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OPTIMAL PACKINGS FOR FILLED RINGS OF CIRCLES

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Abstract. General circle packings are arrangements of circles on a given surface such that no two circles overlap except at tangent points. In this paper, we examine the optimal arrangement of circles centered on concentric annuli, in what we term rings. Our motivation for this is two-fold: first, certain industrial applications of circle packing naturally allow for filled rings of circles; second, any packing of circles within a circle admits a ring structure if one allows for irregular spacing of circles along each ring. As a result, the optimization problem discussed herein will be extended in a subsequent paper to a more general setting. With this framework in mind, we present properties of concentric rings that have common points of tangency, the exact solution for the optimal arrangement of filled rings along with its symmetry group, and applications to construction of aluminum-conductor steel reinforced cables.

Keywords: optimization; minimal separation; dense packing

MSC 2010: 52C15, 52C26

1. Introduction

Circle packing in a circle is a two dimensional problem of packing n equal circles into the smallest possible larger circle. In the cases of $n = 7, 19, 37, 61, 91$, the optimal solution ($n = 7$ and 19, see [2]) or the conjectured optimal solution ($n = 37, 61$ and 91, see [3]) contain filled rings of circles as shown in Figure 1. Such an arrangement is particularly useful when we want to arrange circles into layers. Here we explicitly study the optimal packing of discs in filled rings with minimal separation.

One may consider minimal separation of rings with either disjoint or tangential discs and for identical or unequal discs in separate rings. Optimal packing of discs in rings with such variety of possible arrangements has an array of applications. For example, optimal construction of submarine communication cables or high voltage

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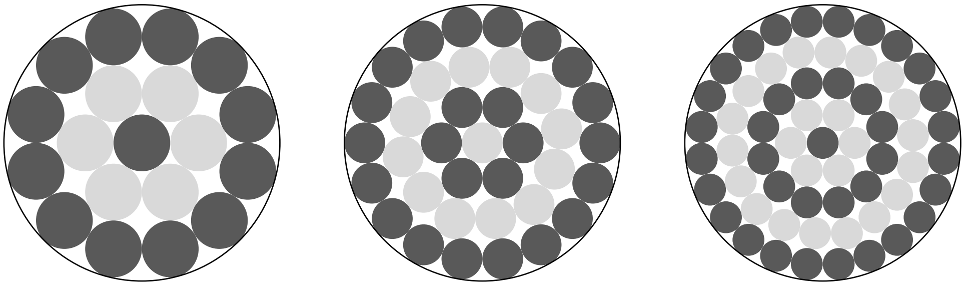

Figure 1. The optimal arrangement of circles in a circle contains filled rings of circles for: (a) $n = 19$ (see [2]), (b) $n = 37$ (conjectured) (see [3]), and (c) $n = 61$ (conjectured) (see [3]).

power cables may require cables composed of fiber optics in rings of tubes, or rings of strands of steel and aluminum, each having a different radius [10], [11]. A circular constellation diagram that represents a signal modulated by a digital modulation scheme [4], [7] may require identifying the centers of disjoint discs that are on minimally separated rings to improve the noise tolerance of the transmission. A sequential or recursive circle packing problem on circles [8] may be solved as an optimal packing of rings. Subsequently, the finding of minimally separated rings is an important class of circle packing problems.

Packing equal circles into an annulus has been studied under packing with circular prohibited areas [5], [9]. Generally, only the computational optimal solution is found by using iteration schemes such as the Zoutendijk method, which generates improved feasible directions at each iteration [12]. In this paper, we formulate the optimization problem related to minimal separation of filled rings. We show that when the discs in each ring are externally tangent, the exact solution can be found by solving an integer optimization problem on a finite set of integers. As an example, we demonstrate the application of the solution to the design of aluminum conductor steel-reinforced (ACSR) cables.

1.1. Problem formulation. Define $\mathrm{Ring}(m, r, s)$ as the set of $m \geqslant 3$ congruent discs with circular boundaries of radius $r > 0$, whose centers are regularly spaced on a common core circle of radius $s > r$ centered at a point O, and whose interiors are disjoint; see both part (a) and part (b) of Figure 2. As indicated in part (b) of that figure, although the interiors of the discs are disjoint, the discs may be externally tangent to each other on their boundary circles, in which case we say the ring is *filled*. In a filled ring, the core circle necessarily has radius $r \csc(\pi/m)$, as indicated in part (b) of Figure 2, and thus we will write just $\mathrm{Ring}(m, r)$ for a filled ring. This filled case is of primary interest in the current paper; however, certain geometric properties which are established in this paper hold for general rings which may not be filled, and these will find further applications in a subsequent paper.

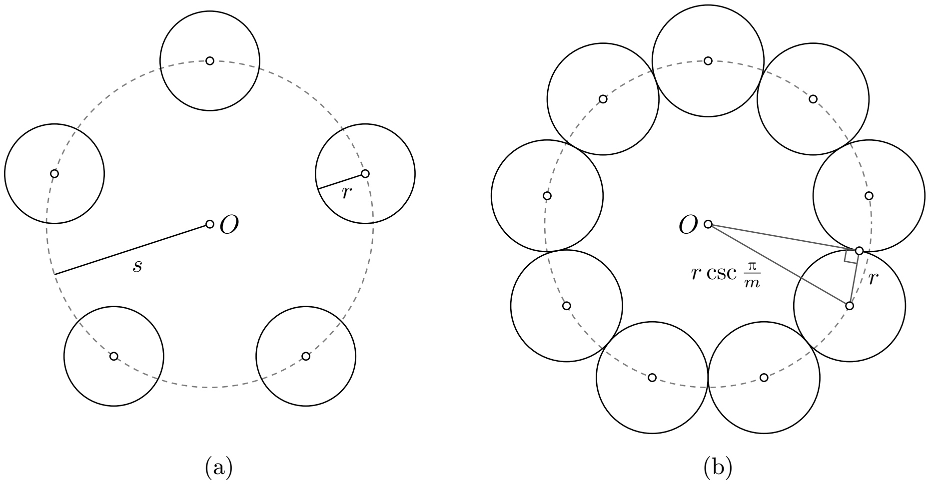

Figure 2. (a) The set of m congruent discs of radius r with centers regularly spaced on a core circle of radius s centered at O, denoted by Ring(m, r, s); here $m = 5$.
(b) A filled ring denoted by Ring(m, r), where the radius of the core circle is $r \csc(\pi/m)$ and consecutive discs are externally tangent to each other; here $m = 9$.

Because the discs in Ring(m, r, s) have boundary circles and the circles likewise bound discs, we will at times speak both of the *discs of* Ring(m, r, s) or the *circles of* Ring(m, r, s); if a sharp distinction is needed at any point, we will provide clarification.

Consider two concentric rings Ring(m, r, s) and Ring(n, ϱ, σ) such that the inequality $\sigma > s$ holds. With this constraint, the core circle of Ring(n, ϱ, σ) lies outside of the core circle of Ring(m, r, s). We say the two rings are arranged *orderly* if the interiors of the discs from the two rings are disjoint, but there exists at least one point of tangency between discs from the two rings; see Figure 3 which shows both the general case in part (a) as well as the filled case in part (b).

In this paper we will be primarily interested in a set of filled rings Ring(m_i, r_i), $i = 1, 2, \ldots, p$, such that $r_{i+1} \csc(\pi/m_{i+1}) > r_i \csc(\pi/m_i)$ for all $i = 1, \ldots, p-1$, and where each successive pair Ring(m_i, r_i) and Ring(m_{i+1}, r_{i+1}) are arranged orderly. We say the orderly packing has *minimal separation* with respect to a fixed initial radius r_1 if the packing minimizes successive radii of the rings Ring(m_i, r_i), $i = 2, \ldots, p$, over all such orderly packings.

