A HIGHER ORDER PRESSURE SEGREGATION SCHEME FOR THE TIME-DEPENDENT MAGNETOHYDRODYNAMICS EQUATIONS

Yun-Bo Yang, Yao-Lin Jiang, Qiong-Xiang Kong, Xi'an

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Abstract. A higher order pressure segregation scheme for the time-dependent incompressible magnetohydrodynamics (MHD) equations is presented. This scheme allows us to decouple the MHD system into two sub-problems at each time step. First, a coupled linear elliptic system is solved for the velocity and the magnetic field. And then, a Poisson-Neumann problem is treated for the pressure. The stability is analyzed and the error analysis is accomplished by interpreting this segregated scheme as a higher order time discretization of a perturbed system which approximates the MHD system. The main results are that the convergence for the velocity and the magnetic field are strongly second-order in time while that for the pressure is strongly first-order in time. Some numerical tests are performed to illustrate the theoretical predictions and demonstrate the efficiency of the proposed scheme.

Keywords: magnetohydrodynamics equations; pressure segregation method; higher order scheme; stability; error estimate

MSC 2010: 65N15, 65N30, 65N12

1. Introduction

In this paper, we consider the numerical approximation of the incompressible magnetohydrodynamics (MHD) equations

$$
\begin{aligned}
&\mathbf{u}_t - (Re)^{-1}\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + S\mathbf{b}\times \operatorname{curl}\mathbf{b} = \mathbf{f} && \text{in } \Omega\times[0,T],\\
&\nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{b} = 0 && \text{in } \Omega\times[0,T],\\
&\mathbf{b}_t + (Rm)^{-1}\operatorname{curl}(\operatorname{curl}\mathbf{b}) - \operatorname{curl}(\mathbf{u}\times\mathbf{b}) = \mathbf{g} && \text{in } \Omega\times[0,T],
\end{aligned} \tag{1.1}
$$

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where Ω is an open bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$, $T > 0$ is the given final time, the unknowns are the velocity $\mathbf{u}$, the pressure p and the magnetic field $\mathbf{b}$, the vector-valued functions $\mathbf{f}$ and $\mathbf{g}$ represent the body forces applied to the fluid and the known applied current with $\nabla \cdot \mathbf{g} = 0$, respectively. Moreover, Re, Rm, and S are three positive constants which denote the Reynolds number, the magnetic Reynolds number, and the coupling number, respectively. These constants are defined by $Re := \varrho U_0 L/\mu$, $Rm := \eta\sigma U_0 L$, $S := B_0^2/(\eta\varrho U_0^2)$, where B_0 denotes the characteristic value of the magnetic field, U_0 the characteristic value of the velocity, ϱ the density, η the magnetic permeability, μ the viscosity of the fluid, σ the electric conductivity and L the characteristic length scale. We consider the initial conditions

$$\mathbf{u}(x, 0) = \mathbf{u}_0, \quad \mathbf{b}(x, 0) = \mathbf{b}_0 \quad \text{in } \Omega, \tag{1.2}$$

and assume the no-slip boundary condition for the velocity, and the perfectly conducting boundary condition for the magnetic field, namely,

$$\mathbf{u} = 0, \quad \mathbf{b} \cdot \mathbf{n} = 0, \quad \operatorname{curl} \mathbf{b} \times \mathbf{n} = 0 \quad \text{on } \partial\Omega \times [0, T], \tag{1.3}$$

where $\mathbf{n}$ denotes the unit outward normal vector on $\partial\Omega$. It is necessary to require that $\mathbf{u}_0$ and $\mathbf{b}_0$ satisfy the compatibility conditions $\nabla \cdot \mathbf{u}_0 = 0$ and $\nabla \cdot \mathbf{b}_0 = 0$, respectively.

The MHD equations (1.1)–(1.3) model incompressible, resistive and electrically conducting fluids under electromagnetic fields. It can be seen that testing the first equation and the third equation of (1.1) by $\mathbf{u}$ and $S\mathbf{b}$, respectively, and adding the resulting equations yields the energy identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\mathbf{u}\|^2 + S\|\mathbf{b}\|^2) + \frac{1}{Re}\|\nabla\mathbf{u}\|^2 + \frac{S}{Rm}\|\operatorname{curl}\mathbf{b}\|^2 = (\mathbf{f}, \mathbf{u}) + S(\mathbf{g}, \mathbf{b}) \quad \forall\, t > 0.$$

About the regularities of the weak solutions, we can see [28].

In the last decades, more and more attention has been attracted to the numerical methods of the incompressible MHD equations. We can refer to [11] for a review of the numerical methods of the MHD equations. For the stationary MHD equations, various numerical approximations have been proposed, mostly concentrated on the stabilized finite element method [10], [18], [27]. For the numerical methods of the time-dependent MHD equations, we can refer to [3], [4], [13], [21], [26], [25], [32], [34], [38], [39] for more details. About the long-term dissipativity of time-stepping schemes for the time-dependent MHD equations, we can see [3], [32]. Recently, Ravindran [25] proposed and analyzed a fully implicit, linearly extrapolated second order backward difference time stepping scheme for solving the time dependent non-homogeneous MHD system. Roughly speaking, the MHD system includes not only the incompressibility and strong nonlinearity, but also the coupling between the Navier-Stokes equations and the Maxwell's equations, which causes that solving the numerical solutions of the MHD equations becomes a very difficult task. If we solve the system (1.1)–(1.3) directly, even by using the time-stepping extrapolation methods [3], [32], it means that we need to find the unknown variables $\mathbf{u}$, p and $\mathbf{b}$ simultaneously, and a large discrete algebraic system is formed. Thus, it is expensive to find the numerical solutions of such a large coupled system directly by the standard Galerkin methods. For dealing with these difficulties, in the 1960s, Chorin [9] and Temam [33] proposed the origin of the projection method for the Navier-Stokes problems, which is a two-step scheme, solving firstly an intermediate velocity via a linear elliptic problem and secondly a velocity-pressure pair via a solenoidal (divergence-free) L^2-projection problem. For the variants of the projection methods, we can refer to [16], [15], [17], [14], [22], [31], [36]. However, there are also some drawbacks of such projection method, for example, the end-of-step velocity does not satisfy the exact boundary conditions and the discrete pressure satisfies an "artificial" Neumann boundary condition. To avoid using artificial boundary conditions of pressure type, some fractional-step schemes for the Navier-Stokes problems were introduced and studied in [7], [6]. It is a two-step scheme in which the incompressibility and the nonlinearities of the Navier-Stokes problems are split into different steps, and it allows the enforcement of the original boundary conditions in all sub-steps. To the author's best knowledge, inspired by these projection methods, for the time-dependent MHD equations (1.1)–(1.3), Prohl [23] proposed a projection scheme. Moreover, it was proved that this scheme provides the weakly order $\frac{1}{2}$ approximations of the velocity and the magnetic field in $\mathbf{H}^1(\Omega)$. Badia et al. [5] proposed several very interesting splitting procedures based on double projection steps. Some unconditionally energy stable numerical schemes based on the standard and rotational pressure-correction schemes were proposed by Choi and Shen [8]. An [2] proposed a new linearized projection scheme, which is unconditionally stable, and it was proved that it provides the weakly first-order approximations of the velocity and the magnetic field in $\mathbf{H}^1(\Omega)$, the strongly first-order approximations of the velocity and the magnetic field in $\mathbf{L}^2(\Omega)$, and the weakly first-order approximation of the pressure in $L^2(\Omega)$ under some additional regularity assumptions.

On the contrary, inspired by the higher order projection methods for the Navier-Stokes equations proposed by Shen [30] and for the natural convection problem proposed by Qian and Zhang [24], we propose a higher order pressure segregation scheme for the time-dependent MHD equations in this article. This scheme allows us to decouple the MHD system into two sub-problems at each time step. First, a coupled linear elliptic system is solved for the velocity and the magnetic field. And then, a Poisson-Neumann problem is treated for the pressure. The stability is analyzed and the error analysis is accomplished by interpreting this segregated scheme as a higher order time discretization of a perturbed system which approximates the MHD system. We prove that it provides the strongly second-order approximations of the velocity and the magnetic field in $\mathbf{L}^2(\Omega)$, the strongly order $\frac{3}{2}$ approximations of the velocity and the magnetic field in $\mathbf{H}^1(\Omega)$, while the strongly first-order approximations of the pressure in $H^1(\Omega)$. Numerical results in both the two and three dimensional spaces are presented to illustrate the theoretical predictions and demonstrate the efficiency of the method.

The rest of the paper is organized as follows. Section 2 introduces some notation, preliminary results and assumptions, which will be used throughout this article, and gives out the higher order pressure segregation scheme for the time-dependent MHD equations. Stability of the second-order pressure segregation scheme is presented in Section 3. In Section 4, we establish rigorously the convergence of the proposed scheme and prove that it provides the strongly second-order approximations of the velocity and the magnetic field in $\mathbf{L}^2(\Omega)$, the strongly order $\frac{3}{2}$ approximations of the velocity and the magnetic field in $\mathbf{H}^1(\Omega)$, while the strongly first-order approximations of the pressure in $H^1(\Omega)$. Numerical experiments are shown to confirm the theoretical predictions and demonstrate the efficiency of the method in Section 5. Finally, we conclude the article.

2. Mathematical preliminaries

We now describe some of the notation, definitions and preliminary lemmas which will be used in the analysis. Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be an open, bounded convex polygonal or polyhedral domain with Lipschitz-continuous boundary $\partial\Omega$. We will use $(\cdot,\cdot)$ and $\|\cdot\|$ to denote the scalar product and the norm in $L^2(\Omega)$, respectively. Let $W^{k,p}(\Omega)$ $(k \in \mathbb{N},\ 1 \leqslant p \leqslant \infty)$ denote the standard Sobolev space. The space $H^k(\Omega)$ is the standard Hilbert Sobolev space of order k with norm $\|\cdot\|_{H^k}$. All other norms will be clearly labeled. In addition, the vector spaces and vector functions will be indicated by boldface type letters, e.g., the spaces $\mathbf{H}^k(\Omega)$, $\mathbf{W}^{k,p}(\Omega)$ and $\mathbf{L}^p(\Omega)$ represent the vector Sobolev spaces $H^k(\Omega)^d$, $W^{k,p}(\Omega)^d$ and $L^p(\Omega)^d$, respectively. Let Z be a Banach space, we denote by $L^p(0,T;Z)$ the time-space function space equipped with the norm

$$\|v\|_{L^p(0,T;Z)} = \begin{cases} \left(\displaystyle\int_0^T \|v\|_Z^p \,\mathrm{d}t\right)^{1/p} & \text{if } 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}_{t\in[0,T]} \|v\|_Z & \text{if } p = \infty. \end{cases}$$

For convenience, we often use the abbreviated form $L^p(Z) := L^p(0,T;Z)$. We also introduce the time discrete space $l^p(Z)$ associated with $L^p(Z)$; $l^p(Z)$ is the space of sequences $\omega := \{\omega^n\colon\ n = 1,\dots,N\}$ with the norm $\|\cdot\|_{l^p(Z)}$ defined by

$$\|\omega\|_{l^p(Z)} = \begin{cases} \left(\Delta t \sum\limits_{n=0}^{N-1} \|\omega^{n+1}\|_Z^p\right)^{1/p} & \text{if } 1 \leqslant p < \infty, \\ \max\limits_{0\leqslant n\leqslant N-1} \|\omega^{n+1}\|_Z & \text{if } p = \infty. \end{cases}$$

Hereafter H^k and L^2 denote the vector spaces $\mathbf{H}^k(\Omega)$ and $\mathbf{L}^2(\Omega)$ or the scalar spaces $H^k(\Omega)$ and $L^2(\Omega)$, respectively.

