THEOREMS ON SOME FAMILIES OF FRACTIONAL DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS

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Abstract. We use the Laplace transform method to solve certain families of fractional order differential equations. Fractional derivatives that appear in these equations are defined in the sense of Caputo fractional derivative or the Riemann-Liouville fractional derivative. We first state and prove our main results regarding the solutions of some families of fractional order differential equations, and then give examples to illustrate these results. In particular, we give the exact solutions for the vibration equation with fractional damping and the Bagley-Torvik equation.

Keywords: fractional calculus; fractional differential equation; Caputo derivative; Laplace transform

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1. Introduction

The origin of fractional calculus rests in the 17th century. However, the first use of fractional operations was made by Niels Henrik Abel in 1823. Abel used fractional calculus to obtain the solution of the tautochrone problem and this area rapidly progressed from that day in [13].

In the last few decades, fractional calculus has been used to model many processes in physics and engineering with the use of fractional order differential equations. Signal processing, fluid mechanics, diffusion process, capacitor theory, electro chemistry, continuum and statistical mechanics are some important areas in which fractional calculus is used [1], [9], [11], [12].

There are several ways to get exact or numerical solutions of fractional order differential equations. In this paper, we use the Laplace transform method to obtain exact solutions of some families of fractional order differential equations [13].

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In the remainder of the paper, we give the definitions of the Laplace transform operator and some related operators which are the Riemann-Liouville fractional integral, the fractional derivative operators and the Caputo fractional derivative operator. Also, we give place to the Laplace transform of the Riemann-Liouville fractional derivative and the Caputo sense fractional derivative of a function. Then, we introduce various theorems about the solutions of some families of fractional order differential equations which include the Riemann-Liouville fractional derivative or the Caputo sense fractional derivative. Finally, we use the introduced theorems to obtain the solutions for the vibration equation with fractional damping, the Bagley-Torvik equation and some other fractional differential equations.

2. Preliminaries

Definition 2.1. The Laplace transform of a function $f(t)$ is defined by [4]

$$\mathcal{L}\{f(t)\} = \bar{f}(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t, \tag{2.1}$$

where $s \in \mathbb{C}$ and $\mathrm{Re}(s) > 0$.

Definition 2.2. The inverse Laplace transform of $\bar{f}(s)$ is defined by [4]

$$\mathcal{L}^{-1}\{\bar{f}(s)\} = f(t) = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{st} \bar{f}(s)\,\mathrm{d}s, \tag{2.2}$$

where $s \in \mathbb{C}$ and $\mathrm{Re}(s) > 0$.

Definition 2.3. The quantity $(f * g)(t)$ is called the convolution of the functions $f(t)$ and $g(t)$ and defined by the integral

$$(f * g)(t) = \int_0^t f(t-\tau) g(\tau)\,\mathrm{d}\tau. \tag{2.3}$$

Theorem 2.4. *If $\mathcal{L}\{f(t)\} = \bar{f}(s)$ and $\mathcal{L}\{g(t)\} = \bar{g}(s)$, then the following equation holds true* [4], *p.* 146,

$$\mathcal{L}\{(f * g)(t)\} = \bar{f}(s)\bar{g}(s). \tag{2.4}$$

Definition 2.5. The Gamma Euler function is given by the integral

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t} t^{z-1}\,\mathrm{d}t, \quad \mathrm{Re}(z) > 0. \tag{2.5}$$

Definition 2.6. The Riemann-Liouville fractional integral of $f(t)$ of order ν is defined by [13]

$$D_{0,t}^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-x)^{\nu-1} f(x)\,\mathrm{d}x, \tag{2.6}$$

where $x > 0$, $\nu \in \mathbb{C}$, and $\mathrm{Re}(\nu) > 0$.

Definition 2.7. The Riemann-Liouville fractional derivative of $f(t)$ of order μ is defined by [13]

$$D_{0,t}^{\mu} f(t) = D^m [D_{0,t}^{-\nu} f(t)], \quad \mu > 0,\ t > 0, \tag{2.7}$$

where $m \geqslant \mathrm{Re}(\mu) > m-1$ and $\nu = m - \mu$.

In equation (2.7), D^m is defined as the so-called Newton-Leibniz differential operator, where $m \in \mathbb{N}$,

$$D^m f(t) = \frac{\mathrm{d}^m f(t)}{\mathrm{d}x^m}. \tag{2.8}$$

Definition 2.8. The fractional derivative of $f(t)$ of order μ in the Caputo sense is defined by [9]

$$D^{\mu} f(t) = D_{0,t}^{-\nu} [D^m f(t)], \quad \mu > 0,\ t > 0, \tag{2.9}$$

where $m \geqslant \mathrm{Re}(\mu) > m-1$ and $\nu = m - \mu$.

It can be easily seen that, if we choose $\mu = m \in \mathbb{Z}^+$ in equations (2.7) and (2.9), then we get the classical derivative that is defined in equation (2.8). So, the Riemann-Liouville fractional derivative and the Caputo sense fractional derivative coincide with the classical derivative under the choice of $\mu = m \in \mathbb{Z}^+$.

Theorem 2.9. *If $f^{(n)}(t)$ is the nth derivative of $f(t)$ for $n \in \mathbb{N}$, then the Laplace transform of $f^{(n)}(t)$ is given by the formula* [4], *p.* 144,

$$\begin{aligned} \mathcal{L}\{f^{(n)}(t)\} &= s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0) \\ &= s^n \bar{f}(s) - \sum_{r=0}^{n-1} f^{(r)}(0) s^{n-r-1}, \end{aligned} \tag{2.10}$$

where $f^{(r)}(0)$ is the value of $f^{(r)}(t)$ at $t = 0$ for $r = 0, 1, \ldots, n-1$.

Theorem 2.10. *The Laplace transform of the Riemann-Liouville fractional derivative of $f(x)$ of order μ is given by* [5],

$$\mathcal{L}\{D^{\mu}_{0,t}f(t)\} = s^{\mu}f(s) - \sum_{k=0}^{m-1} s^{k} D^{(\mu-k-1)}_{0,t} f(0), \tag{2.11}$$

where $\mu \in \mathbb{C}$, $m \geqslant \operatorname{Re}(\mu) > m-1$ *and* $\nu = m - \mu$ *for* $\operatorname{Re}(\mu) > 0$.

Theorem 2.11. *The Laplace transform of the fractional derivative of $f(x)$ of order μ in Caputo sense is given by* [12],

$$\mathcal{L}\{D^{\mu}f(t)\} = s^{\mu}\bar{f}(s) - \sum_{k=1}^{m} s^{\mu-k} D^{k-1} f(0), \tag{2.12}$$

where $\mu \in \mathbb{C}$, $m-1 < \operatorname{Re}(\mu) \leqslant m$, *and* $\nu = m - \mu$ *for* $\operatorname{Re}(\mu) > 0$.

The vibration equation with fractional damping with one degree of freedom is stated as

$$mD^2y(t) + qD^{\mu}y(t) + ky(t) = f(t), \tag{2.13}$$

where m represents the mass of the oscillator, q represents the fractional damping coefficient and k represents the stiffness coefficient [8], [15].

The solution of the equation (2.13) depends on the interval to which μ belongs. In this paper, we give the solutions to the equation (2.13) for the intervals $1 \geqslant \operatorname{Re}(\mu) > 0$ and $2 \geqslant \operatorname{Re}(\mu) > 1$.

The general form of the Bagley-Torvik equation is written as

$$aD^2y(t) + bD^{3/2}y(t) + cDy(t) = f(t), \tag{2.14}$$

with the initial conditions

$$y(0) = c_0,\ y'(0) = c_1, \tag{2.15}$$

where $a \neq 0$, $b, c \in \mathbb{R}$, $t > 0$, and $f(t)$ is a given function from the interval $[0, T]$ into $\mathbb{R}$, see [17].

The parameters in equation (2.14) and the initial conditions in equation (2.15) can be varied to attain different responses.

Linearly damped fractional oscillator that includes a damping term contains a fractional derivative of order $\mu = 3/2$ and can be represented by the Bagley-Torvik equation with the initial conditions $c_0 = 0$ and $c_1 = 0$, see [2].

