SOLVING SECOND-ORDER SINGULARLY PERTURBED ODE BY THE COLLOCATION METHOD BASED ON ENERGETIC ROBIN BOUNDARY FUNCTIONS

Chein-Shan Liu, Nanjing, Botong Li, Beijing

Received March 19, 2019. Published online October 31, 2019.

Abstract. For a second-order singularly perturbed ordinary differential equation (ODE) under the Robin type boundary conditions, we develop an energetic Robin boundary functions method (ERBFM) to find the solution, which automatically satisfies the Robin boundary conditions. For the non-singular ODE the Robin boundary functions consist of polynomials, while the normalized exponential trial functions are used for the singularly perturbed ODE. The ERBFM is also designed to preserve the energy, which can quickly find accurate numerical solutions for the highly singularly perturbed problems by a simple collocation technique.

Keywords: singularly perturbed ODE; Robin boundary function; energetic Robin boundary function; collocation method

MSC 2010: 34B60

1. Introduction

A lot of engineering problems can be described by ordinary differential equations (ODEs), which are subjected to certain boundary conditions, and resulted to the boundary value problems (BVPs). It is better that the solution of BVP can satisfy the boundary conditions exactly, but in the case of Robin type boundary conditions and singularity appeared in the solution, it might be a difficult task. There are many computational methods that have been developed for solving the BVPs [4], [5], [6], [10], [1].

In the paper we propose an energetic Robin boundary functions method for solving the singularly perturbed ODE under the Robin boundary conditions. The highest

The financial support to the first author by the Fundamental Research Funds for the Central Universities under the Grant Number 2017B05714 is highly appreciated.

DOI: 10.21136/AM.2019.0066-19

order derivative term in the ODE is multiplied by a small parameter. When the boundary conditions are imposed, the resulting BVP is a singularly perturbed BVP (SPBVP). It is always so that the SPBVP exhibits a boundary layer, which is a narrow region, where the solution varies rapidly.

For the SPBVP it is difficult to exactly satisfy the Robin boundary conditions, unless one designs the algorithm to satisfy the Robin boundary conditions. Inspired by the works in [17], we solve the second-order ODE with strong singularity by designing an algorithm to automatically satisfy the Robin boundary conditions and also preserving the energy via a new concept of energetic Robin boundary functions. The readers may refer to [14], [2], [19], [22], [12], [16], [11], [7] for the numerical methods to solve the SPBVPs.

The paper is arranged as follows. In Section 2, we derive the homogenization function for the Robin boundary conditions and introduce a new variable, for which the Robin boundary conditions become homogeneous. The idea of polynomial Robin boundary functions which automatically satisfy the homogeneous Robin boundary conditions, is introduced, and then the energetic Robin boundary functions are constructed in Section 3. In Section 4, we derive a linear system to determine the expansion coefficients by a simple collocation technique, where the energetic Robin boundary functions act as the bases of numerical solutions. In Section 5, we introduce the normalized exponential trial functions supplemented by a second-order polynomial as the bases for the numerical solutions of the SPBVPs. Numerical examples are given in Section 6. Finally, the conclusions are drawn in the last section.

2. Homogenization and variable transformation

In the paper we propose a new method for the solution of the following second order boundary value problem (BVP) under the Robin type boundary conditions:

$$\varepsilon u''(x) + p(x)u'(x) + q(x)u(x) = H(x), \quad 0 < x < 1, \tag{1}$$

$$a_1 u(0) + b_1 u'(0) = c_1, \quad a_2 u(1) + b_2 u'(1) = c_2, \tag{2}$$

where a_1, b_1 satisfy $a_1^2 + b_1^2 > 0$, a_2, b_2 satisfy $a_2^2 + b_2^2 > 0$, c_1, c_2 are given constants, and $[0, 1]$ is an interval of our problem. We suppose that $p(x)$, $q(x)$ and $H(x) \in \mathcal{C}[0, 1]$. However, in many applications the independent variable t may be in an interval $[a, b]$, of which after taking the variable transform $x = (t - a)/(b - a)$ we have the problem in the interval $x \in [0, 1]$ again, and the ODE and the Robin boundary conditions should be adjusted accordingly. When $\varepsilon = 1$ we have the usual ODE, while for $0 < \varepsilon \ll 1$ we have a singularly perturbed ODE.

In the construction of the energy method, the first step is the homogenization technique, such that for the new variable

$$y(x) = u(x) - B_0(x), \tag{3}$$

the Robin boundary conditions are homogeneous. If $c_1^2 + c_2^2 = 0$ we can skip the following processes and go to the next section directly.

We divide the derivations of the homogenization function $B_0(x)$ into two parts.

(I) $a_1 = 0$ (hence $b_1 \neq 0$); we can derive

$$B_0(x) = a_0 x + b_0 x^\nu, \quad \nu \geqslant 2, \tag{4}$$

$$a_0 = \frac{c_1}{b_1}, \tag{5}$$

$$b_0 = \frac{b_1 c_2 - a_2 c_1 - b_2 c_1}{b_1 a_2 + b_1 b_2 \nu}. \tag{6}$$

There are many values of ν such that $a_2 + b_2\nu \neq 0$ (hence, $b_1 a_2 + b_1 b_2 \nu \neq 0$) and one can choose it easily.