The following Sobolev spaces are introduced by

$$\begin{aligned} &\mathbf{V} = \mathbf{H}_0^1(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega)\colon\ \mathbf{v} = 0 \text{ on } \partial\Omega\}, \\ &\mathbf{X} = \{\mathbf{v} \in \mathbf{H}^1(\Omega)\colon\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}, \\ &M = L_0^2(\Omega) = \left\{q \in L^2(\Omega)\colon\ \int_\Omega q\,\mathrm{d}x = 0\right\}. \end{aligned}$$

The symbols $C, \overline{C}, C_0, C_1, \dots$ are used to denote generic positive constants independent of the time step size Δt.

In addition, we recall that the following two formulas hold:

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\cdot\mathbf{d} = (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = -(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{d}\times\mathbf{c}), \tag{2.1}$$

$$\int_\Omega \operatorname{curl}\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}x = \int_\Omega \mathbf{u}\cdot\operatorname{curl}\mathbf{v}\,\mathrm{d}x + \int_{\partial\Omega} \mathbf{n}\times\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}S. \tag{2.2}$$

Under the above two formulas, the weak formulation of the MHD system (1.1)–(1.3) is derived by: Find $(\mathbf{u}, p, \mathbf{b}) \in (\mathbf{V}, M, \mathbf{X})$ such that for all $(\mathbf{v}, q, \mathbf{w}) \in (\mathbf{V}, M, \mathbf{X})$,

$$\begin{aligned} &(\mathbf{u}_t, \mathbf{v}) + a_1(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - d(p, \mathbf{v}) + S(\mathbf{b}\times\operatorname{curl}\mathbf{b}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \\ &d(q, \mathbf{u}) = 0, \\ &(\mathbf{b}_t, \mathbf{w}) + a_2(\mathbf{b}, \mathbf{w}) - (\mathbf{u}\times\mathbf{b},\ \operatorname{curl}\mathbf{w}) = (\mathbf{g}, \mathbf{w}), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} a_1(\mathbf{u}, \mathbf{v}) &= \frac{1}{Re}\int_\Omega \nabla\mathbf{u}\colon\ \nabla\mathbf{v}\,\mathrm{d}x && \forall\,\mathbf{u}, \mathbf{v} \in \mathbf{V}, \\ a_2(\mathbf{u}, \mathbf{v}) &= \frac{1}{Rm}\int_\Omega (\operatorname{curl}\mathbf{u}\cdot\operatorname{curl}\mathbf{v} + (\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{v}))\,\mathrm{d}x && \forall\,\mathbf{u}, \mathbf{v} \in \mathbf{X}, \\ d(p, \mathbf{v}) &= \int_\Omega p\operatorname{div}\mathbf{v}\,\mathrm{d}x \quad \forall\,\mathbf{v} \in \mathbf{V}, && p \in M, \\ b(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\,\mathrm{d}x + \frac{1}{2}\int_\Omega (\nabla\cdot\mathbf{u})\mathbf{v}\cdot\mathbf{w}\,\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\,\mathrm{d}x - \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mathbf{v}\,\mathrm{d}x && \forall\,\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}. \end{aligned}$$

It is easy to verify that the trilinear form $b(\cdot,\cdot,\cdot)$ is skew-symmetric with respect to its last two arguments, i.e.,

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = -b(\mathbf{u},\mathbf{w},\mathbf{v}) \quad \forall\,\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{V}. \tag{2.4}$$

We will frequently use the algebraic relations

$$\begin{aligned}&2(a-b,a) = |a|^2-|b|^2+|a-b|^2,\ 2(a-b,b) = |a|^2-|b|^2-|a-b|^2,\\ &(a-b,a+b) = |a|^2-|b|^2.\end{aligned} \tag{2.5}$$

The following standard skew-symmetric form for the convection term will be used:

$$B(\mathbf{u},\mathbf{v}) = (\mathbf{u}\cdot\nabla)\mathbf{v}+\frac{1}{2}(\nabla\cdot\mathbf{u})\mathbf{v} \quad \forall\,\mathbf{u},\mathbf{v}\in\mathbf{H}^1(\Omega).$$

We recall the inequality ([12], page 52, Lemma 3.4)

$$\|\mathbf{v}\| \leqslant \overline{C}(\|\operatorname{curl}\mathbf{v}\|+\|\operatorname{div}\mathbf{v}\|) \quad \forall\,\mathbf{v}\in\mathbf{X}, \tag{2.6}$$

and the continuous embeddings [1]

$$\begin{aligned}&H^1(\Omega)\hookrightarrow L^q(\Omega),\ q\in[1,\infty) && \text{if } d=2,\\ &H^1(\Omega)\hookrightarrow L^q(\Omega),\ q\in[2,6] && \text{if } d=3,\\ &H^2(\Omega)\hookrightarrow L^\infty(\Omega), && d=2,3,\\ &H^2(\Omega)\hookrightarrow W^{1,3}(\Omega), && d=2,3,\end{aligned} \tag{2.7}$$

which will be frequently used in the analysis.

Furthermore, the Young's and Poincaré's inequalities as follows will be used frequently

$$\begin{aligned}&ab\leqslant\frac{\varepsilon}{p}a^p+\frac{\varepsilon^{-q/p}}{q}b^q, \quad a,b,p,q,\varepsilon\in\mathbb{R},\ \frac{1}{p}+\frac{1}{q}=1,\ p,q\in(1,\infty),\ \varepsilon>0,\\ &\|\mathbf{v}\|\leqslant C_p\|\nabla\mathbf{v}\| \quad \forall\,\mathbf{v}\in\mathbf{V},\ C_p=C_p(\Omega).\end{aligned}$$

Let $0=t_0<t_1<\ldots<t_N=T$ be a uniform partition of the time interval $[0,T]$ with time step size $\Delta t=T/N$ $(0<\Delta t<1)$ and $t_{n+1}=(n+1)\Delta t$ for $0\leqslant n\leqslant N-1$. The proposed higher order pressure segregation scheme is the following two-step scheme:

Algorithm 2.1 (Higher order pressure segregation scheme). Let p^0 be given, and taking $\mathbf{u}^0 = \mathbf{u}_0$, $\mathbf{b}^0 = \mathbf{b}_0$, we find $\mathbf{u}^{n+1}$, p^{n+1}, $\mathbf{b}^{n+1}$ by the following two-step scheme:

Step 1. Find $\mathbf{u}^{n+1}$ and $\mathbf{b}^{n+1}$ such that

$$(2.8)\quad \begin{aligned}
&\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} - \frac{1}{Re}\Delta \mathbf{u}^{n+1/2} + B(\mathbf{u}^n, \mathbf{u}^{n+1/2}) + S\mathbf{b}^n \\
&\qquad\qquad\qquad \times \operatorname{curl} \mathbf{b}^{n+1/2} + \nabla p^n = \mathbf{f}(t_{n+1/2}) \quad \text{in } \Omega,\\
&\frac{\mathbf{b}^{n+1}-\mathbf{b}^n}{\Delta t} + \frac{1}{Rm} \operatorname{curl}(\operatorname{curl} \mathbf{b}^{n+1/2}) - \operatorname{curl}(\mathbf{u}^{n+1/2} \times \mathbf{b}^n) = \mathbf{g}(t_{n+1/2}) \quad \text{in } \Omega,\\
&\nabla \cdot \mathbf{b}^{n+1} = 0 \quad \text{in } \Omega,\\
&\mathbf{u}^{n+1} = 0,\ \mathbf{b}^{n+1}\cdot \mathbf{n} = 0,\ \operatorname{curl} \mathbf{b}^{n+1} \times \mathbf{n} = 0 \quad \text{on } \partial\Omega,
\end{aligned}$$

where $\mathbf{u}^{n+1/2} = \frac{1}{2}(\mathbf{u}^{n+1} + \mathbf{u}^n)$, $\mathbf{b}^{n+1/2} = \frac{1}{2}(\mathbf{b}^{n+1} + \mathbf{b}^n)$.

Step 2. Given $\mathbf{u}^{n+1}$ from Step 1, find p^{n+1} such that

$$(2.9)\quad \begin{aligned}
&\alpha \Delta t(\Delta p^{n+1} - \Delta p^n) = \nabla \cdot \mathbf{u}^{n+1} \quad \text{in } \Omega,\\
&\nabla(p^{n+1} - p^n) \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega,
\end{aligned}$$

where α is a constant to be determined.

Remark 2.1. Notice that p^0 is not part of the initial data of our problem. Under suitable compatibility and smoothness assumptions on the initial data and forcing terms, this quantity can be computed by solving a Poisson equation

$$\begin{aligned}
&\Delta p^0 = \operatorname{div}(\mathbf{f}_0 - (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 - S\mathbf{b}_0 \times \operatorname{curl} \mathbf{b}_0) \quad \text{in } \Omega,\\
&\nabla p^0 \cdot \mathbf{n} = \Big(\mathbf{f}_0 + \frac{1}{Re}\Delta \mathbf{u}_0 - (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 - S\mathbf{b}_0 \times \operatorname{curl} \mathbf{b}_0\Big) \cdot \mathbf{n} \quad \text{on } \partial\Omega.
\end{aligned}$$

For convenience, we assume p^0 is exact or $p_0 = p^0$ in our analysis, here p_0 denotes the initial value of pressure.

Remark 2.2. The numerical solution of the velocity $\mathbf{u}^{n+1}$ solved by (2.8) may not belong to the divergence-free function space. Then we improve it from (2.9). Since p^n and p^{n+1} are two successive iterative solutions, the difference between Δp^n and Δp^{n+1} tends to zero as $n \to \infty$, therefore we know that $\nabla \cdot \mathbf{u}^{n+1} \to 0$ as $n \to \infty$.

Remark 2.3. Algorithm 2.1 avoids solving an intermediate velocity, which causes that it can save a large amount of computation cost. What's more, the constant α only must satisfy $\alpha > \frac{1}{4}$, this choice is more flexible than the other projection schemes [29].

R e m a r k 2.4. The linear extrapolation method is used in (2.8), we only need to solve a linear problem at each time step, which avoids using the nonlinear iteration. Noticing that the velocity and pressure in (2.8)–(2.9) are decoupled from each other, the space discretizations for the velocity and the pressure can be chosen independently, and they need not satisfy the inf-sup stable condition.

The purpose of this paper is to prove some temporal error estimates for the time-discrete scheme (2.8)–(2.9), and hence we assume that the initial data and the regularities of the solution $(\mathbf{u}, p, \mathbf{b})$ satisfy:

(A1) $\|\mathbf{u}_0\|_{H^2} + \|\mathbf{b}_0\|_{H^2} + \int_0^T (\|\mathbf{f}(t)\| + \|\mathbf{g}(t)\|)\,\mathrm{d}t \leqslant C,$

(A2) $\sup\limits_{t\in[0,T]} (\|\mathbf{u}(t)\|_{H^2} + \|\mathbf{b}(t)\|_{H^2} + \|\mathbf{u}_t(t)\|_{H^1} + \|\mathbf{b}_t(t)\|_{H^1} + \|\nabla p_t(t)\|) \leqslant C,$

(A3) $\int_0^T (\|\mathbf{u}_{tt}(t)\|_{H^2} + \|\mathbf{b}_{tt}(t)\|_{H^2} + \|\nabla p_{tt}(t)\|_{H^1} + \|\mathbf{u}_{ttt}(t)\|_{L^2} + \|\mathbf{b}_{ttt}(t)\|_{L^2})\,\mathrm{d}t \leqslant C.$

3. Stability analysis

Now, we give out the a priori energy estimates of Algorithm 2.1 in the following theorem, which shows that Algorithm 2.1 is conditionally stable.

