FRACTIONAL-ORDER BESSEL FUNCTIONS WITH VARIOUS APPLICATIONS

Haniye Dehestani, Yadollah Ordokhani, Tehran,
Mohsen Razzaghi, Starkville

Received October 14, 2018. Published online November 18, 2019.

Abstract. We introduce fractional-order Bessel functions (FBFs) to obtain an approximate solution for various kinds of differential equations. Our main aim is to consider the new functions based on Bessel polynomials to the fractional calculus. To calculate derivatives and integrals, we use Caputo fractional derivatives and Riemann-Liouville fractional integral definitions. Then, operational matrices of fractional-order derivatives and integration for FBFs are derived. Also, we discuss an error estimate between the computed approximations and the exact solution and apply it in some examples. Applications are given to three model problems to demonstrate the effectiveness of the proposed method.

Keywords: fractional-order Bessel functions; fractional operational matrix; error estimation

MSC 2010: 34A08, 65M70, 65L70

1. Introduction

Fractional derivatives started from a question when L'Hopital asked "what would be the result of half-differentiating a function." Then it was followed by Leibniz (1695) and Heaviside (1871). To learn more about the history of fractional derivatives, we refer to [27]. Recently, fractional derivatives have played a major role in many areas of science, applied mathematics, engineering, and economics. The applications are now far too many to list here. We propose only a few of them: colored noise [26], earthquake [15], economics [2], electromagnetism [12], fluid-dynamic models [14], [28], seepage flow in porous media [14], and continuum and statistical mechanics [25]. We describe the application of fractional calculus in some of the implement models.

DOI: 10.21136/AM.2019.0279-18

- ▷ The DC motor is a power actuator which converts direct current electrical energy into rotational mechanical energy. The armature-controlled DC motor utilizes a constant field current. This kind of DC motor will be controlled by a nonconventional control technique which is known as a fractional-order control [4].
- ▷ A theoretical model of the spatiotemporal behavior of complex liquid-solid interfaces bordered by the contact line formed between the liquid-like particle and solid-rough substrate is presented by the fractional derivative. This model involves a parameter α, which is the order of the fractional derivative with respect to time (t) and can be related to the roughness exponent of substrate later [34].

Many researchers have shown great interest in using effective techniques to deal with various kinds of fractional problems. Therefore, they have presented many numerical and analytical methods to find a more accurate approximate solution of these problems such as Adomian decomposition method [29], Chebyshev wavelet method [22], [39], homotopy perturbation method [48], Legendre wavelet method [16], [17], Laplace transform method [18] and CAS wavelet method [40] to study more about this topic (see for example [9], [21], [24], [44] and references therein).

Orthogonal functions and polynomial series have been used when dealing with various problems of the dynamical systems. The approach in using orthogonal functions and polynomial series is based on transforming the underlying differential equation into an integral equation through integration, approximating different signals involved in the equation by truncated orthogonal functions and polynomial series and using the operational matrix of integration to eliminate the integral operations.

In 2013, Kazem et al. [19] introduced the fractional-order Legendre functions by change of the variable t to x^α $(0 < \alpha < 1)$ to get an efficient approach for solving fractional differential equations. The paper [43] applied this definition and presented the operational matrix of fractional derivative and integration for such functions to construct a new Tau method for solving fractional partial differential equations. Bhrawy et al. [3] defined the fractional-order generalized Laguerre functions based on the generalized Laguerre polynomials for finding numerical solution of systems of fractional differential equations. Yuzbasi [46] constructed the truncated fractional Bernstein series by changing t to t^α $(0 < \alpha < 1)$ for solving the fractional Riccati type differential equations. In addition, the authors in [6] expanded the fractional Legendre functions to an interval $[0, h]$ and obtained a numerical solution of fractional partial differential equations. Rahimkhani et al. constructed fractional-order Bernoulli wavelets by using the change of variable $t = x^\alpha$ $(0 < \alpha < 1)$ in Bernoulli wavelets, and solved selected problems [36], [37]. Dehestani et al. [8] introduced fractional-order Legendre-Laguerre functions for solving fractional partial differential equations. More recently, the authors in [10] constructed Genocchi-fractional Laguerre functions for solving variable-order time-fractional partial differential equations.

In this paper, we apply fractional-order Bessel functions to solve several problems of fractional order. In the past, many authors have used Bessel polynomials, for example Yuzbasi et al. [45], [47] solved linear differential, integral and integro-differential equations, Parand et al. [33] applied Bessel functions to solving nonlinear Lane-Emden equations, Tohidi et al. [41] presented the Bessel collocation method for solving fractional optimal control problems.

1.1. The aim of this work.

The aim of this work is to introduce a new function for approximating the solution of fractional differential equations, fractional delay differential equations and system of fractional differential equations. Moreover, we discuss the error bound and the rate of convergence for the proposed method.

The advantages of the proposed method are:

(1) Fractional-order Bessel functions (FBFs) constructed by change of variable t to t^α $(0 < \alpha < 1)$, which approximate the fractional function with more accuracy. This feature has made the FBFs more effective than Bessel functions in solving the fractional problems.
(2) Operational matrix of fractional-order derivatives is a sparse matrix, which makes a less error in computation.
(3) Since the coefficients in Bessel polynomials are smaller than the coefficients of Chebyshev, Legendre and Bernoulli polynomials, the computational error in the current method is less.

The effects of these features are shown in seven numerical examples.

The outline of the current paper is as follows. In the following section, we express the basic definitions and properties of the fractional calculus theory. In Section 3, we introduce FBFs and it's properties. Section 4 is devoted to operational matrices of fractional derivative and fractional integration of FBFs for solving fractional problems. In Section 5, we construct an algorithm for solving various kinds of problems by using the FBFs. Error analysis is given in Section 6. In Section 7, we illustrate the accuracy of the proposed scheme by considering numerical examples. Also, a conclusion is given in Section 8.

2. Preliminaries

We consider the essential definitions which are used further in this paper.

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha \geqslant 0$ is defined as (see [1], [5], [30], [31], [35])

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t, \quad x \geqslant 0,\ \alpha \geqslant 0, \quad I^0 f(x) = f(x).$$

Below, we consider a number of properties for $\alpha, \beta \geqslant 0$, $\gamma > -1$ and constants μ_1, μ_2, as

$$I^\alpha(\mu_1 f(x) + \mu_1 g(x)) = \mu_1 I^\alpha f(x) + \mu_2 I^\alpha g(x), \quad I^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$$

Definition 2.2. The fractional derivative of $f(x)$ in the Caputo sense is defined as (see [1], [5], [30], [31], [35])

$$D^\alpha f(x) = I^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)\,\mathrm{d}t,$$

for $m-1 < \alpha \leqslant m$, $m \in \mathbb{N}$, $x > 0$, where $D = \mathrm{d}/\mathrm{d}t$. It has the following properties:

$$D^\alpha C = 0, \quad (C \text{ is a constant})$$
$$D^\alpha x^\gamma = \begin{cases} 0, & \alpha \in \mathbb{N}_0,\ \gamma < \alpha, \\ \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}, & \text{otherwise.} \end{cases}$$

Definition 2.3 (Generalized Taylor's formula)**.** Suppose that $D^{n\alpha} f(x) \in C(0,1]$ for $n = 0, 1, \ldots, N$. Then we have (see [36])

$$f(x) = \sum_{n=0}^{N} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} D^{n\alpha} f(0^+) + \frac{x^{N+1}}{\Gamma(N\alpha+\alpha+1)} D^{(N+1)\alpha} f(\zeta),$$

with $0 < \zeta \leqslant x$ for all $x \in (0,1]$. Also, we get

$$\left| f(x) - \sum_{n=0}^{N} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} D^{n\alpha} f(0^+) \right| \leqslant M_\alpha \frac{x^{N+1}}{\Gamma(N\alpha+\alpha+1)},$$

where $M_\alpha \geqslant \sup\limits_{x\in[0,1]} |D^{(N+1)\alpha} f(x)|$.