To get another response for expression (2.14), let m denote the mass of a thin rigid plate, S denote the area of the rigid, and $\bar{\mu}$ and ϱ denote the viscosity and the density of a Newtonian fluid, respectively. Moreover, let k denote the stiffness of a spring. In equation (2.14) if we let $a = m$, $b = 2S\sqrt{\bar{\mu}\varrho}$, and $c = k$, then the Bagley-Torvik equation represents the modelling of motion of a rigid sheet that submerged in the Newtonian fluid [2], [6].

In this paper, we give the solution to the equation (2.14) with the initial conditions $y(0) = c_0 \neq 0$ and $y'(0) = c_1 \neq 0$.

Before proceeding to the main theorems, we give the definitions of two generalizations of the Mittag-Leffler function since we frequently use these generalizations for the representation of the solutions of the fractional differential equations.

Definition 2.12. The Wiman's function $E_{\mu,\nu}(z)$ is defined by the series

$$E_{\mu,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + \nu)}, \quad \mathrm{Re}(\mu) \geqslant 0, \tag{2.16}$$

where $\nu, \mu \in \mathbb{C}$, see [16].

Definition 2.13. The function $E^{\gamma}_{\mu,\nu}(z)$ is defined by the series

$$E^{\gamma}_{\mu,\nu}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\mu n + \nu) n!}, \quad \mathrm{Re}(\mu) \geqslant 0, \tag{2.17}$$

where $\nu, \mu, \gamma \in \mathbb{C}$ and the function $(\gamma)_n$ is defined by the equations $(\gamma)_n = \Gamma(\gamma + n)/\Gamma(\gamma)$ and $(\gamma)_0 = 1$, see [16].

The function $E^{\gamma}_{\mu,\nu}(z)$ is an entire function of order $[\mathrm{Re}(\mu)]^{-1}$ and some special functions can be written as particular cases of this function [16].

If we take $\gamma = 1$ in the equation (2.17), we obtain the Wiman's function. If we take $\gamma = 1$ and $\nu = 1$ in the equation (2.17), we obtain the function $E_{\mu}(z)$ which is the Mittag-Leffler function [16]. If we take $\gamma = 1$, $\nu = 1$, and $\mu = 1$ in the equation (2.17), we obtain the exponential function e^z, see [10].

Remark 2.14. For $\mathrm{Re}(\nu) \geqslant 0, \mathrm{Re}(s) \geqslant 0$ and $|s| > |\lambda|^{[\mathrm{Re}(\alpha)]^{-1}}$ the Laplace transform of the function $t^{\beta-1}E^{\varrho}_{\alpha,\beta}(\lambda t^{\alpha})$ is given by [16], Eq. (2.5),

$$\mathcal{L}\{t^{\beta-1}E^{\varrho}_{\alpha,\beta}(\lambda t^{\alpha})\} = \frac{1}{s^{\beta}(1 - \lambda s^{-\alpha})^{\varrho}}. \tag{2.18}$$

Remark 2.15. The Laplace transform of the function $t^{\beta-1}$ is given by [7], p. 137, Eq. (1),

$$\mathcal{L}\{t^{\beta-1}\} = \Gamma(\beta) s^{-\beta}, \tag{2.19}$$

for $\mathrm{Re}(s) \geqslant 0$.

3. Solutions for various fractional differential equations

Lemma 3.1. *If* $a, b \in \mathbb{R}$, $b \neq 0$, $\alpha, \mu \in \mathbb{C}$, $\mathrm{Re}(\alpha) > \mathrm{Re}(\mu) > 0$, *and*

$$\Big|\frac{bs^{-\mu}}{s^{\alpha-\mu}+a}\Big| < 1,$$

then

$$\frac{1}{s^{\alpha}+as^{\mu}+b} = \sum_{k=0}^{\infty}(-b)^k s^{-\alpha k-\alpha}(1+as^{\mu-\alpha})^{-k-1}, \quad \Big|\frac{bs^{-\alpha}}{1+as^{\mu-\alpha}}\Big| < 1. \tag{3.1}$$

Proof. Note that we have the equality

$$\begin{aligned} \frac{1}{s^{\alpha}+as^{\mu}+b} &= \frac{s^{-\mu}}{s^{\alpha-\mu}+a+bs^{-\mu}} \\ &= \frac{s^{-\mu}}{(s^{\alpha-\mu}+a)\Big(1+\dfrac{bs^{-\mu}}{s^{\alpha-\mu}+a}\Big)}. \end{aligned} \tag{3.2}$$

By using the series expansion [9]

$$\Big(1+\frac{bs^{-\mu}}{s^{\alpha-\mu}+a}\Big)^{-1} = \sum_{k=0}^{\infty}\Big(\frac{-bs^{-\mu}}{s^{\alpha-\mu}+a}\Big)^k, \tag{3.3}$$

we obtain the equation

$$\begin{aligned} \frac{1}{s^{\alpha}+as^{\mu}+b} &= \frac{s^{-\mu}}{s^{\alpha-\mu}+a}\sum_{k=0}^{\infty}\Big(\frac{-bs^{-\mu}}{s^{\alpha-\mu}+a}\Big)^k \\ &= \sum_{k=0}^{\infty}\frac{(-b)^k s^{-\mu k-\mu}}{(s^{\alpha-\mu}+a)^{k+1}} \\ &= \sum_{k=0}^{\infty}\frac{(-b)^k s^{-\mu k-\mu}}{s^{(\alpha-\mu)(k+1)}(1+as^{\mu-\alpha})^{k+1}} \\ &= \sum_{k=0}^{\infty}\frac{(-b)^k s^{-\alpha k-\alpha}}{(1+as^{\mu-\alpha})^{k+1}}, \end{aligned} \tag{3.4}$$

as desired. □

Lemma 3.2. *If $b \in \mathbb{R}$, $b \neq 0$, $\alpha \in \mathbb{C}$, and $|bs^{-\alpha}| < 1$, then the equation*

$$\frac{1}{s^{\alpha}+b} = \sum_{k=0}^{\infty}(-b)^k s^{-\alpha k-\alpha} \tag{3.5}$$

is a special case of Lemma 3.1.

P r o o f. Equation (3.5) could be easily obtained by letting $a = 0$ in Lemma 3.1. $\square$

Theorem 3.3. *If $a, b \in \mathbb{R}$, $b \neq 0$, $\alpha, \mu \in \mathbb{C}$, $m, n \in \mathbb{N}$, $n \geqslant \mathrm{Re}(\alpha) > n-1 \geqslant m \geqslant \mathrm{Re}(\mu) > m-1 \geqslant 0$, and $|bs^{-\mu}/(s^{\alpha-\mu}+a)| < 1$, then the solution of the fractional differential equation,*

$$D^{\alpha}y(t) + aD^{\mu}y(t) + by(t) = f(t), \tag{3.6}$$

with the initial conditions $y^{(r)}(0) = c_r$ for $r = 0, 1, 2, \ldots, n-1$ is

$$\begin{aligned} y(t) = {} & \int_0^t f(x)\sum_{k=0}^{\infty}(-b)^k(t-x)^{\alpha k+\alpha-1}E^{k+1}_{\alpha-\mu,\alpha k+\alpha}(-a(t-x)^{\alpha-\mu})\,\mathrm{d}x \\ & + \sum_{r=0}^{n-1} c_r \sum_{k=0}^{\infty}(-b)^k t^{\alpha k+r}E^{k+1}_{\alpha-\mu,\alpha k+r+1}(-at^{\alpha-\mu}) \\ & + a\sum_{j=0}^{m-1} c_j \sum_{k=0}^{\infty}(-b)^k t^{\alpha k+\alpha-\mu+j}E^{k+1}_{\alpha-\mu,\alpha k+\alpha-\mu+j+1}(-at^{\alpha-\mu}). \end{aligned} \tag{3.7}$$

P r o o f. Applying the Laplace operator to the differential equation (3.6), we get

$$\mathcal{L}\{D^{\alpha}y(t) + aD^{\mu}y(t) + by(t)\} = \mathcal{L}\{f(t)\}. \tag{3.8}$$