(II) $a_1 \neq 0$; we can derive

$$B_0(x) = a_0 + b_0 x^\nu, \quad \nu \geqslant 2, \tag{7}$$

$$a_0 = \frac{c_1}{a_1}, \tag{8}$$

$$b_0 = \frac{a_1 c_2 - a_2 c_1}{a_1 a_2 + a_1 b_2 \nu}. \tag{9}$$

There are many values of ν such that $a_2 + b_2\nu \neq 0$ (hence, $a_1 a_2 + a_1 b_2 \nu \neq 0$) and one can choose it easily.

The above function $B_0(x)$ includes a parameter ν. Let

$$B_0(x) = a_0 + b_0 x, \tag{10}$$

and through some derivations we can obtain

$$a_0 = \frac{c_1 b_2 + c_1 a_2 - c_2 b_1}{a_1 b_2 + a_1 a_2 - a_2 b_1}, \tag{11}$$

$$b_0 = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 + a_1 a_2 - a_2 b_1}. \tag{12}$$

In the case with $a_1 b_2 + a_1 a_2 - a_2 b_1 = 0$, we must employ the above (I) or (II) to set up the function $B_0(x)$.

Through the variable transformation (3), we obtain a new BVP with the homogeneous Robin boundary conditions:

$$\begin{aligned}(13)\qquad &\varepsilon y''(x) + p(x)y'(x) + q(x)y(x)\\ &\qquad = F(x) = H(x) - \varepsilon B_0''(x) - B_0'(x)p(x) - B_0(x)q(x), \quad 0 < x < 1,\\ (14)\qquad &a_1 y(0) + b_1 y'(0) = 0, \quad a_2 y(1) + b_2 y'(1) = 0.\end{aligned}$$

3. Energetic Robin boundary functions method

By multiplying both sides of (13) by $y(x)$, integrating it from $x = 0$ to $x = 1$, one can derive

$$\int_0^1 [\varepsilon y''(x)y(x) + p(x)y'(x)y(x) + q(x)y^2(x)]\,\mathrm{d}x = \int_0^1 F(x)y(x)\,\mathrm{d}x. \tag{15}$$

If there exists an exact solution $y(x)$ of (13) and (14), it must satisfy the above equation. The resulting equation is an energy equation and we will use it as a mathematical tool to solve $y(x)$.

The next step is searching the Robin boundary functions which automatically satisfy (14). In terms of polynomials we can derive

$$B_j(x) = 1 - \frac{a_1}{b_1}x + \frac{a_1 b_2 + a_1 a_2 - a_2 b_1}{b_1 a_2 + (j+1)b_1 b_2}x^{j+1}, \quad j \geqslant 1 \text{ if } b_1 \neq 0, \tag{16}$$

$$B_j(x) = x - \frac{a_2 + b_2}{a_2 + (j+1)b_2}x^{j+1}, \quad j \geqslant 1 \text{ if } b_1 = 0. \tag{17}$$

For the homogeneous Robin boundary conditions in (14) we may encounter the case that there exists a positive integer j_0 such that $a_2 + (j_0+1)b_2 = 0$, for example, when $a_2 = 4$, $b_2 = -1$, $j_0 = 3$. With this situation we can skip this j_0 in (16) and (17), and they are modified to

$$\begin{aligned}(18)\quad B_j(x) &= 1 - \frac{a_1}{b_1}x + \frac{a_1 b_2 + a_1 a_2 - a_2 b_1}{b_1 a_2 + (j+1)b_1 b_2}x^{j+1},\\ &\qquad\qquad j = 1, \dots, j_0 - 1,\ j_0 + 1,\ j_0 + 2, \dots, \text{ if } b_1 \neq 0,\\ (19)\quad B_j(x) &= x - \frac{a_2 + b_2}{a_2 + (j+1)b_2}x^{j+1},\\ &\qquad\qquad j = 1, \dots, j_0 - 1,\ j_0 + 1,\ j_0 + 2, \dots, \text{ if } b_1 = 0.\end{aligned}$$

They are at least second-order polynomial functions which satisfy the following homogeneous Robin boundary conditions:

$$a_1 B_j(0) + b_1 B_j'(0) = 0,\ a_2 B_j(1) + b_2 B_j'(1) = 0, \quad j \geqslant 1. \tag{20}$$

For a BVP if the boundary conditions make the coefficient preceding x^{j+1} be zero, then (16) and (17) are not applicable. For this case we can enrich the boundary functions by including other type functions.

From (16)–(20) it is obvious that when $B_j(x)$ is a Robin boundary function, $\beta B_j(x)$, $\beta \in \mathbb{R}$, is also a Robin boundary function, and when $B_j(x)$ and $B_k(x)$ are Robin boundary functions, $B_j(x) + B_k(x)$ is also a Robin boundary function. The Robin boundary functions are closed under scalar multiplication and addition. Therefore, the set of

$$\{B_j(x)\}, \quad j \geqslant 1, \tag{21}$$

and the zero element constitute a linear space of the Robin boundary functions, denoted by $\mathcal{B}$.