3. Fractional-order Bessel functions

3.1. Bessel equation. In 1732, Daniel Bernoulli credited the concept of Bessel functions for the first time, however, the names of these functions are taken from Friedrich Wilhelm Bessel. The Bessel equation is a special case of the Sturm-Liouville problem written as (see [13], [32])

$$x^2y'' + xy' + (x^2 - n^2)y = 0, \tag{3.1}$$

where n is any real number. The solution to the Bessel equation yields Bessel functions of the first and second kind. The Bessel functions of the first kind $J_n(x)$ are defined as follows:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)} \Big(\frac{x}{2}\Big)^{2k+n}. \tag{3.2}$$

The Bessel functions of the first kind are orthogonal with respect to the weight function $w(x) = x$ in the interval $[0,1]$ with the orthogonality property

$$\int_0^1 xJ_n(\lambda x)J_n(\mu x)\,\mathrm{d}x = \tfrac{1}{2}[J_{n+1}(\lambda)]^2\delta_{\lambda\mu}, \tag{3.3}$$

such that in the relation λ, μ are roots of the equation $J_n(x) = 0$, and $\delta_{\lambda\mu}$ is the Kronecker function.

3.2. Fractional-order Bessel equation. Consider the fractional Bessel equation

$$x^{2\alpha}y'' + x^{\alpha}y' + \alpha^2(x^{2\alpha} - n^2)y = 0, \tag{3.4}$$

where $0 \leqslant \alpha < 1$ and n is any real number. If $\alpha = 1$, then (3.4) is a classical Bessel equation. We can investigate solutions by a fractional Frobenius series as follows:

$$y = B(\tilde{J}_\alpha)_n(x) + C(Y_\alpha)_n(x),$$

where $(\tilde{J}_\alpha)_n(x)$ are FBFs of the first kind of order n, so that

$$(\tilde{J}_\alpha)_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)} \Big(\frac{x^\alpha}{2}\Big)^{2k+n}. \tag{3.5}$$

The set of FBFs of the first kind $(\tilde{J}_\alpha)_n(x)$ are orthogonal with respect to the weight function $w_\alpha(x) = x^{2\alpha-1}$ in the interval $[0,1]$. Using the change of variables $t = x^\alpha$, $\alpha > 0$, we have

$$\int_0^1 x^{2\alpha-1}(\tilde{J}_\alpha)_n(\lambda x)(\tilde{J}_\alpha)_n(\mu x)\,\mathrm{d}x = \frac{1}{2\alpha}[J_{n+1}(\lambda)]^2\delta_{\lambda\mu}. \tag{3.6}$$

We introduce a number of FBFs features in the following, where n is a non-negative integer number

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^{n\alpha}(\tilde{J}_\alpha)_n(x)) = \alpha x^{n\alpha}(\tilde{J}_\alpha)_{n-1}(x),$$
$$\frac{\mathrm{d}}{\mathrm{d}x}((\tilde{J}_\alpha)_n(x)) = \alpha(\tilde{J}_\alpha)_{n-1}(x) - \frac{n\alpha}{x^\alpha}(\tilde{J}_\alpha)_n(x),$$
$$(\tilde{J}_\alpha)_{n-1}(x) + (\tilde{J}_\alpha)_{n+1}(x) = \frac{2n}{x^\alpha}(\tilde{J}_\alpha)_n(x).$$

The nth degree truncated FBFs of the first kind are defined by

$$(J_\alpha)_n(x) = \sum_{k=0}^{[(N-n)/2]} \frac{(-1)^k}{k!\,\Gamma(k+n+1)}\Big(\frac{x^\alpha}{2}\Big)^{2k+n}, \quad 0 \leqslant x < \infty,\ n \in \mathbb{N}, \tag{3.7}$$

where N is a positive integer such that $N \geqslant n$ and $n = 0, 1, \ldots, N$. For $N = 2$, we have

$$(J_\alpha)_0(x) = 1 - \frac{x^{2\alpha}}{4}, \quad (J_\alpha)_1(x) = \frac{x^\alpha}{2}, \quad (J_\alpha)_2(x) = \frac{x^{2\alpha}}{8}.$$

Figure 1 illustrates graphs of fractional-order Bessel functions for various values of α for $N = 2$.

A function $f(x) \in L^2[0,1]$ may be expanded into FBFs as

$$f(x) = \sum_{n=0}^{\infty} a_n(J_\alpha)_n(x).$$

Also, we can consider the following truncated series for $f(x)$

$$f(x) \simeq \sum_{n=0}^{N} a_n(J_\alpha)_n(x) = A^\top J_\alpha(x), \quad N \geqslant n,$$

where

$$A = \Big(\int_0^1 f(x)J_\alpha(x)\,\mathrm{d}x\Big)Q_\alpha^{-1}, \quad Q_\alpha = \int_0^1 x^{2\alpha-1}J_\alpha(x)J_\alpha^\top(x)\,\mathrm{d}x. \tag{3.8}$$

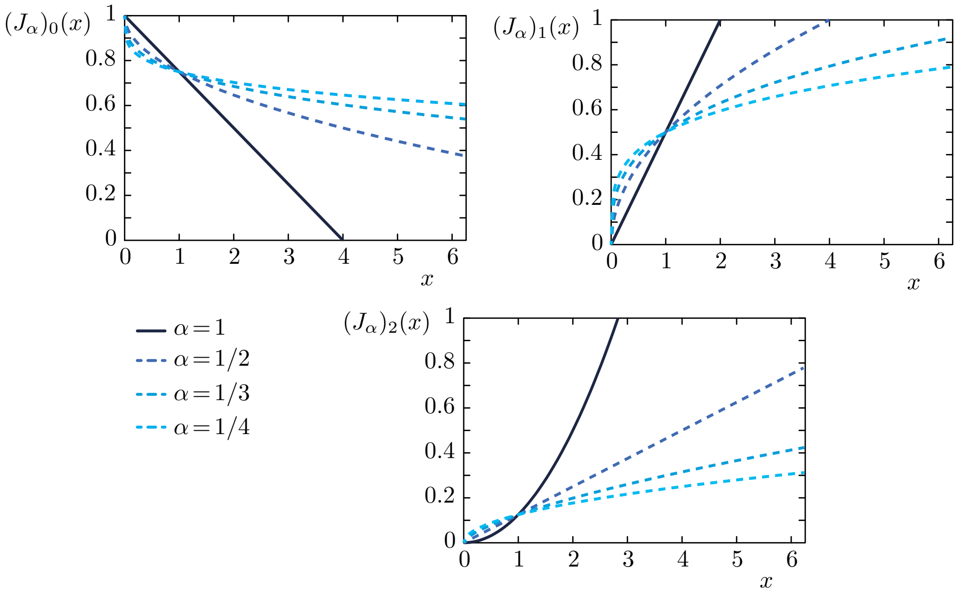

Figure 1. Fractional-order Bessel functions of first kind for $N = 2$.

4. Operational matrices of fractional derivative and integration

The main purpose of this section is to introduce the operational matrices of fractional derivative and the integration of FBFs.