Our problem is therefore to find orderly packings of filled rings that exhibit minimal separation. To this end, the outline of the paper is as follows: In Section 2, we prove initial geometric properties of concentric orderly packed pairs of rings, both

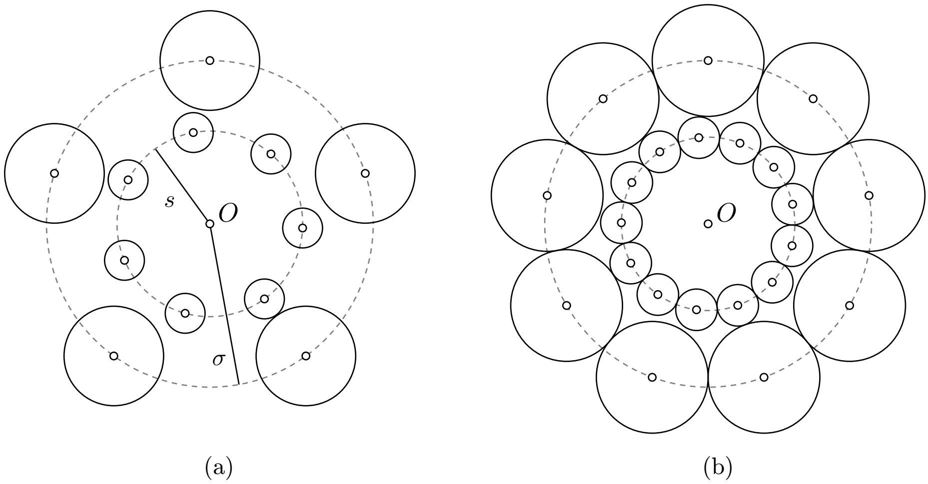

Figure 3. (a) An orderly packing of two concentric rings $\mathrm{Ring}(m, r, s)$ and $\mathrm{Ring}(n, \varrho, \sigma)$. (b) An orderly packing of two filled rings $\mathrm{Ring}(m, r)$ and $\mathrm{Ring}(n, \varrho)$.

filled and otherwise. In Section 3 we provide an exact solution for the problem of minimizing the radius of an outer filled ring with respect to a fixed inner filled ring, and determine the symmetry group of this optimal solution. In Section 4 we implement this solution computationally to generate examples of minimally separated orderly packed rings, in particular some that are relevant to the construction of high voltage power line cables.

2. Geometry of concentric rings

In this section we prove several observations of orderly packed concentric rings; these hold for general rings, not just filled rings, except where noted. All of these observations in and of themselves are basic, but nevertheless their combined effect will allow the solution of our optimization problem in Section 3.

Proposition 2.1. *Consider two fixed concentric orderly packed rings* $\mathrm{Ring}(m, r, s)$ *and* $\mathrm{Ring}(n, \varrho, \sigma)$ *centered at a point* O*. Let* C *be a circle centered at a point* X *on* $\mathrm{Ring}(m, r, s)$*. Then the following four statements are true:*

1. *No more than two circles from* $\mathrm{Ring}(n, \varrho, \sigma)$ *are tangent to* C*.*
2. *If two circles from* $\mathrm{Ring}(n, \varrho, \sigma)$ *are tangent to* C*, those circles are consecutive along their ring, and are reflections of each other in the line* $\overline{OX}$*.*
3. *If any circle from* $\mathrm{Ring}(m, r, s)$ *centered at a point* X_i *is tangent to a circle from* $\mathrm{Ring}(n, \varrho, \sigma)$ *centered at a point* Y_j*, then the central* $\angle X_i O Y_j = \theta$ *for a fixed value* $\theta \geqslant 0$ *that is uniquely determined by* r, s, ϱ *and* σ*.*

4. *The points of tangency between any circles of the two rings are themselves on a circle with center O.*

P r o o f. The proof is a compass and straight-edge construction in Euclidean geometry. Fix a circle C centered at a point X on $\mathrm{Ring}(m, r, s)$. Then OX is a fixed value s. If a circle C_Y centered at a point Y on $\mathrm{Ring}(n, \varrho, \sigma)$ is tangent to C at a point P, then OY is a fixed value of σ and XY is a fixed value, namely $r + \varrho$. This is shown in Figure 4, where Y must be both on a gray circle c_O of fixed radius centered at O, and on a gray circle c_X of fixed radius centered at X. Circles intersect each other in at most two points, hence there are at most two points where Y can occur, and these are reflections of each other through the line $\overline{OX}$, as indicated by Y and Y' in Figure 4. This establishes item 1.

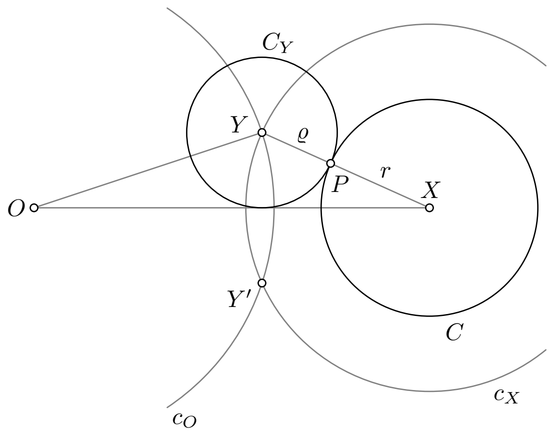

Figure 4. There are only two points Y and Y' which could be centers for the circle tangent to C, and these are reflections of each other through the line $\overline{OX}$.

Next, observe that if both the circle C_Y centered at Y and the circle $C_{Y'}$ centered at Y' are tangent to C, with the latter tangent at a point P', by the reasoning in the above paragraph, $C_{Y'}$ is a reflection of C_Y through the line $\overline{OX}$, and P' is a reflection of P, as indicated by the circle $C_{Y'}$ in Figure 5. Moreover, these circles must be consecutive along $\mathrm{Ring}(n, \varrho, \sigma)$, and in the case when the rings are filled, have $\overline{OX}$ as a common tangent line as indicated in Figure 5. The reason for this is as follows: if there were an intermediate circle along $\mathrm{Ring}(n, \varrho, \sigma)$ between these two circles, its center would be along the subarc of c_O which intersects the disc bounded by c_X. The points along this subarc have distance to X less than $r + \varrho$, and thus this intermediate circle with radius ϱ would be forced to intersect C at two points, since the distance from X to C is r; this contradicts the orderly packing of the two rings. This establishes item 2.

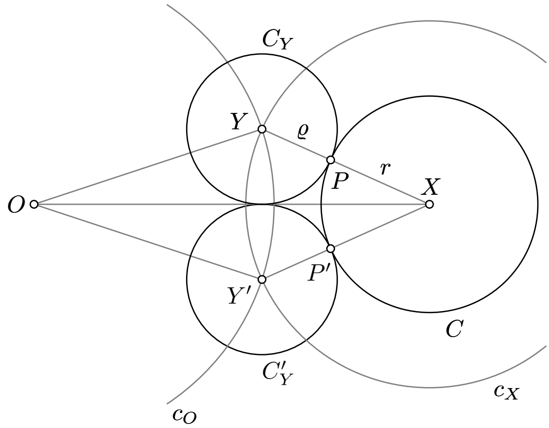

Figure 5. Two circles tangent to C are reflections of each other through the line $\overline{OX}$ and consecutive in their ring.

Finally, nothing special was assumed for C, and the above two paragraphs show that the construction in Figure 4 is in fact generic for any point of tangency P between circles on $\mathrm{Ring}(m, r, s)$ and circles on $\mathrm{Ring}(n, \varrho, \sigma)$, and only depends on the fixed lengths $OX = s$, $OY = \sigma$, and $XY = r + \varrho$. Thus, any point of tangency between circles in the two rings will look precisely like that in Figure 4 after some series of rigid transformations which fix O. As a result, there is only one magnitude for any resulting central angle $\angle XOY$, and only one distance from O to points of tangency P, and this establishes items 3 and 4. $\square$

For a given collection of orderly packed rings, we will denote by S the *symmetry group of the packing*, namely the group of isometries of $\mathbb{R}^2$ which keep the packing fixed. Rings are bounded subsets of $\mathbb{R}^2$ and can be invariant under reflections and rotations. Recall that the *dihedral group* is denoted by D_n, which is the symmetry group of a regular n-gon, made up of n rotations and n reflections. We thus have the following basic lemma, whose proof we provide for completeness.

Lemma 2.1. *Consider a single ring* $\mathrm{Ring}(m, r, s)$ *centered at a point* O.