By the equation (2.12), this implies that we have

$$s^{\alpha}\bar{y}(s) - \sum_{r=0}^{n-1} s^{\alpha-r-1}c_r + as^{\mu}\bar{y}(s) - \sum_{j=0}^{m-1} as^{\mu-j-1}c_j + b\bar{y}(s) = \bar{f}(s),$$

where $c_r = D^r y(0)$ for $r = 0, 1, 2, \ldots, n-1$, $c_j = D^j y(0)$ for $j = 0, 1, 2, \ldots, m-1$, $\bar{y}(s) = \mathcal{L}\{y(t)\}$, and $\bar{f}(s) = \mathcal{L}\{f(t)\}$. Therefore, by solving this equation for $\bar{y}(s)$, we get

$$\bar{y}(s) = \frac{\bar{f}(s) + \sum_{r=0}^{n-1} c_r s^{\alpha-r-1} + a\sum_{j=0}^{m-1} c_j s^{\mu-j-1}}{s^{\alpha}+as^{\mu}+b}. \tag{3.9}$$

Now, by using Lemma 3.1, we have

$$\bar{y}(s) = \left(\bar{f}(s) + \sum_{r=0}^{n-1} s^{\alpha-r-1} c_r + \sum_{j=0}^{m-1} a s^{\mu-j-1} c_j\right) \times \sum_{k=0}^{\infty} (-b)^k s^{-\alpha k-\alpha} (1 + a s^{\mu-\alpha})^{-k-1}$$
$$= \bar{f}(s) \sum_{k=0}^{\infty} (-b)^k s^{-\alpha k-\alpha} (1 + a s^{\mu-\alpha})^{-k-1} + \sum_{r=0}^{n-1} c_r \sum_{k=0}^{\infty} (-b)^k s^{-\alpha k-r-1} (1 + a s^{\mu-\alpha})^{-k-1} + a \sum_{j=0}^{m-1} c_j \sum_{k=0}^{\infty} (-b)^k s^{-\alpha k-\alpha+\mu-j-1} (1 + a s^{\mu-\alpha})^{-k-1}. \tag{3.10}$$

If we take the inverse Laplace transform of $\bar{y}(s)$, by using Theorem 2.4 and equation (2.18), we get

$$y(t) = f(t) * \sum_{k=0}^{\infty} (-b)^k t^{\alpha k+\alpha-1} E^{k+1}_{\alpha-\mu,\alpha k+\alpha}(-a t^{\alpha-\mu}) + \sum_{r=0}^{n-1} c_r \sum_{k=0}^{\infty} (-b)^k t^{\alpha k+r} E^{k+1}_{\alpha-\mu,\alpha k+r+1}(-a t^{\alpha-\mu}) + a \sum_{j=0}^{m-1} c_j \sum_{k=0}^{\infty} (-b)^k t^{\alpha k+\alpha-\mu+j} E^{k+1}_{\alpha-\mu,\alpha k+\alpha-\mu+j+1}(-a t^{\alpha-\mu}). \tag{3.11}$$

Then the equation (3.7) directly follows from the equation (3.11) by using the definition of convolution (2.3). □

Corollary 3.4. *If $a, b \in \mathbb{R}$, $b \neq 0$, $\alpha, \mu \in \mathbb{C}$, $n, m \in \mathbb{N}$, $n \geqslant \mathrm{Re}(\alpha) > n-1 \geqslant m \geqslant \mathrm{Re}(\mu) > m-1 \geqslant 0$, and $|b s^{-\mu}/(s^{\alpha-\mu} + a)| < 1$, then the solution of the fractional differential equation*

$$D^{\alpha} y(t) + a D^{\mu} y(t) + b y(t) = 0 \tag{3.12}$$

with the initial conditions $y^{(r)}(0) = c_r$ for $r = 0, 1, 2, \ldots, n-1$ is

$$y(t) = \sum_{r=0}^{n-1} c_r \sum_{k=0}^{\infty} (-b)^k t^{\alpha k+r} E^{k+1}_{\alpha-\mu,\alpha k+r+1}(-a t^{\alpha-\mu}) + a \sum_{j=0}^{m-1} c_j \sum_{k=0}^{\infty} (-b)^k t^{\alpha k+\alpha-\mu+j} E^{k+1}_{\alpha-\mu,\alpha k+\alpha-\mu+j+1}(-a t^{\alpha-\mu}). \tag{3.13}$$

P r o o f. Equation (3.13) can be easily obtained by substituting $f(t)=0$ in Theorem 3.3. $\square$

Theorem 3.5. *If $a,b\in\mathbb{R}$, $\alpha,\mu\in\mathbb{C}$, $n\in\mathbb{N}$, $n\geqslant \operatorname{Re}(\alpha)>n-1\geqslant 0$, and $|bs^{-\alpha}|<1$, then the solution of the fractional differential equation*

$$D^{\alpha}y(t)+by(t)=f(t) \tag{3.14}$$

with the initial conditions $y^{(r)}(0)=c_r$ for $r=0,1,2,\ldots,n-1$ is

$$y(t)=\int_0^t f(x)(t-x)^{\alpha-1}E_{\alpha,\alpha}(-b(t-x)^{\alpha})\,\mathrm{d}x+\sum_{r=0}^{n-1}c_r t^r E_{\alpha,r+1}(-bt^{\alpha}). \tag{3.15}$$

P r o o f. Applying the Laplace operator to the differential equation (3.14), we get

$$\mathcal{L}\{D^{\alpha}y(t)+by(t)\}=\mathcal{L}\{f(t)\}. \tag{3.16}$$

By the equation (2.12), this implies that we have

$$s^{\alpha}\bar{y}(s)-\sum_{r=0}^{n-1}s^{\alpha-r-1}c_r+b\bar{y}(s)=\bar{f}(s), \tag{3.17}$$

where $c_r=D^r y(0)$ for $r=0,1,2,\ldots,n-1$, $\bar{y}(s)=\mathcal{L}\{y(t)\}$, and $\bar{f}(s)=\mathcal{L}\{f(t)\}$. Therefore, we can easily get

$$\bar{y}(s)=\Big[\bar{f}(s)+\sum\nolimits_{r=0}^{n-1}c_r s^{\alpha-r-1}\Big](s^{\alpha}+b)^{-1}. \tag{3.18}$$

Now, by using Lemma 3.2, we obtain

$$\begin{aligned}\bar{y}(s)&=\Big(\bar{f}(s)+\sum_{r=0}^{n-1}s^{\alpha-r-1}c_r\Big)\sum_{k=0}^{\infty}(-b)^k s^{-\alpha k-\alpha}\\&=\bar{f}(s)\sum_{k=0}^{\infty}(-b)^k s^{-\alpha k-\alpha}+\sum_{r=0}^{n-1}c_r\sum_{k=0}^{\infty}(-b)^k s^{-\alpha k-r-1}.\end{aligned} \tag{3.19}$$

If we take the inverse Laplace transform of $\bar{y}(s)$ by using Theorem 2.4, equation (2.19), and Definition 2.12, we get

$$\mathcal{L}^{-1}\{\bar{y}(s)\}=y(t)=f(t)*t^{\alpha-1}E_{\alpha,\alpha}(-bt^{\alpha})+\sum_{r=0}^{n-1}c_r t^r E_{\alpha,r+1}(-bt^{\alpha}). \tag{3.20}$$

Then, equation (3.15) directly follows from equation (3.20) by using the definition of convolution (2.3). $\square$

Theorem 3.6. *If $a, b \in \mathbb{R}$, $b \neq 0$, $\mu \in \mathbb{C}$, $n, m \in \mathbb{N}$, $n \geqslant m \geqslant \operatorname{Re}(\mu) > m-1 \geqslant 0$ and $|bs^{-\mu}/(s^{n-\mu}+a)| < 1$, then the solution of the fractional differential equation*

$$D^n y(t) + aD^\mu y(t) + by(t) = 0 \tag{3.21}$$

with the initial conditions $y^{(r)}(0) = c_r$ for $r = 0, 1, 2, \ldots, n-1$ is

$$\begin{aligned} y(t) = {} & \sum_{r=0}^{n-1} c_r \sum_{k=0}^{\infty} (-b)^k t^{nk+r} E^{k+1}_{n-\mu, nk+r+1}(-at^{n-\mu}) \\ & + a \sum_{j=0}^{m-1} c_j \sum_{k=0}^{\infty} (-b)^k t^{nk+n-\mu+j} E^{k+1}_{n-\mu, nk+n-\mu+j+1}(-at^{n-\mu}). \end{aligned} \tag{3.22}$$

