla \overline{v}_\varepsilon\cdot\nabla(v_\varepsilon-\overline{v}_\varepsilon)^+\,\mathrm{d}x = \int_{Z_k^{k+\varepsilon}} \overline{h}(v_\varepsilon-\overline{v}_\varepsilon)^+\,\mathrm{d}x. \tag{3.6}$$

Subtracting (3.6) from (3.5), and using (1.6), we get

$$\int_{Z_k^{k+\varepsilon}} \Big(\frac{a(|\nabla v_\varepsilon|)}{|\nabla v_\varepsilon|}\nabla v_\varepsilon - \frac{a(|\nabla \overline{v}_\varepsilon|)}{|\nabla \overline{v}_\varepsilon|}\nabla \overline{v}_\varepsilon\Big)\cdot \nabla(v_\varepsilon - \overline{v}_\varepsilon)^+ \,\mathrm{d}x = \int_{Z_k^{k+\varepsilon}} (\mathrm{div}(H) - \bar{h})(v_\varepsilon - \overline{v}_\varepsilon)^+ \,\mathrm{d}x \leqslant 0. \tag{3.7}$$

Taking into account (3.7) and (1.2), we obtain $\nabla(v_\varepsilon - v_\varepsilon)^+ = 0$ a.e. in $Z_k^{k+\varepsilon}$. Since $(v_\varepsilon - \overline{v}_\varepsilon)^+ = 0$ on $\partial Z_k^{k+\varepsilon}$, we get $(v_\varepsilon - \overline{v}_\varepsilon)^+ = 0$ in $Z_k^{k+\varepsilon}$. Hence, $v_\varepsilon \leqslant \overline{v}_\varepsilon$ in $Z_k^{k+\varepsilon}$. □

Lemma 3.2. *After extending v_ε by 0 to $Z_{k+\varepsilon}$, we obtain*

$$\int_{Z_k} \Big(\frac{a(|\nabla v_\varepsilon|)}{|\nabla v_\varepsilon|}\nabla v_\varepsilon + \chi([v_\varepsilon > 0])H(x)\Big)\cdot\nabla\zeta \,\mathrm{d}x \geqslant 0 \quad \forall\, \zeta \in W^{1,A}(Z_k),\ \zeta \geqslant 0,\ \zeta = 0 \text{ on } \partial Z_k \cap U.$$

Proof. First we have $\Delta_A v_\varepsilon = -\mathrm{div} H \leqslant 0$ in $Z_k^{k+\varepsilon}(w_1, w_2)$, and by (3.3), $v_\varepsilon \geqslant 0$ in $Z_k^{k+\varepsilon}(w_1, w_2)$. By Lemma 2.1 we obtain $v_\varepsilon > 0$ in $Z_k^{k+\varepsilon}(w_1, w_2)$.

Let us point out that $v_\varepsilon = 0$ on $L = \partial Z_k^{k+\varepsilon}(w_1, w_2) \cap \{x_2 = k + \varepsilon\}$ and $v_\varepsilon \in C^{1,\alpha}_{\mathrm{loc}}(Z_k^{k+\varepsilon}(w_1, w_2) \cup L)$ for some $\alpha \in (0,1)$ (see [17]). Moreover, we have

$$|\nabla v_\varepsilon(x)| \leqslant a^{-1}(2\bar{h}\varepsilon) \quad \forall\, x \in L. \tag{3.8}$$

Indeed, from Lemma 3.1 we have $v_\varepsilon \leqslant \overline{v}_\varepsilon$ in $Z_k^{k+\varepsilon}(w_1, w_2)$, and since $v_\varepsilon = \overline{v}_\varepsilon = 0$ on L and $v_\varepsilon, \overline{v}_\varepsilon \geqslant 0$, we obtain

$$\forall\,(x_1, x_2) \in Z_k^{k+\varepsilon}(w_1, w_2) \Big|\frac{v_\varepsilon(x_1, x_2) - v_\varepsilon(x_1, k+\varepsilon)}{x_2 - k - \varepsilon}\Big| \leqslant \Big|\frac{\overline{v}_\varepsilon(x_1, x_2) - \overline{v}_\varepsilon(x_1, k+\varepsilon)}{x_2 - k - \varepsilon}\Big|.$$

Letting x_2 go to $k+\varepsilon$, we get $|v_{\varepsilon x_2}(x_1, k+\varepsilon)| \leqslant |\overline{v}_{\varepsilon x_2}(x_1, k+\varepsilon)|$ on L, which is equivalent to $|\nabla v_\varepsilon(x_1, k+\varepsilon)| \leqslant |\nabla \overline{v}_\varepsilon(x_1, k+\varepsilon)|$ on L, since $v_\varepsilon = \overline{v}_\varepsilon = 0$ on L.