1. *The rotations which fix* $\mathrm{Ring}(m, r, s)$ *are precisely those around* O *which are integer multiples of* $2\pi/m$.
2. *The reflections that fix a filled* $\mathrm{Ring}(m, r)$ *are precisely across those* m *lines which contain* O *as well as either the center of a circle on* $\mathrm{Ring}(m, r)$ *or a point of tangency between two circles on* $\mathrm{Ring}(m, r)$.
3. *The symmetry group* S *of a filled* $\mathrm{Ring}(m, r)$ *is* D_m, *the dihedral group of order* $2m$.

Proof. For item 1 we observe that the m circles which comprise $\mathrm{Ring}(m, r, s)$ all have the same radii, and their m centers are regularly spaced along the core circle of the ring with angular separation $2\pi/m$ between successive centers. Thus, the rotations that take centers of congruent circles to centers of congruent circles, and hence fix $\mathrm{Ring}(m, r, s)$, are precisely those that are integer multiples of $2\pi/m$.

For item 2 we consider two cases, namely when m is even or odd. When m is even, the line $\overline{OX}$, which passes through the center X of a circle on $\mathrm{Ring}(m, r)$, must also extend through O to a diameter of the core circle that intersects an antipodal center X' of a circle on $\mathrm{Ring}(m, r)$; see part (a) of Figure 6. This line then divides $\mathrm{Ring}(m, r)$ evenly in half and is a line of reflection, and there are $m/2$ such distinct lines. Similarly, the line $\overline{OP}$ which passes through a point of tangency between two consecutive circles must extend through O to a diameter of the core circle that intersects an antipodal point of tangency P' on the opposite side of the filled $\mathrm{Ring}(m, r)$; see part (b) of Figure 6. This line then divides $\mathrm{Ring}(m, r)$ in half and is a line of reflection, and there are $m/2$ such distinct lines, yielding a total of m lines of reflection.

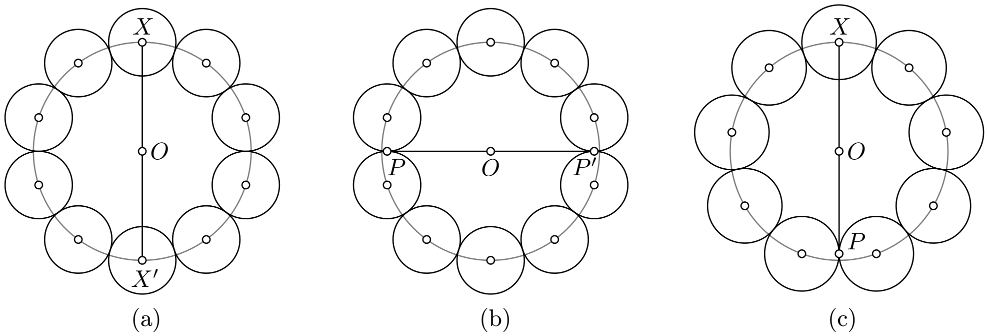

Figure 6. Lines of reflection for a filled ring $\mathrm{Ring}(m, r)$. Parts (a) and (b) are for the case of m even, and part (c) is for the case of m odd.

When m is odd, the line $\overline{OX}$ which passes through the center X of a circle on $\mathrm{Ring}(m, r)$ can be extended through O to a diameter which intersects the core circle halfway between the centers of two circles on $\mathrm{Ring}(m, r)$, and hence passes through a point of tangency P; see part (c) of Figure 6. A reflection across $\overline{OXP}$ thus fixes $\mathrm{Ring}(m, r)$, and there are m such lines, one for each center of a circle on $\mathrm{Ring}(m, r)$.

Finally, no other lines of reflection exist, as any other line which passes through O but does not intersect a center or a point of tangency, will intersect some circle in a non-diameter chord. Reflection across the chord yields two points of intersection of that circle with its image, hence does not fix the ring. This proves items 2 and 3. $\square$

We now consider the symmetry group for two orderly packed rings.

Lemma 2.2. *Let* $\mathrm{Ring}(m, r, s)$ *and* $\mathrm{Ring}(n, \varrho, \sigma)$ *be two concentric orderly packed rings centered at a point* O*. Let* S *be the symmetry group of the packing, and also let* $k = \gcd(m, n)$*.*

1. *S contains as its rotations precisely all rotations about O by integer multiples of the angle $2\pi/k$, hence exactly k rotations.*
2. *For filled rings* $\mathrm{Ring}(m, r)$ *and* $\mathrm{Ring}(n, \varrho)$*, the reflections in the group S are precisely through those lines which pass through the center O, and which contain either a center of a circle or a point of tangency between circles on* $\mathrm{Ring}(m, r)$*, as well as either a center of a circle or a point of tangency between circles on* $\mathrm{Ring}(n, \varrho)$*.*
3. *The symmetry group S for the orderly packing of filled rings* $\mathrm{Ring}(m, r)$ *and* $\mathrm{Ring}(n, \varrho)$ *contains at most k reflections.*

Proof. For item 1, observe that since $k|m$ and $k|n$, we have $k = m/m'$ and $k = n/n'$ for some $m', n' \in \mathbb{N}$. Therefore, $2\pi/k = m' \cdot 2\pi/m = n' \cdot 2\pi/n$, so by item 1 in Lemma 2.1, a rotation of $2\pi/k$ about O fixes both $\mathrm{Ring}(m, r, s)$ and $\mathrm{Ring}(n, \varrho, \sigma)$. Thus, the symmetry group S contains all rotations about O by multiples of $2\pi/k$, hence at least k rotations. Moreover, by Lemma 2.1 item 3 and Lagrange's theorem for finite groups, this rotational subgroup of S must be a subgroup of both the m rotations in D_m and the n rotations in D_n, and thus its order must divide both m and n. Since $k = \gcd(m, n)$, there are at most k rotations in S, and item 1 follows.

For item 2, we observe that by Lemma 2.1 item 2 any line containing O as well as either a center of a circle or a point of tangency on the filled $\mathrm{Ring}(m, r)$, and either a center of a circle or a point of tangency on the filled $\mathrm{Ring}(n, \varrho)$, will be a line of reflection for the packing. Moreover, again by Lemma 2.1 item 2, these are the only possible lines of reflection. This establishes item 2 of the current lemma.

For item 3, again by Lemma 2.1 item 3 the symmetry group S for the orderly packing of filled rings $\mathrm{Ring}(m, r)$ and $\mathrm{Ring}(n, \varrho)$ must be a subgroup of both dihedral groups D_m and D_n, so again by Lagrange's theorem the order of S must divide both $2m$ and $2n$. We know that $\gcd(2m, 2n) = 2k$, with the result that S can contain at most $2k$ elements. By item 1 in the current lemma we know S already has exactly k rotations; it must then have at most k reflections, and this establishes item 3. $\square$

In the next section we will formulate and solve the optimization problem related to the orderly arrangement of two concentric filled rings, $\mathrm{Ring}(m, r)$ and $\mathrm{Ring}(n, \varrho)$, with center O such that $\varrho \csc(\pi/n) > r \csc(\pi/m)$. To do so, by Lemma 2.2 item 1, we only need to consider the minimization problem over a circular sector with a central angle of $2\pi/k$. We therefore conclude this section by establishing notation as well as an initial lemma before formalizing the minimization problem in the next section.

Referring to Figure 7, we denote the centers of the circles in $\mathrm{Ring}(m, r)$ and $\mathrm{Ring}(n, \varrho)$ by $X_0, \ldots, X_{m-1}$ and $Y_0, \ldots, Y_{n-1}$, respectively, in a cyclic counterclockwise ordering along their respective rings. Let X_0 and Y_0 be the centers of two circles which share a point of tangency P, with X_0 positioned on the polar axis, and Y_0 positioned at a central angle θ, which we may assume is non-negative after possibly reflecting both rings in $\overline{OX_0}$. We emphasize we are not claiming this is a line of reflection for the symmetry group, just that the solution to our minimization problem will not depend on this initial setup.