P r o o f. Applying the Laplace operator to the differential equation (3.21), we get

$$\mathcal{L}\{D^n y(t) + aD^\mu y(t) + by(t)\} = 0. \tag{3.23}$$

By the equations (2.10) and (2.12), this implies that we have

$$s^n \bar{y}(s) - \sum_{r=0}^{n-1} c_r s^{n-r-1} + as^\mu \bar{y}(s) - a \sum_{j=0}^{m-1} c_j s^{\mu-j-1} + b\bar{y}(s) = 0, \tag{3.24}$$

where $c_r = D^r y(0)$ for $r = 0, 1, 2, \ldots, n-1$, $c_j = D^j y(0)$ for $j = 0, 1, 2, \ldots, m-1$, and $\bar{y}(s) = \mathcal{L}\{y(t)\}$. Therefore, by solving this equation for $\bar{y}(s)$, we get

$$\bar{y}(s) = \frac{\sum_{r=0}^{n-1} c_r s^{n-r-1} + a\sum_{j=0}^{m-1} c_j s^{\mu-j-1}}{s^n + as^\mu + b}. \tag{3.25}$$

The rest of the proof is similar to the proof of Theorem 3.4. □

Theorem 3.7. *If $a, b \in \mathbb{R}$, $\alpha, \mu \in \mathbb{C}$, $m, n \in \mathbb{N}$, $n \geqslant \operatorname{Re}(\alpha) > n-1 \geqslant m \geqslant \operatorname{Re}(\mu) > m-1 \geqslant 0$, and $|bs^{-\mu}/(s^{\alpha-\mu}+a)| < 1$, then the solution of the fractional differential equation*

$$D^\alpha_{0,t} y(t) + aD^\mu_{0,t} y(t) + by(t) = f(t) \tag{3.26}$$

with the initial conditions $D^{\alpha-r-1}_{0,t} y(0) = c_r$ for $r = 0, 1, 2, \ldots, n-1$ and $D^{\mu-j-1}_{0,t} y(0) = b_j$ for $j = 0, 1, 2, \ldots, m-1$ is

$$\begin{aligned} y(t) = {} & \int_0^t f(x) \sum_{k=0}^{\infty} (-b)^k (t-x)^{\alpha k+\alpha-1} E^{k+1}_{\alpha-\mu, \alpha k+\alpha}(-a(t-x)^{\alpha-\mu}) \, \mathrm{d}x \\ & + \sum_{r=0}^{n-1} c_r \sum_{k=0}^{\infty} (-b)^k t^{\alpha k+\alpha-r-1} E^{k+1}_{\alpha-\mu, \alpha k+\alpha-r}(-at^{\alpha-\mu}) \\ & + a \sum_{j=0}^{m-1} b_j \sum_{k=0}^{\infty} (-b)^k t^{\alpha k+\alpha-j-1} E^{k+1}_{\alpha-\mu, \alpha k+\alpha-j}(-at^{\alpha-\mu}). \end{aligned} \tag{3.27}$$

P r o o f. Applying the Laplace operator to the fractional differential equation (3.26), we obtain

$$\mathcal{L}\{D_{0,t}^{\alpha}y(t)+aD_{0,t}^{\mu}y(t)+by(t)\}=\mathcal{L}\{f(t)\}. \tag{3.28}$$

By using equation (2.11), we have

$$s^{\alpha}\bar{y}(s)-\sum_{r=0}^{n-1}c_r s^r+as^{\mu}\bar{y}(s)-a\sum_{j=0}^{m-1}b_j s^j+b\bar{y}(s)=\bar{f}(s), \tag{3.29}$$

where $D_{0,t}^{\alpha-r-1}y(0)=c_r$ if $r=0,1,\ldots,n-1$, $D_{0,t}^{\mu-j-1}y(0)=b_j$ if $j=0,1,\ldots,m-1$, $\bar{y}(s)=\mathcal{L}\{y(t)\}$ and $\bar{f}(s)=\mathcal{L}\{f(t)\}$. Then, by solving equation (3.29) for $\bar{y}(s)$, we get

$$\bar{y}(s)=\frac{\bar{f}(s)+\sum_{r=0}^{n-1}c_r s^r+a\sum_{j=0}^{m-1}b_j s^j}{s^{\alpha}+s^{\mu}+b}. \tag{3.30}$$

Now, by using Lemma 3.1, we find

$$\begin{aligned}\bar{y}(s)&=\Bigl(\bar{f}(s)+\sum_{r=0}^{n-1}s^r c_r+a\sum_{j=0}^{m-1}s^j b_j\Bigr)\sum_{k=0}^{\infty}(-b)^k s^{-\alpha k-\alpha}(1+as^{\mu-\alpha})^{-k-1}\\&=\bar{f}(s)\sum_{k=0}^{\infty}(-b)^k s^{-\alpha k-\alpha}(1+as^{\mu-\alpha})^{-k-1}\\&\quad+\sum_{r=0}^{n-1}c_r\sum_{k=0}^{\infty}(-b)^k s^{-\alpha k-\alpha+r}(1+as^{\mu-\alpha})^{-k-1}\\&\quad+a\sum_{j=0}^{m-1}b_j\sum_{k=0}^{\infty}(-b)^k s^{-\alpha k-\alpha+j}(1+as^{\mu-\alpha})^{-k-1}.\end{aligned} \tag{3.31}$$

If we take the inverse Laplace transform of $\bar{y}(s)$ by using Theorem 2.4 and equation (2.18), we get

$$\begin{aligned}y(t)&=f(t)*\sum_{k=0}^{\infty}(-b)^k t^{\alpha k+\alpha-1}E_{\alpha-\mu,\alpha k+\alpha}^{k+1}(-at^{\alpha-\mu})\\&\quad+\sum_{r=0}^{n-1}c_r\sum_{k=0}^{\infty}(-b)^k t^{\alpha k+\alpha-r-1}E_{\alpha-\mu,\alpha k+\alpha-r}^{k+1}(-at^{\alpha-\mu})\\&\quad+a\sum_{j=0}^{m-1}b_j\sum_{k=0}^{\infty}(-b)^k t^{\alpha k+\alpha-j-1}E_{\alpha-\mu,\alpha k+\alpha-j}^{k+1}(-at^{\alpha-\mu}).\end{aligned} \tag{3.32}$$

Then, the equation (3.27) directly follows from the equation (3.32) by using the definition of convolution (2.3). □

Corollary 3.8. *If $a,b\in\mathbb{R}$, $\alpha,\mu\in\mathbb{C}$, $m,n\in\mathbb{N}$, $n\geqslant \operatorname{Re}(\alpha)\geqslant n-1\geqslant m\geqslant \operatorname{Re}(\mu)>m-1\geqslant 0$ and $|bs^{-\mu}/(s^{\alpha-\mu}+a)|<1$, then the solution of the fractional differential equation*

$$D^{\alpha}_{0,t}y(t)+aD^{\mu}_{0,t}y(t)+by(t)=0 \tag{3.33}$$

with the initial conditions $D^{\alpha-r-1}_{0,t}y(0)=c_r$ for $r=0,1,2,\ldots,n-1$ and $D^{\mu-j-1}_{0,t}y(0)=b_j$ for $j=0,1,2,\ldots,m-1$ is

$$\begin{aligned} y(t)={}&\sum_{r=0}^{n-1}c_r\sum_{k=0}^{\infty}(-b)^k t^{\alpha k+\alpha-r-1}E^{k+1}_{\alpha-\mu,\alpha k+\alpha-r}(-at^{\alpha-\mu})\\ &+a\sum_{j=0}^{m-1}b_j\sum_{k=0}^{\infty}(-b)^k t^{\alpha k+\alpha-j-1}E^{k+1}_{\alpha-\mu,\alpha k+\alpha-j}(-at^{\alpha-\mu}).\end{aligned} \tag{3.34}$$

Proof. Equation (3.34) can be easily obtained by substituting $f(t)=0$ in Theorem 3.7. □