The following result can help us in solving (13) and (14).

Theorem 1. *In the linear space $\mathcal{B}$ there exist Robin boundary functions*

$$E_j(x) = \gamma_j B_j(x), \quad j \geqslant 1, \ j \text{ not summed}, \tag{22}$$

where

$$e_2 = \int_0^1 [\varepsilon B_j''(x) B_j(x) + p(x) B_j'(x) B_j(x) + q(x) B_j^2(x)] \, \mathrm{d}x, \tag{23}$$

$$e_1 = \int_0^1 B_j(x) F(x) \, \mathrm{d}x,$$

$$\gamma_j = \frac{e_1}{e_2}, \tag{24}$$

are such that $E_j(x)$ satisfies the following energy integral equation:

$$\int_0^1 [\varepsilon E_j''(x) E_j(x) + p(x) E_j'(x) E_j(x) + q(x) E_j^2(x)] \, \mathrm{d}x = \int_0^1 F(x) E_j(x) \, \mathrm{d}x. \tag{25}$$

Proof. Because $B_j(x) \in \mathcal{B}$ is an element of the linear space $\mathcal{B}$, the multiplication in (22) renders $E_j(x) \in \mathcal{B}$, an element in the linear space $\mathcal{B}$, which satisfies the homogeneous Robin boundary conditions:

$$a_1 E_j(0) + b_1 E_j'(0) = 0, \quad a_2 E_j(1) + b_2 E_j'(1) = 0, \tag{26}$$

due to (20).

Because $E_j(x)$ already satisfies the boundary conditions (26), we impose the energy identity (15) on $E_j(x)$ and derive (25), which is an energy equation in terms of the Robin boundary function $E_j(x)$ defined in the linear space.

Inserting (22) for $E_j(x)$ and

$$E_j'(x) = \gamma_j B_j'(x),\ E_j''(x) = \gamma_j B_j''(x) \tag{27}$$

for $E_j'(x)$ and $E_j''(x)$ into (25), one can derive a quadratic equation to determine the multiplier γ_j:

$$e_2 \gamma_j^2 = e_1 \gamma_j, \tag{28}$$

where the coefficients e_1 and e_2 were defined in (23). Then the solution of γ_j is derived in (24). This ends the proof of the theorem. □

The Robin boundary function $E_j(x)$ in (22) endowed with the multiplier γ_j in (24) not only satisfies the homogeneous Robin boundary conditions but also preserves the energy in (25). The multiplier γ_j is determined by using the energy identity (25). Hence, $E_j(x)$ is an energetic Robin boundary function, and correspondingly the numerical method based on $E_j(x)$ is an energetic Robin boundary functions method (ERBFM).

4. Deriving the linear system by collocation method

The numerical procedure for solving $y(x)$ is given in the following form: to find the expansion coefficients c_j in

$$y(x) = \sum_{j=1}^{n} c_j s_j E_j(x), \quad \big[u(x) = B_0(x) + \sum_{j=1}^{n} c_j s_j E_j(x)\big], \tag{29}$$

where $E_j(x)$ acts as the basis in the numerical solution of $y(x)$. It can be seen that $y(x)$ in (29) automatically satisfies (14), due to (26).

Because the boundary conditions are automatically satisfied by (29), we only need to guarantee that the governing equation (13) is satisfied. First we set $s_j = 1$. Inside the interval $(0, 1)$ we can collocate n_q points $x_i = i/(n_q + 1)$, $i = 1, \ldots, n_q$, to satisfy (13) by inserting (29) for $y(x)$, so that we have a linear system:

$$\mathbf{Ac} = \mathbf{F}, \tag{30}$$

which can be used to determine the expansion coefficients $\mathbf{c} := \{c_j\}$, whose number is n. In the above, the components of $\mathbf{A}$ and $\mathbf{F}$ are given, respectively, by $a_{ij} =$

$\varepsilon E_j''(x_i) + p(x_i)E_j'(x_i) + q(x_i)E_j(x_i)$ and $F_i = F(x_i)$. The dimension of $\mathbf{A}$ is $n_q \times n$, and (30) is an over-determined system with $n_q > n$.

In general, the norms of the columns of the coefficient matrix $\mathbf{A}$ are not equal. If one asks the norms of the columns of the coefficient matrix of $\mathbf{A}$ to be equal, the multiple-scale s_j is determined by [15]

$$s_j = \frac{R_0}{\|\mathbf{a}_j\|}, \tag{31}$$

where $\mathbf{a}_j$ denotes the jth column of $\mathbf{A}$ in (30) and R_0 is a parameter. Hence, we have $\|\mathbf{a}_j\| = R_0$, $j = 1, \ldots, n$.