Given that $|\nabla \overline{v}_\varepsilon| = \vartheta'_\varepsilon(k+\varepsilon-x_2) \leqslant \vartheta'_\varepsilon(0) = a^{-1}(2\bar{h}\varepsilon)$, (3.8) holds.

Now since the outward unit normal vector to L is $\nu = e_2 = (0,1)$, we get by (1.3) and (3.8), since $\varepsilon \in (0, \underline{h}/2\bar{h})$

$$\frac{a(|\nabla v_\varepsilon|)}{|\nabla v_\varepsilon|}\nabla v_\varepsilon\cdot\nu + H(x)\cdot\nu = \frac{a(|\nabla v_\varepsilon|)}{|\nabla v_\varepsilon|}\nabla v_\varepsilon\cdot e_2 + H_2(x) \geqslant -a(|\nabla v_\varepsilon|) + \underline{h} \geqslant -2\bar{h}\varepsilon + \underline{h} \geqslant 0 \quad \text{on } L. \tag{3.9}$$

Finally, for $\zeta \in W^{1,A}(Z_k)$, $\zeta \geqslant 0$, $\zeta = 0$ on $\partial Z_k \cap U$, we obtain from (3.2) and (3.9)

$$\int_{Z_k} \Big(\frac{a(|\nabla v_\varepsilon|)}{|\nabla v_\varepsilon|}\nabla v_\varepsilon + \chi(\{v_\varepsilon > 0\})H(x)\Big)\cdot\nabla\zeta \,\mathrm{d}x = \int_L \Big(\frac{a(|\nabla v_\varepsilon|)}{|\nabla v_\varepsilon|}\nabla v_\varepsilon\cdot\nu + H(x)\cdot\nu\Big)\zeta \,\mathrm{d}\sigma \geqslant 0.$$

□

Lemma 3.3. *Assume that*

$$u \circ T_h(t_k(w_1), w_1) = u \circ T_h(t_k(w_2), w_2) = 0,$$
$$u \circ T_h(t_k(w), w) \leqslant \vartheta_\varepsilon(\varepsilon) = v_\varepsilon(t_k(w), w) \quad \forall\, w \in (w_1, w_2).$$

Then we have

$$\lim_{\delta\to 0} \frac{1}{\delta} \int_{Z_k^{k+\varepsilon}(w_1,w_2)\cap\{0<u-v_\varepsilon<\delta\}} \Big(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u - \frac{a(|\nabla v_\varepsilon|)}{|\nabla v_\varepsilon|}\nabla v_\varepsilon\Big) \cdot \nabla(u - v_\varepsilon)\,\mathrm{d}x = 0.$$

Proof. For $\delta, \eta > 0$, let $F_\delta(s)$ be as in the proof of Lemma 2.4, $\mathrm{d}_\eta(x_2) = F_\eta(x_2 - \overline{k})$ and $\overline{k} = k + \varepsilon$. By applying Lemma 2.3 and Lemma 3.2 respectively for $\zeta = F_\delta(u - v_\varepsilon) + d_\eta(1 - F_\delta(u))$ and $\zeta = F_\delta(u - v_\varepsilon)$, we get

$$\begin{aligned}
&\int_{Z_k} \Big(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u + \chi H(x)\Big) \cdot \nabla(F_\delta(u - v_\varepsilon))\,\mathrm{d}x \\
&\quad\leqslant -\int_{Z_k} \Big(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u + \chi H(x)\Big) \cdot \nabla(d_\eta(1 - F_\delta(u)))\,\mathrm{d}x \\
&\qquad - \int_{Z_k} \Big(\frac{a(|\nabla v_\varepsilon|)}{|\nabla v_\varepsilon|}\nabla v_\varepsilon + \chi(\{v_\varepsilon > 0\})H(x)\Big) \cdot \nabla(F_\delta(u - v_\varepsilon))\,\mathrm{d}x \leqslant 0.
\end{aligned}$$