Theorem 3.1. *Under the assumption* (A1), *let* $(\mathbf{u}^{n+1}, p^{n+1}, \mathbf{b}^{n+1}) \in (\mathbf{V}, M, \mathbf{X}_0)$ *be the solution of Algorithm* 2.1 *if the initial discrete pressure* p^0 *satisfies* $\Delta t\|\nabla p^0\|^2 \leqslant C_0$, *where* C_0 *is a constant. Then, for* $\alpha > \frac14$ *and all* $0 \leqslant l \leqslant N-1$, *there exists a constant* $C_1 = C_1(Re, Rm, S, \mathbf{f}, \mathbf{g}, \mathbf{u}_0, \mathbf{b}_0, C_0, C_p, \alpha, T) > 0$ *such that*

$$(3.1)\quad \Big(1 - \frac{1}{4\alpha}\Big)\|\mathbf{u}^{l+1}\|^2 + S\|\mathbf{b}^{l+1}\|^2 + \Delta t \sum_{n=1}^{l}\Big(\frac{1}{Re}\|\nabla\mathbf{u}^{n+1/2}\|^2 + \frac{S}{Rm}\|\operatorname{curl}\mathbf{b}^{n+1/2}\|^2\Big) \leqslant C_1.$$

P r o o f. Testing the first equation and the second equation of (2.8) by $2\Delta t\mathbf{u}^{n+1/2}$ and $2S\Delta t\mathbf{b}^{n+1/2}$, respectively, and applying the identities (2.5) and the formula (2.2) along with (2.4), one has

$$(3.2)\quad \|\mathbf{u}^{n+1}\|^2 - \|\mathbf{u}^n\|^2 + \frac{2\Delta t}{Re}\|\nabla\mathbf{u}^{n+1/2}\|^2 + 2\Delta t S(\mathbf{b}^n \times \operatorname{curl}\mathbf{b}^{n+1/2}, \mathbf{u}^{n+1/2}) + 2\Delta t(\nabla p^n, \mathbf{u}^{n+1/2}) = 2\Delta t(\mathbf{f}(t_{n+1/2}), \mathbf{u}^{n+1/2}),$$

and

$$
\begin{aligned}
(3.3)\quad & S[\|\mathbf{b}^{n+1}\|^2 - \|\mathbf{b}^n\|^2] + \frac{2S\Delta t}{Rm}\|\mathrm{curl}\,\mathbf{b}^{n+1/2}\|^2 \\
& - 2\Delta t S(\mathbf{u}^{n+1/2} \times \mathbf{b}^n,\ \mathrm{curl}\ \mathbf{b}^{n+1/2}) = 2S\Delta t(\mathbf{g}(t_{n+1/2}), \mathbf{b}^{n+1/2}).
\end{aligned}
$$

Next, taking the sum of (2.9) for two consecutive time-steps, we have

$$
\begin{aligned}
& \alpha\Delta t(\Delta p^{n+1} - \Delta p^{n-1}) = \nabla \cdot (\mathbf{u}^{n+1} + \mathbf{u}^n) \quad \text{in } \Omega, \\
& \nabla(p^{n+1} - p^{n-1}) \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega.
\end{aligned}
$$

Thus, by integration by parts, (2.4) and (2.8), we deduce

$$
\begin{aligned}
(3.4)\qquad & 2\Delta t(\nabla p^n, \mathbf{u}^{n+1/2}) \\
& = \Delta t(\nabla p^n, \mathbf{u}^{n+1} + \mathbf{u}^n) \\
& = -\Delta t(p^n, \nabla \cdot (\mathbf{u}^{n+1} + \mathbf{u}^n)) \quad \text{(by integration by parts)} \\
& = -\alpha\Delta t^2(p^n, \Delta p^{n+1} - \Delta p^{n-1}) \quad \text{(by (2.8))} \\
& = \alpha\Delta t^2(\nabla p^n, \nabla p^{n+1} - \nabla p^{n-1}) \quad \text{(by integration by parts)} \\
& = \frac{\alpha\Delta t^2}{2}[(\|\nabla p^{n+1}\|^2 + \|\nabla p^n\|^2) - (\|\nabla p^n\|^2 + \|\nabla p^{n-1}\|^2) \\
& \qquad - \|\nabla p^{n+1} - \nabla p^n\|^2 + \|\nabla p^n - \nabla p^{n-1}\|^2] \quad \text{(by (2.4))}.
\end{aligned}
$$

Making use of the vector identity

$$
(3.5)\qquad (\mathbf{u} \times \mathrm{curl}\ \mathbf{v}, \mathbf{w}) = (\mathbf{w} \times \mathbf{u},\ \mathrm{curl}\ \mathbf{v}) \quad \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega)
$$

and (3.4), and adding equations (3.2) and (3.3), we obtain

$$
\begin{aligned}
(3.6)\qquad & \|\mathbf{u}^{n+1}\|^2 - \|\mathbf{u}^n\|^2 + \frac{2\Delta t}{Re}\|\nabla \mathbf{u}^{n+1/2}\|^2 \\
& \quad + S[\|\mathbf{b}^{n+1}\|^2 - \|\mathbf{b}^n\|^2] + \frac{2S\Delta t}{Rm}\|\mathrm{curl}\,\mathbf{b}^{n+1/2}\|^2 \\
& \quad + \frac{\alpha\Delta t^2}{2}[(\|\nabla p^{n+1}\|^2 + \|\nabla p^n\|^2) - (\|\nabla p^n\|^2 + \|\nabla p^{n-1}\|^2)] \\
& = \frac{\alpha\Delta t^2}{2}(\|\nabla p^{n+1} - \nabla p^n\|^2 - \|\nabla p^n - \nabla p^{n-1}\|^2) \\
& \quad + 2\Delta t(\mathbf{f}(t_{n+1/2}), \mathbf{u}^{n+1/2}) + 2S\Delta t(\mathbf{g}(t_{n+1/2}), \mathbf{b}^{n+1/2}).
\end{aligned}
$$

For the last two terms on the right-hand side of (3.6), applying the Cauchy-Schwarz, Young's and Poincaré's inequalities and (2.6), we arrive at

$$\begin{aligned}(3.7)\qquad 2\Delta t(\mathbf{f}(t_{n+1}),\mathbf{u}^{n+1}) &\leqslant 2C_p\Delta t\|\mathbf{f}(t_{n+1/2})\|\|\nabla\mathbf{u}^{n+1/2}\|\\ &\leqslant \frac{\Delta t}{Re}\|\nabla\mathbf{u}^{n+1/2}\|^2 + ReC_p^2\Delta t\|\mathbf{f}(t_{n+1/2})\|^2,\\ (3.8)\qquad 2S\Delta t(\mathbf{g}(t_{n+1}),\mathbf{b}^{n+1}) &\leqslant 2S\Delta t\|\mathbf{g}(t_{n+1/2})\|\|\operatorname{curl}\mathbf{b}^{n+1/2}\|\\ &\leqslant 2S\overline{C}\Delta t\|\mathbf{g}(t_{n+1/2})\|\|\mathbf{b}^{n+1/2}\|\\ &\leqslant \frac{S\Delta t}{Rm}\|\operatorname{curl}\mathbf{b}^{n+1/2}\|^2 + CRmS\Delta t\|\mathbf{g}(t_{n+1/2})\|^2.\end{aligned}$$

Taking the sum of (3.6) from $n=1$ to l $(1\leqslant l\leqslant N-1)$ and using (3.7) and (3.8), we have

$$\begin{aligned}(3.9)\qquad &\|\mathbf{u}^{l+1}\|^2 + S\|\mathbf{b}^{l+1}\|^2 + \Delta t\sum_{n=1}^{l}\Big(\frac{1}{Re}\|\nabla\mathbf{u}^{n+1/2}\|^2 + \frac{S}{Rm}\|\operatorname{curl}\mathbf{b}^{n+1/2}\|^2\Big)\\ &\quad + \frac{\alpha\Delta t^2}{2}(\|\nabla p^{l+1}\|^2 + \|\nabla p^l\|^2)\\ &\leqslant \|\mathbf{u}^1\|^2 + S\|\mathbf{b}^1\|^2 + \frac{\alpha\Delta t^2}{2}\|\nabla p^{l+1} - \nabla p^l\|^2 - \frac{\alpha\Delta t^2}{2}\|\nabla p^1 - \nabla p^0\|^2\\ &\quad + \frac{\alpha\Delta t^2}{2}(\|\nabla p^1\|^2 + \|\nabla p^0\|^2) + ReC_p^2\Delta t\sum_{n=1}^{l}\|\mathbf{f}(t_{n+1/2})\|^2\\ &\quad + CRmS\Delta t\sum_{n=1}^{l}\|\mathbf{g}(t_{n+1/2})\|^2.\end{aligned}$$

Choosing $n=l$ in (2.9), and testing (2.9) by $p^{l+1}-p^l$, one has

$$\begin{aligned}\alpha\Delta t\|\nabla p^{l+1} - \nabla p^l\|^2 = (\mathbf{u}^{l+1},\nabla p^{l+1} - \nabla p^l) &\leqslant \|\mathbf{u}^{l+1}\|\|\nabla p^{l+1} - \nabla p^l\|\\ &\leqslant \frac{\alpha\Delta t}{2}\|\nabla p^{l+1} - \nabla p^l\|^2 + \frac{1}{2\alpha\Delta t}\|\mathbf{u}^{l+1}\|^2.\end{aligned}$$

This leads to

$$(3.10)\qquad \alpha\Delta t^2\|\nabla p^{l+1} - \nabla p^l\|^2 \leqslant \frac{1}{\alpha}\|\mathbf{u}^{l+1}\|^2.$$

Thus, we find

$$(3.11)\qquad \frac{\alpha\Delta t^2}{2}\|\nabla p^{l+1} - \nabla p^l\|^2 \leqslant \frac{1}{4\alpha}\|\mathbf{u}^{l+1}\|^2 + \frac{\alpha\Delta t^2}{2}(\|\nabla p^{l+1}\|^2 + \|\nabla p^l\|^2).$$

Noticing that

$$-\frac{\alpha\Delta t^2}{2}\|\nabla p^1-\nabla p^0\|^2 \leqslant \alpha\Delta t^2(\|\nabla p^1\|^2+\|\nabla p^0\|^2), \tag{3.12}$$

and then substituting (3.11) and (3.12) into (3.9), yields that

$$\begin{aligned}\Big(1-\frac{1}{4\alpha}\Big)\|\mathbf{u}^{l+1}\|^2 &+ S\|\mathbf{b}^{l+1}\|^2 \\ &+\Delta t\sum_{n=1}^{l}\Big(\frac{1}{Re}\|\nabla\mathbf{u}^{n+1/2}\|^2+\frac{S}{Rm}\|\mathrm{curl}\,\mathbf{b}^{n+1/2}\|^2\Big)\\ &\leqslant \|\mathbf{u}^1\|^2+S\|\mathbf{b}^1\|^2+\frac{3\alpha\Delta t^2}{2}(\|\nabla p^1\|^2+\|\nabla p^0\|^2)\\ &\quad + ReC_p^2\|\mathbf{f}\|^2_{l^2(L^2)}+CRmS\|\mathbf{g}\|^2_{l^2(L^2)}.\end{aligned} \tag{3.13}$$