Theorem 3.9. *If $a,b\in\mathbb{R}$, $\alpha,\mu\in\mathbb{C}$, $n\in\mathbb{N}$, $n\geqslant\operatorname{Re}(\alpha)>n-1\geqslant 0$ and $|bs^{-\alpha}|<1$, then the solution of the fractional differential equation*

$$D^{\alpha}_{0,t}y(t)+by(t)=f(t) \tag{3.35}$$

with the initial conditions $D^{\alpha-r-1}_{0,t}y(0)=c_r$ for $r=0,1,2,\ldots,n-1$ is

$$\begin{aligned} y(t)={}&\int_0^t f(x)\sum_{k=0}^{\infty}(-b)^k(t-x)^{\alpha k+\alpha-1}E^{k+1}_{\alpha-\mu,\alpha k+\alpha}(0)\,\mathrm{d}x\\ &+\sum_{r=0}^{n-1}c_r\sum_{k=0}^{\infty}(-b)^k t^{\alpha k+\alpha-r-1}E^{k+1}_{\alpha-\mu,\alpha k+\alpha-r}(0).\end{aligned} \tag{3.36}$$

Proof. Applying the Laplace operator to the fractional differential equation (3.26), we obtain

$$\mathcal{L}\{D^{\alpha}_{0,t}y(t)+by(t)\}=\mathcal{L}\{f(t)\}. \tag{3.37}$$

By using equation (2.11), we have

$$s^{\alpha}\bar{y}(s)-\sum_{r=0}^{n-1}c_r s^r+b\bar{y}(s)=\bar{f}(s), \tag{3.38}$$

where $D_{0,t}^{\alpha-r-1}y(0) = c_r$ for $r = 0, 1, 2, \ldots, n-1$, $\bar{y}(s) = \mathcal{L}\{y(t)\}$ and $\bar{f}(s) = \mathcal{L}\{f(t)\}$. Then, by solving equation (3.38) for $\bar{y}(s)$, we get

$$\bar{y}(s) = \left[\bar{f}(s) + \sum_{r=0}^{n-1} c_r s^r\right](s^\alpha + b)^{-1}. \tag{3.39}$$

Now, by using Lemma 3.2, we find

$$\begin{aligned} \bar{y}(s) &= \left[\bar{f}(s) + \sum_{r=0}^{n-1} c_r s^r\right] \sum_{k=0}^{\infty} (-b)^k s^{-\alpha k-\alpha} \\ &= \bar{f}(s) \sum_{k=0}^{\infty} (-b)^k s^{-\alpha k-\alpha} + \sum_{r=0}^{n-1} c_r \sum_{k=0}^{\infty} (-b)^k s^{-\alpha k-\alpha+r}. \end{aligned} \tag{3.40}$$

If we take the inverse Laplace transform of $\bar{y}(s)$ by using Theorem 2.4, equation (2.19) and Definition 2.12, we get

$$y(t) = f(t) * t^{\alpha-1} E_{\alpha,\alpha}(-bt^\alpha) + \sum_{r=0}^{n-1} c_r t^{\alpha-r-1} E_{\alpha,\alpha+r}(-bt^\alpha). \tag{3.41}$$

Then, the equation (3.36) directly follows from the equation (3.41) by using the definition of convolution (2.3). $\square$

Theorem 3.10. *If $a, b \in \mathbb{R}$, $\mu \in \mathbb{C}$, $m, n \in \mathbb{N}$, $n \geqslant m \geqslant \mathrm{Re}(\mu) > m-1 \geqslant 0$ and $|bs^{-\mu}/(s^{n-\mu} + a)| < 1$, then the solution of the fractional differential equation*

$$D^n y(t) + a D_{0,t}^\mu y(t) + by(t) = 0 \tag{3.42}$$

with the initial conditions $y^{(n-r-1)}(0) = c_r$ for $r = 0, 1, 2, \ldots, n-1$ and $D_{0,t}^{\mu-j-1} y(0) = b_j$ for $j = 0, 1, 2, \ldots, m-1$ is

$$\begin{aligned} y(t) = &\sum_{r=0}^{n-1} c_r \sum_{k=0}^{\infty} (-b)^k t^{nk+n-r-1} E_{n-\mu,nk+n-r}^{k+1}(-at^{n-\mu}) \\ &+ a \sum_{j=0}^{m-1} b_j \sum_{k=0}^{\infty} (-b)^k t^{nk+n-j-1} E_{n-\mu,nk+n-j}^{k+1}(-at^{n-\mu}). \end{aligned} \tag{3.43}$$

Proof. Applying the Laplace operator to the differential equation (3.42), we get

$$\mathcal{L}\{D^n y(t) + a D_{0,t}^\mu y(t) + by(t)\} = 0. \tag{3.44}$$

By the equations (2.10) and (2.12), this implies that we have

$$s^n\bar{y}(s) - \sum_{r=0}^{n-1} s^r c_r + as^\mu \bar{y}(s) - \sum_{j=0}^{m-1} as^j b_j + b\bar{y}(s) = 0, \tag{3.45}$$

where $D_{0,t}^{\alpha-r-1}y(0) = c_r$ for $r = 0, 1, \ldots, n-1$, $D_{0,t}^{\mu-j-1}y(0) = b_j$ for $j = 0, 1, \ldots, m-1$ and $\bar{y}(s) = \mathcal{L}\{y(t)\}$. Therefore, we get

$$\bar{y}(s) = \frac{\sum_{r=0}^{n-1} c_r s^r + a\sum_{j=0}^{m-1} b_j s^j}{s^n + s^\mu + b}. \tag{3.46}$$

The rest of the proof is similar to the proof of Theorem 3.7. □

4. Illustrative examples

As applications of the theorems that we give, we have the following results. By the first two examples, it could be seen how the solution of the fractional differential equation differs for the different interval values that the fractional order derivative part of the differential equation belongs to.

E x a m p l e 4.1. If $1 \geqslant \mathrm{Re}(\mu) > 0$, then the solution for the vibration equation with fractional damping with one degree of freedom

$$\widetilde{m}D^2y(t) + \tilde{q}D^\mu y(t) + \tilde{k}y(t) = f(t), \tag{4.1}$$

with the initial conditions $y(0) = c_0$ and $y'(0) = c_1$, is

$$\begin{aligned} y(t) = {} & \int_0^t f(x)\sum_{k=0}^{\infty}(-b)^k(t-x)^{2k+1}E_{2-\mu,2k+2}^{k+1}(-a(t-x)^{2-\mu})\,\mathrm{d}x \\ & + c_0\sum_{k=0}^{\infty}(-b)^k t^{2k}E_{2-\mu,2k+1}^{k+1}(-at^{2-\mu}) \\ & + c_1\sum_{k=0}^{\infty}(-b)^k t^{2k+1}E_{2-\mu,2k+2}^{k+1}(-at^{2-\mu}) \\ & + ac_0\sum_{k=0}^{\infty}(-b)^k t^{2k+2-\mu}E_{2-\mu,2k+3-\mu}^{k+1}(-at^{2-\mu}). \end{aligned} \tag{4.2}$$

P r o o f. If we divide both sides of equation (4.1) by $\widetilde{m}$, we get the equation

$$D^2y(t) + \frac{\tilde{q}}{\widetilde{m}}D^\mu y(t) + \frac{\tilde{k}}{\widetilde{m}}y(t) = f(t). \tag{4.3}$$

Now, if we make choices of $\alpha = 2$, $a = \tilde{q}/\widetilde{m}$ and $b = \tilde{k}/\widetilde{m}$ in equation (3.6), we obtain equation (4.3). So, equation (4.2) directly follows from Theorem 3.3. □

R e m a r k 4.2. If $1 \geqslant \operatorname{Re}(\mu) > 0$, then the solution for the fractional differential equation

$$\widetilde{m} D^2 y(t) + \tilde{q} D^\mu y(t) + \tilde{k} y(t) = 0, \tag{4.4}$$

with the initial conditions $y(0) = c_0 \neq 0$ and $y'(0) = c_1 \neq 0$, is

$$\begin{aligned} y(t) = {} & c_0 \sum_{k=0}^{\infty} \Big(-\frac{\tilde{k}}{\widetilde{m}}\Big)^k t^{2k} E^{k+1}_{2-\mu,2k+1}\Big(-\frac{\tilde{q}}{\widetilde{m}} t^{2-\mu}\Big) \\ & + c_1 \sum_{k=0}^{\infty} \Big(-\frac{\tilde{k}}{\widetilde{m}}\Big)^k t^{2k+1} E^{k+1}_{2-\mu,2k+2}\Big(-\frac{\tilde{q}}{\widetilde{m}} t^{2-\mu}\Big) \\ & + \frac{\tilde{q}}{\widetilde{m}} c_0 \sum_{k=0}^{\infty} \Big(-\frac{\tilde{k}}{\widetilde{m}}\Big)^k t^{2k+2-\mu} E^{k+1}_{2-\mu,2k+3-\mu}\Big(-\frac{\tilde{q}}{\widetilde{m}} t^{2-\mu}\Big). \end{aligned} \tag{4.5}$$

Note that this is the special case of the vibration equation with fractional damping defined in the equation (2.13) with the choice of $1 \geqslant \operatorname{Re}(\mu) > 0$ and $f(t) = 0$.