Adding these inequalities, we get since $d_\eta = 0$ in $\{v_\varepsilon > 0\}$

$$\begin{aligned}
&\int_{Z_k\cap\{v_\varepsilon>0\}} F_\delta'(u - v_\varepsilon)\Big(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u - \frac{a(|\nabla v_\varepsilon|)}{|\nabla v_\varepsilon|}\nabla v_\varepsilon\Big) \cdot \nabla(u - v_\varepsilon)\,\mathrm{d}x \\
&\quad\leqslant -\int_{Z_k\cap\{v_\varepsilon=0\}} (1 - d_\eta)\Big(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u + \chi H(x)\Big) \cdot \nabla(F_\delta(u))\,\mathrm{d}x \\
&\qquad - \int_{Z_k\cap\{v_\varepsilon=0\}} (1 - F_\delta(u))\Big(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u + \chi H(x)\Big) \cdot \nabla\,\mathrm{d}_\eta\,\mathrm{d}x = I_1^{\delta\eta} + I_2^{\delta\eta}.
\end{aligned}$$

Since

$$|I_1^{\delta\eta}| \leqslant \int_{D_{k\cap\{\overline{k}<x_2<\overline{k}+\eta\}}} (a(|\nabla u|) + |H(x)|)|\nabla(F_\delta(u))|\,\mathrm{d}x,$$

we obtain $\lim\limits_{\eta\to 0} I_1^{\delta\eta} = 0$. As for $I_2^{\delta\eta}$, we have

$$\begin{aligned}
I_2^{\delta\eta} &= -\int_{Z_k\cap[u=v_\varepsilon=0]} \chi H(x) \cdot \nabla\,\mathrm{d}_\eta\,\mathrm{d}x \\
&\quad - \int_{Z_k\cap[u>v_\varepsilon=0]} (1 - F_\delta(u))\Big(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u + H(x)\Big) \cdot \nabla\,\mathrm{d}_\eta\,\mathrm{d}x \\
&\leqslant -\int_{Z_k\cap[u>v_\varepsilon=0]} (1 - F_\delta(u))\Big(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u + H(x)\Big) \cdot \nabla\,\mathrm{d}_\eta\,\mathrm{d}x = I_3^{\delta\eta},
\end{aligned}$$

since we have by (1.3) $\chi H(x)\cdot\nabla\,\mathrm{d}_\eta=\chi H_2(x)\partial_{x_2}d_\eta=\eta^{-1}\chi H_2(x)\chi_{\{\overline{k}<x_2<\overline{k}+\eta\}}\geqslant 0$ in $Z_k\cap\{u=v_\varepsilon=0\}$.

Let $J=\{w\in(w_1,w_2)/\varphi_h(w)>t_{\overline{k}}(w)\}$. Then given that $u\in C^{0,1}_{\mathrm{loc}}(U)$, one has for some positive constant C

$$\begin{aligned}|I_3^{\delta\eta}|&\leqslant\frac{C}{\eta}\int_{Z_k\cap\{u>v_\varepsilon=0\}\cap\{\overline{k}<x_2<\overline{k}+\eta\}}(1-F_\delta(u))\,\mathrm{d}x\\&=\frac{C}{\eta}\int_J\int_{t_{\overline{k}}(w)}^{\min(\varphi_h(w),t_{\overline{k}+\eta}(w))}(1-F_\delta(u\circ T_h))(t,w)\cdot(-Y_h(t,w))\,\mathrm{d}t\,\mathrm{d}w\\&\leqslant C\bar{h}\int_J\Big(\frac{1}{\eta}\int_{t_{\overline{k}}(w)}^{t_{\overline{k}}(w)+\eta}(1-F_\delta(u\circ T_h))\,\mathrm{d}t\Big)\,\mathrm{d}w.\end{aligned}$$