5. Normalized exponential trial functions

In the strong-form formulation of differential equations it is known that the selection of trial functions is very important, for which we suppose that the set of trial functions is complete, linearly independent, and satisfying the boundary conditions exactly. In general, the polynomial basis in Section 3 is hard to match the singularity behavior for the SPBVP. We will give a different set of trial functions which are not used in the literature to treat the second-order singularly perturbed problems

$$\varphi_j(x) = \frac{\mathrm{e}^{jx} - 1}{\mathrm{e}^j - 1}, \quad \varphi_j(0) = 0, \quad \varphi_j(1) = 1, \tag{32}$$

$$\varphi_0(x) = x, \ \varphi_0(0) = 0, \quad \varphi_0(1) = 1. \tag{33}$$

To avoid the divergence of e^{jx}, we have introduced a normalized factor $\mathrm{e}^j - 1$ in the denominator. Therefore, $\varphi_j(x)$ is a normalized exponential trial function. Liu et al. [18] extended the above trial functions in the weak-form formulation of the fourth-order singular beam equation to find the numerical solution.

In order to let $\varphi_j(x)$ satisfy the homogeneous Robin boundary conditions we can derive

$$B_j(x) = 1 - \frac{x^2}{a_2 + 2b_2}\left[a_2 - \frac{a_1 a_2(\mathrm{e}^j - 1)}{jb_1} - \frac{a_1 b_2 \mathrm{e}^j}{b_1}\right] - \frac{a_1(\mathrm{e}^j - 1)}{jb_1}\varphi_j(x), \quad j \in \mathbb{Z} \text{ if } b_1 \neq 0, \tag{34}$$

$$B_j(x) = x^2 - \frac{(a_2 + 2b_2)(\mathrm{e}^j - 1)}{a_2(\mathrm{e}^j - 1) + jb_2\mathrm{e}^j}\varphi_j(x), \quad j \in \mathbb{Z} \text{ if } b_1 = 0. \tag{35}$$

The special case $B_0(x)$ can be obtained by applying the L'Hospital rule to the above equations.

Then, by applying Theorem 1 to the above $B_j(x)$, we can derive the trial functions $E_j = \gamma_j B_j(x)$. We suppose that the solution $y(x)$ can be expanded by

$$y(x) = \sum_{j=-m_1}^{m_2} a_j s_j E_j(x), \quad \big[u(x) = B_0(x) + \sum_{j=-m_1}^{m_2} a_j s_j E_j(x)\big], \tag{36}$$

where $n = m_1 + m_2 + 1$ and the unknown coefficients a_j have to be determined.

6. Numerical examples

In order to assess the performance of the newly developed ERBFM let us investigate the following examples.

Example 1. Let us consider the following BVP [13]:

$$\ddot{u}(t) + \frac{1}{t}\dot{u}(t) + \Big(1 - \frac{1}{4t^2}\Big)u(t) = \sqrt{t}\cos t, \quad u(1) = 1, \quad u(6) = -0.5. \tag{37}$$

The exact solution is

$$u(t) = \Big(\frac{A_0}{\sqrt{t}} + \frac{\sqrt{t}}{4}\Big)\cos t + \Big(\frac{B_0}{\sqrt{t}} + \frac{t^{3/2}}{4}\Big)\sin t, \tag{38}$$

where

$$\begin{aligned} A_0 &= \frac{\sqrt{6}}{\cos 1 \sin 6 - \sin 1 \cos 6}\Big[\Big(1 - \frac{\cos 1 + \sin 1}{4}\Big)\frac{\sin 6}{\sqrt{6}} \\ &\quad + \sin 1\Big(\frac{1}{2} + \frac{\sqrt{6}}{4}\cos 6 + \frac{\sqrt{6}^3}{4}\sin 6\Big)\Big] \approx 0.0588713, \\ B_0 &= \frac{\sqrt{6}}{\cos 1 \sin 6 - \sin 1 \cos 6}\Big[\Big(\frac{\cos 1 + \sin 1}{4} - 1\Big)\frac{\cos 6}{\sqrt{6}} \\ &\quad - \cos 1\Big(\frac{1}{2} + \frac{\sqrt{6}}{4}\cos 6 + \frac{\sqrt{6}^3}{4}\sin 6\Big)\Big] \approx 0.740071. \end{aligned} \tag{39}$$

With $n = 50$, $n_q = 200$, and $R_0 = 1$ we apply the ERBFM to find the numerical solution, which is compared to the exact solution (38), and the numerical error is shown in Fig. 1 by blue color solid line. While the maximum error is 1.84×10^{-7}, the root-mean-square-error (RMSE) at totally 300 points is 6.37×10^{-8}. Under the same parameter values, the boundary functions method (BFM) leads to the maximum error being 6.17×10^{-7}, and the RMSE being 2×10^{-7}. In Fig. 1, we show the numerical errors. The accuracy of the ERBFM is slightly better than the BFM.