4.1. Operational matrix of fractional derivatives. The Caputo fractional derivatives operator of order $\nu > 0$ of the vector $J_\alpha(x)$ can be expressed by

$$D^\nu J_\alpha(x) \simeq \eta(\alpha, \nu, x) J_\alpha(x), \tag{4.1}$$

where $\eta(\alpha, \nu, x)$ is called the operational matrix of Caputo fractional derivatives of order $\nu > 0$ for $J_\alpha(x)$. In this case, we apply (3.7) and the properties of the Caputo fractional derivative to obtain all elements of $\eta(\alpha, \nu, x)$ as follows:

$$\begin{aligned} D^\nu (J_\alpha)_n(x) &= \sum_{k=0}^{s} \frac{(-1)^k}{k!\,\Gamma(k+n+1)2^{2k+n}} D^\nu\big(x^{\alpha(2k+n)}\big) \\ &= \sum_{k=\lceil(\nu-n\alpha)/2\alpha\rceil}^{s} \frac{(-1)^k}{k!\,\Gamma(k+n+1)2^{2k+n}} \frac{\Gamma(2k\alpha+n\alpha+1)}{\Gamma(2k\alpha+n\alpha-\nu+1)} x^{2k\alpha+n\alpha-\nu} \\ &= x^{n\alpha-\nu} \sum_{k=\lceil(\nu-n\alpha)/2\alpha\rceil}^{s} c_{n,k}^{\alpha,\nu} x^{2k\alpha}, \qquad s = \left[\frac{N-n}{2}\right], \end{aligned} \tag{4.2}$$

where

$$c_{n,k}^{\alpha,\nu} = \frac{(-1)^k \Gamma(2k\alpha + n\alpha + 1)}{k!\,\Gamma(k+n+1) 2^{2k+n} \Gamma(2k\alpha + n\alpha - \nu + 1)}.$$

The approximation of $x^{2k\alpha}$ by fractional-order Bessel series yields

$$x^{2k\alpha} \simeq \sum_{j=0}^{N} d_{k,j}^{\alpha,\nu} (J_\alpha)_j(x).$$

By replacing the above equation in (4.2), we obtain

$$\begin{aligned} D^\nu (J_\alpha)_n(x) & \\ &\simeq x^{n\alpha-\nu} \sum_{k=\lceil(\nu-n\alpha)/2\alpha\rceil}^{s} c_{n,k}^{\alpha,\nu} \sum_{j=0}^{N} d_{k,j}^{\alpha,\nu} (J_\alpha)_j(x) \\ &= x^{n\alpha-\nu} \sum_{j=0}^{N} \Bigg(\sum_{k=\lceil(\nu-n\alpha)/2\alpha\rceil}^{s} \varrho_{n,k,j}^{\alpha,\nu} \Bigg) (J_\alpha)_j(x) \quad (\text{where } \varrho_{n,k,j}^{\alpha,\nu} = c_{n,k}^{\alpha,\nu} d_{k,j}^{\alpha,\nu}) \\ &= x^{n\alpha-\nu} \sum_{j=0}^{N} \eta_{n,j}^{\alpha,\nu} (J_\alpha)_j(x), \end{aligned} \tag{4.3}$$

where

$$\eta_{nj}^{\alpha,\nu} = \sum_{k=\lceil(\nu-n\alpha)/2\alpha\rceil}^{s} \frac{(-1)^k}{k!\,\Gamma(n+k+1) 2^{2k+n}} \frac{\Gamma(2k\alpha + n\alpha + 1)}{\Gamma(2k\alpha + n\alpha - \nu + 1)} d_{k,j}^{\alpha,\nu}. \tag{4.4}$$

The fractional derivatives of FBFs given by (4.4) can be written in the matrix form as

$$\begin{aligned} D^\nu (J_\alpha)_n(x) \simeq x^{n\alpha-\nu} \Bigg[& \sum_{k=\lceil(\nu-n\alpha)/2\alpha\rceil}^{s} \varrho_{n,k,0}^{\alpha,\nu}, \\ & \sum_{k=\lceil(\nu-n\alpha)/2\alpha\rceil}^{s} \varrho_{n,k,1}^{\alpha,\nu}, \ldots, \sum_{k=\lceil(\nu-n\alpha)/2\alpha\rceil}^{s} \varrho_{n,k,N}^{\alpha,\nu} \Bigg] J_\alpha(x). \end{aligned} \tag{4.5}$$

4.2. Operational matrix of fractional integration. The Riemann-Liouville fractional integration of the vector $J_\alpha(x)$ can be obtained as follows

$$I^\nu J_\alpha(x) \simeq \xi(\alpha, \nu, x) J_\alpha(x), \tag{4.6}$$

where $\xi(\alpha, \nu, x)$ denotes the operational matrix of fractional integration of order $\nu > 0$ for the FBFs. Due to (3.7) and the properties of the Riemann-Liouville fractional integration, we have

$$I^{\nu}(J_{\alpha})_n(x) = \sum_{k=0}^{s} \frac{(-1)^k}{k!\,\Gamma(k+n+1)2^{2k+n}} I^{\nu}(x^{\alpha(2k+n)}) \tag{4.7}$$
$$= \sum_{k=0}^{s} \frac{(-1)^k}{k!\,\Gamma(k+n+1)2^{2k+n}} \frac{\Gamma(2k\alpha+n\alpha+1)}{\Gamma(2k\alpha+n\alpha+\nu+1)} x^{2k\alpha+n\alpha+\nu}$$
$$= x^{n\alpha+\nu} \sum_{k=0}^{s} p_{n,k}^{\alpha,\nu} x^{2k\alpha},$$

where

$$p_{n,k}^{\alpha,\nu} = \frac{(-1)^k}{k!\,\Gamma(k+n+1)2^{2k+n}} \frac{\Gamma(2k\alpha+n\alpha+1)}{\Gamma(2k\alpha+n\alpha+\nu+1)}.$$

Also, $x^{2k\alpha}$ can be expanded in $N+1$ terms of FBFs as

$$x^{2k\alpha} \simeq \sum_{j=0}^{N} q_{k,j}^{\alpha,\nu} (J_{\alpha})_j(x). \tag{4.8}$$

By substituting the above equation in (4.7), we obtain

$$I^{\nu}(J_{\alpha})_n(x) \simeq x^{n\alpha+\nu} \sum_{k=0}^{s} p_{n,k}^{\alpha,\nu} \sum_{j=0}^{N} q_{k,j}^{\alpha,\nu} (J_{\alpha})_j(x) \tag{4.9}$$
$$= x^{n\alpha+\nu} \sum_{j=0}^{N} \left(\sum_{k=0}^{s} R_{n,k,j}^{\alpha,\nu} \right) (J_{\alpha})_j(x)$$
$$= x^{n\alpha+\nu} \sum_{j=0}^{N} \xi_{n,j}^{\alpha,\nu} (J_{\alpha})_j(x),$$

where $R_{n,k,j}^{\alpha,\nu} = p_{n,k}^{\alpha,\nu} q_{k,j}^{\alpha,\nu}$. Hence, each element of $\xi_{n,j}^{\alpha,\nu}$ can be expressed as

$$\xi_{n,j}^{\alpha,\nu} = \sum_{k=0}^{s} \frac{(-1)^k}{k!\,\Gamma(n+k+1)2^{2k+n}} \frac{\Gamma(2k\alpha+n\alpha+1)}{\Gamma(2k\alpha+n\alpha+\nu+1)} q_{k,j}^{\alpha,\nu}. \tag{4.10}$$

The fractional integration of FBFs given by (4.9) can be written in the matrix form as

$$I^{\nu}(J_{\alpha})_n(x) \simeq x^{n\alpha+\nu} \left[\sum_{k=0}^{s} R_{n,k,0}^{\alpha,\nu}, \sum_{k=0}^{s} R_{n,k,1}^{\alpha,\nu}, \ldots, \sum_{k=0}^{s} R_{n,k,N}^{\alpha,\nu} \right] J_{\alpha}(x). \tag{4.11}$$

4.3. Error bound for the operational matrix of fractional integration.

Lemma 4.1. *Suppose that H is a Hilbert space and Y is a closed subspace of H such that $\dim Y < \infty$ and $y_1, y_2, \ldots, y_m$ is any basis for Y. Let z be an arbitrary element in H and y the unique best approximation to z out of Y. Then* [20]

$$\|z - y^*\|_2^2 = \frac{G(z, y_1, y_2, \ldots, y_m)}{G(y_1, y_2, \ldots, y_m)},$$

where

$$G(z, y_1, y_2, \ldots, y_m) = \begin{vmatrix} \langle z, z\rangle & \langle z, y_1\rangle & \ldots & \langle z, y_m\rangle \\ \langle y_1, z\rangle & \langle y_1, y_1\rangle & \ldots & \langle y_1, y_m\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle y_m, z\rangle & \langle y_m, y_1\rangle & \ldots & \langle y_m, y_m\rangle \end{vmatrix}.$$