Since the function $t\mapsto 1-F_\delta(u\circ T_h(t,w))$ is continuous, we obtain

$$\limsup_{\eta\to0}|I_3^{\delta\eta}|\leqslant C\bar{h}\int_J(1-F_\delta(u\circ T_h(t_{\overline{k}}(w),w)))\,\mathrm{d}w.$$

Hence,

$$\begin{aligned}\int_{Z_k^{k+\varepsilon}(w_1,w_2)\cap\{0<u-v_\varepsilon<\delta\}}&\frac{1}{\delta}\Big(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u-\frac{a(|\nabla v_\varepsilon|)}{|\nabla v_\varepsilon|}\nabla v_\varepsilon\Big)\cdot\nabla(u-v_\varepsilon)^+\,\mathrm{d}x\\&\leqslant C\int_J(1-F_\delta(u\circ T_h(t_{\overline{k}}(w),w)))\,\mathrm{d}w.\end{aligned}$$

The lemma follows by letting $\delta\to0$. □

Lemma 3.4. *Assume that the assumptions of Lemma* 3.3 *hold. Then we have*

$$\int_{Z_k^{k+\varepsilon}(w_1,w_2)}\mathbb{A}(x)\nabla(u-v_\varepsilon)^+\cdot\nabla\zeta\,\mathrm{d}x=0\quad\forall\,\zeta\in\mathcal{D}(Z_k^{k+\varepsilon}(w_1,w_2)),\tag{3.10}$$

where

$$\mathbb{A}(\xi)=(\mathbb{A}_{ij}),\quad\mathbb{A}_{ij}=\frac{\partial\mathcal{A}^i}{\partial x_j}\quad\textit{and}\quad\mathcal{A}^i(\xi)=\frac{a(|\xi|)}{|\xi|}\xi_i.$$

P r o o f. First, we observe that we have for any $\zeta\in\mathcal{D}(Z_k^{k+\varepsilon}(w_1,w_2))$

$$\begin{aligned}&\int_{Z_k^{k+\varepsilon}(w_1,w_2)}\chi(\{u>v_\varepsilon\})\Big(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u-\frac{a(|\nabla v_\varepsilon|)}{|\nabla v_\varepsilon|}\nabla v_\varepsilon\Big)\cdot\nabla\zeta\,\mathrm{d}x\\&=\lim_{\delta\to0}\int_{Z_k^{k+\varepsilon}(w_1,w_2)}F_\delta(u-v_\varepsilon)\Big(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u-\frac{a(|\nabla v_\varepsilon|)}{|\nabla v_\varepsilon|}\nabla v_\varepsilon\Big)\cdot\nabla\zeta\,\mathrm{d}x=\lim_{\delta\to0}I_\delta,\end{aligned}\tag{3.11}$$

where

(3.12)
$$
\begin{aligned}
I_\delta = &\int_{Z_k^{k+\varepsilon}(w_1,w_2)} \Big(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u - \frac{a(|\nabla v_\varepsilon|)}{|\nabla v_\varepsilon|}\nabla v_\varepsilon\Big) \cdot \nabla(F_\delta(u-v_\varepsilon)\zeta)\,\mathrm{d}x \\
&- \frac{1}{\delta}\int_{Z_k^{k+\varepsilon}(w_1,w_2)\cap[0<u-v_\varepsilon<\delta]} \zeta\Big(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u - \frac{a(|\nabla v_\varepsilon|)}{|\nabla v_\varepsilon|}\nabla v_\varepsilon\Big) \cdot \nabla(u-v_\varepsilon)\,\mathrm{d}x \\
= &\, I_\delta^1 - I_\delta^2.
\end{aligned}
$$

By Lemma 3.3 and (1.2) we have

$$
\lim_{\delta\to 0} I_\delta^2 = 0. \tag{3.13}
$$

Regarding the integral I_δ^1, we have from (P) (ii) and the problem (3.2), because $(F_\delta(u-v_\varepsilon)\zeta) \in W_0^{1,A}(Z_k^{k+\varepsilon}(w_1,w_2))$ and $\chi = 1$ a.e. in $\{u > v_\varepsilon\}$ that