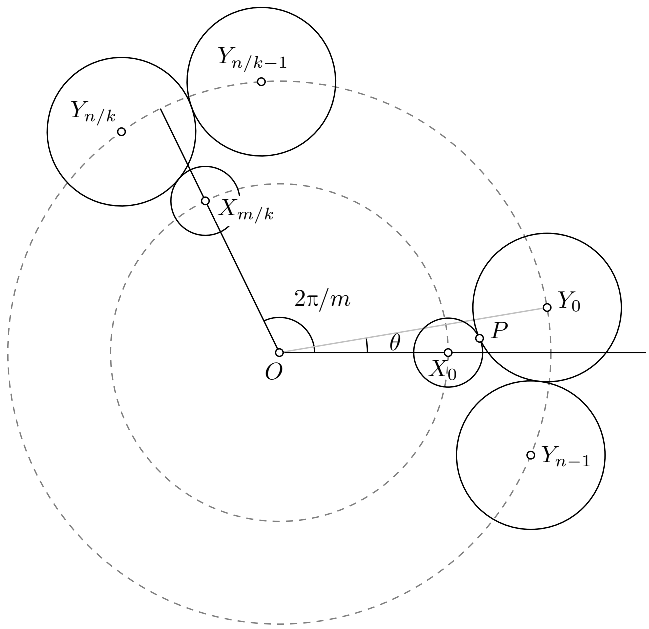

Figure 7. The rotationally invariant sector of angle $2\pi/k$.

Consider the half-open infinite circular sector Φ having vertex at the origin and with central angle $2\pi/k$, which contains all points having polar angles $\varphi \in [0, 2\pi/k)$, with X_0 positioned at $\varphi = 0$. We then have the following initial lemma.

Lemma 2.3. *In the sector Φ described above, we have $\angle X_0 O Y_0 = \theta \leqslant \pi/m$.*

Proof. We think of r, m, ϱ and n as all fixed and examine the local configuration of the circles centered at X_0, X_1 and Y_0; we consider what values of θ can be geometrically realized for these circles in an orderly packing while maintaining the point of tangency P between the circles centered at X_0 and Y_0. To this end we refer to part (a) of Figure 8 which shows local configuration. The centers X_0 and X_1 are on their core circle indicated in dashed gray, and by hypothesis we must have a point of tangency P between the circle centered at X_0 and the gray circle centered at Y_0, at some angle θ which a priori could be as small as zero. In order to maintain this point of tangency, we require Y_0 to be on a dotted black circle c of radius $r + \varrho$ centered at X_0. Thus, any possible increase in θ in this local configuration is effectively accomplished by rolling the gray circle centered at Y_0 counterclockwise along the circle centered at X_0, as indicated in the movement from part (a) to part (b) in Figure 8, keeping all other aspects of the figure fixed. At $\theta = \pi/m$ the gray circle centered at Y_0 will have its center on the line containing the point of tangency between the circles centered at X_0 and X_1, and by the construction in Proposition 2.1 the gray circle centered at Y_0 will then have the second point of tangency P' with the circle centered at X_1, as indicated in part (b) of Figure 8. It is then evident that θ can no longer increase, since doing so would roll the gray circle further counterclockwise so as to intersect the circle centered at X_1 twice, thus breaking the orderly packing. Hence, $\theta \leqslant \pi/m$. □

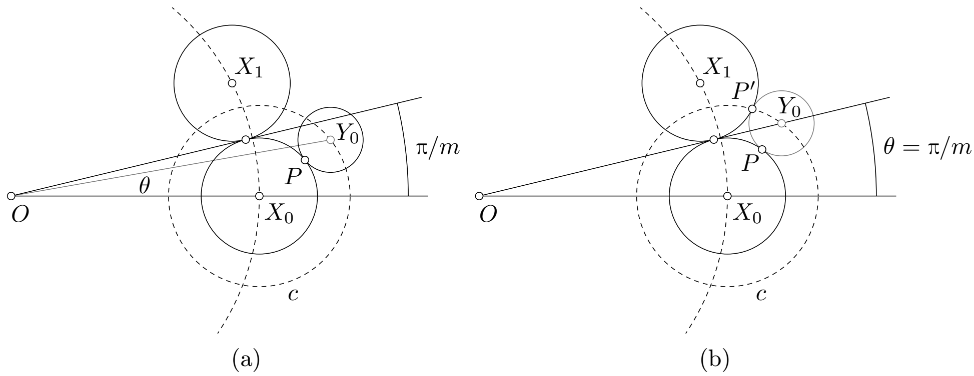

Figure 8. Possible increase of θ while maintaining the point of tangency P.

3. Minimal separation of filled orderly packed rings

Here we formulate and solve the optimization problem related to the orderly arrangement of two filled rings, $\mathrm{Ring}(m, r)$ and $\mathrm{Ring}(n, \varrho)$, with center O such that $\varrho \csc(\pi/n) > r \csc(\pi/m)$. If the rings are arranged orderly, from Proposition 2.1 we have at most $\min\{2m, 2n\}$ tangent points between them. For given r, m and n, consider the minimization of the objective function $f(\varrho) = \varrho$ constrained by finitely many tangent points between the rings. Since $f'(\varrho) = 1$, from the first order necessary conditions (see [6]), any optimal solution is a boundary point. Subsequently, the constrained set is given by

$$0 < |\mathrm{Ring}(m, r) \cap \mathrm{Ring}(n, \varrho)| \leqslant \min\{2m, 2n\} \tag{3.1}$$

with at least one constraint active (at least one tangent point).

With $k = \gcd(m, n)$, as discussed in the end of Section 2 we only need to consider the minimization problem over the circular sector Φ with a central angle of $2\pi/k$. Using the notation and positioning described in Figure 7, we then have the following initial proposition which shows in our minimization problem we will necessarily have multiple points of tangency between rings within the sector Φ.

Proposition 3.1. *If P is the only point of tangency in the circular sector Φ, then ϱ is not minimized for the orderly packing.*

P r o o f. We examine the entire orderly packing under the condition that only one point of tangency occurs in the circular sector Φ; we do this examination first for fixed r, m, ϱ and n, and then show that in fact ϱ is not minimized and can be decreased while maintaining the orderliness of the packing. To this end, by Lemma 2.2 item 1 we know there will be precisely k points of tangency over the entire packing, namely one each in their own circular sector that is rotationally symmetric to Φ by rotations that are integer multiples of $2\pi/k$. By Proposition 2.1, all of these points of tangency must occur on a circle of fixed radius R from O, where R is determined by the construction in the proof of that proposition, which in turn depends only on the fixed values r, m, ϱ and n. Furthermore, no circle in the packing has two points of tangency on it, and if a point of tangency occurs between a circle with center X_i and a circle of center Y_j, then $\angle X_iOY_j = \theta$.

We therefore consider how the circle of radius R around O intersects both rings. We imagine the circles in $\mathrm{Ring}(n, \varrho)$ as bounding discs colored in light gray, and the circles in $\mathrm{Ring}(m, r)$ as bounding discs colored in darker gray, as in part (a) of Figure 9; there the points of tangency are indicated by P's, and the circle C of radius R containing these points of tangency is indicated as well. We first consider the case when $\theta > 0$, which is depicted in Figure 9; once establishing this case, that of $\theta = 0$ will follow quickly.