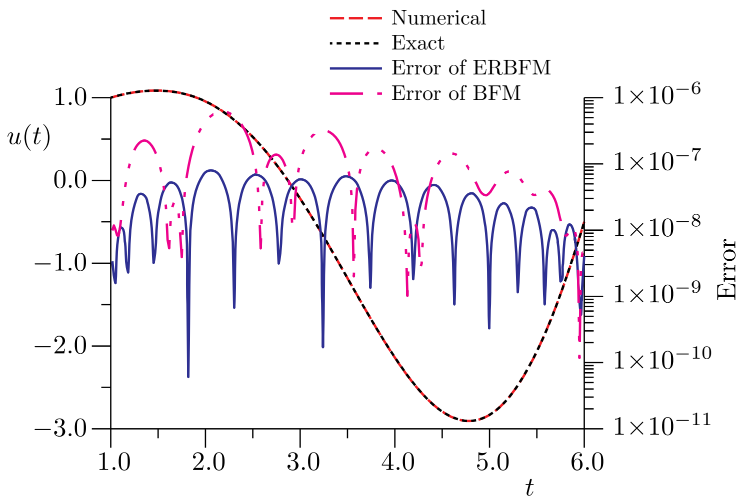

Figure 1. Comparing the numerical solutions obtained by the ERBFM with the exact solution and showing the errors obtained by the ERBFM and BFM.

In Table 1, we investigate the maximum error and the RMSE obtained by the BFM and the ERBFM with different values of n, the other parameters being $n_q = 100$ and $R_0 = 1$.

n	10	20	30	40	50
BFM (ME)	7.66×10^{-7}	1.03×10^{-7}	6.18×10^{-8}	7.03×10^{-8}	6.09×10^{-8}
ERBFM (ME)	7.66×10^{-7}	1.06×10^{-7}	5.70×10^{-8}	6.59×10^{-8}	7.14×10^{-8}
BFM (RMSE)	4.00×10^{-7}	5.43×10^{-8}	3.08×10^{-8}	3.43×10^{-8}	2.90×10^{-8}
ERBFM (RMSE)	4.00×10^{-7}	5.54×10^{-8}	2.63×10^{-8}	3.51×10^{-8}	3.17×10^{-8}

Table 1. For Example 1 comparing the maximum error (ME) and the root-mean-square-error (RMSE) obtained by the BFM and the ERBFM with different values of n.

From Table 1, we can observe that the ERBFM is convergent stably, while the BFM at $n = 30$ is abnormal. For small n, the performances of the BFM and the ERBFM are the same, but with large n, the performance of the ERBFM is better than that of the BFM.

Example 2. We solve

$$\varepsilon u''(x) + u'(x) - u(x) = 0, \quad u(0) = 1, \quad u(1) = 1, \tag{40}$$

which has been calculated by Reddy and Chakravarthy [20] and Ilicasu and Schultz [8] by using different methods, and has a closed-form solution:

$$u(x) = \frac{1}{e^{p_2} - e^{p_1}}[(e^{p_2} - 1)e^{p_1 x} + (1 - e^{p_1})e^{p_2 x}], \tag{41}$$

where

$$p_1 = \frac{-1 + \sqrt{1 + 4\varepsilon}}{2\varepsilon}, \quad p_2 = \frac{-1 - \sqrt{1 + 4\varepsilon}}{2\varepsilon}. \tag{42}$$

We expand the solution $u(x)$ by (36). Under the parameters $\varepsilon = 0.01$, $m_1 = 100$, $m_2 = 1$, $n_q = 200$, and $R_0 = 0.1$, we can find that the solution $u(x)$ is very close to the exact one with the maximum error being 8.09×10^{-8} as shown in Fig. 2(a). Obviously, our maximum error is much smaller than that calculated by Varner and Choudhury [21], and by Reddy and Chakravarthy [20], who used a smaller stepsize $h = 0.001$.

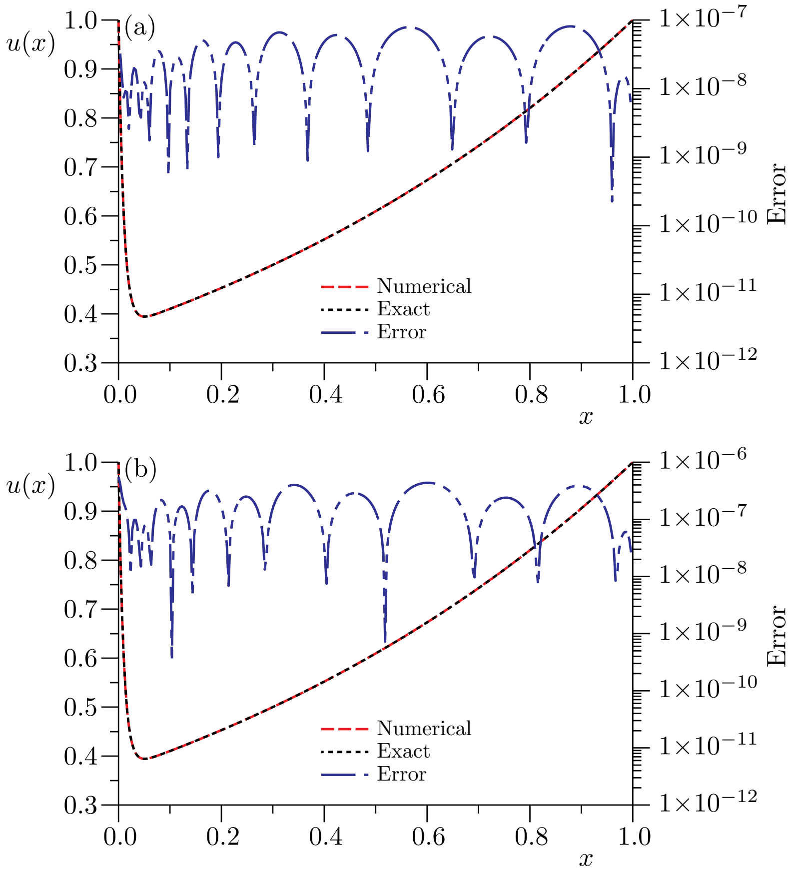

Figure 2. For (a) Example 2 and (b) Example 3, comparing the numerical solutions obtained by the ERBFM with the exact solutions and showing the errors.