To complete the proof of the theorem, we need to bound $\|\mathbf{u}^1\|^2$, $\|\mathbf{b}^1\|^2$, and $\Delta t^2\|\nabla p^1\|^2$. Testing the first equation and the second equation of (2.8) at $n=0$ by $2\Delta t\mathbf{u}^{1/2}$ and $2\Delta t\mathbf{b}^{1/2}$, respectively, and then adding the resulting equations and applying (3.5), we have

$$\begin{aligned}\|\mathbf{u}^1\|^2-\|\mathbf{u}^0\|^2+\frac{2\Delta t}{Re}\|\nabla\mathbf{u}^{1/2}\|^2+S(\|\mathbf{b}^1\|^2-\|\mathbf{b}^0\|^2)+\frac{2S\Delta t}{Rm}\|\mathrm{curl}\,\mathbf{b}^{1/2}\|^2&\\ = -2\Delta t(\nabla p^0,\cdot\mathbf{u}^{1/2})+2\Delta t(\mathbf{f}(t_{1/2}),\mathbf{u}^{1/2})+2S\Delta t(\mathbf{g}(t_{1/2}),\mathbf{b}^{1/2})&\\ \leqslant \frac{\Delta t}{Re}\|\nabla\mathbf{u}^{1/2}\|^2+\frac{S\Delta t}{Rm}\|\mathrm{curl}\,\mathbf{b}^{1/2}\|^2+2ReC_p^2\Delta t\|\nabla p^0\|^2&\\ +2ReC_p^2\Delta t\|\mathbf{f}(t_{1/2})\|^2+CRmS\Delta t\|\mathbf{g}(t_{1/2})\|^2.&\end{aligned} \tag{3.14}$$

Thus, it holds that

$$\begin{aligned}\|\mathbf{u}^1\|^2+S\|\mathbf{b}^1\|^2+\frac{\Delta t}{Re}\|\nabla\mathbf{u}^{1/2}\|^2+\frac{S\Delta t}{Rm}\|\mathrm{curl}\,\mathbf{b}^{1/2}\|^2&\\ \leqslant \|\mathbf{u}^0\|^2+S\|\mathbf{b}^0\|^2+2ReC_p^2\Delta t\|\nabla p^0\|^2&\\ +2ReC_p^2\Delta t\|\mathbf{f}(t_{1/2})\|^2+CRmS\Delta t\|\mathbf{g}(t_{1/2})\|^2.&\end{aligned} \tag{3.15}$$

Under the condition $\Delta t\|\nabla p^0\|^2\leqslant C_0$, we obtain

$$\begin{aligned}\|\mathbf{u}^1\|^2+S\|\mathbf{b}^1\|^2+\frac{\Delta t}{Re}\|\nabla\mathbf{u}^{1/2}\|^2+\frac{S\Delta t}{Rm}\|\mathrm{curl}\,\mathbf{b}^{1/2}\|^2&\\ \leqslant \|\mathbf{u}^0\|^2+S\|\mathbf{b}^0\|^2+2ReC_p^2C_0&\\ +2ReC_p^2\|\mathbf{f}\|^2_{l^2(L^2)}+CRmS\|\mathbf{g}\|^2_{l^2(L^2)}.&\end{aligned} \tag{3.16}$$

Taking $l=0$ for (3.10), we can deduce

$$\Delta t^2\|\nabla p^1\|^2-\Delta t^2\|\nabla p^0\|^2\leqslant \Delta t^2\|\nabla p^1-\nabla p^0\|^2\leqslant\frac{1}{\alpha^2}\|\mathbf{u}^1\|^2. \tag{3.17}$$

Combining (3.16) and (3.17) with (3.13), we obtain the desired estimate and complete the proof. □

4. Error analysis

In this section, we will obtain error estimates for the velocity, the pressure and the magnetic field. For this, let $t_{n+1/2} = (n + \frac{1}{2})\Delta t$. For any function $w(t)$ and any sequence of functions $\{f^n\}_{n=0}^N$ we define

$$\widetilde{w}(t_{n+1/2}) = \frac{1}{2}(w(t_{n+1}) + w(t_n)), \quad f^{n+1/2} = \frac{1}{2}(f^{n+1} + f^n),$$

and the errors at $t = t_n$ $(n = 0, 1, \ldots, N)$ by

$$\mathbf{e}_u^n := \mathbf{u}(t_n) - \mathbf{u}^n, \quad \mathbf{e}_b^n := \mathbf{b}(t_n) - \mathbf{b}^n, \quad e_p^n := p(t_n) - p^n.$$

We can rewrite the first equation and the second equation of (1.1) at $t = t_{n+1/2}$, respectively, by

$$\begin{aligned}\frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)}{\Delta t} &- \frac{1}{Re}\Delta\tilde{\mathbf{u}}(t_{n+1/2}) + (\mathbf{u}(t_n)\cdot\nabla)\tilde{\mathbf{u}}(t_{n+1/2}) + \nabla p(t_n) \\ &+ S\mathbf{b}(t_n) \times \operatorname{curl} \widetilde{\mathbf{b}}(t_{n+1/2}) = \mathbf{f}(t_{n+1/2}) + \mathcal{R}_u^{n+1},\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}\frac{\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n)}{\Delta t} &+ \frac{1}{Rm} \operatorname{curl}(\operatorname{curl} \widetilde{\mathbf{b}}(t_{n+1/2})) \\ &- \operatorname{curl}(\tilde{\mathbf{u}}(t_{n+1/2}) \times \mathbf{b}(t_n)) = \mathbf{g}(t_{n+1/2}) + \mathcal{R}_b^{n+1},\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}\mathcal{R}_u^{n+1} &= \Big[\frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)}{\Delta t} - \mathbf{u}_t(t_{n+1/2})\Big] + \frac{1}{Re}(\Delta\mathbf{u}(t_{n+1/2}) - \Delta\tilde{\mathbf{u}}(t_{n+1/2})) \\ &\quad + [(\mathbf{u}(t_n)\cdot\nabla)\tilde{\mathbf{u}}(t_{n+1/2}) - (\mathbf{u}(t_{n+1/2})\cdot\nabla)\mathbf{u}(t_{n+1/2})] \\ &\quad + [\nabla p(t_n) - \nabla p(t_{n+1/2})] \\ &\quad + [S\mathbf{b}(t_n) \times \operatorname{curl} \widetilde{\mathbf{b}}(t_{n+1/2}) - S\mathbf{b}(t_{n+1/2}) \times \operatorname{curl} \mathbf{b}(t_{n+1/2})] \\ &= \mathcal{R}_{u1}^{n+1} + \mathcal{R}_{u2}^{n+1} + \mathcal{R}_{u3}^{n+1} + \mathcal{R}_{u4}^{n+1} + \mathcal{R}_{u5}^{n+1},\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}_b^{n+1} &= \Big[\frac{\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n)}{\Delta t} - \mathbf{b}_t(t_{n+1/2})\Big] \\ &\quad + \frac{1}{Rm}[\operatorname{curl}(\operatorname{curl} \widetilde{\mathbf{b}}(t_{n+1/2})) - \operatorname{curl}(\operatorname{curl} \mathbf{b}(t_{n+1/2}))] \\ &\quad + \operatorname{curl}(\mathbf{u}(t_{n+1/2}) \times \mathbf{b}(t_{n+1/2}) - \tilde{\mathbf{u}}(t_{n+1/2}) \times \mathbf{b}(t_n)) \\ &= \mathcal{R}_{b1}^{n+1} + \mathcal{R}_{b2}^{n+1} + \mathcal{R}_{b3}^{n+1}.\end{aligned}$$

Subtracting the first equation and the second equation of (2.8) from (4.1) and (4.2), respectively, we arrive at

$$\tag{4.3} \begin{aligned} \frac{\mathbf{e}_u^{n+1}-\mathbf{e}_u^n}{\Delta t} - \frac{1}{Re}\Delta \mathbf{e}_u^{n+1/2} + B(\mathbf{e}_u^n, \tilde{\mathbf{u}}(t_{n+1/2})) + B(\mathbf{u}^n, \mathbf{e}_u^{n+1/2}) + \nabla e_p^n \\ + S\mathbf{e}_b^n \times \operatorname{curl} \widetilde{\mathbf{b}}(t_{n+1/2}) + S\mathbf{b}^n \times \operatorname{curl} \mathbf{e}_b^{n+1/2} = R_u^{n+1}, \end{aligned}$$

and

$$\tag{4.4} \begin{aligned} \frac{\mathbf{e}_b^{n+1}-\mathbf{e}_b^n}{\Delta t} + \frac{1}{Rm} \operatorname{curl}(\operatorname{curl} \mathbf{e}_b^{n+1/2}) - \operatorname{curl}(\mathbf{e}_u^{n+1/2} \times \mathbf{b}^n) \\ - \operatorname{curl}(\tilde{\mathbf{u}}(t_{n+1/2}) \times \mathbf{e}_b^n) = \mathcal{R}_b^{n+1}. \end{aligned}$$

Lemma 4.1. *Under the assumptions* (A1)–(A3), *there exists a positive constant* $C>0$ *such that for all* $0 \leqslant n \leqslant N-1$, *the following estimates hold:*

$$\tag{4.5} \|\mathcal{R}_u^{n+1}\| \leqslant C\Delta t, \ \ \|\mathcal{R}_b^{n+1}\| \leqslant C\Delta t.$$

Proof. Set $\varphi \in C^3(0,T)$. By the Taylor series expansion with integral remainder, we know

$$\begin{aligned} &\frac{\varphi(t_{n+1})-\varphi(t_n)}{\Delta t} - \varphi_t(t_{n+1/2}) \\ &\qquad = \frac{1}{2\Delta t}\Big[\int_{t_{n+1/2}}^{t_{n+1}} (t_{n+1}-t)^2 \varphi_{ttt}(t)\,\mathrm{d}t - \int_{t_{n+1/2}}^{t_n} (t_n-t)^2\varphi_{ttt}(t)\,\mathrm{d}t\Big], \\ &\frac{\varphi(t_{n+1})+\varphi(t_n)}{2} - \varphi(t_{n+1/2}) \\ &\qquad = \frac{1}{2}\int_{t_{n+1/2}}^{t_{n+1}} (t_{n+1}-t)\varphi_{tt}(t)\,\mathrm{d}t + \frac{1}{2}\int_{t_{n+1/2}}^{t_n} (t_n-t)\varphi_{tt}(t)\,\mathrm{d}t, \\ &\varphi(t_{n+1})-\varphi(t_n) = \int_{t_n}^{t_{n+1}} \varphi_t(t)\,\mathrm{d}t, \end{aligned}$$

where φ can be replaced by $\mathbf{u}$, p or $\mathbf{b}$, respectively. Thus, we have

$$\begin{aligned} \|\mathcal{R}_{u1}^{n+1}\| &\leqslant \Big\| \frac{1}{2\Delta t}\int_{t_{n+1/2}}^{t_{n+1}} (t_{n+1}-t)^2 \mathbf{u}_{ttt}\,\mathrm{d}t - \frac{1}{2\Delta t}\int_{t_{n+1/2}}^{t_n} (t_n-t)^2\mathbf{u}_{ttt}\,\mathrm{d}t \Big\| \\ &\leqslant C\Delta t^{3/2}\|\mathbf{u}_{ttt}\|_{L^2(t_n,t_{n+1};L^2)}. \end{aligned}$$