In the case of $\theta > 0$, referring back to Figure 8 it is clear we will have that $\varrho(\csc(\pi/n) - 1) < R < r(1 + \csc(\pi/m))$, meaning that the circle C will intersect $\mathrm{Ring}(m, r)$ in m dark gray arcs $\alpha_1, \ldots, \alpha_m$, one for each disc in $\mathrm{Ring}(m, r)$, and C will intersect $\mathrm{Ring}(n, \varrho)$ in n light gray arcs $\beta_1, \ldots, \beta_n$, one for each disc in $\mathrm{Ring}(n, \varrho)$. These arcs are shown in part (b) of Figure 9, where circle C has been magnified for ease of visualization. The only places where the arcs are not disjoint are the points of tangency P, and since no circle has two points of tangency, these points occur at most

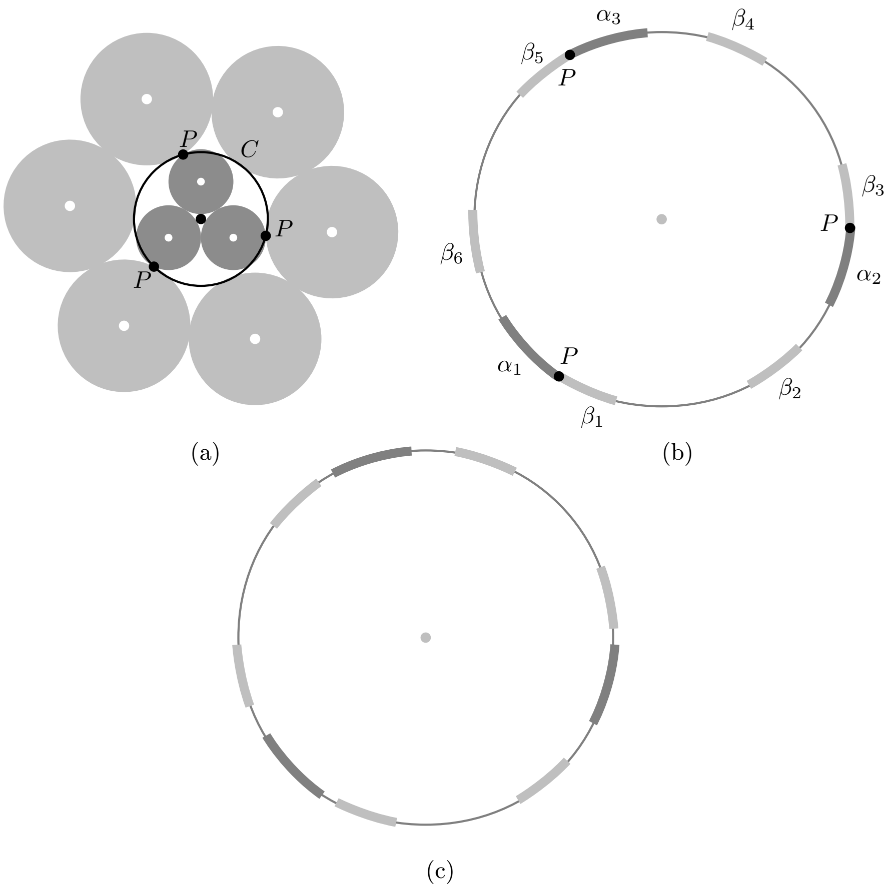

Figure 9. Making room to decrease ϱ in the case where only one point of tangency occurs in each sector.

once for each arc, in particular at most once for the arcs $\beta_1, \ldots, \beta_n$. Furthermore, if we think of each arc as oriented in the direction of increasing polar angle φ, each arc has a front endpoint at greater φ-value, and a back endpoint at lesser φ-value. This has the following consequence: First, recall that if a circle centered at Y_j in $\mathrm{Ring}(n, \varrho)$ is tangent to a circle centered at X_i in $\mathrm{Ring}(m, r)$, then Y_j is at greater polar angle relative to X_i. Therefore, the corresponding light gray arc β_j will necessarily intersect the dark gray arc α_i at β_j's back endpoint. Therefore on C, every front endpoint of β_j is disjoint from $\mathrm{Ring}(m, r)$, and there exists an $\varepsilon > 0$ such that rotating $\mathrm{Ring}(n, \varrho)$ about O for ε radians will induce rotation on β_j so as to keep the lengths of all β_j fixed, but rotate them to be entirely disjoint from $\mathrm{Ring}(m, r)$; this is indicated in the movement from part (b) to part (c) in Figure 9. As a result, the new rotated $\mathrm{Ring}(n, \varrho)$ will be entirely disjoint from $\mathrm{Ring}(m, r)$, since any point of intersection between the two rings necessarily includes points on C by construction, with fixed ϱ. With this new rotated $\mathrm{Ring}(n,\varrho)$ disjoint from $\mathrm{Ring}(m,r)$, ϱ can be decreased, along with the radius $\varrho\csc(\pi/n)$ of the core circle, to obtain a new orderly packing with smaller ϱ.

Finally, when $\theta=0$, the above argument applies, but where each arc α_i and β_j is in fact just a point; as a result there is still room to rotate $\mathrm{Ring}(n,\varrho)$ forward so as to make the two rings disjoint, and then decrease ϱ. $\square$

We can now formulate the precise minimization problem. Returning to Figure 7, the polar coordinates of the centers in the sector $\overline{\Phi}$, which includes all angles in the *closed* interval $\varphi\in[0,2\pi/k]$, are given by

$$\begin{aligned} X_i &= \Big(r\csc\frac{\pi}{m}, \frac{2\pi}{m}i\Big), & i&=0,1,2,\ldots,\frac{m}{k}, \quad \text{and}\\ Y_j &= \Big(\varrho\csc\frac{\pi}{n}, \frac{2\pi}{n}j+\theta\Big), & j&=0,1,2,\ldots,\frac{n}{k}-1. \end{aligned}$$

Since $\angle X_iOY_j = 2\pi j/n - 2\pi i/m + \theta$, by applying the cosine rule for $\triangle OX_iY_j$, we get

$$(r+\varrho)^2 \leqslant r^2\csc^2\frac{\pi}{m} + \varrho^2\csc^2\frac{\pi}{n} - 2r\varrho\csc\frac{\pi}{m}\csc\frac{\pi}{n}\cos\Big(\frac{2\pi j}{n}-\frac{2\pi i}{m}+\theta(\varrho)\Big).$$

For the special case of $\triangle OX_0Y_0$ this will in fact be an equality, namely

$$(r+\varrho)^2 = r^2\csc^2\frac{\pi}{m} + \varrho^2\csc^2\frac{\pi}{n} - 2r\varrho\csc\frac{\pi}{m}\csc\frac{\pi}{n}\cos\theta(\varrho).$$

Then the minimization problem can be formulated as:

$$\begin{aligned}
&(3.2) &&\text{minimize} && f(\varrho)=\varrho,\\
&(3.3) &&\text{subject to} && \varrho\csc\frac{\pi}{n} \geqslant r\csc\frac{\pi}{m},\\
&(3.4) && && 0\leqslant\theta(\varrho)\leqslant\frac{\pi}{m}, \quad \text{(by Lemma 2.3)}\\
&(3.5) && && (r+\varrho)^2 = r^2\csc^2\frac{\pi}{m} + \varrho^2\csc^2\frac{\pi}{n} - 2r\varrho\csc\frac{\pi}{m}\csc\frac{\pi}{n}\cos\theta(\varrho),\\
&(3.6) && && (r+\varrho)^2 \leqslant r^2\csc^2\frac{\pi}{m} + \varrho^2\csc^2\frac{\pi}{n}r\\
& && && \qquad - 2r\varrho\csc\frac{\pi}{m}\csc\frac{\pi}{n}\cos\Big(\frac{2\pi j}{n}-\frac{2\pi i}{m}+\theta(\varrho)\Big),\\
& && && \qquad \forall\, i=1,2,\ldots,\frac{m}{k},\ j=0,1,2,\ldots,\frac{n}{k}-1.
\end{aligned}$$

Next we show that the solution can be found by solving a linear integer programing problem.

Theorem 3.1. *The solution for the minimization problem* (3.2)–(3.6) *with* $n, m > 2$ *is given by* (3.5) *with* $\theta = \pi(i_0/m - j_0/n)$, *where* (i_0, j_0) *is the solution to:*

$$\begin{aligned} &\textit{minimize}_{i,j} && \frac{i}{m} - \frac{j}{n}, \\ &\textit{subject to} && \frac{i}{m} - \frac{j}{n} > 0, \quad i \in 1, 2, \ldots, \frac{m}{k}, \; j \in 0, 1, \ldots, \frac{n}{k} - 1. \end{aligned}$$

P r o o f. Throughout this proof we utilize the fact that $0 < \pi/n, \pi/m < \pi/2$.