$$
\begin{aligned}
I_\delta^1 = &\int_{Z_k^{k+\varepsilon}(w_1,w_2)} \frac{a(|\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla(F_\delta(u-v_\varepsilon)\zeta)\,\mathrm{d}x \\
&- \int_{Z_k^{k+\varepsilon}(w_1,w_2)} \frac{a(|\nabla v_\varepsilon|)}{|\nabla v_\varepsilon|}\nabla v_\varepsilon \cdot \nabla(F_\delta(u-v_\varepsilon)\zeta)\,\mathrm{d}x \\
= &- \int_{Z_k^{k+\varepsilon}(w_1,w_2)} \chi H(x) \cdot \nabla(F_\delta(u-v_\varepsilon)\zeta)\,\mathrm{d}x \\
&+ \int_{Z_k^{k+\varepsilon}(w_1,w_2)} H(x) \cdot \nabla(F_\delta(u-v_\varepsilon)\zeta)\,\mathrm{d}x = 0.
\end{aligned} \tag{3.14}
$$

It follows from (3.11)–(3.14) that

$$
\int_{Z_k^{k+\varepsilon}(w_1,w_2)} \chi(\{u>v_\varepsilon\})\Big(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u - \frac{a(|\nabla v_\varepsilon|)}{|\nabla v_\varepsilon|}\nabla v_\varepsilon\Big) \cdot \nabla\zeta\,\mathrm{d}x = 0 \quad \forall\,\zeta \in \mathcal{D}(Z_k^{k+\varepsilon}(w_1,w_2)),
$$

which can be written as

$$
\int_{Z_k^{k+\varepsilon}(w_1,w_2)} \chi(\{u>v_\varepsilon\})\bigg(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{A}(\nabla w_t))\,\mathrm{d}t\bigg) \cdot \nabla\zeta\,\mathrm{d}x = 0 \quad \forall\,\zeta \in \mathcal{D}(Z_k^{k+\varepsilon}(w_1,w_2)), \tag{3.15}
$$

where

$$
\mathcal{A}(\xi) = (\mathcal{A}^1, \mathcal{A}^2)(\xi) = \frac{a(|\xi|)}{|\xi|}\xi
$$

and $w_t = tu + (1-t)v_\varepsilon$. Now observe that

$$
\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{A}(\nabla w_t))\,\mathrm{d}t = \bigg(\int_0^1 \frac{\partial \mathcal{A}^i}{\partial x_j}(\nabla w_t)\bigg)_{i,j=1,2} \nabla(u-v) = \mathbb{A}(x)\nabla(u-v). \tag{3.16}
$$

Hence, we obtain (3.10) from (3.15) and (3.16). □

Lemma 3.5. *We have*

$$\min(1,a_0)\frac{a(z)}{z}|\xi|^2 \leqslant \mathbb{A}_{ij}(z)\xi_i\xi_j \leqslant \max(1,a_1)\frac{a(z)}{z}|\xi|^2 \quad \forall\, z \neq 0\ \forall\, \xi \in \mathbb{R}^2. \tag{3.17}$$

Proof. Let $z \neq 0$ and $\xi = (\xi_1,\xi_2) \in \mathbb{R}^2$. Since $\mathbb{A} \in C^1(\mathbb{R}^2 \setminus \{(0,0)\})$, we get by direct calculation

$$\mathbb{A}_{ij}(z) = \frac{\partial(\mathcal{A}^i(z))}{\partial z_j} = \frac{a'(z)z - a(z)}{z^3} z_i z_j + \frac{a(z)}{z}\delta_{ij}$$
$$\mathbb{A}_{ij}(z)\xi_i\xi_j = \frac{a'(z)z - a(z)}{z^3}(z_1\xi_1 + z_2\xi_2)^2 + \frac{a(z)}{z}|\xi|^2.$$

Using (1.1), we obtain

$$\frac{a(z)}{z}\Big((a_0-1)\frac{|z\cdot\xi|^2}{z^2} + |\xi|^2\Big) \leqslant \mathbb{A}_{ij}(z)\xi_i\xi_j \leqslant \frac{a(z)}{z}\Big((a_1-1)\frac{|z\cdot\xi|^2}{z^2} + |\xi|^2\Big).$$

Then, if $a_0 \geqslant 1$, the left-hand side of inequality (3.17) holds. When $a_0 < 1$, we use the Cauchy-Schwarz inequality $|z\cdot\xi| \leqslant |z||\xi|$, to conclude. We proceed in the same way for the right-hand side. □

Lemma 3.6. *Assume that the assumptions of Lemma* 3.3 *hold. Then we have:*

If u is not positive in $Z_k^{k+\varepsilon}(w_1,w_2)$, then $u = 0$ in $Z_{k+\varepsilon}$.