Similarly, $\|\mathcal{R}_{u2}^{n+1}\| \leqslant C\Delta t^{3/2}\|\mathbf{u}_{tt}\|_{L^2(t_n,t_{n+1};H^2)}$. Notice that

$$\begin{aligned}
\mathcal{R}_{u3}^{n+1} &= (\tilde{\mathbf{u}}(t_{n+1/2})\cdot\nabla)\tilde{\mathbf{u}}(t_{n+1/2}) - (\mathbf{u}(t_{n+1/2})\cdot\nabla)\mathbf{u}(t_{n+1/2}) \\
&\quad + (\mathbf{u}(t_n) - \tilde{\mathbf{u}}(t_{n+1/2}))\cdot\nabla)\tilde{\mathbf{u}}(t_{n+1/2}) \\
&= (\tilde{\mathbf{u}}(t_{n+1/2}) - \mathbf{u}(t_{n+1/2}))\cdot\nabla\tilde{\mathbf{u}}(t_{n+1/2}) \\
&\quad + \mathbf{u}(t_{n+1/2})\cdot\nabla(\tilde{\mathbf{u}}(t_{n+1/2}) - \mathbf{u}(t_{n+1/2})) \\
&\quad - \frac{1}{2}(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n))\cdot\nabla\tilde{\mathbf{u}}(t_{n+1/2}), \\
\mathcal{R}_{u4}^{n+1} &= (\nabla p(t_n) - \nabla\tilde{p}(t_{n+1/2})) + (\nabla\tilde{p}(t_{n+1/2}) - \nabla p(t_{n+1/2})) \\
&= -\frac{1}{2}(\nabla p(t_{n+1}) - \nabla p(t_n)) + (\nabla\tilde{p}(t_{n+1/2}) - \nabla p(t_{n+1/2})),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{R}_{u5}^{n+1} &= S\widetilde{\mathbf{b}}(t_{n+1/2})\times\operatorname{curl}\widetilde{\mathbf{b}}(t_{n+1/2}) - S\mathbf{b}(t_{n+1/2})\times\operatorname{curl}\mathbf{b}(t_{n+1/2}) \\
&\quad + S(\mathbf{b}(t_n) - \widetilde{\mathbf{b}}(t_{n+1/2}))\times\operatorname{curl}\widetilde{\mathbf{b}}(t_{n+1/2}) \\
&= S(\widetilde{\mathbf{b}}(t_{n+1/2}) - \mathbf{b}(t_{n+1/2}))\times\operatorname{curl}\widetilde{\mathbf{b}}(t_{n+1/2}) \\
&\quad + S\mathbf{b}(t_{n+1/2})\times\operatorname{curl}(\widetilde{\mathbf{b}}(t_{n+1/2}) - \mathbf{b}(t_{n+1/2})) \\
&\quad - \frac{S}{2}(\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n))\times\operatorname{curl}\widetilde{\mathbf{b}}(t_{n+1/2}).
\end{aligned}$$

By the Hölder inequality and the Sobolev embeddings (2.7), we have

$$\begin{aligned}
\|\mathcal{R}_{u3}^{n+1}\| &\leqslant \|\tilde{\mathbf{u}}(t_{n+1/2}) - \mathbf{u}(t_{n+1/2})\|_{L^6}\|\nabla\tilde{\mathbf{u}}(t_{n+1/2})\|_{L^3} \\
&\quad + \|\mathbf{u}(t_{n+1/2})\|_{L^\infty}\|\nabla(\tilde{\mathbf{u}}(t_{n+1/2}) - \mathbf{u}(t_{n+1/2}))\| \\
&\quad + \frac{1}{2}\|\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)\|_{L^6}\|\nabla\tilde{\mathbf{u}}(t_{n+1/2})\|_{L^3} \\
&\leqslant C\Delta t^{3/2}\|\mathbf{u}_{tt}\|_{L^2(t_n,t_{n+1};H^1)}\|\tilde{\mathbf{u}}(t_{n+1/2})\|_{H^2} \\
&\quad + C\Delta t\|\mathbf{u}_t\|_{l^\infty(H^1)}\|\tilde{\mathbf{u}}(t_{n+1/2})\|_{H^2},
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{R}_{u4}^{n+1}\| &\leqslant \frac{\Delta t}{2}\|\nabla p_t\|_{l^\infty(L^2)} + C\Delta t^{3/2}\|\mathbf{p}_{tt}\|_{L^2(t_n,t_{n+1};H^1)}, \\
\|\mathcal{R}_{u5}^{n+1}\| &\leqslant C\Delta t^{3/2}\|\mathbf{b}_{tt}\|_{L^2(t_n,t_{n+1};H^1)}\|\widetilde{\mathbf{b}}(t_{n+1/2})\|_{H^2} \\
&\quad + C\Delta t\|\mathbf{b}_t\|_{l^\infty(H^1)}\|\widetilde{\mathbf{b}}(t_{n+1/2})\|_{H^2}.
\end{aligned}$$

By the regularity assumptions (A2) and (A3), we obtain $\|\mathcal{R}_u^{n+1}\| \leqslant C\Delta t$.

Next, we bound $\mathcal{R}_b^{n+1}$. With the same techniques as above, we know $\|\mathcal{R}_{b1}^{n+1}\| \leqslant \frac{1}{24}\Delta t^2\|\mathbf{u}_{ttt}\|_{l^\infty(L^2)}$. Using the identity

$$\operatorname{curl}\operatorname{curl}\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \Delta\mathbf{A},$$

we find

$$\mathcal{R}_{b2}^{n+1} = -\frac{1}{Rm}\Delta(\widetilde{\mathbf{b}}(t_{n+1/2}) - \mathbf{b}(t_{n+1/2}))$$

and thus

$$\|\mathcal{R}_{b2}^{n+1}\| \leqslant C\Delta t^{3/2}\|\mathbf{b}_{tt}\|_{L^2(t_n,t_{n+1};H^2)}.$$

Next, we decompose R_{b3}^{n+1} as

$$\begin{aligned}\mathcal{R}_{b3}^{n+1} &= \mathrm{curl}((\mathbf{u}(t_{n+1/2}) - \tilde{\mathbf{u}}(t_{n+1/2})) \times \mathbf{b}(t_{n+1/2})) \\ &\quad + \mathrm{curl}(\tilde{\mathbf{u}}(t_{n+1/2}) \times (\mathbf{b}(t_{n+1/2}) - \widetilde{\mathbf{b}}(t_{n+1/2}))) \\ &\quad + \frac{1}{2}\ \mathrm{curl}(\tilde{\mathbf{u}}(t_{n+1/2}) \times (\mathbf{b}(t_{n+1}) - \mathbf{b}(t_n))) = R_{b31}^{n+1} + R_{b32}^{n+1} + R_{b33}^{n+1}.\end{aligned}$$

Using the formula

$$\mathrm{curl}(\mathbf{u} \times \mathbf{v}) = (\mathrm{div}\,\mathbf{v})\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{u} - (\mathrm{div}\,\mathbf{u})\mathbf{v}, \tag{4.6}$$

we have

$$\begin{aligned}\|\mathcal{R}_{b31}^{n+1}\| &= \| - (\mathbf{u}(t_{n+1/2}) - \tilde{\mathbf{u}}(t_{n+1/2})) \cdot \nabla\mathbf{b}(t_{n+1/2}) \\ &\qquad + (\mathbf{b}(t_{n+1/2}) \cdot \nabla)(\mathbf{u}(t_{n+1/2}) - \tilde{\mathbf{u}}(t_{n+1/2}))\| \\ &\leqslant C\|\mathbf{u}(t_{n+1/2}) - \tilde{\mathbf{u}}(t_{n+1/2})\|_{H^1}\|\mathbf{b}(t_{n+1/2})\|_{H^2} \\ &\leqslant C\Delta t^{3/2}\|\mathbf{u}_{tt}\|_{L^2(t_n,t_{n+1};H^2)}\|\mathbf{b}(t_{n+1/2})\|_{H^2}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\|\mathcal{R}_{b32}^{n+1}\| \leqslant C\Delta t^{3/2}\|\mathbf{b}_{tt}\|_{L^2(t_n,t_{n+1};H^1)}\|\tilde{\mathbf{u}}(t_{n+1/2})\|_{H^2}, \\ &\|\mathcal{R}_{b33}^{n+1}\| \leqslant C\Delta t\|\mathbf{b}_t\|_{l^\infty(H^1)}\|\tilde{\mathbf{u}}(t_{n+1/2})\|_{H^2}.\end{aligned}$$

Thus, we know $\|\mathcal{R}_b^{n+1}\| \leqslant C\Delta t$ and complete the proof. □

Theorem 4.1. *Under the assumptions* (A1)–(A3) *and* $\alpha > \frac{1}{4}$, *there exists a positive constant* $C > 0$ *such that for all* $0 \leqslant n \leqslant N-1$, *the following error estimates hold:*

$$\|\mathbf{e}_u^{n+1}\|^2 + \|\mathbf{e}_b^{n+1}\|^2 + \|\Delta t\nabla e_p^{n+1}\|^2 \leqslant C\Delta t^4,\ \|\nabla\mathbf{e}_u^{n+1}\|^2 + \|\mathbf{e}_b^{n+1}\|_{H^1}^2 \leqslant C\Delta t^3, \tag{4.7}$$

where C *depends on* Re, Rm, S, $\alpha, \mathbf{f}, \mathbf{g}, \mathbf{u}_0$ *and* $\mathbf{b}_0$.

P r o o f. Testing (4.3) by $2\Delta t\mathbf{e}_u^{n+1/2}$ and noticing that $b(\mathbf{u},\mathbf{v},\mathbf{v}) = 0$ for all $\mathbf{u},\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, we deduce

$$\begin{aligned}(4.8)\quad \|\mathbf{e}_u^{n+1}\|^2 - \|\mathbf{e}_u^n\|^2 + \frac{2\Delta t}{Re}\|\nabla \mathbf{e}_u^{n+1/2}\|^2 + \Delta t(\nabla e_p^n, \mathbf{e}_u^{n+1} + \mathbf{e}_u^n) \\ = -2\Delta t b(\mathbf{e}_u^n, \tilde{\mathbf{u}}(t_{n+1/2}), \mathbf{e}_u^{n+1/2}) - 2S\Delta t(\mathbf{e}_b^n \times \text{curl } \widetilde{\mathbf{b}}(t_{n+1/2}), \mathbf{e}_u^{n+1/2}) \\ - 2\Delta t S(\mathbf{b}^n \times \text{curl } \mathbf{e}_b^{n+1/2}, \mathbf{e}_u^{n+1/2}) + 2\Delta t(\mathcal{R}_u^{n+1}, \mathbf{e}_u^{n+1/2}).\end{aligned}$$

Similarly, testing (4.4) by $2S\Delta t\mathbf{e}_b^{n+1/2}$ leads to

$$\begin{aligned}(4.9)\quad S[\|\mathbf{e}_b^{n+1}\|^2 - \|\mathbf{e}_b^n\|^2] + \frac{2S\Delta t}{Rm}\|\text{curl}\, \mathbf{e}_b^{n+1/2}\|^2 \\ = 2S\Delta t(\mathbf{e}_u^{n+1/2} \times \mathbf{b}^n, \text{curl } \mathbf{e}_b^{n+1/2}) \\ + 2S\Delta t(\tilde{\mathbf{u}}(t_{n+1/2}) \times \mathbf{e}_b^n, \text{curl } \mathbf{e}_b^{n+1/2}) + 2S\Delta t(\mathcal{R}_b^{n+1}, \mathbf{e}_b^{n+1/2}).\end{aligned}$$

Adding and subtracting the terms $\mathbf{u}(t_{n+1})$, $p(t_{n+1})$ in (2.9), we find

$$\alpha\Delta t(-\Delta e_p^{n+1} + \Delta e_p^n) + \alpha\Delta t(\Delta p(t_{n+1}) - \Delta p(t_n)) = -\nabla \cdot \mathbf{e}_u^{n+1}.$$