E x a m p l e 4.3. If $2 \geqslant \operatorname{Re}(\mu) > 1$, then the solution for the vibration equation with fractional damping with one degree of freedom

$$\widetilde{m} D^2 y(t) + \tilde{q} D^\mu y(t) + \tilde{k} y(t) = f(t), \tag{4.6}$$

with the initial conditions $y(0) = c_0$ and $y'(0) = c_1$, is

$$\begin{aligned} y(t) = {} & \int_0^t f(x) \sum_{k=0}^{\infty} \Big(-\frac{\tilde{k}}{\widetilde{m}}\Big)^k (t-x)^{2k+1} E^{k+1}_{2-\mu,2k+2}\Big(-\frac{\tilde{q}}{\widetilde{m}} (t-x)^{2-\mu}\Big) \, \mathrm{d}x \\ & + c_0 \sum_{k=0}^{\infty} \Big(-\frac{\tilde{k}}{\widetilde{m}}\Big)^k t^{2k} E^{k+1}_{2-\mu,2k+1}\Big(-\frac{\tilde{q}}{\widetilde{m}} t^{2-\mu}\Big) \\ & + c_1 \sum_{k=0}^{\infty} \Big(-\frac{\tilde{k}}{\widetilde{m}}\Big)^k t^{2k+1} E^{k+1}_{2-\mu,2k+2}\Big(-\frac{\tilde{q}}{\widetilde{m}} t^{2-\mu}\Big) \\ & + \frac{\tilde{q}}{\widetilde{m}} c_0 \sum_{k=0}^{\infty} \Big(-\frac{\tilde{k}}{\widetilde{m}}\Big)^k t^{2k+2-\mu} E^{k+1}_{2-\mu,2k+2-\mu+1}\Big(-\frac{\tilde{q}}{\widetilde{m}} t^{2-\mu}\Big) \\ & + \frac{\tilde{q}}{\widetilde{m}} c_1 \sum_{k=0}^{\infty} \Big(-\frac{\tilde{k}}{\widetilde{m}}\Big)^k t^{2k+3-\mu} E^{k+1}_{2-\mu,2k+2-\mu+2}\Big(-\frac{\tilde{q}}{\widetilde{m}} t^{2-\mu}\Big). \end{aligned} \tag{4.7}$$

P r o o f. If we divide both sides of equation (4.6) by $\widetilde{m}$, we get the following equation

$$D^2 y(t) + \frac{\tilde{q}}{\widetilde{m}} D^\mu y(t) + \frac{\tilde{k}}{\widetilde{m}} y(t) = f(t). \tag{4.8}$$

Now, if we make choices of $\alpha = 2$, $a = \tilde{q}/\widetilde{m}$ and $b = \tilde{k}/\widetilde{m}$ in equation (3.6), we obtain equation (4.8). So, the equation (4.7) directly follows from Theorem 3.3. $\square$

Remark 4.4. If $2 \geqslant \mathrm{Re}(\mu) > 1$, then the solution for the fractional differential equation,

$$\widetilde{m}D^2 y(t) + \tilde{q}D^{\mu} y(t) + \tilde{k}y(t) = 0, \tag{4.9}$$

with the initial conditions $y(0) = c_0$ and $y'(0) = c_1$, is

$$\begin{aligned} y(t) = {} & c_0 \sum_{k=0}^{\infty} \Big(-\frac{\tilde{k}}{\widetilde{m}}\Big)^k t^{2k} E^{k+1}_{2-\mu,2k+1}\Big(-\frac{\tilde{q}}{\widetilde{m}} t^{2-\mu}\Big) \\ & + c_1 \sum_{k=0}^{\infty} \Big(-\frac{\tilde{k}}{\widetilde{m}}\Big)^k t^{2k+1} E^{k+1}_{2-\mu,2k+2}\Big(-\frac{\tilde{q}}{\widetilde{m}} t^{2-\mu}\Big) \\ & + \frac{\tilde{q}}{\widetilde{m}} c_0 \sum_{k=0}^{\infty} \Big(-\frac{\tilde{k}}{\widetilde{m}}\Big)^k t^{2k+2-\mu} E^{k+1}_{2-\mu,2k+3-\mu}\Big(-\frac{\tilde{q}}{\widetilde{m}} t^{2-\mu}\Big) \\ & + \frac{\tilde{q}}{\widetilde{m}} c_1 \sum_{k=0}^{\infty} \Big(-\frac{\tilde{k}}{\widetilde{m}}\Big)^k t^{2k+3-\mu} E^{k+1}_{2-\mu,2k+4-\mu}\Big(-\frac{\tilde{q}}{\widetilde{m}} t^{2-\mu}\Big). \end{aligned} \tag{4.10}$$

Note that this is the special case of the vibration equation with fractional damping defined in equation (2.13) with the choice of $2 \geqslant \mathrm{Re}(\mu) > 1$ and $f(t) = 0$.

Example 4.5. The solution for the Bagley-Torvik equation,

$$\tilde{a}D^2 y(t) + \tilde{b}D^{3/2} y(t) + \tilde{c}y(t) = f(t), \tag{4.11}$$

with the initial conditions $y(0) = c_0 \neq 0$ and $y'(0) = c_1 \neq 0$, is

$$\begin{aligned} y(t) = {} & \int_0^t f(x) \sum_{k=0}^{\infty} \Big(-\frac{\tilde{c}}{\tilde{a}}\Big)^k (t-x)^{2k+1} E^{k+1}_{2-\mu,2k+2}\Big(-\frac{\tilde{b}}{\tilde{a}} (t-x)^{1/2}\Big)\,\mathrm{d}x \\ & + c_0 \sum_{k=0}^{\infty} \Big(-\frac{\tilde{c}}{\tilde{a}}\Big)^k t^{2k} E^{k+1}_{1/2,2k+1}\Big(-\frac{\tilde{b}}{\tilde{a}} t^{1/2}\Big) \\ & + c_1 \sum_{k=0}^{\infty} \Big(-\frac{\tilde{c}}{\tilde{a}}\Big)^k t^{2k+1} E^{k+1}_{1/2,2k+2}\Big(-\frac{\tilde{b}}{\tilde{a}} t^{1/2}\Big) \\ & + \frac{\tilde{b}}{\tilde{a}} c_0 \sum_{k=0}^{\infty} \Big(-\frac{\tilde{c}}{\tilde{a}}\Big)^k t^{2k+1/2} E^{k+1}_{1/2,2k+3/2}\Big(-\frac{\tilde{b}}{\tilde{a}} t^{1/2}\Big) \\ & + \frac{\tilde{b}}{\tilde{a}} c_1 \sum_{k=0}^{\infty} \Big(-\frac{\tilde{c}}{\tilde{a}}\Big)^k t^{2k+3/2} E^{k+1}_{1/2,2k+5/2}\Big(-\frac{\tilde{b}}{\tilde{a}} t^{1/2}\Big). \end{aligned} \tag{4.12}$$