Proof. Assume that u is not positive in $Z_k^{k+\varepsilon}(w_1,w_2)$. Then

$$\exists (t_0,w_0) \text{ such that } T_h(t_0,w_0) \in Z_k^{k+\varepsilon}(w_1,w_2) \text{ and } u\circ T_h(t_0,w_0) = 0.$$

This leads by Theorem 2.2 (ii) to

$$u\circ T_h(t,w_0) = 0 \quad \forall\, t \in [t_0, t_{k+\varepsilon}]. \tag{3.18}$$

From Lemmas 3.4 and 3.5 we know that

$$\operatorname{div}(\mathbb{A}(x)\nabla(u - v_\varepsilon)^+) = 0 \quad \text{in } Z_k^{k+\varepsilon}(w_1,w_2). \tag{3.19}$$

Moreover, by Lemma 3.5, the matrix $\mathbb{A}(x)$ satisfies for all $x \in Z_k^{k+\varepsilon}(w_1,w_2)$ and $\xi \in \mathbb{R}^2$

$$\min(1,a_0)\lambda(x)|\xi|^2 \leqslant \mathbb{A}(x)\xi\cdot\xi \leqslant \max(1,a_1)\lambda(x)|\xi|^2 \tag{3.20}$$
$$\text{with } \lambda(x) = \int_0^1 \frac{a(|\nabla w_t(x)|)}{|\nabla w_t(x)|}\,\mathrm{d}t, \quad w_t = tu + (1-t)v_\varepsilon.$$

Next, we have $v_\varepsilon \in C^{1,\alpha}(Z_k^{k+\varepsilon}(w_1, w_2) \cup L)$, where

$$L = \partial Z_k^{k+\varepsilon}(w_1, w_2) \cap \{x_2 = k + \varepsilon\}.$$

We also have $v_\varepsilon = 0$ on L and $v_\varepsilon > 0$ in $Z_k^{k+\varepsilon}(w_1, w_2)$. So v_ε achieves its minimum value on the line segment L. By Lemma 3.2 of [12], we must have $|\nabla v_\varepsilon| > 0$ along L. Therefore, for δ small enough such that $w_1 + \delta < w_2 - \delta$ there exist two positive constants c_0, c_1 such that

$$\forall\, x \in \overline{Z}_k^{k+\varepsilon}(w_1, w_2) \cap \{k + \varepsilon - \delta \leqslant x_2 \leqslant k + \varepsilon\} \cap \{w_1 + \delta \leqslant w \leqslant w_2 - \delta\} = Z_{k+\varepsilon-\delta}^{k+\varepsilon}$$

$$c_0 \leqslant |\nabla v_\varepsilon(x)| \leqslant c_1. \tag{3.21}$$

On the other hand, $|\nabla u|$ is also bounded in $Z_{k+\varepsilon-\delta}^{k+\varepsilon}$, since by Theorem 2.1, $u \in C^{0,1}(\overline{Z}_k^{k+\varepsilon}(w_1, w_2))$. It follows from (3.20)–(3.21) that we have for two positive constants λ_0 and λ_1

$$\lambda_0 \leqslant \lambda(x) \leqslant \lambda_1 \quad \text{in} \quad Z_{k+\varepsilon-\delta}^{k+\varepsilon}$$

and therefore, we get from (3.20)

$$\min(1, a_0)\lambda_0|\xi|^2 \leqslant \mathbb{A}(x)\xi \cdot \xi \leqslant \max(1, a_1)\lambda_1|\xi|^2 \quad \forall\, x \in Z_{k+\varepsilon-\delta}^{k+\varepsilon} \quad \forall\, \xi \in \mathbb{R}^2. \tag{3.22}$$

Taking into account (3.18), we see that

$$Z_{k+\varepsilon-\delta}^{k+\varepsilon} \cap \{u = 0\} \neq \emptyset. \tag{3.23}$$