Testing the above equation by $q \in M$, we obtain

$$(4.10)\quad \alpha\Delta t(\nabla e_p^{n+1} - \nabla e_p^n, \nabla q) - \alpha\Delta t(\nabla p(t_{n+1}) - \nabla p(t_n), \nabla q) = (\mathbf{e}_u^{n+1}, \nabla q).$$

Choosing $q = \Delta t e_p^n$ and using (2.5), we know

$$\begin{aligned}(4.11)\quad \frac{\alpha\Delta t^2}{2}(\|\nabla e_p^{n+1}\|^2 - \|\nabla e_p^n\|^2 - \|\nabla e_p^{n+1} - \nabla e_p^n\|^2) - \alpha\Delta t^2(\nabla p(t_{n+1}) \\ - \nabla p(t_n), \nabla e_p^n) = \Delta t(\mathbf{e}_u^{n+1}, \nabla e_p^n).\end{aligned}$$

By adding (4.8), (4.9) and (4.11), and applying (3.5), one has

$$\begin{aligned}(4.12)\quad \|\mathbf{e}_u^{n+1}\|^2 - \|\mathbf{e}_u^n\|^2 + \frac{2\Delta t}{Re}\|\nabla \mathbf{e}_u^{n+1/2}\|^2 + \frac{\alpha\Delta t^2}{2}(\|\nabla e_p^{n+1}\|^2 - \|\nabla e_p^n\|^2) \\ + S(\|\mathbf{e}_b^{n+1}\|^2 - \|\mathbf{e}_b^n\|^2) + \frac{2S\Delta t}{Rm}\|\text{curl}\, \mathbf{e}_b^{n+1/2}\|^2 \\ = -\Delta t(\nabla e_p^n, \mathbf{e}_u^n) + \frac{\alpha\Delta t^2}{2}\|\nabla e_p^{n+1} - \nabla e_p^n\|^2 \\ + \alpha\Delta t^2(\nabla p(t_{n+1}) - \nabla p(t_n), \nabla e_p^n) - 2\Delta t b(\mathbf{e}_u^n, \tilde{\mathbf{u}}(t_{n+1/2}), \mathbf{e}_u^{n+1/2}) \\ - 2S\Delta t(\mathbf{e}_b^n \times \text{curl } \widetilde{\mathbf{b}}(t_{n+1/2}), \mathbf{e}_u^{n+1/2}) + 2\Delta t(\mathcal{R}_u^{n+1}, \mathbf{e}_u^{n+1/2}) \\ + 2S\Delta t(\tilde{\mathbf{u}}(t_{n+1/2}) \times \mathbf{e}_b^n, \text{curl } \mathbf{e}_b^{n+1/2}) + 2S\Delta t(\mathcal{R}_b^{n+1}, \mathbf{e}_b^{n+1/2}) \\ = \sum_{i=1}^{8} \mathcal{I}_i.\end{aligned}$$

Now, we estimate $\mathcal{I}_1, \ldots, \mathcal{I}_8$, below. By the Cauchy-Schwarz and Young's inequalities, we estimate $\mathcal{I}_1$ as:

$$\mathcal{I}_1 \leqslant \|\mathbf{e}_u^n\|^2 + C\Delta t^2 \|\nabla e_p^n\|^2.$$

For bounding $\mathcal{I}_2$, we set $\varepsilon = \alpha - \frac{1}{4}$ and choose $q = e_p^{n+1} - e_p^n$ in (4.10), and then we have

$$\begin{aligned}
&\alpha\Delta t \|\nabla e_p^{n+1} - \nabla e_p^n\|^2 \\
&\quad = (\mathbf{e}_u^{n+1}, \nabla e_p^{n+1} - \nabla e_p^n) + \alpha\Delta t(\nabla p(t_{n+1}) - \nabla p(t_n), \nabla e_p^{n+1} - \nabla e_p^n) \\
&\quad \leqslant \frac{\Delta t(4\alpha - 3\varepsilon)}{8}\|\nabla e_p^{n+1} - \nabla e_p^n\|^2 + \frac{2}{\Delta t(4\alpha - 3\varepsilon)}\|\mathbf{e}_u^{n+1}\|^2 \\
&\qquad + \frac{3\varepsilon\Delta t}{8}\|\nabla e_p^{n+1} - \nabla e_p^n\|^2 + C\Delta t\|\nabla(p(t_{n+1}) - p(t_n))\|^2 \\
&\quad \leqslant \frac{\alpha\Delta t}{2}\|\nabla e_p^{n+1} - \nabla e_p^n\|^2 + \frac{2}{(1+\varepsilon)\Delta t}\|\mathbf{e}_u^{n+1}\|^2 + C\Delta t^3\|p_t\|^2_{l^\infty(H^1)},
\end{aligned}$$

which leads to

$$\alpha\Delta t^2 \|\nabla e_p^{n+1} - \nabla e_p^n\|^2 \leqslant \frac{4}{(1+\varepsilon)}\|\mathbf{e}_u^{n+1}\|^2 + C\Delta t^4 \|p_t\|^2_{l^\infty(H^1)}.$$

Hence,

$$\begin{aligned}
\mathcal{I}_2 &= \frac{(1+\frac{1}{2}\varepsilon)\alpha\Delta t^2}{4}\|\nabla e_p^{n+1} - \nabla e_p^n\|^2 + \frac{(1-\frac{1}{2}\varepsilon)\alpha\Delta t^2}{4}\|\nabla e_p^{n+1} - \nabla e_p^n\|^2 \\
&\leqslant \frac{1+\frac{1}{2}\varepsilon}{(1+\varepsilon)}\|\mathbf{e}_u^{n+1}\|^2 + C\Delta t^4\|p_t\|^2_{l^\infty(H^1)} + \frac{(1-\frac{1}{2}\varepsilon)\alpha\Delta t^2}{2}(\|\nabla e_p^{n+1}\|^2 + \|\nabla e_p^n\|^2).
\end{aligned}$$

By the Hölder, Cauchy-Schwarz and Young's inequalities and the Sobolev embeddings (2.7), we bound the remaing terms as follows:

$$\begin{aligned}
\mathcal{I}_3 &\leqslant \alpha\Delta t^2\|\nabla p(t_{n+1}) - \nabla p(t_n)\|\|\nabla e_p^n\| \leqslant C\Delta t^2\|\nabla e_p^n\|^2 + C\Delta t^4\|p_t\|^2_{l^\infty(H^1)}, \\
\mathcal{I}_4 &\leqslant C\Delta t\|\mathbf{e}_u^n\|^2_{L^6}\|\nabla\tilde{\mathbf{u}}(t_{n+1/2})\|_{L^3}\|\mathbf{e}_u^{n+1/2}\| \\
&\leqslant \frac{\varepsilon}{6(1+\varepsilon)}\|\mathbf{e}_u^{n+1/2}\|^2 + C\Delta t^2\|\nabla\mathbf{e}_u^n\|^2 \\
&\leqslant \frac{\varepsilon}{12(1+\varepsilon)}\|\mathbf{e}_u^{n+1}\|^2 + \frac{\varepsilon}{12(1+\varepsilon)}\|\mathbf{e}_u^n\|^2 + C\Delta t^2\|\nabla\mathbf{e}_u^n\|^2, \\
\mathcal{I}_5 &\leqslant 2S\Delta t\|\mathbf{e}_b^n\|_{L^6}\|\mathrm{curl}\,\widetilde{\mathbf{b}}(t_{n+1/2})\|_{L^3}\|\mathbf{e}_u^{n+1/2}\| \\
&\leqslant \frac{\varepsilon}{6(1+\varepsilon)}\|\mathbf{e}_u^{n+1/2}\|^2 + C\Delta t^2\|\mathbf{e}_b^n\|^2_{H^1}\|\widetilde{\mathbf{b}}(t_{n+1/2})\|^2_{H^2} \\
&\leqslant \frac{\varepsilon}{12(1+\varepsilon)}\|\mathbf{e}_u^{n+1}\|^2 + \frac{\varepsilon}{12(1+\varepsilon)}\|\mathbf{e}_u^n\|^2 + C\Delta t^2\|\mathbf{e}_b^n\|^2_{H^1}.
\end{aligned}$$

To bound the term $\mathcal{I}_7$, we use (2.2) and the following formula (4.6) and derive that

$$\begin{aligned}
\mathcal{I}_7 &= 2S\Delta t(\mathrm{curl}(\tilde{\mathbf{u}}(t_{n+1/2}) \times \mathbf{e}_b^n), \mathbf{e}_b^{n+1/2}) \\
&\leqslant 2S\Delta t\|\mathrm{curl}(\tilde{\mathbf{u}}(t_{n+1/2}) \times \mathbf{e}_b^n)\|\|\mathbf{e}_b^{n+1/2}\| \\
&\leqslant 2S\Delta t(\|\tilde{\mathbf{u}}(t_{n+1/2}) \cdot \nabla \mathbf{e}_b^n\| + \|(\mathbf{e}_b^n \cdot \nabla)\tilde{\mathbf{u}}(t_{n+1/2})\|)\|\mathbf{e}_b^{n+1/2}\| \\
&\leqslant 2S\Delta t(\|\tilde{\mathbf{u}}(t_{n+1/2})\|_{L^\infty}\|\mathbf{e}_b^n\|_{H^1} + \|\mathbf{e}_b^n\|_{L^6}\|\nabla\tilde{\mathbf{u}}(t_{n+1/2})\|_{L^3})\|\mathbf{e}_b^{n+1/2}\| \\
&\leqslant 2S\Delta t(\|\tilde{\mathbf{u}}(t_{n+1/2})\|_{H^2}\|\mathbf{e}_b^n\|_{H^1} + \|\mathbf{e}_b^n\|_{H^1}\|\tilde{\mathbf{u}}(t_{n+1/2})\|_{H^2})\|\mathbf{e}_b^{n+1/2}\| \\
&\leqslant \frac{S}{4}\|\mathbf{e}_b^{n+1}\|^2 + \frac{S}{4}\|\mathbf{e}_b^n\|^2 + C\Delta t^2\|\mathbf{e}_b^n\|_{H^1}^2.
\end{aligned}$$

For the remaining terms, we can deduce that

$$\mathcal{I}_6 \leqslant 2\Delta t\|\mathcal{R}_u^{n+1}\|\|\mathbf{e}_u^{n+1/2}\| \leqslant \frac{\varepsilon}{12(1+\varepsilon)}\|\mathbf{e}_u^{n+1}\|^2 + \frac{\varepsilon}{12(1+\varepsilon)}\|\mathbf{e}_u^n\|^2 + C\Delta t^2\|\mathcal{R}_u^{n+1}\|^2.$$

$$\mathcal{I}_8 \leqslant 2\Delta t\|\mathcal{R}_b^{n+1}\|\|\mathbf{e}_b^{n+1/2}\| \leqslant \frac{S}{4}\|\mathbf{e}_b^{n+1}\|^2 + \frac{S}{4}\|\mathbf{e}_b^n\|^2 + C\Delta t^2\|\mathcal{R}_b^{n+1}\|^2.$$

Combining the estimates $\mathcal{I}_1$ to $\mathcal{I}_8$ with (4.12), we find

$$\begin{aligned}
(4.13)\qquad &\frac{\varepsilon}{4(1+\varepsilon)}\|\mathbf{e}_u^{n+1}\|^2 + \frac{S}{2}\|\mathbf{e}_b^{n+1}\|^2 + \frac{2\Delta t}{Re}\|\nabla \mathbf{e}_u^{n+1/2}\|^2 \\
&\quad + \frac{\alpha\Delta t^2\varepsilon}{4}\|\nabla e_p^{n+1}\|^2 + \frac{2S\Delta t}{Rm}\|\mathrm{curl}\,\mathbf{e}_b^{n+1/2}\|^2 \\
&\leqslant C(\|\mathbf{e}_u^n\|^2 + \|\mathbf{e}_b^n\|^2 + \Delta t^2\|\nabla e_p^n\|^2) + C\Delta t^4\|p_t\|_{l^\infty(H^1)}^2 \\
&\quad + C\Delta t^2(\|\nabla \mathbf{e}_u^n\|^2 + \|\mathrm{curl}\,\mathbf{e}_b^n\|^2) + C\Delta t^2(\|\mathcal{R}_u^{n+1}\|^2 + \|\mathcal{R}_b^{n+1}\|^2).
\end{aligned}$$

Now, we prove the conclusions of Theorem 4.1 by mathematical induction.