Lemma 4.2. *Suppose $g \in L^2[0,1]$ is approximated by g_N as (see* [38]*)*

$$g(x) \simeq g_N(x) = \sum_{n=0}^{N} \kappa_n (J_\alpha)_n(x),$$

and consider

$$L_N(g) = \int_0^1 [g(x) - g_N(x)]^2 \, \mathrm{d}x.$$

Then we have

$$\lim_{N\to\infty} L_N(g) = 0.$$

The operational matrix of fractional integration has the error vector

$$E^\nu = I^\nu J_\alpha - \xi^{\alpha,\nu} J_\alpha,$$

where

$$E^\nu = [e^\nu_{\alpha,n}]_{(N+1)\times 1}, \quad n = 0, 1, \ldots, N.$$

Due to (4.8) and Lemma 4.1, we get

$$\left\| x^{2k\alpha} - \sum_{j=0}^{N} q^{\alpha,\nu}_{k,j} (J_\alpha)_j(x) \right\|_2 = \Big(\frac{G(x^{2k\alpha}, (J_\alpha)_0(x), \ldots, (J_\alpha)_N(x))}{G((J_\alpha)_0(x), (J_\alpha)_1(x), \ldots, (J_\alpha)_N(x))} \Big)^{1/2}. \tag{4.12}$$

Therefore, according to the above equations and (4.12), we obtain

$$
\begin{aligned}
(4.13)\quad \|e^{\nu}_{\alpha,n}\|_2 &= \left\| I^{\nu}(J_\alpha)_n(x) - \sum_{k=0}^{s} p^{\alpha,\nu}_{n,k}\Big(\sum_{j=0}^{N} q^{\alpha,\nu}_{k,j}(J_\alpha)_j(x)\Big)\right\|_2, \quad n = 0,1,\ldots,N,\\
&\leqslant \sum_{k=0}^{s} \frac{\Gamma(2k\alpha+n\alpha+1)}{k!\,\Gamma(k+n+1)2^{2k+n}\Gamma(2k\alpha+n\alpha+\nu+1)}\\
&\quad\times \left\| x^{2k\alpha} - \sum_{j=0}^{N} q^{\alpha,\nu}_{k,j}(J_\alpha)_j(x)\right\|_2\\
&\leqslant \sum_{k=0}^{s} \frac{\Gamma(2k\alpha+n\alpha+1)}{k!\,\Gamma(k+n+1)2^{2k+n}\Gamma(2k\alpha+n\alpha+\nu+1)}\\
&\quad\times \Big(\frac{G(x^{2k\alpha},(J_\alpha)_0(x),\ldots,(J_\alpha)_N(x))}{G((J_\alpha)_0(x),(J_\alpha)_1(x),\ldots,(J_\alpha)_N(x))}\Big)^{1/2}.
\end{aligned}
$$

As a result, by considering the above discussion, we can conclude that by increasing the number of the fractional Bessel bases, the error vector E^{ν} tends to zero.

5. Method of solution

In this section we use the FBFs of the first kind to solve various kinds of fractional-order differential equations such as

$$
(5.1)\qquad D^{\nu}y(x) = F(x, y(x), y(\tau_1 x), y(\tau_2 x), \ldots, y(\tau_k x)), \quad n-1<\nu\leqslant n,\ n\in\mathbb{N},
$$

with initial conditions

$$
y^{(i)}(0) = \delta_i, \quad i = 0,1,\ldots,n-1,
$$

where τ_l, $l = 1,\ldots,k$, are constants, y is the unknown function and F is the known continuous linear or nonlinear function. To solve the problem, we expand the function $y^{(n)}(x)$ by FBFs as

$$
(5.2)\qquad y^{(n)}(x) \simeq A^{\top}J_\alpha(x).
$$

By using the operational matrix of integration in (4.6), we have

$$
\begin{aligned}
(5.3)\qquad y^{(n-1)}(x) &\simeq A^{\top}\xi(\alpha,1,x)J_\alpha(x) + \delta_{n-1},\\
y^{(n-2)}(x) &\simeq A^{\top}(\xi(\alpha,1,x))^2J_\alpha(x) + x\delta_{n-1} + \delta_{n-2},\\
&\ \,\vdots\\
y(x) &\simeq A^{\top}(\xi(\alpha,1,x))^nJ_\alpha(x) + \sum_{i=0}^{n-1}\frac{x^i}{i!}\delta_i.
\end{aligned}
$$

Therefore, we have

$$y(\tau_j x) \simeq A^\top(\xi(\alpha,1,x))^n J_\alpha(\tau_j x) + \sum_{i=0}^{n-1} \frac{(\tau_j x)^i}{i!}\delta_i. \tag{5.4}$$

On the other hand, by applying the fractional operational matrix of the derivative and the properties of the Caputo fractional derivative, we obtain

$$D^\nu y(x) \simeq A^\top(\xi(\alpha,1,x))^n \eta(\alpha,\nu,x) J_\alpha(x) + \sum_{i=0}^{n-1} \frac{\delta_i}{i!}\frac{\Gamma(i+1)}{\Gamma(i+1+\nu)} x^{i-\nu}. \tag{5.5}$$

As a result, by substituting (5.2)–(5.5) in (5.1), we achieve the algebraic equation with $N+1$ unknown coefficients. Then we use collocation points (see [9]) defined by

$$x_i = \frac{1}{N}i, \quad i = 0,1,\ldots,N.$$

Consequently, we can obtain the unknown vector A by solving the above system and using Newton's iterative method.

6. Error analysis

In this section we examine the upper bound of error for a sufficiently smooth function, which is expanded in terms of FBFs.

Theorem 6.1. *Suppose that $D^{n\alpha}f(x) \in C(0,1]$ for $n = 0,1,\ldots,N$, $2\alpha(N+2) \geqslant 1$ and $Y_N^\alpha = \operatorname{span}\{(J_\alpha)_0(x),(J_\alpha)_1(x),\ldots,(J_\alpha)_N(x)\}$. If $p_N^* = A^\top J_\alpha$ is the best approximation to f from Y_N^α, then the error bound is presented as*

$$\|f(x) - p_N^*(x)\|_{L_w^2[0,1]} \leqslant \frac{M_\alpha}{\Gamma(N\alpha+\alpha+1)}\sqrt{\frac{1}{2\alpha(N+2)}}, \tag{6.1}$$

where $M_\alpha = \sup_{x\in[0,1]} |D^{(N+1)\alpha}f(x)|$.

P r o o f. Due to the generalized Taylor's formula introduced in Definition 2.3, we have

$$p_N(x) = \sum_{n=0}^{N} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} D^{n\alpha}f(0^+),$$

for which we know that

$$|f(x) - p_N(x)| \leqslant \frac{M_\alpha x^{(N+1)\alpha}}{\Gamma(N\alpha+\alpha+1)}.$$

Since $A^\top J_\alpha$ is the best approximation to f from Y_N^α and $p_N f \in Y_N^\alpha$, one has

$$\begin{aligned}
\|f(x)-p_N^*(x)\|^2_{L^2_w[0,1]} &\leqslant \|f(x)-p_N(x)\|^2_{L^2_w[0,1]} \\
&= \int_0^1 |f(x)-p_N(x)|^2 x^{2\alpha-1}\,\mathrm{d}x \\
&\leqslant \frac{M_\alpha^2}{\Gamma(N\alpha+\alpha+1)^2}\int_0^1 x^{(2N+2)\alpha}x^{2\alpha-1}\,\mathrm{d}x \\
&= \frac{M_\alpha^2}{\Gamma(N\alpha+\alpha+1)^2(2\alpha(N+2))}.
\end{aligned}$$

□

Now by taking the square roots, the theorem can be proved.

This theorem shows the approximate solution computed by the FBFs converges to the exact solution.