Claim 1: $\arg\min_{\varrho} f(\varrho) = \arg\max_{\varrho} \theta(\varrho)$. Differentiating (3.5) with respect to ϱ gives

$$\frac{\mathrm{d}\theta}{\mathrm{d}\varrho} = \frac{r + \varrho - \varrho \csc^2(\pi/n) + r \csc(\pi/m) \csc(\pi/n) \cos\theta}{r\varrho \csc(\pi/m) \csc(\pi/n) \sin\theta}.$$

Suppose $\mathrm{d}\theta/\mathrm{d}\varrho = 0$. Then $r + \varrho - \varrho \csc^2(\pi/n) + r \csc(\pi/m) \csc(\pi/n) \cos\theta = 0$ and we have

$$(r + \varrho)^2 \sin^2 \frac{\pi}{n} = \varrho^2 \csc^2 \frac{\pi}{n} + r^2 \csc^2 \frac{\pi}{m} \cos^2\theta - 2r\varrho \csc\frac{\pi}{m} \csc\frac{\pi}{n} \cos\theta. \tag{3.7}$$

We observe (refer back to Figure 2 part (b)) that all the consecutive tangent points of $\mathrm{Ring}(m, r)$ are on a circle of radius $r\cot(\pi/m)$. If $\varrho\cot(\pi/n) = r\csc(\pi/m)$, $\mathrm{Ring}(n, \varrho)$ contains the core circle of $\mathrm{Ring}(m, r)$. Then with the rings viewed as collections of discs, $\mathrm{Ring}(m, r) \cap \mathrm{Ring}(n, \varrho)$ is infinite, which contradicts (3.1). We therefore must have $\varrho\cot(\pi/n) > r\csc(\pi/m)$. Subtracting equation (3.7) from equation (3.5) and using the fact that $0 \leqslant \pi/n \leqslant \pi/2$ and $\varrho\cos(\pi/n) > r\csc(\pi/m)\sin(\pi/n)$,

$$\begin{aligned} \sin\theta &= \sin\frac{\pi}{m}\Big(\cos\frac{\pi}{n} + \frac{\varrho}{r}\cos\frac{\pi}{n}\Big) \\ &> \sin\frac{\pi}{m}\Big(\cos\frac{\pi}{n} + \sin\frac{\pi}{n}\csc\frac{\pi}{m}\Big) \\ &\geqslant \sin\frac{\pi}{m}\Big(\cos\frac{\pi}{n} + \sin\frac{\pi}{n}\Big) \geqslant \sin\frac{\pi}{m}. \end{aligned}$$

Since $0 \leqslant \theta \leqslant \pi/m < \pi/2$ by Lemma 2.3, we have a contradiction. We conclude that $\mathrm{d}\theta/\mathrm{d}\varrho \neq 0$ in the feasible region.

When $\theta = 0$ and $n > 2$, we have that $r\csc(\pi/m) = \varrho\csc(\pi/n) - r - \varrho$ and

$$r + \varrho - \varrho\csc^2\frac{\pi}{n} + r\csc\frac{\pi}{m}\csc\frac{\pi}{n} = (r + \varrho)\Big(1 - \csc\frac{\pi}{n}\Big) < 0.$$

Then $\lim_{\theta\to 0^+} \mathrm{d}\theta/\mathrm{d}\varrho = -\infty$. Hence $\mathrm{d}\theta/\mathrm{d}\varrho < 0$ for all $0 < \theta \leqslant \pi/m$, which concludes the proof of the claim.

Claim 2: $i/m < j/n$ or $0 \leqslant \theta \leqslant \pi(i/m - j/n)$, with equality holding for at most one pair of i, j in the feasibility region.

First, we show that for any given parameter set (n, m, r, ϱ), at most one of the set of inequality constraints given by (3.6) is active. That is, we have at most two tangency points between rings over the circular sector with a central angle of $2\pi/k$. For any two tangent circles with centers at X_i and Y_j, by Proposition 2.1 item 3 we know that $\angle X_i O Y_j = 2\pi j/n - 2\pi i/m + \theta = \pm\theta$. If $2\pi j/n - 2\pi i/m + \theta = \theta$, then $j/i = n/m = (n/k)/(m/k)$. Since $(n/k)/(m/k)$ is an irreducible fraction, along with $0 \leqslant j < n/k$ and $0 < i \leqslant m/k$, there is no integer pair (i, j) satisfying the equation. We can only have

$$\frac{2\pi j}{n} - \frac{2\pi i}{m} + \theta = -\theta. \tag{3.8}$$

If there are two pairs of integers (i_1, j_1) and (i_2, j_2) satisfying (3.8), then $(j_1 - j_2)/(i_1 - i_2) = (n/k)/(m/k)$. The same argument as above shows that there are no such two pairs. We must have at most one pair (i, j), and hence one active inequality constraint, satisfying (3.8).

From equation (3.5) and inequalities (3.6) we obtain $\cos\theta \geqslant \cos(2\pi j/n - 2\pi i/m + \theta)$ for all i, j. We know $\theta \in [0, \pi/2)$, but the argument in the cosine function on the right-hand side of this inequality could be positive or negative; as a result we obtain

$$\frac{i}{m} \leqslant \frac{j}{n} \quad \text{or} \quad 0 \leqslant \theta \leqslant \pi\Big(\frac{i}{m} - \frac{j}{n}\Big).$$

However, $i/m \neq j/n$, which completes the proof of the claim.

Since $\varrho \cot(\pi/n) > r \csc(\pi/m)$, inequality constraint (3.3) is inactive. Let $[0, \theta_0]$ be the feasibility region for θ. Then from Claim 2 and inequality constraint (3.4),

$$[0, \theta_0] = \bigcap_{i/m - j/n > 0} \Big[0, \min\Big\{\frac{\pi}{m} i - \frac{\pi}{n} j, \frac{\pi}{m}\Big\}\Big].$$

We have $\max\theta = \theta_0$. From Proposition 3.1 we know there are at least two tangency points between rings over the circular sector with a central angle of $2\pi/k$. Subsequently, the feasible region is completely characterized by constraints (3.5)–(3.6). From Claim 2 and Proposition 3.1, $\theta_0 = \pi(i/m - j/n)$ for exactly one positive (i, j) pair given by the integer optimization problem in the theorem statement. From Claim 1, the optimal solution for the minimization problem (3.2)–(3.6) is given by (3.5) with $\theta = \theta_0$. □

As an immediate consequence we can confirm that this optimal solution for the orderly packing of two concentric rings has the symmetry of a dihedral group.

Proposition 3.2. *Let* $\mathrm{Ring}(m, r)$ *and* $\mathrm{Ring}(n, \varrho)$ *be two concentric filled orderly packed rings centered at a point* O*, with minimal separation. Let* $k = \gcd(m, n)$*. Then the symmetry group* S *of this minimal packing is* D_k*, the dihedral group of order* $2k$*.*

P r o o f. By Lemma 2.2 item 1 we already know that S contains precisely k rotations. Furthermore, by Lemma 2.2 item 3 we also know that S contains at most k reflections. We thus need to show that there are at least k distinct lines of reflection.

To see this, consider the following: From Theorem 3.1 we know that in each of the k copies of the rotationally invariant sector Φ there are exactly two points of tangency between $\mathrm{Ring}(m, r)$ and $\mathrm{Ring}(n, \varrho)$, with the first point of tangency (proceeding counterclockwise) occurring for an angle $\angle X_0OY_0 = \theta$ and the second occurring for an angle $\angle X_iOY_j = -\theta$, for some $\theta > 0$. As a result, the ray γ which extends from O and bisects $\angle X_0OX_i$ must also bisect $\angle Y_0OY_j$; see Figure 10.

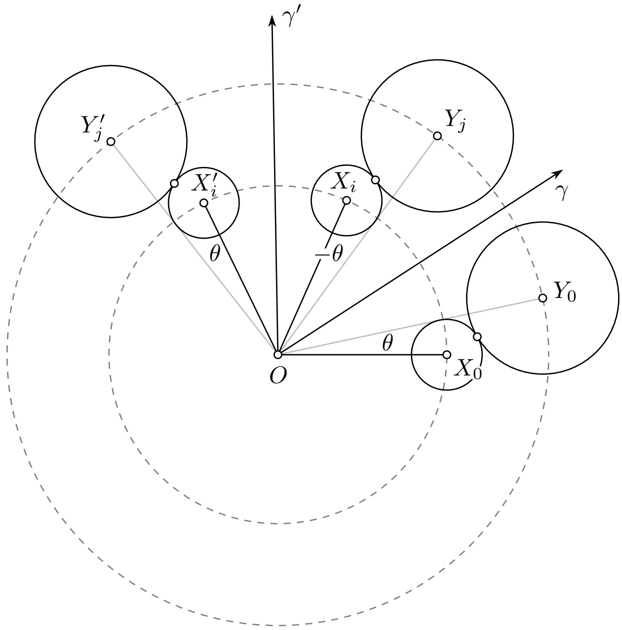

Figure 10. Identifying rays which lie on lines of reflection.