P r o o f. If we divide both sides of the equation (4.11) by $\tilde{a}$, we get the equation

$$D^2 y(t) + \frac{\tilde{b}}{\tilde{a}} D^{3/2} y(t) + \frac{\tilde{c}}{\tilde{a}} y(t) = f(t). \tag{4.13}$$

Now, equation (4.12) directly follows from Theorem (3.3) by the choices of $\alpha = 2$, $\mu = 3/2$, $a = \tilde{b}/\tilde{a}$, $b = \tilde{c}/\tilde{a}$ and the fact that $m = 2$ for $\mu = 3/2$. □

R e m a r k 4.6. The solution for the fractional differential equation

$$\tilde{a} D^2 y(t) + \tilde{b} D^{3/2} y(t) + \tilde{c} y(t) = 0, \tag{4.14}$$

with the initial conditions $y(0) = c_0 \neq 0$ and $y'(0) = c_1 \neq 0$, is

$$\begin{aligned} y(t) = {} & c_0 \sum_{k=0}^{\infty} \Big(-\frac{\tilde{c}}{\tilde{a}}\Big)^k t^{2k} E^{k+1}_{1/2,2k+1}\Big(-\frac{\tilde{b}}{\tilde{a}} t^{1/2}\Big) \\ & + c_1 \sum_{k=0}^{\infty} \Big(-\frac{\tilde{c}}{\tilde{a}}\Big)^k t^{2k+1} E^{k+1}_{1/2,2k+2}\Big(-\frac{\tilde{b}}{\tilde{a}} t^{1/2}\Big) \\ & + \frac{\tilde{b}}{\tilde{a}} c_0 \sum_{k=0}^{\infty} \Big(-\frac{\tilde{c}}{\tilde{a}}\Big)^k t^{2k+1/2} E^{k+1}_{1/2,2k+3/2}\Big(-\frac{\tilde{b}}{\tilde{a}} t^{1/2}\Big) \\ & + \frac{\tilde{b}}{\tilde{a}} c_1 \sum_{k=0}^{\infty} \Big(-\frac{\tilde{c}}{\tilde{a}}\Big)^k t^{2k+3/2} E^{k+1}_{1/2,2k+5/2}\Big(-\frac{\tilde{b}}{\tilde{a}} t^{1/2}\Big). \end{aligned} \tag{4.15}$$

Note that this is the special case of Bagley-Torvik equation defined in (2.14) with the choice of $f(t) = 0$.

For the next example, we choose $f(t) = t^{\varrho-1}$ in Theorem 3.3 and obtain the solution for a particular case of the fractional differential equation (3.6).

E x a m p l e 4.7. The solution for the fractional differential equation

$$D^\alpha y(t) + a D^\mu y(t) + b y(t) = t^{\varrho-1} \tag{4.16}$$

with the initial conditions $y^{(r)}(0) = c_r$ for $r = 0, 1, 2, \ldots, m-1$ is

$$\begin{aligned} y(t) = {} & \sum_{k=0}^{\infty} (-b)^k t^{\alpha k+\alpha+\varrho-1} E^{k+1}_{\alpha-\mu,\alpha k+\alpha}(-a t^{\alpha-\mu}) \\ & + \sum_{r=0}^{n-1} c_r \sum_{k=0}^{\infty} (-b)^k t^{\alpha k+r} E^{k+1}_{\alpha-\mu,\alpha k+r+1}(-a t^{\alpha-\mu}) \\ & + a \sum_{j=0}^{m-1} c_j \sum_{k=0}^{\infty} (-b)^k t^{\alpha k+\alpha-\mu+j} E^{k+1}_{\alpha-\mu,\alpha k+\alpha-\mu+j+1}(-a t^{\alpha-\mu}). \end{aligned} \tag{4.17}$$

P r o o f. By taking $f(t)=t^{\varrho-1}$ in Theorem 3.3, we get the following solution for the equation (4.16):

$$y(t)=\int_0^t x^{\varrho-1}\sum_{k=0}^{\infty}(-b)^k(t-x)^{\alpha k+\alpha-1}E^{k+1}_{\alpha-\mu,\alpha k+\alpha}(-a(t-x)^{\alpha-\mu})\,\mathrm{d}x$$
$$+\sum_{r=0}^{n-1}c_r\sum_{k=0}^{\infty}(-b)^k t^{\alpha k+r}E^{k+1}_{\alpha-\mu,\alpha k+r+1}(-at^{\alpha-\mu})$$
$$+a\sum_{j=0}^{m-1}c_j\sum_{k=0}^{\infty}(-b)^k t^{\alpha k+\alpha-\mu+j}E^{k+1}_{\alpha-\mu,\alpha k+\alpha-\mu+j+1}(-at^{\alpha-\mu}). \tag{4.18}$$

Now, by changing the order of the integration and the summation in equation (4.18) and then calculating the integration by using the relation [16], pp. 10, Eq. (2.10),

$$\int_t^x (s-t)^{\mu-1}(x-s)^{\beta-1}E^{\varrho}_{\alpha,\beta}(\lambda(x-s)^{\alpha})\,\mathrm{d}s=\Gamma(\mu)(x-t)^{\beta+\mu-1}E^{\varrho}_{\alpha,\beta}(\lambda(x-t)^{\alpha}), \tag{4.19}$$

we obtain equation (4.17). $\square$

R e m a r k 4.8. If $1\geqslant \operatorname{Re}(\mu)>1/2$, then the solution for the fractional differential equation

$$D^{2\mu}y(t)+aD^{\mu}y(t)+by(t)=t^{\varrho-1} \tag{4.20}$$

with the initial conditions $y(0)=c_0$ and $y'(0)=c_1$ is

$$y(t)=\sum_{k=0}^{\infty}(-b)^k t^{2\mu k+2\mu+\varrho-1}E^{k+1}_{\mu,2\mu k+2\mu}(-at^{\mu})$$
$$+c_0\sum_{k=0}^{\infty}(-b)^k t^{2\mu k}E^{k+1}_{\mu,2\mu k+1}(-at^{\mu})$$
$$+c_1\sum_{k=0}^{\infty}(-b)^k t^{2\mu k+1}E^{k+1}_{\mu,2\mu k+2}(-at^{\mu})$$
$$+ac_0\sum_{k=0}^{\infty}(-b)^k t^{2\mu k+\mu}E^{k+1}_{\mu,2\mu k+\mu+1}(-at^{\mu}). \tag{4.21}$$

Note that this is the special case of the equation (4.16) with the choices of $\alpha=2\mu$ and $1\geqslant\operatorname{Re}(\mu)>1/2$.

As is known, ordinary differential equations and partial differential equations are widely used to solve physics problems. However, due to the distinctive behaviour of materials, classical damping descriptions may fail. The reason for this is that in classical theory, the operators are local ones. One of many well-known techniques that are applied to overcome this kind of difficulties is to use differential equations with fractional time derivatives. For details see [14]. The next example is given for this reason.

Example 4.9. In fractional mechanics, Newton second law of motion could be defined as

$$F = ma = mD^{\nu}v(t), \tag{4.22}$$

where "m" is the mass of the body. If the force "F" is constant, the body in motion moves with a constant fractional acceleration of F/m. Now, we consider the vertical motion of a body in a resisting medium in which there is a resisting force proportional to the fractional velocity. If the body is projected downward with zero initial velocity in a uniform gravitational field, then the equation of motion is given by the fractional differential equation

$$mD^{\nu}v(t) = mg - kv(t), \quad 1 > \nu > 0, \tag{4.23}$$

and the solution for this fractional differential equation is

$$v(t) = \frac{gm}{k}\Big[1 - E_{\nu}\Big(-\frac{k}{m}t^{\nu}\Big)\Big]. \tag{4.24}$$

Proof. Dividing the equation (4.23) by m, we get

$$D^{\nu}v(t) + \frac{k}{m}v(t) = g. \tag{4.25}$$

Choosing $f(t) = g$, $b = km^{-1}$, $\alpha = \nu$ and $y(0) = c_0 = 0$ in Theorem 3.5 and using the fact that $n = 1$ for $1 > \nu > 0$, we have

$$\begin{aligned} v(t) &= \int_0^t g(t-x)^{\nu-1}E_{\nu,\nu}\Big(-\frac{k}{m}(t-x)^{\nu}\Big)\,\mathrm{d}x \\ &= g\int_0^t \sum_{n=0}^{\infty}\Big(-\frac{k}{m}\Big)^n \frac{(t-x)^{\nu n+\nu-1}}{\Gamma(\nu n+\nu)}\,\mathrm{d}x. \end{aligned} \tag{4.26}$$