Firstly, we prove the results hold for $n = 0$. Noticing that $\mathbf{e}_u^0 = \mathbf{e}_b^0 = e_p^0 = 0$, and applying (4.5), we deduce

$$\begin{aligned}
(4.14)\qquad &\frac{\varepsilon}{4(1+\varepsilon)}\|\mathbf{e}_u^1\|^2 + \frac{S}{2}\|\mathbf{e}_b^1\|^2 + \frac{2\Delta t}{Re}\|\nabla \mathbf{e}_u^{1/2}\|^2 \\
&\quad + \frac{\alpha\Delta t^2\varepsilon}{4}\|\nabla e_p^1\|^2 + \frac{2S\Delta t}{Rm}\|\mathrm{curl}\,\mathbf{e}_b^{1/2}\|^2 \\
&\leqslant C(\|\mathbf{e}_u^0\|^2 + \|\mathbf{e}_b^0\|^2 + \Delta t^2\|\nabla e_p^0\|^2) + C\Delta t^4 \leqslant C\Delta t^4.
\end{aligned}$$

From (4.14) and the triangle inequality, we deduce that

$$\begin{aligned}
&\|\mathbf{e}_u^1\|^2 + \|\mathbf{e}_b^1\|^2 + \Delta t^2\|\nabla e_p^1\|^2 \leqslant C\Delta t^4, \\
&\|\nabla \mathbf{e}_u^1\|^2 \leqslant 2(\|2\nabla \mathbf{e}_u^{1/2}\|^2 + \|\nabla \mathbf{e}_u^0\|^2) \leqslant C\Delta t^3, \\
&\|\mathrm{curl}\,\mathbf{e}_b^1\|^2 \leqslant 2(\|2\,\mathrm{curl}\,\mathbf{e}_b^{1/2}\|^2 + \|\mathrm{curl}\,\mathbf{e}_b^0\|^2) \leqslant C\Delta t^3.
\end{aligned}$$

Secondly, assuming the conclusion holds for all $n \leqslant m-1$ $(m \leqslant N-1)$, we have

$$\begin{aligned}
&\|\mathbf{e}_u^n\|^2 + \|\mathbf{e}_b^n\|^2 + \Delta t^2 \|\nabla e_p^n\|^2 \leqslant C\Delta t^4,\\
&\|\nabla \mathbf{e}_u^n\|^2 \leqslant 2(\|2\nabla \mathbf{e}_u^{n-\frac{1}{2}}\|^2 + \|\nabla \mathbf{e}_u^{n-1}\|^2) \leqslant C\Delta t^3,\\
&\|\mathrm{curl}\,\mathbf{e}_b^n\|^2 \leqslant 2(\|2\,\mathrm{curl}\;\mathbf{e}_b^{n-\frac{1}{2}}\|^2 + \|\mathrm{curl}\,\mathbf{e}_b^{n-1}\|^2) \leqslant C\Delta t^3.
\end{aligned}$$

Finally, we need to prove the conclusion holds for $n = m$ $(m \leqslant N-1)$. Let $n = m$ in (4.13), we know

$$\begin{aligned}
\frac{\varepsilon}{4(1+\varepsilon)}\|\mathbf{e}_u^{m+1}\|^2 &+ \frac{S}{2}\|\mathbf{e}_b^{m+1}\|^2 + \frac{2\Delta t}{Re}\|\nabla \mathbf{e}_u^{m+1/2}\|^2\\
&+ \frac{\alpha\Delta t^2\varepsilon}{4}\|\nabla e_p^{m+1}\|^2 + \frac{2S\Delta t}{Rm}\|\mathrm{curl}\,\mathbf{e}_b^{m+1/2}\|^2\\
&\leqslant C(\|\mathbf{e}_u^m\|^2 + \|\mathbf{e}_b^m\|^2 + \Delta t^2\|\nabla e_p^m\|^2)\\
&\quad + C\Delta t^4\|p_t\|^2_{l^\infty(H^1)} + C\Delta t^2(\|\nabla \mathbf{e}_u^m\|^2 + \|\mathrm{curl}\,\mathbf{e}_b^m\|^2)\\
&\quad + C\Delta t^2(\|\mathcal{R}_u^{m+1}\|^2 + \|\mathcal{R}_b^{m+1}\|^2) \leqslant C\Delta t^4.
\end{aligned}$$

Hence, we arrive at

$$\begin{aligned}
&\|\nabla \mathbf{e}_u^{m+1}\|^2 \leqslant 2(\|2\nabla \mathbf{e}_u^{m+1/2}\|^2 + \|\nabla \mathbf{e}_u^m\|^2) \leqslant C\Delta t^3,\\
&\|\mathrm{curl}\,\mathbf{e}_b^{m+1}\|^2 \leqslant 2(\|2\,\mathrm{curl}\;\mathbf{e}_b^{m+1/2}\|^2 + \|\mathrm{curl}\,\mathbf{e}_b^{m+1}\|^2) \leqslant C\Delta t^3,
\end{aligned}$$

and the proof is completed. □

Remark 4.1. If we assume the initial data $(\mathbf{u}^0, p^0, \mathbf{b}^0)$ for the scheme (2.8)–(2.9) satisfies

$$\begin{aligned}
&\|\mathbf{u}^0 - \mathbf{u}_0\| + \|\mathbf{b}^0 - \mathbf{b}_0\| \leqslant C\Delta t^2,\\
&\|\nabla(\mathbf{u}^0 - \mathbf{u}_0)\| + \|\mathbf{b}^0 - \mathbf{b}_0\|_{H^1} \leqslant C\Delta t^{3/2},\\
&\|\nabla(p^0 - p(0))\| \leqslant C\Delta t,
\end{aligned}$$

then we know that the convergence for the velocity and the magnetic field are strongly second-order in $\mathbf{L}^2(\Omega)$, strongly of order $\frac{3}{2}$ in $\mathbf{H}^1(\Omega)$, and for the pressure is strongly first-order $H^1(\Omega)$, and these results are consistent with those in Theorem 4.1.

5. Numerical experiments

In this section, we perform some numerical experiments both in two and three dimensional spaces to validate the effectiveness of the proposed higher order pressure segregation scheme. We first introduce the finite element approximation of the time discrete scheme (2.8)–(2.9). Let $\tau_h = \{\Omega_h\}$ be a quasi-uniform partition of Ω into triangles or tetrahedrons with diameters h $(0 < h < 1)$. We consider the finite element spaces $\mathbf{V}_h \subset \mathbf{V}$, $\mathbf{X}_h \subset \mathbf{X}$ and $M_h \subset M$ associated with a regular family of triangulations τ_h. Define I_h and J_h to be the interpolation operators from $\mathbf{L}^2$ to $\mathbf{V}_h$ and $\mathbf{X}_h$, respectively. Now, we give out the full discrete scheme corresponding to the time discrete scheme (2.8)–(2.9) as follows:

Algorithm 5.1 (Full discrete scheme). Choosing p_h^0 as an approximation of p^0, taking $\mathbf{u}_h^0 = I_h\mathbf{u}_0$, $\mathbf{b}_h^0 = J_h\mathbf{b}_0$, we find $\mathbf{u}_h^{n+1}, p_h^{n+1}, \mathbf{b}_h^{n+1}$ by the following two steps:

Step 1: Find $\mathbf{u}_h^{n+1}$ and $\mathbf{b}_h^{n+1}$ such that

$$\begin{aligned}
(5.1)\qquad & \Big(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v}_h\Big) + a_1(\mathbf{u}_h^{n+1/2}, \mathbf{v}_h) + b(\mathbf{u}_h^n, \mathbf{u}_h^{n+1/2}, v_h) + (\nabla p_h^n, \mathbf{v}_h) \\
& \quad + S(\mathbf{b}_h^n \times \operatorname{curl} \mathbf{b}_h^{n+1/2}, \mathbf{v}_h) = (\mathbf{f}(t_{n+1/2}), \mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h, \\
& \Big(\frac{\mathbf{b}_h^{n+1} - \mathbf{b}_h^n}{\Delta t}, \mathbf{w}_h\Big) + a_2(\mathbf{b}_h^{n+1/2}, \mathbf{w}_h) - (\mathbf{u}_h^{n+1/2} \times \mathbf{b}_h^n, \operatorname{curl} \mathbf{w}_h) \\
& \quad = (\mathbf{g}(t_{n+1/2}), \mathbf{w}_h) \quad \forall\, \mathbf{w}_h \in \mathbf{X}_h,
\end{aligned}$$

where $\mathbf{u}_h^{n+1/2} = \frac{1}{2}(\mathbf{u}_h^{n+1} + \mathbf{u}_h^n)$, $\mathbf{b}_h^{n+1/2} = \frac{1}{2}(\mathbf{b}_h^{n+1} + \mathbf{b}_h^n)$.

Step 2: Let $\mathbf{u}_h^{n+1}$ be given from Step 1, then find p_h^{n+1} such that

$$(5.2)\qquad \alpha\Delta t(\nabla(p_h^{n+1} - p_h^n), \nabla q_h) = (\mathbf{u}_h^{n+1}, \nabla q_h) \quad \forall\, q_h \in M_h,$$

where α is a constant to be determined.

We first test the problem with the known analytical solution to confirm the established theoretical findings. Next, the lid driven cavity flow problem in both two and three dimensional space is simulated, and we use this physical model to verify the efficiency of the developed numerical scheme. We choose linear polynomials to approximate the velocity, the pressure and the magnetic field. The domain Ω is subdivided into triangles. All simulations were run using the public finite element software package freefem++ [19].

Firstly, we set the parameters $Re = 1.0$, $Rm = 1.0$, $S = 1.0$, and the domain $\Omega = [0,1] \times [0,1]$. The forcing functions $\mathbf{f}$ and $\mathbf{g}$ and boundary values of $(\mathbf{u}, p, \mathbf{b})$ are given so that problem (1.1) has the analytical solution

$$\mathbf{u} = (\sin(t)\sin(2\pi y)\sin(\pi x)^2, -\sin(t)\sin(2\pi x)\sin(\pi y)^2)^\top,$$
$$p = (\sin(2\pi x) + \sin(2\pi y))\exp(-t),$$
$$\mathbf{b} = (\sin(t)\sin(\pi x)\cos(\pi y), -\sin(t)\sin(\pi y)\cos(\pi x))^\top.$$

We examine the errors and the convergence orders of the higher order pressure segregation scheme. The purpose of this test is to verify the time convergence order, so we fix the mesh width $h = 1/128$ and the final time $T = 1.0$. In Table 1, we present the numerical results with varying time step size Δt. From these data, we can see that the errors become smaller and smaller as the time step size decrease. The results in Table 1 suggest that the convergence for the velocity and the magnetic field are second-order in $\mathbf{L}^2(\Omega)$ and of order $\frac{3}{2}$ in $\mathbf{H}^1(\Omega)$, and for the pressure it is first-order in $H^1(\Omega)$. The numerical results are in good agreement with the convergence rates predicted by the theoretical analysis.