E x a m p l e 3. We revisit Example 2 again; however, we consider the Robin type boundary conditions:

$$\begin{aligned} (43)\qquad & \varepsilon u''(x) + u'(x) - u(x) = 0, \\ & u(0) + u'(0) = 1 + \frac{1}{e^{p_2} - e^{p_1}}[p_1(e^{p_2} - 1) + p_2(1 - e^{p_1})], \quad u(1) = 1, \end{aligned}$$

where p_1 and p_2 were defined by (42).

Under the parameters $\varepsilon = 0.01$, $m_1 = 100$, $m_2 = 1$, $n_q = 300$, $\nu = 2$, and $R_0 = 1$, we can find that the solution $u(x)$ is very close to the exact one with the maximum error being 5.53×10^{-7} as shown in Fig. 2(b). The accuracy is slightly worse than that in Example 2.

E x a m p l e 4. We calculate this example by adding a non-homogeneous term on the right-hand side, which is subjected to the Robin type boundary conditions:

$$\begin{aligned} (44)\qquad & \varepsilon u''(x) + u'(x) = 1 + 2x, \\ & u(0) + u'(0) = \frac{2\varepsilon - 1}{\varepsilon[1 - \exp(-1/\varepsilon)]} + 1 - 2\varepsilon, \\ & u(1) + u'(1) = 1 + \frac{(2\varepsilon - 1)\exp(-1/\varepsilon)}{\varepsilon[1 - \exp(-1/\varepsilon)]} + 3 - 2\varepsilon, \end{aligned}$$

whose exact solution is

$$(45)\qquad u(x) = \frac{(2\varepsilon - 1)[1 - \exp(-x/\varepsilon)]}{1 - \exp(-1/\varepsilon)} + x(x + 1 - 2\varepsilon).$$

We employ the parameters $\varepsilon = 0.01$, $m_1 = 110$, $m_2 = 10$, $n_q = 300$, $\nu = 2$ and $R_0 = 1$. From Fig. 3, we can find that the solution $u(x)$ is very close to the exact one with the maximum error being 2.68×10^{-7}. The maximum error is smaller than that in [8], [21], where the Dirichlet boundary conditions $u(0) = 0$, $u(1) = 1$ were considered.

E x a m p l e 5. We consider an internal boundary layer case [3]:

$$(46)\qquad \varepsilon \ddot{u}(t) + 2t\dot{u}(t) + (1 + t^2)u(t) = 0, \quad u(-1) = 2, \quad u(1) = 1,$$

where $\varepsilon = 0.2$.

Upon letting $x = (t + 1)/2$ and $w(x) = u(t)$, we have

$$(47)\quad \varepsilon w''(x) + (8x - 4)w'(x) + (16x^2 - 16x + 8)w(x) = 0, \quad w(0) = 2, \quad w(1) = 1.$$

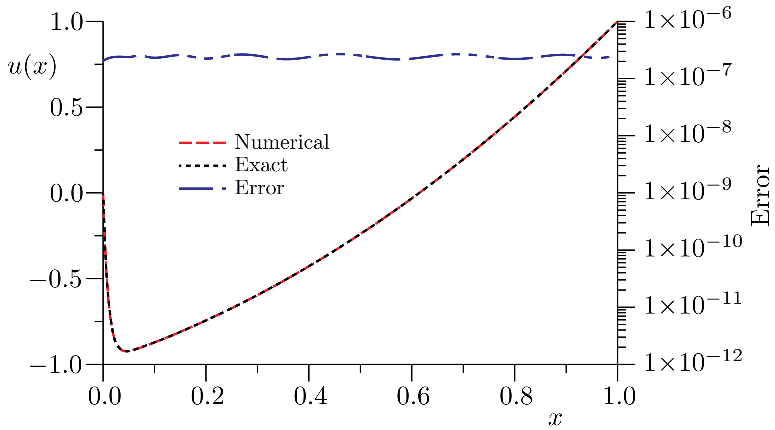

Figure 3. For Example 4, comparing the numerical solution obtained by the ERBFM with the exact solution and showing the error.

For the parameters $\varepsilon = 0.2$, $m_1 = 15$, $m_2 = 15$, $n_q = 150$, and $R_0 = 1$, the numerical result obtained by the ERBFM is shown in Fig. 4. For the purpose of comparison we also apply the fourth-order Runge-Kutta (RK4) method to integrate (47) starting from the initial conditions $w(0) = 2$ and $w'(0) = 4.85305042$ and with a step-size $\Delta x = 0.005$, which can match the final value $w(1) = 1$ with an error 2.644×10^{-8}. The maximum difference between the presented numerical solution and the RK4 solution as shown in Fig. 4 is 6.61×10^{-2}.