Theorem 6.2. *Assume that $y_N(x) = A^\top J_\alpha(x)$ is the approximate solution obtained by the method presented in the previous section. If $\tilde{y}_N(x) = \tilde{A}^\top \tilde{J}_\alpha(x)$ is the FBFs of the first kind expansion of the exact solution $y(x)$, where*

$$\tilde{A} = [\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_N]^\top, \quad \tilde{J}_\alpha(x) = [(\tilde{J}_\alpha)_0(x), (\tilde{J}_\alpha)_1(x), \ldots, (\tilde{J}_\alpha)_N(x)]^\top,$$

and $(\tilde{J}_\alpha)_n(x)$, $n = 0, 1, \ldots, N$, is the fractional order of Bessel polynomials of the first kind, which is defined in (3.5), *then we obtain the upper bound of the error for the solution obtained by the present method as*

$$\begin{aligned}
\|y(x)-y_N(x)\|_{L^2_w[0,1]} \\
\leqslant \frac{M_\alpha}{\Gamma(N\alpha+\alpha+1)}\sqrt{\frac{1}{2\alpha(N+2)}} + \Theta_{\alpha,N}\|\tilde{A}-A\|_2 + \Upsilon_{\alpha,N}\|A\|_2,
\end{aligned} \tag{6.2}$$

where

$$\begin{aligned}
\Theta_{\alpha,N} &= \left[\sum_{n=0}^{N}\frac{1}{2\alpha}[J_{n+1}(1)]^2\right]^{1/2}, \\
\Upsilon_{\alpha,N} &= \left[\sum_{n=0}^{N}\sum_{k=[(N-n)/2]}^{\infty}\frac{1}{(k!\,\Gamma(k+n+1)2^{2k+n})^2(4k\alpha+2(n+1)\alpha)}\right]^{1/2}.
\end{aligned}$$

P r o o f. To prove (6.2), we write

$$\|y(x)-y_N(x)\|_{L^2_w[0,1]} \leqslant \|y(x)-\tilde{y}_N(x)\|_{L^2_w[0,1]} + \|\tilde{y}_N(x)-y_N(x)\|_{L^2_w[0,1]}. \tag{6.3}$$

According to (6.1), we have

$$\|y(x)-\tilde{y}_N(x)\|_{L^2_w[0,1]} \leqslant \frac{M_\alpha}{\Gamma(N\alpha+\alpha+1)}\sqrt{\frac{1}{2\alpha(N+2)}}. \tag{6.4}$$

Also, we obtain

$$\begin{aligned}
&\|\tilde{y}_N(x)-y_N(x)\|_{L^2_w[0,1]} \\
&\quad= \Big\|\sum_{n=0}^{N}\tilde{a}_n(\tilde{J}_\alpha)_n(x) - \sum_{n=0}^{N} a_n(J_\alpha)_n(x)\Big\|_{L^2_w[0,1]} \\
&\quad\leqslant \Big\|\sum_{n=0}^{N}\tilde{a}_n(\tilde{J}_\alpha)_n(x) - \sum_{n=0}^{N} a_n(\tilde{J}_\alpha)_n(x)\Big\|_{L^2_w[0,1]} \\
&\qquad+ \Big\|\sum_{n=0}^{N} a_n(\tilde{J}_\alpha)_n(x) - \sum_{n=0}^{N} a_n(J_\alpha)_n(x)\Big\|_{L^2_w[0,1]} \\
&\quad\leqslant \Big\|\sum_{n=0}^{N}[\tilde{a}_n - a_n](\tilde{J}_\alpha)_n(x)\Big\|_{L^2_w[0,1]} \\
&\qquad+ \Big\|\sum_{n=0}^{N} a_n[(\tilde{J}_\alpha)_n(x) - (J_\alpha)_n(x)]\Big\|_{L^2_w[0,1]} \\
&\quad= \Big(\int_0^1\Big|\sum_{n=0}^{N}(\tilde{a}_n - a_n)(\tilde{J}_\alpha)_n(x)\Big|^2 x^{2\alpha-1}\,\mathrm{d}x\Big)^{1/2} \\
&\qquad+ \Big(\int_0^1\Big|\sum_{n=0}^{N} a_n((\tilde{J}_\alpha)_n(x) - (J_\alpha)_n(x))\Big|^2 x^{2\alpha-1}\,\mathrm{d}x\Big)^{1/2} \\
&\quad\leqslant \Big(\int_0^1\Big[\sum_{n=0}^{N}|\tilde{a}_n - a_n|^2\Big]\Big[\sum_{n=0}^{N}|(\tilde{J}_\alpha)_n(x)|^2\Big] x^{2\alpha-1}\,\mathrm{d}x\Big)^{1/2} \\
&\qquad+ \Big(\int_0^1\Big[\sum_{n=0}^{N}|a_n|^2\Big]\Big[\sum_{n=0}^{N}|(\tilde{J}_\alpha)_n(x) - (J_\alpha)_n(x)|^2\Big] x^{2\alpha-1}\,\mathrm{d}x\Big)^{1/2} \\
&\quad\leqslant \|\tilde{A}-A\|_2\Big[\sum_{n=0}^{N}\int_0^1 x^{2\alpha-1}|(\tilde{J}_\alpha)_n(x)|^2\,\mathrm{d}t\Big]^{1/2} \\
&\qquad+ \|A\|_2\Big[\sum_{n=0}^{N}\int_0^1 |(\tilde{J}_\alpha)_n(x) - (J_\alpha)_n(x)|^2 x^{2\alpha-1}\,\mathrm{d}x\Big]^{1/2}.
\end{aligned} \tag{6.5}$$

Then, by using the orthogonality property of FBFs of the first kind, we get

$$
\begin{aligned}
(6.6)\quad & \|\tilde{y}_N(x) - y_N(x)\|_{L^2_w[0,1]} \\
& \leqslant \|\tilde{A} - A\|_2 \Bigg[\sum_{n=0}^{N} \frac{1}{2\alpha}[J_{n+1}(1)]^2\Bigg]^{1/2} \\
& \quad + \|A\|_2 \Bigg[\sum_{n=0}^{N} \int_0^1 \Bigg|\sum_{k=[(N-n)/2]}^{\infty} \frac{(-1)^k x^{(2k+n)\alpha}}{k!\,\Gamma(k+n+1)2^{2k+n}}\Bigg|^2 x^{2\alpha-1}\,\mathrm{d}x\Bigg]^{1/2} \\
& \leqslant \|\tilde{A} - A\|_2 \Bigg[\sum_{n=0}^{N} \frac{1}{2\alpha}[J_{n+1}(1)]^2\Bigg]^{1/2} \\
& \quad + \|A\|_2 \Bigg[\sum_{n=0}^{N} \int_0^1 \sum_{k=[(N-n)/2]}^{\infty} \frac{x^{4k\alpha+2(n+1)\alpha-1}}{(k!\,\Gamma(k+n+1)2^{2k+n})^2}\,\mathrm{d}x\Bigg]^{1/2} \\
& \leqslant \|\tilde{A} - A\|_2 \Bigg[\sum_{n=0}^{N} \frac{1}{2\alpha}[J_{n+1}(1)]^2\Bigg]^{1/2} \\
& \quad + \|A\|_2 \Bigg[\sum_{n=0}^{N} \sum_{k=[(N-n)/2]}^{\infty} \frac{1}{(k!\,\Gamma(k+n+1)2^{2k+n})^2(4k\alpha+2(n+1)\alpha)}\Bigg]^{1/2}.
\end{aligned}
$$

□

By means of (6.3)–(6.6), we determine the upper bound of the error.

According to FBFs properties and the above results, it can be inferred that by increasing the number of FBFs, the upper bound of error tends to zero.

7. Illustrative examples

In this section we test the performance of the scheme on some examples. The computational results are presented in examples that were performed using MATLAB.