Since γ bisects $\angle X_0OX_i$ between the centers of two circles on $\mathrm{Ring}(m, r)$, by the individual rotational symmetry of $\mathrm{Ring}(m, r)$ we know that γ must intersect either a center of a circle or a point of tangency between circles on $\mathrm{Ring}(m, r)$, depending on whether an odd or even number of circles occur along $\mathrm{Ring}(m, r)$ between the circles corresponding to X_0 and X_i. Likewise since γ bisects $\angle Y_0OY_j$, it must also intersect either a center of a circle on $\mathrm{Ring}(n, \varrho)$ or a point of tangency between circles on Ring(n, ϱ). As a result, by Lemma 2.2 item 2 the ray γ must lie on a line of reflection for the packing. This gives at least k distinct rays which lie on lines of reflection for the packing.

We require a bit more to show that there are at least k lines of reflection. We continue further along both rings, and now consider the sector of the rings which begins at the $\angle X_i O Y_j = -\theta$ described above, and then proceeds further counterclockwise to the next point of tangency which will be at an angle $\angle X_i' O Y_j' = \theta$; again refer to Figure 10. Again, there is now a ray γ' which extends from O and bisects $\angle X_i O X_i'$ as well as $\angle Y_j O Y_j'$, and thus by the exact same reasoning as in the above paragraph, γ' must lie on a line of reflection for the packing. We thus obtain at least $2k$ distinct rays which lie on lines of reflection for the packing, hence at least k distinct lines of reflection, and this concludes the proof of the proposition. $\square$

4. Results and applications

Here we discuss several examples of finding ϱ for the orderly packing with minimal separation of filled Ring(m, r) and Ring(n, ϱ), where $m, n \in \mathbb{N}$ and r are given. From equation (3.5), the optimal ϱ is given by the quadratic equation

$$\varrho^2 \cot^2 \frac{\pi}{n} - 2r\varrho\Big(1 + \csc\frac{\pi}{m}\csc\frac{\pi}{n}\cos\theta_0\Big) + r^2 \cot^2 \frac{\pi}{m} = 0, \tag{4.1}$$

where θ_0 is the optimal θ given in Theorem 3.1. The larger solution is achieved when Ring(n, ϱ) is outside of Ring(m, r).

In packing problems, the general objective is to obtain a packing of the greatest possible density, calculated as the ratio of the total area occupied by circles to the container area. For two orderly packed rings Ring(m, r) and Ring(n, ϱ), a higher number of tangency points between rings results in a more compact packing. In this example, we numerically demonstrate the local optimality of the density when every circle in Ring(m, r) has at least one point of tangency with the circles of Ring(n, ϱ). Below we identify two such groups of orderly packed rings with minimal separation.

(1) Suppose $n = pm$, $p \in \mathbb{N}$. Then $k = m$ and the symmetry group of the packing is D_m. We have $i = 1$ and $j = 0, 1, \ldots, p-1$. Angle θ is maximized when $j = p-1$ and $\theta_0 = \pi/n$. The circles with centers X_0 and Y_0 and the circles with centers X_1 and Y_{p-1} are tangent (see Figure 7). With rotational symmetry for every angle $2\pi/m$, every circle in Ring(m, r) has exactly two points of tangency with the circles of Ring(n, ϱ). See Figure 11 for some examples.

(2) Suppose $m = 2p$ and $n = (2q+1)p$, where $p, q \in \mathbb{N}$. Then $k = p$ and the symmetry group of the packing is D_p. We have $i = 1, 2$ and $j = 0, 1, \ldots, 2q$. Angle θ

is maximized when $i = 1$ and $j = q$, and the maximum is $\theta_0 = \pi/[2p(2q+1)]$. Every disk in $\mathrm{Ring}(m, r)$ has exactly one point of tangency with the discs of $\mathrm{Ring}(n, \varrho)$. See Figure 12 for some examples.

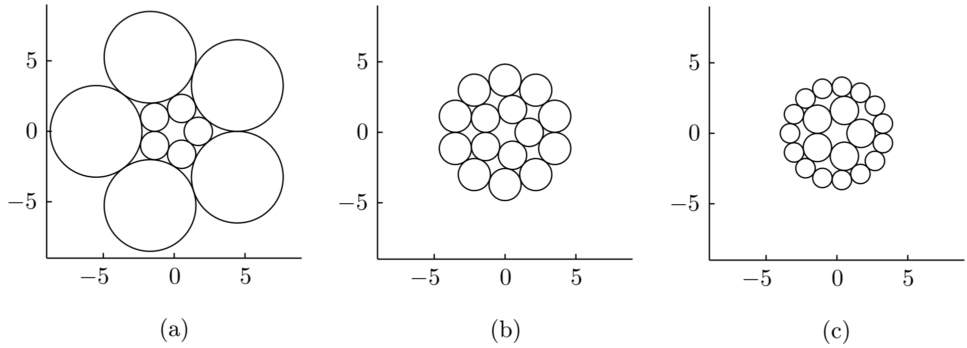

Figure 11. Minimally separated orderly packed rings, $\mathrm{Ring}(m, r)$ and $\mathrm{Ring}(n, \varrho)$, with (a) $m = 5$, $n = 5$, $r = 1$ and $\varrho = 3.217$, (b) $m = 5$, $n = 10$, $r = 1$ and $\varrho = 1.1415$, and (c) $m = 5$, $n = 15$, $r = 1$ and $\varrho = 0.6895$.

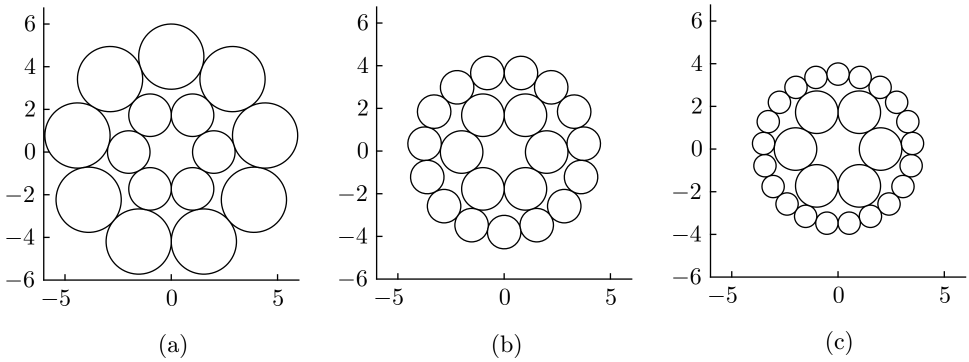

Figure 12. Minimally separated orderly packed rings, $\mathrm{Ring}(m, r)$ and $\mathrm{Ring}(n, \varrho)$, with (a) $m = 6$, $n = 9$, $r = 1$ and $\varrho = 1.5312$, (b) $m = 6$, $n = 15$, $r = 1$, and $\varrho = 0.7813$ and (c) $m = 6$, $n = 21$, $r = 1$ and $\varrho = 0.5232$.