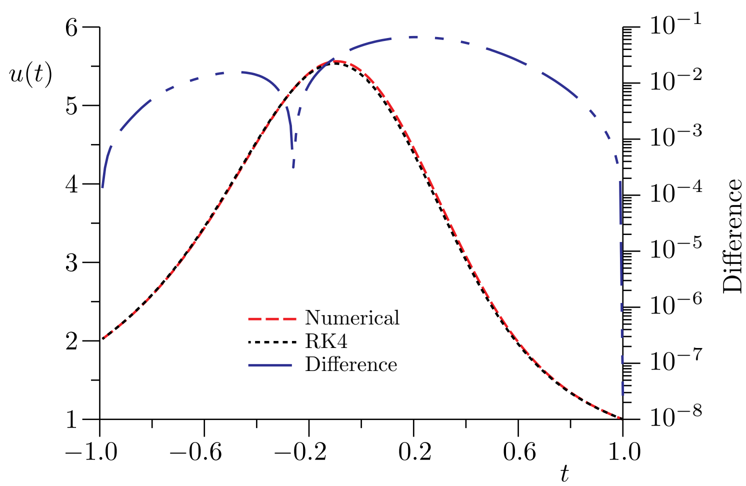

Figure 4. For Example 5, comparing the numerical solution obtained by the ERBFM and the RK4 and showing the difference.

E x a m p l e 6. Finally, we consider the case with two boundary layers [11]:

$$-\varepsilon u''(x) + (1 + x - x^2)u(x) = g(x), \quad u(-1) = 0, \quad u(1) = 0, \tag{48}$$

where $\varepsilon = 0.001$. The exact solution $u(x)$ and $g(x)$ are given by

$$\begin{aligned} u(x) &= 1 + (x-1)\mathrm{e}^{-x/\sqrt{\varepsilon}} - x\mathrm{e}^{(x-1)/\sqrt{\varepsilon}}, \\ g(x) &= 1 + x(1-x) + [2\sqrt{\varepsilon} - x^2(1-x)]\mathrm{e}^{-x/\sqrt{\varepsilon}} \\ &\quad + [2\sqrt{\varepsilon} - x(1-x)^2]\mathrm{e}^{(x-1)/\sqrt{\varepsilon}}. \end{aligned} \tag{49}$$

For the parameters $m_1 = 40$, $m_2 = 40$, $n_q = 200$ and $R_0 = 1$, the numerical result obtained by the ERBFM is shown in Fig. 5, which is very close to the exact solution (49), whose maximum error is 1.65×10^{-7}. The accuracy is better than that calculated by Kadalbajoo and Aggarwal [9], and Khuri and Sayfy [11].

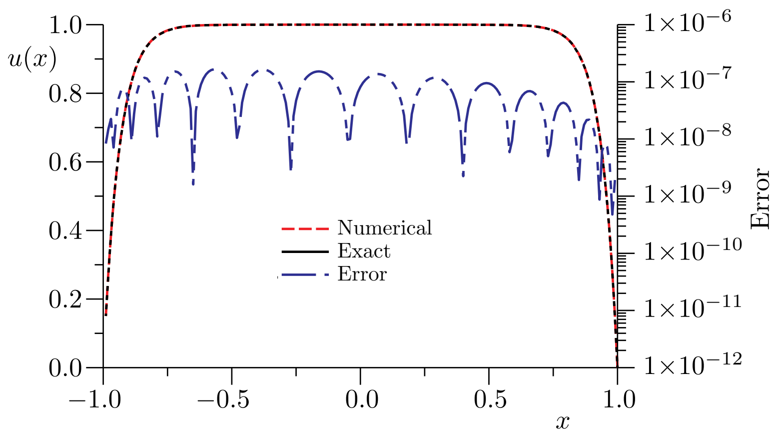

Figure 5. For Example 6, comparing the numerical solution obtained by the ERBFM with the exact one and showing the error.

7. Conclusions

Owing to the existence of a boundary layer in the singularly perturbed BVP, it is utmost important to design the numerical method to exactly match the given Robin boundary conditions. We have developed two types of bases for the polynomials used in the non-singularly perturbed ODE and the normalized exponential functions used in the SPBVP. The main contributions of the present paper are the introduction of a new concept of Robin boundary functions and then deriving an energy identity in terms of the energetic Robin boundary functions, which not only satisfy the homogeneous Robin boundary conditions, but also preserve the energy. Furthermore, the energetic Robin boundary functions were adopted as the bases to expand the numerical solution of the SPBVP, and then we have transformed the highly singular problem into solving a well-conditioned linear system to determine the expansion coefficients by a simple collocation technique. Numerical examples showed that the novel algorithm ERBFM is highly accurate and stable.