Then, by making the change of variable $t-x=u$ and changing the order of integration under the condition of absolute convergence, we obtain

$$v(t) = g\sum_{n=0}^{\infty}\Big(-\frac{k}{m}\Big)^n \frac{1}{\Gamma(\nu n+\nu)}\int_0^t u^{\nu n+\nu-1}\,\mathrm{d}u \tag{4.27}$$
$$= g\sum_{n=0}^{\infty}\Big(-\frac{k}{m}\Big)^n \frac{t^{\nu n+\nu}}{\Gamma(\nu n+\nu+1)}$$
$$= -\frac{gm}{k}\Big[\sum_{n=0}^{\infty}\Big(-\frac{k}{m}\Big)^{n+1}\frac{t^{\nu n+\nu}}{\Gamma(\nu n+\nu+1)}\Big]$$
$$= \frac{gm}{k}\Big[1-\sum_{n=-1}^{\infty}\Big(-\frac{k}{m}\Big)^{n+1}\frac{t^{\nu n+\nu}}{\Gamma(\nu n+\nu+1)}\Big]$$
$$= \frac{gm}{k}\Big[1-\sum_{n=0}^{\infty}\Big(-\frac{k}{m}\Big)^{n}\frac{t^{\nu n}}{\Gamma(\nu n+1)}\Big].$$

Finally, we can get the equation (4.24) by using Definition 2.13. □

The fractional differential equation (4.23) could also be solved in [3] by using another method.

By the following examples, we give solutions for some fractional differential equations that contain the Riemann-Liouville fractional derivative.

Example 4.10. The solution for the fractional differential equation

$$D^{\alpha}_{0,t}y(t) + aD^{\mu}_{0,t}y(t) + by(t) = t^{\varrho-1} \tag{4.28}$$

with the initial conditions $D^{\alpha-r-1}_{0,t}y(0) = c_r$ for $r = 0,1,2,\ldots,n-1$ and $D^{\mu-j-1}_{0,t}y(0) = b_j$ for $j = 0,1,2,\ldots,m-1$, is

$$y(t) = \sum_{k=0}^{\infty}(-b)^k t^{\alpha k+\alpha+\varrho-1}E^{k+1}_{\alpha-\mu,\alpha k+\alpha}(-at^{\alpha-\mu}) \tag{4.29}$$
$$+\sum_{r=0}^{n-1}c_r\sum_{k=0}^{\infty}(-b)^k t^{\alpha k+\alpha-r-1}E^{k+1}_{\alpha-\mu,\alpha k+\alpha-r}(-at^{\alpha-\mu})$$
$$+a\sum_{j=0}^{m-1}b_j\sum_{k=0}^{\infty}(-b)^k t^{\alpha k+\alpha-j-1}E^{k+1}_{\alpha-\mu,\alpha k+\alpha-j}(-at^{\alpha-\mu}).$$

P r o o f. By taking $f(t)=t^{\varrho-1}$ in Theorem 3.7, we get the solution for the equation (4.28)

$$y(t)=\int_0^t x^{\varrho-1}\sum_{k=0}^{\infty}(-b)^k(t-x)^{\alpha k+\alpha-1}E^{k+1}_{\alpha-\mu,\alpha k+\alpha}(-a(t-x)^{\alpha-\mu})\,\mathrm{d}x \\ +\sum_{r=0}^{n-1}c_r\sum_{k=0}^{\infty}(-b)^k t^{\alpha k+\alpha-r-1}E^{k+1}_{\alpha-\mu,\alpha k+\alpha-r}(-at^{\alpha-\mu}) \\ +a\sum_{j=0}^{m-1}b_j\sum_{k=0}^{\infty}(-b)^k t^{\alpha k+\alpha-j-1}E^{k+1}_{\alpha-\mu,\alpha k+\alpha-j}(-at^{\alpha-\mu}). \tag{4.30}$$

Now, by changing the order of the integration and the summation in equation (4.30) and then using equation (4.19), we can directly get the equation (4.29). □

R e m a r k 4.11. If $1\geqslant \mathrm{Re}(\mu)>1/2$, then the solution for the fractional differential equation

$$D^{2\mu}_{0,t}y(t)+aD^{\mu}_{0,t}y(t)+by(t)=t^{\varrho-1} \tag{4.31}$$

with the initial conditions $D^{\alpha-1}_{0,t}y(0)=c_0$, $D^{\alpha-2}_{0,t}y(0)=c_1$ and $D^{\mu-1}_{0,t}y(0)=b_0$, is

$$y(t)=\sum_{k=0}^{\infty}(-b)^k t^{2\mu k+2\mu+\varrho-1}E^{k+1}_{\mu,2\mu k+2\mu}(-at^{\mu}) \\ +c_0\sum_{k=0}^{\infty}(-b)^k t^{2\mu k+2\mu-1}E^{k+1}_{\mu,2\mu k+2\mu}(-at^{\mu}) \\ +c_1\sum_{k=0}^{\infty}(-b)^k t^{2\mu k+2\mu-2}E^{k+1}_{\mu,2\mu k+2\mu-1}(-at^{\mu}) \\ +ab_0\sum_{k=0}^{\infty}(-b)^k t^{2\mu k+2\mu-1}E^{k+1}_{\mu,2\mu k+2\mu}(-at^{\mu}). \tag{4.32}$$

Note that this is the special case of the equation (4.28) with the choices of $\alpha=2\mu$ and $1\geqslant \mathrm{Re}(\mu)>1/2$.

E x a m p l e 4.12. The solution for the fractional differential equation

$$D^{7/2}_{0,t}y(t)+D^{5/2}_{0,t}y(t)+y(t)=f(t), \tag{4.33}$$

with the initial conditions $D_{0,t}^{5/2-r}y(0) = c_r$ for $r = 0,1,2,3$ and $D_{0,t}^{3/2-j}y(0) = b_j$, for $j = 0,1,2$ is

$$
\begin{aligned}
(4.34) \qquad y(t) = &\int_0^t f(x) \sum_{k=0}^{\infty} (-1)^k (t-x)^{7/2k+5/2} E_{1,7/2k+7/2}^{k+1}(x-t)\,\mathrm{d}x \\
&+ c_0 \sum_{k=0}^{\infty} (-1)^k t^{7/2k+5/2} E_{1,7/2k+7/2}^{k+1}(-t) \\
&+ c_1 \sum_{k=0}^{\infty} (-1)^k t^{7/2k+3/2} E_{1,7/2k+5/2}^{k+1}(-t) \\
&+ c_2 \sum_{k=0}^{\infty} (-1)^k t^{7/2k+1/2} E_{1,7/2k+3/2}^{k+1}(-t) \\
&+ c_3 \sum_{k=0}^{\infty} (-1)^k t^{7/2k-1/2} E_{1,7/2k+1/2}^{k+1}(-t) \\
&+ b_0 \sum_{k=0}^{\infty} (-1)^k t^{7/2k+5/2} E_{1,7/2k+7/2}^{k+1}(-t) \\
&+ b_1 \sum_{k=0}^{\infty} (-1)^k t^{7/2k+3/2} E_{1,7/2k+5/2}^{k+1}(-t) \\
&+ b_2 \sum_{k=0}^{\infty} (-1)^k t^{7/2k+1/2} E_{1,7/2k+3/2}^{k+1}(-t).
\end{aligned}
$$

Proof. By taking $\alpha = 7/2$, $\mu = 5/2$ and $a = b = 1$ in equation (3.26), we get the equation (4.33). Then, the equation (4.34) directly follows from Theorem 3.7. $\square$

5. Conclusion

Laplace transform method is widely used and very effective tool for solving fractional differential equations. Throughout the paper, we use this method to obtain general solutions for various families of fractional differential equations and solutions for some specific members of these families. Solutions for several different fractional differential equations that contain the Caputo sense fractional derivative or the Riemann-Liouville fractional derivative could also be achieved by the theorems that are presented in this paper.

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