Example 7.1. Consider the fractional differential equation [36], [33]

$$D^\nu y(x) + 3y(x) = 3x^3 + \frac{8}{\Gamma(0.5)}x^{1.5}, \quad 1 < \nu \leqslant 2,\ 0 \leqslant x \leqslant 1,$$

subject to initial conditions

$$y(0) = 0, \quad y'(0) = 0.$$

The exact solution of this problem when $\nu = \frac{3}{2}$ is $y(x) = x^3$. According to the proposed method, we have

$$y''(x) \simeq A^\top J_\alpha(x).$$

Then

$$y'(x) \simeq A^\top \xi(\alpha, 1, x) J_\alpha(x),$$

and

$$y(x) \simeq A^\top (\xi(\alpha, 1, x))^2 J_\alpha(x).$$

Also, from (5.5) we have

$$D^\nu y(x) \simeq A^\top (\xi(\alpha, 1, x))^2 \eta(\alpha, \nu, x) J_\alpha(x).$$

Therefore, we get

$$A^\top (\xi(\alpha, 1, x))^2 \eta(\alpha, \nu, x) J_\alpha(x) + 3A^\top (\xi(\alpha, 1, x))^2 J_\alpha(x) = 3x^3 + \frac{8}{\Gamma(0.5)} x^{1.5}.$$

Due to the above process and the collocation points, we obtain the numerical results, which are illustrated in Table 1 and Figure 2. Table 1 shows the absolute errors

x	Present Method	Method in [36]	Method in [33]	
	$N = 3$	$k = 1,\ M = 3$	$N = 7$	$N = 9$
0.1	1.4183×10^{-17}	2.26837×10^{-15}	1.67093×10^{-10}	3.65474×10^{-14}
0.3	1.4734×10^{-17}	4.75314×10^{-16}	1.67093×10^{-10}	4.21589×10^{-14}
0.5	2.0193×10^{-17}	9.43690×10^{-16}	1.06212×10^{-9}	9.14555×10^{-13}
0.7	4.2303×10^{-17}	1.16573×10^{-15}	7.03957×10^{-10}	8.21475×10^{-14}
0.9	1.0250×10^{-16}	5.55112×10^{-16}	1.92731×10^{-10}	1.32512×10^{-13}

Table 1. Comparison of the absolute errors obtained by the present method with the methods in [36], [33] for $\alpha = \frac{1}{2}$, $\nu = \frac{3}{2}$ for Example 7.1.

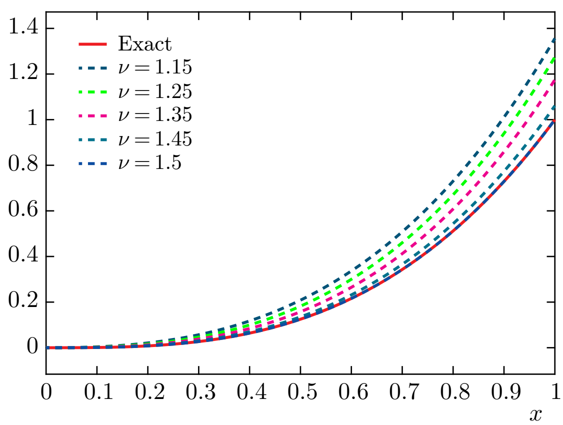

Figure 2. Approximate solutions for various values of ν with $N = 3$, $\alpha = 0.5$ for Example 7.1.

obtained by the fractional-order Bernoulli wavelets method [36], Bessel functions [33] and the present method. The comparisons in Table 1 show that the present method is more accurate compared to other methods. Also, Figure 2 shows the approximate solution obtained for different values of α, ν with $N = 3$.

Example 7.2. Consider the initial value problem [17]

$$D^{\nu}y(x) + y(x) = x^4 - \frac{1}{2}x^3 - \frac{3}{\Gamma(4-\nu)}x^{3-\nu} + \frac{24}{\Gamma(5-\nu)}x^{4-\nu}, \quad 0 \leqslant \nu \leqslant 1,\ 0 \leqslant x \leqslant 1,$$

subject to initial condition

$$y(0) = 0.$$

The exact solution of this problem is $y(x) = x^4 - \frac{1}{2}x^3$. Absolute errors between the exact solution and the numerical solutions for different values of α and ν are given in Table 2. Results in Table 2 suggest the present method works well and is more efficient than the Legendre wavelets method in [17].

x	Present method $N = 12$ $\alpha = \nu = \frac{1}{3}$	Present method $N = 12$ $\alpha = \nu = \frac{1}{2}$	Present method $N = 12$ $\alpha = \nu = 1$	Present method $N = 8$ $\alpha = \nu = \frac{1}{2}$	Legendre wavelets method [17] $M = 8,\ k = 1$ $\nu = \frac{1}{2}$
0	1.03×10^{-16}	1.66×10^{-17}	9.07×10^{-16}	1.72×10^{-16}	1.81×10^{-12}
0.1	4.98×10^{-12}	7.46×10^{-14}	8.54×10^{-16}	1.63×10^{-15}	3.09×10^{-12}
0.2	2.52×10^{-12}	4.04×10^{-14}	7.50×10^{-16}	7.25×10^{-16}	2.07×10^{-11}
0.3	1.69×10^{-12}	2.80×10^{-14}	6.89×10^{-16}	5.48×10^{-16}	3.64×10^{-11}
0.4	1.27×10^{-12}	2.14×10^{-14}	6.08×10^{-16}	3.97×10^{-16}	8.29×10^{-12}
0.5	1.02×10^{-12}	1.72×10^{-14}	5.64×10^{-16}	2.89×10^{-16}	1.34×10^{-10}
0.6	8.48×10^{-13}	1.43×10^{-14}	5.10×10^{-16}	2.70×10^{-16}	4.93×10^{-10}
0.7	7.25×10^{-13}	1.22×10^{-14}	4.57×10^{-16}	2.35×10^{-16}	1.20×10^{-9}
0.8	6.32×10^{-13}	1.06×10^{-14}	4.16×10^{-16}	1.11×10^{-16}	2.41×10^{-9}
0.9	5.60×10^{-13}	9.32×10^{-15}	4.44×10^{-16}	5.55×10^{-17}	4.32×10^{-9}
1.0	5.10×10^{-13}	1.09×10^{-14}	3.77×10^{-15}	8.88×10^{-16}	7.15×10^{-9}

Table 2. Absolute errors for different values of N and α, ν for Example 7.2.

Example 7.3. Consider the fractional Riccati equation [17]

$$D^{\nu}y(x) = -y^2(x) + 1, \quad 0 \leqslant \nu \leqslant 1,\ 0 \leqslant x \leqslant 1,$$

subject to initial condition

$$y(0) = 0,$$

The exact solution of this problem, with $\nu = 1$ is

$$y(x) = \frac{\exp(2x) - 1}{\exp(2x) + 1}.$$

The errors of numerical solutions for different values of α and N are given in Table 3. Also, the absolute errors for various values of ν are presented in Table 4. From this table, it is clear that with the values of ν approaching 1, the absolute errors decrease.

α	N	L_∞-error	L_2-error
1	4	3.25×10^{-4}	6.33×10^{-4}
	6	6.80×10^{-6}	1.37×10^{-5}
	9	1.19×10^{-7}	2.69×10^{-7}
0.5	4	9.43×10^{-4}	2.43×10^{-3}
	6	1.17×10^{-4}	2.76×10^{-3}
	9	1.01×10^{-4}	2.42×10^{-4}

Table 3. Errors for different values of N and α with $\nu = 1$ for Example 7.3.

x	$\nu = 0.9$	$\nu = 0.99$	$\nu = 0.999$	$\nu = 1$
0.1	2.76×10^{-2}	2.48×10^{-3}	2.45×10^{-4}	1.19×10^{-7}
0.3	4.25×10^{-2}	4.10×10^{-3}	4.10×10^{-4}	1.02×10^{-7}
0.5	3.53×10^{-2}	3.68×10^{-3}	3.69×10^{-4}	8.76×10^{-8}
0.7	1.87×10^{-2}	2.16×10^{-3}	2.19×10^{-4}	7.08×10^{-8}
0.9	5.11×10^{-4}	3.15×10^{-4}	3.43×10^{-5}	5.49×10^{-8}
CPU	3.6783×10^{-2}	3.4125×10^{-2}	3.2087×10^{-2}	3.2825×10^{-2}

Table 4. Absolute errors for different values of ν with $\alpha = 1$ and $N = 9$ for Example 7.3.