References

[1] *U. M. Ascher, R. M. M. Mattheij, R. D. Russell*: Numerical Solution of Boundary Value Problems for Ordinary Differential Equations. Classics in Applied Mathematics 13, SIAM, Society for Industrial and Applied Mathematics, Philadelphia, 1995. zbl MR doi

[2] *A. Awoke, Y. N. Reddy*: An exponentially fitted special second-order finite difference method for solving singular perturbation problems. Appl. Math. Comput. *190* (2007), 1767–1782. zbl MR doi

[3] *C. M. Bender, S. A. Orszag*: Advanced Mathematical Methods for Scientists and Engineers. International Series in Pure and Applied Mathematics, McGraw-Hill Book, New York, 1978. zbl MR

[4] *J. R. Cash*: Numerical integration of nonlinear two-point boundary value problems using iterated deferred corrections. I. A survey and comparison of some one-step formulae. Comput. Math. Appl., Part A *12* (1986), 1029–1048. zbl MR doi

[5] *J. R. Cash*: On the numerical integration of nonlinear two-point boundary value problems using iterated deferred corrections. II. The development and analysis of highly stable deferred correction formulae. SIAM J. Numer. Anal. *25* (1988), 862–882. zbl MR doi

[6] *J. R. Cash, R. W. Wright*: Continuous extensions of deferred correction schemes for the numerical solution of nonlinear two-point boundary value problems. Appl. Numer. Math. *28* (1998), 227–244. zbl MR doi

[7] *N. Doğan, V. S. Ertürk, Ö. Akın*: Numerical treatment of singularly perturbed two-point boundary value problems by using differential transformation method. Discrete Dyn. Nat. Soc. *2012* (2012), Article ID 579431, 10 pages. zbl MR doi

[8] *F. O. Ilicasu, D. H. Schultz*: High-order finite-difference techniques for linear singular perturbation boundary value problems. Comput. Math. Appl. *47* (2004), 391–417. zbl MR doi

[9] *M. K. Kadalbajoo, V. K. Aggarwal*: Fitted mesh B-spline collocation method for solving self-adjoint singularly perturbed boundary value problems. Appl. Math. Comput. *161* (2005), 973–987. zbl MR doi

[10] *H. B. Keller*: Numerical Methods for Two-Point Boundary Value Problems. Blaisdell Publishing Company, Waltham, 1968. zbl MR

[11] *S. A. Khuri, A. Sayfy*: Self-adjoint singularly perturbed boundary value problems: an adaptive variational approach. Math. Methods Appl. Sci. *36* (2013), 1070–1079. zbl MR doi

[12] *T.-C. Lin, D. H. Schultz, W. Zhang*: Numerical solutions of linear and nonlinear singular perturbation problems. Comput. Math. Appl. *55* (2008), 2574–2592. zbl MR doi

[13] *C.-S. Liu*: Efficient shooting methods for the second-order ordinary differential equations. CMES, Comput. Model. Eng. Sci. *15* (2006), 69–86. zbl MR

[14] *C.-S. Liu*: The Lie-group shooting method for singularly perturbed two-point boundary value problems. CMES, Comput. Model. Eng. Sci. *15* (2006), 179–196. zbl MR

[15] *C.-S. Liu*: An equilibrated method of fundamental solutions to choose the best source points for the Laplace equation. Eng. Anal. Bound. Elem. *36* (2012), 1235–1245. zbl MR doi

[16] *C.-S. Liu*: The Lie-group shooting method for solving nonlinear singularly perturbed boundary value problems. Commun. Nonlinear Sci. Numer. Simul. *17* (2012), 1506–1521. zbl MR doi
[17] *C.-S. Liu, B. Li*: Reconstructing a second-order Sturm-Liouville operator by an energetic boundary function iterative method. Appl. Math. Lett. *73* (2017), 49–55. zbl MR doi
[18] *C.-S. Liu, D. Liu, W.-S. Jhao*: Solving a singular beam equation by using a weak-form integral equation method. Appl. Math. Lett. *64* (2017), 51–58. zbl MR doi
[19] *K. C. Patidar*: High order parameter uniform numerical method for singular perturbation problems. Appl. Math. Comput. *188* (2007), 720–733. zbl MR doi
[20] *Y. N. Reddy, P. Pramod Chakravarthy*: An initial-value approach for solving singularly perturbed two-point boundary value problems. Appl. Math. Comput. *155* (2004), 95–110. zbl MR doi
[21] *T. N. Varner, S. R. Choudhury*: Non-standard difference schemes for singular perturbation problems revisited. Appl. Math. Comput. *92* (1998), 101–123. zbl MR doi
[22] *J. Vigo-Aguiar, S. Natesan*: An efficient numerical method for singular perturbation problems. J. Comput. Appl. Math. *192* (2006), 132–141. zbl MR doi

Authors' addresses: *Chein-Shan Liu*, College of Mechanics and Materials, Hohai University, Nanjing, Jiangsu 210098, China; Center of Excellence for Ocean Engineering, Center of Excellence for the Oceans, National Taiwan Ocean University, Keelung 202-24, Taiwan; *Botong Li* (corresponding author), School of Mathematics and Physics, University of Science and Technology Beijing, Beijing 100083, China, e-mail: `libotong@ustb.edu.cn`.