Figure 13 illustrates the highest density of two orderly packed rings, $\mathrm{Ring}(m, 1)$ and $\mathrm{Ring}(n, \varrho)$, in an annulus of inner and outer radius $\csc(\pi/m) - 1$ and $\varrho \times (\csc(\pi/n) + 1)$, respectively. The peaks in each plot represent local maxima and occur when $n/m = p/2$, $p = 2, 3, 4, \ldots$

Orderly packed rings are useful in high voltage power cable designs. Aluminum conductor steel-reinforced (ACSR) cable is a high-capacity, high-strength stranded conductor used in overhead power lines. The outer strands are aluminum, chosen for its high conductivity, and the center strands are steel, chosen to increase the strength

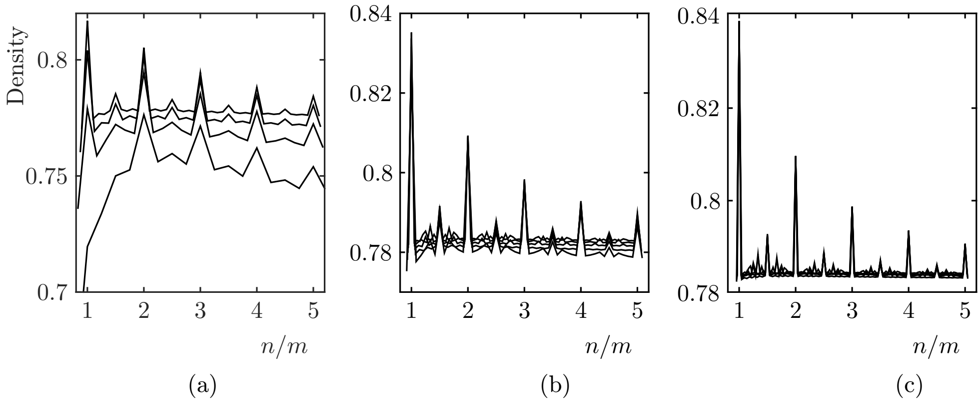

Figure 13. Ratio n/m versus density for two orderly packed rings with minimal separation, $\mathrm{Ring}(m, 1)$ and $\mathrm{Ring}(n, \varrho)$, for (a) $m = 4, 6, 8, 10$ (the lowest curve corresponds to $m = 4$), (b) $m = 12, 14, 16, 18, 20$, and (c) $m = 22, 24, 26, 28, 30$.

of the cable. Each strand has a circular cross section. High strand packing density is achieved by placing the wires in rings (see Figure 14).

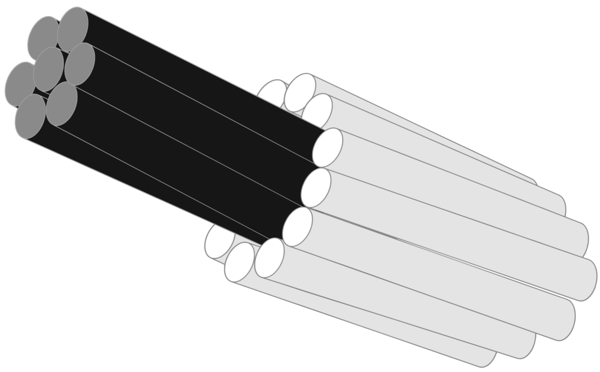

Figure 14. Strands in high voltage power cables. Steel strands are shown in black and aluminum strands are shown in light gray.

Strand conductor rings are often compressed to reduce the diameter. After compression, the aluminum strands no longer have precisely circular cross sections. In practice, it is appropriate to choose a radius that is slightly smaller than the optimal radius for aluminum strands to allow for compression. For orderly packed rings $\mathrm{Ring}(m, 1)$ and $\mathrm{Ring}(m+6, \varrho)$, the inner radius of the outer ring is at most the outer radius of the inner ring, with the result

$$\varrho \leqslant \frac{\csc(\pi/m) + 1}{\csc(\pi/(m+6)) - 1}, \tag{4.2}$$

the right-hand side is monotone decreasing for $m > 1$. From Theorem 3.1 we can obtain the optimal ϱ as 1 for $m = 6$. Combined with inequality (4.2), we obtain $1 \leqslant \varrho \leqslant 1.04$ for minimal separation given $m \geqslant 6$.

In ACSR cables, a sequence of orderly packed rings, $\mathrm{Ring}(m + 6i, \varrho_i)$, $i = 1, \ldots, N$, can be effectively constructed using equal radius aluminum strands. Since $r\csc(\pi/6) = 2r$, we can orderly pack $\mathrm{Ring}(6, 1)$ on a disk with radius 1. Subsequently, 1, 7, or 19 equal radius steel strands for the steel core, combined with 20 (7 and 13 orderly packed rings with minimal separation), 24 (9 and 15), 26 (10 and 16), 30 (12 and 18), or 45 (9, 15, and 21) equal radius aluminum strands in two or three rings, are all appropriate configurations. Most of these configurations are already used in commercial designs of ACSR cables. See [1] for technical data on AcuTech™ ACSR conductors. We can expand this list to many other configurations. Note that among all possible sequences of rings, two sequences, $6, 12, 18, 24, \ldots$ and $9, 15, 21, 27, \ldots$, have the highest number of tangent points when minimally separated and possibly provide the most compact configurations for the aluminum rings. Figure 15 illustrates possible configurations along with the density. Upon request, the authors can provide a set of MATLAB programs to find the optimal ϱ and θ_0 and to plot the orderly packed rings with minimal separation.

A c k n o w l e d g m e n t s. We dedicate this paper to the memory of Dr. Iraj Kalantari who had great interest in tiling problems. The authors are thankful for many useful discussions they had with Iraj on circle packing. The authors wish to thank the anonymous reviewers for their helpful comments to improve the paper.

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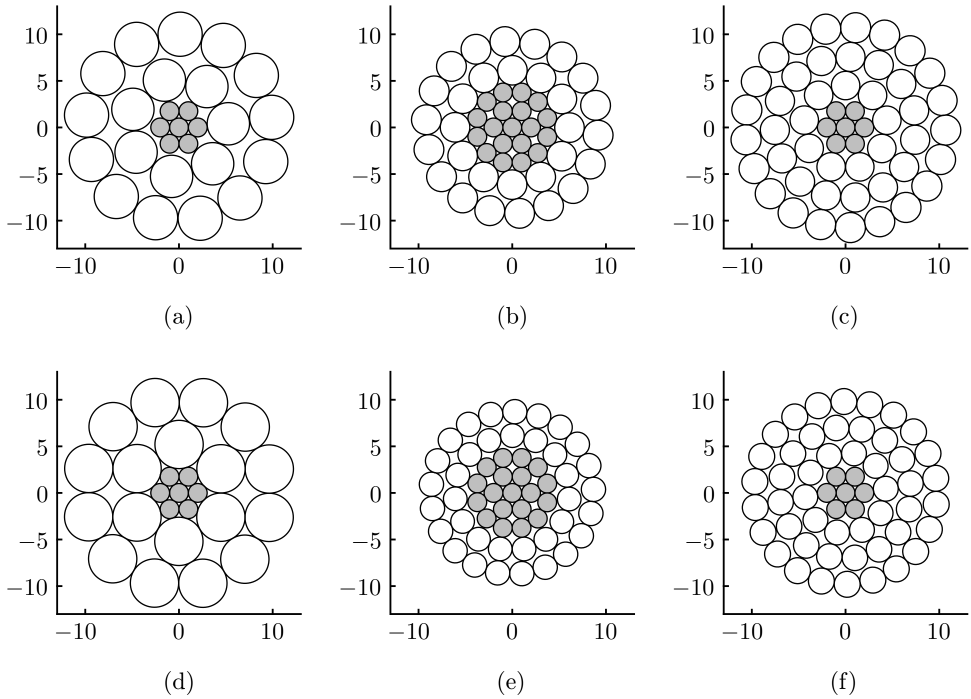

Figure 15. Minimally separated rings useful in ACSR cable designs. Possible steel strands are in gray and aluminum strands are in white.
(a) A configuration with 7 steel and 20 aluminum strands. The packing density is 0.7727. Industrial realization is available with approximately 9/4 ratio for aluminum to steel diameter [1].
(b) A configuration with 19 steel and 30 aluminum strands. The packing density is 0.8120. Industrial realization is available with approximately 5/3 aluminum to steel diameter [1].
(c) A configuration with 7 steel and 45 aluminum strands. The packing density is 0.7834. Industrial realization is available with approximately 3/2 aluminum to steel diameter [1].
(d) A possible configuration with 7 steel and 18 aluminum strands. The packing density is 0.8050.
(e) A possible configuration with 19 steel and 36 aluminum strands. The packing density is 0.7887.
(f) A possible configuration with 7 steel and 48 aluminum strands. The packing density is 0.7816.

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