Example 7.4. Consider the linear pantograph differential equation [23], [11]

$$D^\nu y(x) + y(x) - \frac{1}{10} y\Big(\frac{x}{5}\Big) = \frac{-1}{10} \exp\Big(\frac{-x}{5}\Big), \quad 0 \leqslant \nu \leqslant 1,\ 0 \leqslant x \leqslant 1,$$

subject to initial condition

$$y(0) = 1.$$

The exact solution of this problem, with $\nu = 1$, is $y(x) = \exp(x)$. Table 5 displays the absolute error for different values of N and $\alpha = \nu = 1$ with results in [23], [11]. From Table 5, by increasing the number of FBFs, we see that the absolute error tends to zero. Also, Table 5 shows that our results are more accurate in comparison to results obtained by methods in [11], [23]. Figure 3 illustrates the numerical results for various values of ν with $N=6$ and the exact solution of the problem. Moreover, according to the upper bound of error in Section 6 for $N=6$, we have

$$\alpha=0.5 \;\Rightarrow\; \|y(x)-y_6(x)\|_{L^2_w[0,1]} \leqslant 8.2602\times 10^{-2},$$

and

$$\alpha=1 \;\Rightarrow\; \|y(x)-y_6(x)\|_{L^2_w[0,1]} \leqslant 1.3484\times 10^{-4}.$$

	Present method				Method [23]	Method [11]
x	$N=4$	$N=6$	$N=10$	$N=12$		$N=64$
2^{-1}	3.13×10^{-6}	2.12×10^{-8}	2.52×10^{-14}	1.86×10^{-17}	3.33×10^{-11}	3.93×10^{-14}
2^{-2}	1.56×10^{-5}	2.26×10^{-8}	3.04×10^{-14}	3.29×10^{-17}	4.13×10^{-11}	2.16×10^{-14}
2^{-3}	1.02×10^{-5}	2.92×10^{-8}	3.84×10^{-14}	7.80×10^{-17}	4.62×10^{-11}	1.66×10^{-15}
2^{-4}	3.76×10^{-6}	1.48×10^{-8}	3.53×10^{-14}	4.98×10^{-17}	4.89×10^{-11}	6.21×10^{-15}
2^{-5}	1.12×10^{-6}	5.12×10^{-9}	1.69×10^{-14}	6.38×10^{-17}	5.05×10^{-11}	3.86×10^{-14}
2^{-6}	3.05×10^{-7}	1.49×10^{-9}	5.81×10^{-15}	5.33×10^{-17}	4.74×10^{-11}	4.54×10^{-14}

Table 5. Absolute error for different values of N with $\alpha=\nu=1$ for Example 7.4.

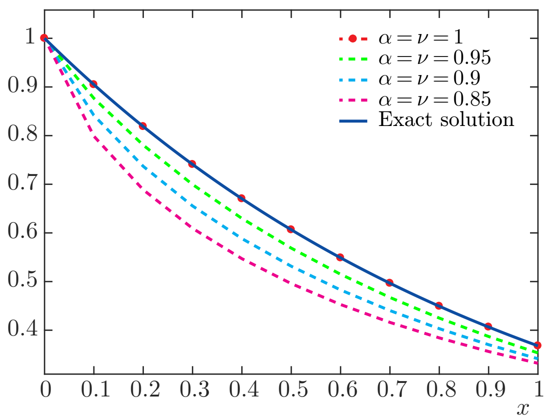

Figure 3. Approximate solutions for various values of $\alpha=\nu$ with $N=6$ for Example 7.4.

Example 7.5. Consider the linear fractional pantograph differential equation (see [37], [42], [7])

$$D^{\nu}y(x) = -y(x) + 0.1y(\tfrac{4}{5}x) + 0.5D^{\nu}y(\tfrac{4}{5}x) + (0.32x-0.5)\exp(-0.8x) + \exp(-x),$$
$$0\leqslant \nu\leqslant 1,\quad 0\leqslant x\leqslant 1,$$

subject to initial condition

$$y(0)=0.$$

The exact solution of this problem, with $\nu = 1$, is $y(x) = x\exp(-x)$. Absolute errors between the exact solution and the numerical solution for different values of N with $\alpha = \nu = 1$ are given in Table 6. Figure 4(a) displays the comparison of the approximate solution with the exact solution for various values of $\alpha = \nu = 0.8, 0.85, 0.9, 0.95, 1$, with $N = 3$. Also, the absolute error between the exact and approximate solutions for $\alpha = \nu = 1$ with $N = 3$ is plotted in Figure 4(b). From Table 6 and Figure 4, it is clear that the approximate solutions converge to the exact solution. Also, these results show that the present method is more accurate in comparison to methods in [37], [42], [7].

x	Present method		Method in [37]	Method in [42]	Method in [7]
	$N=6$	$N=9$	$k=2, M=6$		
0.1	2.6377×10^{-7}	2.5606×10^{-11}	1.98×10^{-8}	4.65×10^{-3}	1.30×10^{-3}
0.3	2.7422×10^{-7}	1.8959×10^{-11}	7.78×10^{-9}	2.57×10^{-2}	2.63×10^{-3}
0.5	1.9451×10^{-7}	1.2631×10^{-11}	6.34×10^{-5}	4.43×10^{-2}	2.83×10^{-3}
0.7	9.2527×10^{-8}	7.9481×10^{-12}	4.36×10^{-5}	5.37×10^{-2}	2.39×10^{-3}
0.9	1.0706×10^{-7}	4.3873×10^{-12}	2.80×10^{-5}	5.35×10^{-2}	1.64×10^{-3}

Table 6. Comparison of the absolute errors obtained by the present method with the methods in [37], [42], [7] for $\alpha = \nu = 1$ for Example 7.5.

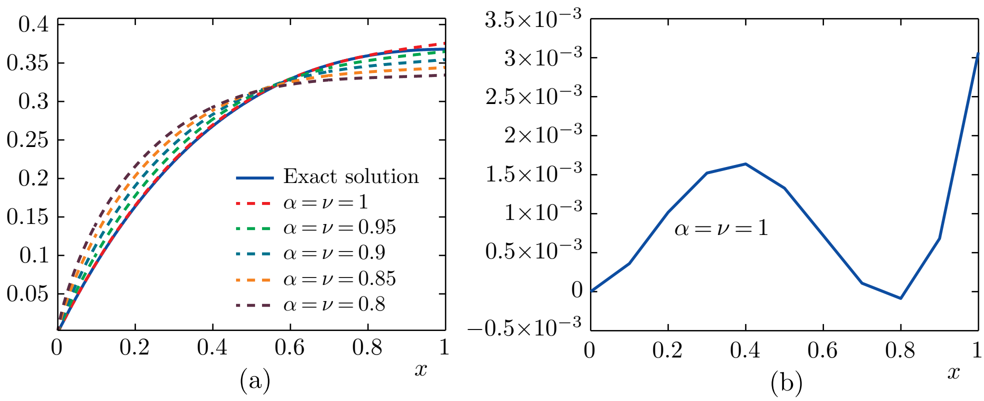

Figure 4. (a) The comparison of the approximate solution with the exact solution for various values of $\alpha = \nu = 0.8, 0.85, 0.9, 0.95, 1$, with $N = 3$ for Example 7.5.
(b) The absolute errors between the exact and approximate solutions for $\alpha = \nu = 1$, with $N = 3$ for Example 7.5.

Example 7.6. Consider the fractional nonlinear pantograph differential equation [16]

$$D^{\nu} y(x) = 1 - 2y^2\Big(\frac{x}{2}\Big), \quad 1 \leqslant \nu \leqslant 2, \quad 0 \leqslant x \leqslant 1,$$

subject to initial conditions

$$y(0) = 1, \quad y'(0) = 0.$$

The exact solution of this problem, with $\nu = 2$, is $y(x) = \cos x$. In Table 7, we compare the absolute errors obtained by the present method with the modified Laguerre wavelets method for various values of N with $\alpha = \nu = 1$. Also, the absolute errors for various values of ν are presented in Table 8. This table illustrates that with the values of ν approaching 2, the absolute error tends to zero. Figure 5 shows the behavior of the approximate solutions for various values of ν with $N = 8$ and $\alpha = 0.5$.