It follows from (3.19), (3.22), (3.23), and the strong maximum principle that $(u - v_\varepsilon)^+ \equiv 0$ in $Z_{k+\varepsilon-\delta}^{k+\varepsilon}$. Consequently, we obtain $u \leqslant v_\varepsilon$ in $Z_{k+\varepsilon-\delta}^{k+\varepsilon}$, and therefore $u \circ T_h(t_{k+\varepsilon}(w), w) = 0$ for all $w \in (w_1 + \delta, w_2 - \delta)$. Since δ is arbitrarily small, we get $u \circ T_h(t_{k+\varepsilon}(w), w) = 0$ for all $w \in (w_1, w_2)$. Hence, by Theorem 2.2 (ii) we obtain $u = 0$ in $Z_{k+\varepsilon}$. □

Lemma 3.7. *Let $w_0 \in (w_*, w^*)$, $x_0 = T_h(t_0, w_0)$ be such that $u(x_0) = 0$ and for some $\eta > 0$, $B_\eta(T_h(t_0, w_0)) \subset\subset U$. Then there exist two sequences $(t_n^-, w_n^-)_n$ and $(t_n^+, w_n^+)_n$ such that $\lim_{n\to\infty}(t_n^+, w_n^+) = \lim_{n\to\infty}(t_n^-, w_n^-) = (t_0, w_0)$ and for all n,*

(i) $T_h(t_n^-, w_n^-) \in B_\eta(T_h(t_0, w_0)) \cap \{w < w_0\}$, $u \circ T_h(t_n^-, w_n^-) = 0$,
(ii) $T_h(t_n^+, w_n^+) \in B_\eta(T_h(t_0, w_0)) \cap \{w > w_0\}$, $u \circ T_h(t_n^+, w_n^+) = 0$.

P r o o f. First we observe that by Lemma 2.6 the following situations cannot occur simultaneously:

(a) $u \circ T_h > 0$ in $B_\eta(T_h(t_0, w_0)) \cap \{w < w_0\}$,

(b) $u \circ T_h > 0$ in $B_\eta(T_h(t_0, w_0)) \cap \{w > w_0\}$.

In fact, to prove the lemma, it is enough to show that neither (a) nor (b) hold. So assume for example that (a) holds. Then by Lemma 2.6 there exists a sequence $(t_n^+, w_n^+)_n$ such that $T_h(t_n^+, w_n^+) \in B_\eta(T_h(t_0, w_0)) \cap \{w > w_0\}$,

$$u \circ T_h(t_n^+, w_n^+) = 0 \quad \text{and} \quad \lim_{n\to\infty} (t_n^+, w_n^+) = (t_0, w_0).$$

Let $k = \max\{T_h^2(t_0, w_0), T_h^2(t_n^+, w_n^+)\}$. Then since $u(x_0) = 0$ and u is continuous at x_0, we may assume that for n large enough we have

$$u \circ T_h(t_k(w), w) \leqslant \vartheta_\varepsilon(\varepsilon) \quad \forall\, w \in (w_0, w_n^+). \tag{3.24}$$

For $\varepsilon > 0$ small enough and n large enough, we may also assume that

$$Z_k^{k+\varepsilon}(w_0, w_n^+) \subset\subset U. \tag{3.25}$$

We observe that because of the sequence $(t_n^+, w_n^+)_n$ and Theorem 2.2 (i), u is not positive in $Z_k^{k+\varepsilon}(w_0, w_n^+)$. Then, by using (3.24), (3.25), and Lemma 3.6, we conclude that for $\varepsilon > 0$ small enough and n large enough we must have $u = 0$ in $Z_{k+\varepsilon} \cap T_h(\{w_0 < w < w_n^+\})$. Now since we have assumed that (a) holds, we are in contradiction with Lemma 2.6.

Similarly, if we assume that (b) holds, we get a contradiction as well. □

We are now ready to prove the main result of this paper.

Theorem 3.1. *The function φ_h is continuous in the interval (w_*, w^*).*

Proof. Let $w_0 \in (w_*, w^*)$. We will prove that φ_h is continuous at w_0. To this end, it is enough to show that φ_h is upper semi-continuous at w_0.