Δt	$\|\mathbf{u}-\mathbf{u}^n\|_{L^2}$	Rate	$\|\mathbf{u}-\mathbf{u}^n\|_{H^1}$	Rate	$\|p-p^n\|_{H^1}$	Rate
0.125	0.0819886		0.605373		2.17762	
0.0625	0.0202001	2.02106	0.227014	1.41504	1.03106	1.07862
0.03125	0.0053656	1.91254	0.0815380	1.47723	0.543186	0.924610
0.015625	0.00145792	1.87984	0.0289970	1.49157	0.292672	0.892162
Δt	$\|\mathbf{b}-\mathbf{b}^n\|_{L^2}$	Rate	$\|\mathbf{b}-\mathbf{b}^n\|_{H^1}$	Rate		
0.125	0.0245068		0.175177			
0.0625	0.00611660	2.00238	0.0640519	1.45150		
0.03125	0.00155896	1.97215	0.0225844	1.49754		
0.015625	0.000442682	1.81624	0.00797788	1.50125		

Table 1. The errors and convergence rates for analytical test problem by using the higher order pressure segregation scheme.

Secondly, we consider a famous test problem used in fluid dynamics, known as driven cavity flow. It is a model of the flow in a cavity with the lid moving in one direction: In this example, both the 2D domain $\Omega = [-1,1]^2$ and the 3D domain $\Omega = [-1,1]^3$ are considered. Set the source terms $\mathbf{f} = \mathbf{0}$, $\mathbf{g} = \mathbf{0}$. For the 2D case, the boundary conditions are imposed as

$$\begin{aligned} &\mathbf{u} = 0 && \text{on } x = \pm 1,\ y = -1, \\ &\mathbf{u} = (1,0) && \text{on } y = 1, \\ &\mathbf{b} \times \mathbf{n} = \widetilde{\mathbf{b}}_0 \times \mathbf{n} && \text{on } \partial\Omega, \end{aligned}$$

where $\widetilde{\mathbf{b}}_0 = (1,0)$. For the 3D case, the boundary conditions are imposed as

$$\begin{aligned} &\mathbf{u} = 0 && \text{on } x = \pm 1,\ y = \pm 1,\ z = -1, \\ &\mathbf{u} = (1,0,0) && \text{on } z = 1, \\ &\mathbf{b} \times \mathbf{n} = \widetilde{\mathbf{b}}_0 \times \mathbf{n} && \text{on } \partial\Omega, \end{aligned}$$

where $\widetilde{\mathbf{b}}_0 = (1,0,0)$.

We choose the parameters $Re = 1.0$, $Rm = 10.0$, $S = 1.0$ and the time step size $\Delta t = 0.01$. The numerical results are shown in Figures 1 and 2. Figure 1 shows the velocity field and the magnetic field at different times ($t = 0.4, 0.8, 1.0$) computed with the mesh width $h = \frac{1}{32}$ in the 2D domain. We can see that, as time goes on, a small vortex of the velocity is formed at the lower left corner of the cavity, and the magnetic field gradually moves to the lower right due to the coupling effect. The solution computed with the mesh width $h = \frac{1}{16}$ in the 3D domain is shown in Figure 2, from which we find that the flow vectors on slice ($y = 0$) demonstrate a behavior similar to the 2D case. These results confirm to the actual physical law (see [13]), hence, the proposed numerical scheme is very efficacious for the MHD equations.

6. Conclusions and future works

A higher order pressure segregation scheme for time-dependent MHD equations was proposed in this article. The stability is analyzed and the error analysis is accomplished by interpreting this segregated scheme as a higher order time discretization of a perturbed system which approximates the MHD system. We have proved that this scheme provides strongly second-order approximations of the velocity and the magnetic field in $\mathbf{L}^2(\Omega)$, strongly order $\frac{3}{2}$ approximations of the velocity and the magnetic field in $\mathbf{H}^1(\Omega)$, while strongly first-order approximations of the pressure in $H^1(\Omega)$. Finally, some numerical tests are performed to validate the theoretical predictions and the efficiency of the numerical scheme. Such segregated scheme also can be extended to other complex models, such as in [20], [35], [36], [37].

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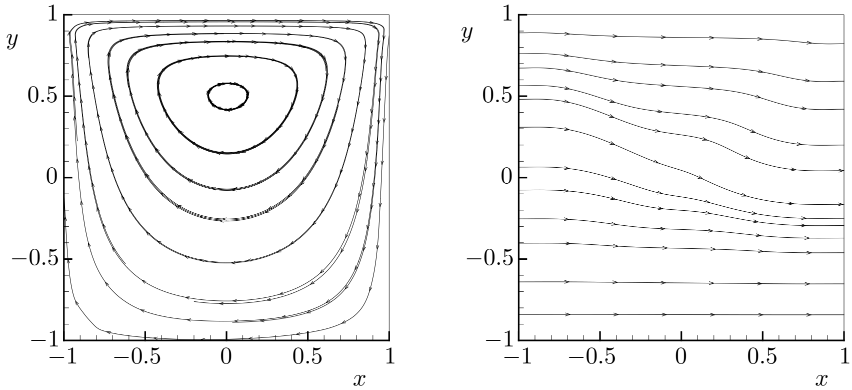
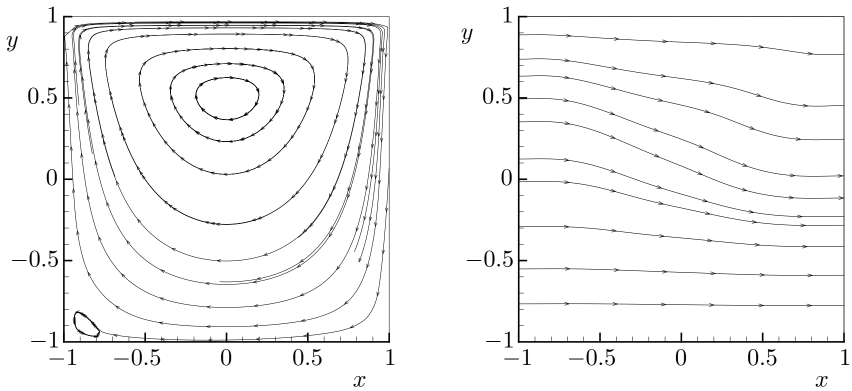
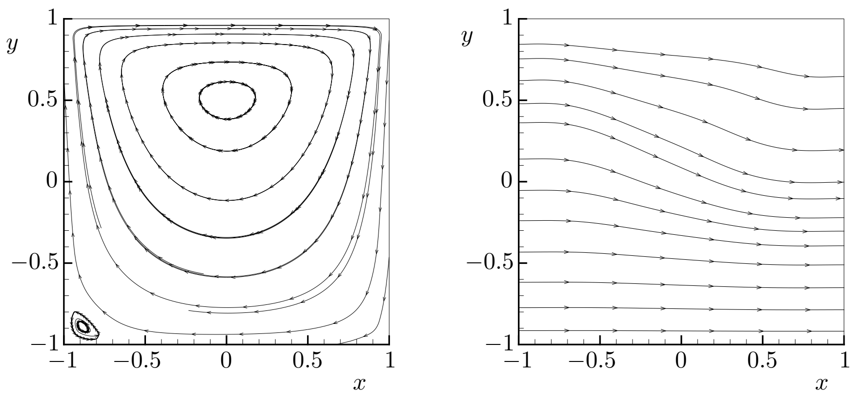


(a) $t = 0.4$

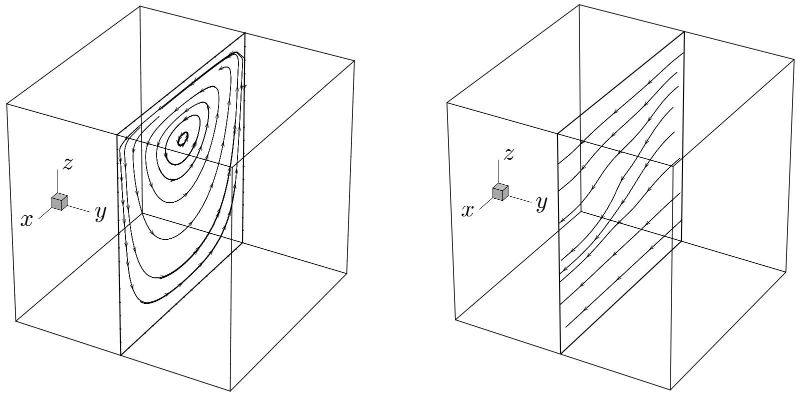


(b) $t = 0.8$

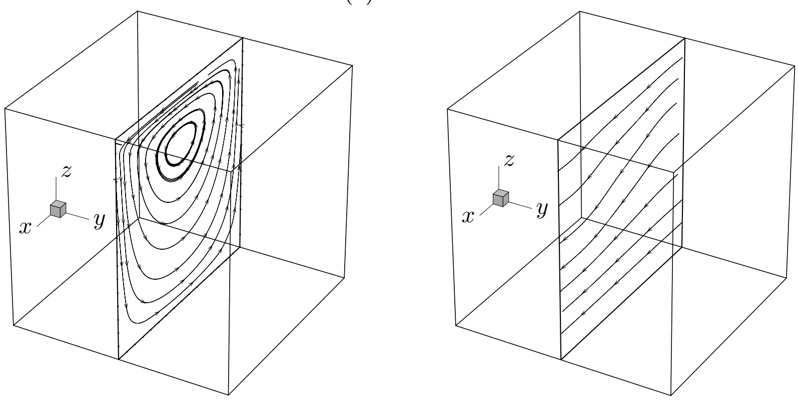


(c) $t = 1.0$

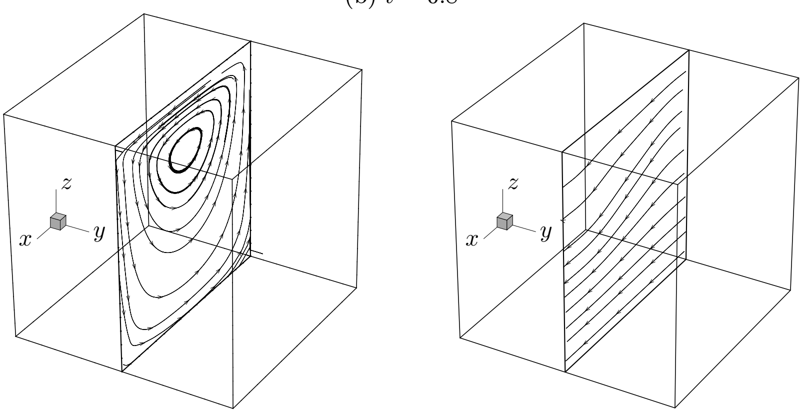
Figure 1. The velocity field (left) and the magnetic field (right) at different times with $Re = 1$, $Rm = 10$, $S = 1$.



(a) $t = 0.4$



(b) $t = 0.8$



(c) $t = 1.0$

Figure 2. The velocity field (left) and the magnetic field (right) in the 3D domain at different times with $Re = 1,\ Rm = 10,\ S = 1$.

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Authors' addresses: *Yun-Bo Yang*, School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi, 710049, P. R. China; Department of Mathematics, Yunnan Normal University, Kunming, Yunnan, 650500, P. R. China, e-mail: `ybyang13@126.com`; *Yao-Lin Jiang*, School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi, 710049, P. R. China, e-mail: `yljiang@mail.xjtu.edu.cn`; *Qiong-Xiang Kong* (corresponding author), School of Human Settlements and Civil Engineering, Xi'an Jiaotong University, Xi'an, Shaanxi, 710049, P. R. China, e-mail: `qxkong@mail.xjtu.edu.cn`.