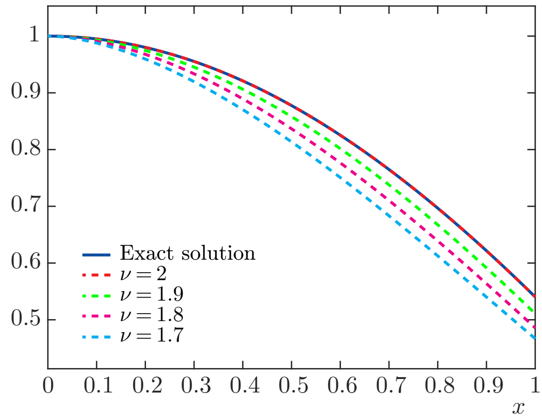

Figure 5. The comparison of approximate solutions for different values of $\nu = 2, 1.9, 1.8, 1.7$ with $N = 8$ and $\alpha = 0.5$ for Example 7.6.

x	Present method			Modified Laguerre wavelets method [16]
	$N=5$	$N=7$	$N=10$	$K=1, N=20$
0	0	0	0	–
0.1	3.35×10^{-8}	6.06×10^{-11}	2.44×10^{-15}	2.11×10^{-8}
0.2	7.56×10^{-8}	1.22×10^{-10}	4.74×10^{-15}	2.09×10^{-8}
0.3	1.06×10^{-7}	1.75×10^{-10}	7.09×10^{-15}	2.09×10^{-8}
0.4	1.35×10^{-7}	2.28×10^{-10}	9.35×10^{-15}	2.08×10^{-8}
0.5	1.63×10^{-7}	2.75×10^{-10}	1.14×10^{-14}	2.06×10^{-8}
0.6	1.88×10^{-7}	3.17×10^{-10}	1.34×10^{-14}	2.04×10^{-3}
0.7	2.06×10^{-7}	3.52×10^{-10}	1.53×10^{-14}	2.03×10^{-8}
0.8	2.21×10^{-7}	3.79×10^{-10}	1.70×10^{-14}	2.00×10^{-8}
0.9	2.33×10^{-7}	3.98×10^{-10}	1.86×10^{-14}	1.99×10^{-8}
1.0	2.39×10^{-7}	4.09×10^{-10}	1.98×10^{-14}	–

Table 7. The comparison of the absolute errors for different values of N and $\alpha = \nu = 1$ with the modified Laguerre wavelets method [16] for Example 7.6.

x	$\nu = 1.65$	$\nu = 1.75$	$\nu = 1.85$	$\nu = 2$
0	0	0	0	0
0.1	7.97×10^{-3}	4.86×10^{-3}	2.51×10^{-3}	1.20×10^{-12}
0.3	4.12×10^{-2}	2.69×10^{-2}	1.47×10^{-2}	3.46×10^{-12}
0.5	7.34×10^{-2}	5.02×10^{-2}	2.87×10^{-2}	5.43×10^{-12}
0.7	9.12×10^{-2}	6.60×10^{-2}	3.92×10^{-4}	6.94×10^{-12}
0.9	8.92×10^{-2}	6.83×10^{-2}	4.28×10^{-2}	7.87×10^{-12}
1	7.83×10^{-2}	6.31×10^{-2}	4.11×10^{-2}	8.05×10^{-12}

Table 8. Absolute errors for different values of ν with $\alpha = 1$ and $N = 8$ for Example 7.6.

Example 7.7. Consider the system of fractional differential equations (see [36])

$$\begin{cases} D^{\nu_1} y_1(x) = y_1(x) + y_2(x), & 0 < \nu_1 \leqslant 1,\ 0 \leqslant x \leqslant 1, \\ D^{\nu_2} y_2(x) = -y_1(x) + y_2(x), & 0 < \nu_2 \leqslant 1, \end{cases}$$

subject to the initial conditions

$$y_1(0) = 0, \quad y_2(0) = 1.$$

The exact solution of this system when $\nu_1 = \nu_2 = 1$ is $y_1(x) = \exp x \sin x$ and $y_2(x) = \exp x \cos x$. We present the results for various values of N in Tables 9 and 10 and see that as the terms of FBFs N increase the absolute error tends to zero. Also, Figure 6 shows the curves of approximate solutions for various values of α, ν with $N = 6$. From this graph it is seen that the approximate solutions converge to the exact solution.

x	$N = 5$	$N = 7$	$N = 9$
0.1	2.3853×10^{-6}	2.1173×10^{-8}	1.0318×10^{-10}
0.3	2.3087×10^{-5}	3.2095×10^{-8}	9.6528×10^{-11}
0.5	4.4635×10^{-5}	4.9916×10^{-8}	9.3743×10^{-11}
0.7	8.2236×10^{-5}	7.2857×10^{-8}	7.9913×10^{-11}
0.9	1.4631×10^{-4}	8.9388×10^{-8}	2.3875×10^{-11}

Table 9. Absolute errors of $y_1(x)$ for different values of N with $\alpha = \nu = 1$ for Example 7.7.

x	$N = 5$	$N = 7$	$N = 9$
0.1	5.7927×10^{-5}	3.9640×10^{-8}	5.7870×10^{-11}
0.3	8.4909×10^{-5}	3.6275×10^{-8}	8.4769×10^{-11}
0.5	1.0162×10^{-4}	3.8998×10^{-8}	1.2399×10^{-10}
0.7	9.8213×10^{-8}	4.2169×10^{-8}	1.6991×10^{-10}
0.9	3.4325×10^{-3}	2.7364×10^{-8}	1.8095×10^{-10}

Table 10. Absolute errors of $y_2(x)$ for different values of N with $\alpha = \nu = 1$ for Example 7.7.

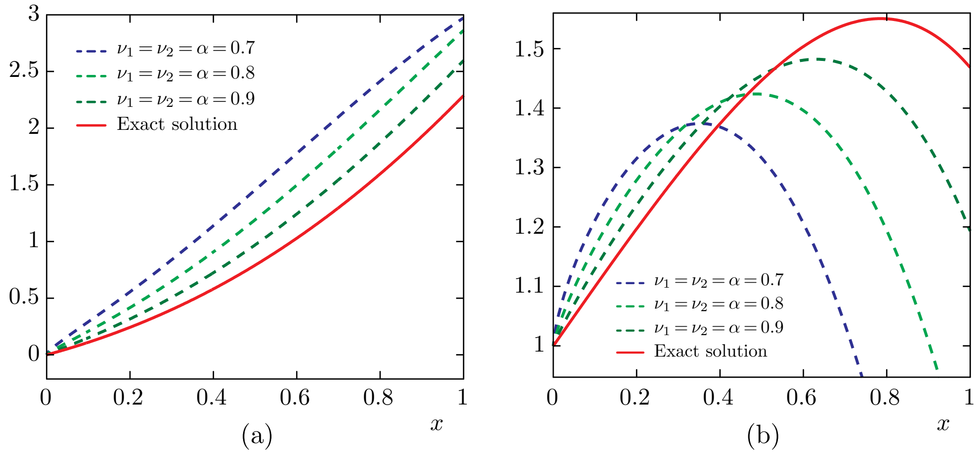

Figure 6. (a) The comparison of $y_1(x)$ with the exact solution for various values of $\alpha = \nu_1 = \nu_2$ with $N = 6$ for Example 7.7.
(b) The comparison of $y_2(x)$ with the exact solution for various values of $\alpha = \nu_1 = \nu_2$, with $N = 6$ for Example 7.7.

8. Conclusion

In the present work we introduced new functions called fractional-order Bessel functions of the first kind. First, we presented the fractional Bessel functions operational matrices of the Caputo fractional derivative and the Riemann-Liouville fractional integration. Then, we used these functions and their operational matrices to approximate the solutions of fractional-order differential equations, fractional pantograph differential equations and systems of fractional differential equations. Numerical examples show the validity and efficiency of the method. Also, it is demonstrated that the present method in comparison to other methods works well and achieves good accuracy.

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Authors' addresses: *Haniye Dehestani*, *Yadollah Ordokhani* (corresponding author), Department of Applied Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran, P. Code: 1993893973, Iran, e-mail: `h.dehestani@alzahra.ac.ir`, `ordokhani@alzahra.ac.ir`; *Mohsen Razzaghi*, Department of Mathematics and Statistics, Mississippi State University, Starkville, MS 39762, USA, e-mail: `razzaghi@math.msstate.edu`.