Let $x_0 = T_h(\varphi_h(w_0), w_0) = T_h(t_0, w_0)$ and let $\varepsilon > 0$. Since $u(x_0) = 0$ and u is continuous at x_0, there exists $\eta \in (0, \varepsilon)$ such that

$$u(x) \leqslant \vartheta_\varepsilon(\varepsilon) \quad \forall\, x \in B_\eta(x_0) \subset\subset U. \tag{3.26}$$

By Lemma 3.7, there exists two sequences $(t_n^-, w_n^-)_n$ and $(t_n^+, w_n^+)_n$ such that $\lim\limits_{n\to\infty} (t_n^+, w_n^+) = \lim\limits_{n\to\infty} (t_n^-, w_n^-) = (t_0, w_0)$ and for all n

(i) $T_h(t_n^-, w_n^-) \in B_\eta(T_h(t_0, w_0)) \cap \{w < w_0\}$, $u \circ T_h(t_n^-, w_n^-) = 0$,

(ii) $T_h(t_n^+, w_n^+) \in B_\eta(T_h(t_0, w_0)) \cap \{w > w_0\}$, $u \circ T_h(t_n^+, w_n^+) = 0$.

Let $k = \max\{T_h^2(t_n^-, w_n^-), T_h^2(t_0, w_0), T_h^2(t_n^+, w_n^+)\}$ and let C be the constant in Lemma 2.2. We observe that we can choose ε small enough and n large enough so that

$$\varepsilon' = \varepsilon/2C < \underline{h}/2\bar{h},$$
$$Z_k^{k+\varepsilon'}(w_n^-, w_n^+) \subset\subset B_\eta(x_0). \tag{3.27}$$

We also observe that because $T_h(t_0, w_0) = 0$, and by Theorem 2.2 (i), u is not positive in $Z_k^{k+\varepsilon'}(w_n^-, w_n^+)$. Then, by using (3.26), (3.27), and Lemma 3.6, we see that for n large enough, we must have

$$u = 0 \quad \text{in } T_h(\{w_n^- < w < w_n^+\}) \cap \{x_2 \geqslant k + \varepsilon'\}.$$

Therefore, we obtain

$$\varphi_h(w) \leqslant t_{k+\varepsilon'}(w) \quad \forall\, w \in (w_n^-, w_n^+). \tag{3.28}$$

From Lemma 2.2, we infer that we have for $\eta < \varepsilon/4C$

$$\begin{aligned} t_{k+\varepsilon'}(w) &\leqslant t_{x_{02}}(w_0) + C(|k + \varepsilon' - x_{02}| + |w - w_0|) \\ &\leqslant t_0 + C(\eta + \varepsilon' + \eta) = t_0 + 2C\eta + \varepsilon/2 \\ &\leqslant t_0 + \varepsilon/2 + \varepsilon/2 = t_0 + \varepsilon. \end{aligned} \tag{3.29}$$

Combining (3.28) and (3.29), we obtain

$$\varphi_h(w) \leqslant \varphi_h(w_0) + \varepsilon \quad \forall\, w \in (w_n^-, w_n^+),$$

which is the upper semi-continuity of φ_h at w_0. □

Corollary 3.1. *We have*

$$\chi = \chi_{\{u>0\}}.$$

P r o o f. We observe that by (2.3), it is enough to show that we have for each h

$$\chi \circ T_h = \chi_{\{t<\varphi_h(w)\}}. \tag{3.30}$$

First, we have by (P) (i) and (2.3)

$$\chi \circ T_h = 1 \quad \text{a.e. in } \{t < \varphi_h(w)\}. \tag{3.31}$$

Next, we have by Lemma 2.5

$$\chi \circ T_h = 0 \quad \text{a.e. in } \mathrm{Int}(\{u \circ T_h = 0\}) = \mathrm{Int}(\{t \geqslant \varphi_h(w)\}). \tag{3.32}$$

Now, the set $\{t = \varphi_h(w)\}$ being of measure zero (since φ_h is continuous at each point w such that $T_h(\varphi_h(w), w) \in \Omega$), we conclude that (3.30) follows from (3.31)–(3.32). □

A c k n o w l e d g e m e n t. The author is grateful for the facilities and excellent research conditions provided by the Fields Institute, where part of this research was carried out.

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