# THE ELLIPTIC PROBLEMS IN A FAMILY OF PLANAR OPEN SETS

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*Abstract.* We propose, on a model case, a new approach to classical results obtained by V. A. Kondrat'ev (1967), P. Grisvard (1972), (1985), H. Blum and R. Rannacher (1980), V. G. Maz'ya (1980), (1984), (1992), S. Nicaise (1994a), (1994b), (1994c), M. Dauge (1988), (1990), (1993a), (1993b), A. Tami (2016), and others, describing the singularities of solutions of an elliptic problem on a polygonal domain of the plane that may appear near a corner. It provides a more precise description of how the solutions decompose, puts into evidence the analogy of such decompositions with standard Taylor expansions, and gives uniform estimates with respect to the angle parameter. This last property allows the treatment of families of elliptic problems on families of open sets.

*Keywords*: biharmonic operator; elliptic problems; nonsmooth boundaries; uniform singularity estimates; Sobolev spaces

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## 1. Introduction

The behaviour of solutions of elliptic problems on polygons near a corner has been investigated in the 60's. The method used by our predecessors in this matter, for example the case of the Laplacian $\Delta$ (see [7]) relies, through appropriate changes of variables, on the idea that singularities appear as poles in the complex plane of some kernel associated to the problem. However, one drawback of this approach is the lack of uniformity in the estimates with respect to the angle parameter. This prevented us to use the same method for our model problem, defined on a family of open sets with a variable angle at the origin.

We indeed consider a family of open sets $\Omega_\omega$ of the plane, whose boundaries are smooth except at one point, the origin $O$, where they are locally polygonal with an angle $\omega \in \,]0, 2\pi[$. To fix the ideas, we assume that the boundary of $\Omega_\omega$ near $O$ contains

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two segments $\Gamma^+ = \{t(1,0);\ t \in [0,1]\}$, on one side, and $\Gamma^- = \{(1-t)(\cos\omega, \sin\omega);\ t \in [0,1]\}$, on the other, and that the truncated sector $\{(r,\theta);\ r \in ]0,1],\ \theta \in ]0,\omega[\}$ is included in $\Omega_\omega$. We want to study the family of problems $(P_\omega)$ $\Delta^2 u_\omega = f_\omega$ with boundary condition $u_\omega = \Delta u_\omega = 0$. There, the right-hand sides (r.h.s.) are assumed to depend smoothly on $\omega$ in $L^2(\Omega_\omega)$.

If $\omega < \pi$ it is known from [1] that the solution $u_\omega$ decomposes as

$$u_\omega = u_{1,\omega} + u_{2,\omega} + u_{3,\omega}, \tag{1.1}$$

where $u_{1,\omega}$, $u_{2,\omega}$ are singular and $u_{3,\omega}$ is regular. Indeed, near the origin, $u_{1,\omega}$, $u_{2,\omega}$, $u_{3,\omega}$ are respectively of regularities $H^{1+\pi/\omega-\varepsilon}$, $H^{2+\pi/\omega-\varepsilon}$, $H^4$ for every $\varepsilon > 0$, while the solution $u_\pi$ is, in the neighborhood of the origin again, of regularity $H^4$. One thus naturally looks for a resolution of the singularity near the angle $\pi$ whose description was the main motivation for this work. Our main result is that there exists a decomposition (1.1) of $u_\omega$ which is uniform with respect to $\omega$, when $\omega \to \pi$, with the best possible topologies for each term, which is analogous to the Taylor expansion of $u_\omega$ near 0 and which converges towards the Taylor expansion of $u_\pi$.

To obtain such a result, we follow a new approach that we will fully describe. Beyond the particular case treated here, our method is generalizable to a much wider class of problems: this is where its main interest lies.

## 2. A new approach to Blum and Rannacher results

We describe in this section our approach to solving problem $(P_\omega)$ with a fixed parameter $\omega$ that we restrict to the interval $]0,\pi[$, in order to avoid too heavy a multiplication of sub-cases. As we announced before, we will pay great attention to deriving uniform estimates with respect to $\omega$.

**2.1. The main steps.** Let $\Omega$ be a planar open set whose boundary is smooth except at one point, the origin, where it is locally polygonal with angle $\omega$. Recall that we assume the boundary of $\Omega$ near $O$ contains two segments $\Gamma^+ = \{t(1,0);\ t \in [0,1]\}$, on one side, and $\Gamma^- = \{(1-t)(\cos\omega, \sin\omega);\ t \in [0,1]\}$, on the other, the truncated sector $\{(r,\theta);\ r \in ]0,1],\ \theta \in ]0,\omega[\}$ being included in $\Omega$. We denote by $L$ the elliptic operator in the problem $(P_\omega)$ above, defined in the variational sense, and by $D(L)$ its domain as a subspace of the Hilbert space $H(\Delta^2,\Omega) = \{u \in H^2(\Omega)\colon\ \Delta^2 u \in L^2(\Omega)\}$ with the norm $\|u\|_{H(\Delta^2,\Omega)} = (\|u\|^2_{H^2(\Omega)} + \|\Delta^2 u\|^2_{L^2(\Omega)})^{1/2}$. More precisely, $D(L) = \{u \in H(\Delta^2,\Omega)\colon\ u = \Delta u = 0 \text{ on } \partial\Omega\}$.

All functions $u$ in $D(L)$ are of regularity $H^4$ outside any neighbourhood of the origin and by the Poincaré inequality $\|\Delta^2 u\|_{L^2(\Omega)}$ is a norm on $D(L)$. Let us define by $D_0(L)$ the closure in $D(L)$ of the set $\{u \in D(L);\ u = 0$ in some neighborhood of $0\}$. Our method starts with the following:

**Lemma 2.1.** *Under the hypothesis above, in particular when $\omega \neq \pi$,*

$$D_0(L) = \{u \in D(L);\ u \in H^4(\Omega)\}.$$

Hence, by the Hahn-Banach theorem, to describe the behavior of those functions $u$ in the domain of $L$ which are singular near the origin, it is enough to understand what are the linear forms on $D(L)$ which vanish on $D_0(L)$. Denote by $\Lambda_0(L)$ the space of all such linear forms. Note that, when $l \in \Lambda_0(L)$ and $u \in D(L)$, we have $l(u) = l(u\chi)$ for any $C^\infty$ function $\chi$, compactly supported in $\overline{\Omega}$ and identically equal to 1 in some neighborhood of $O$.

**Lemma 2.2.** *The space $\Lambda_0(L)$ is of finite dimension.*

- ▷ *It is trivial when $\omega \leqslant \frac{1}{3}\pi$.*
- ▷ *When $\frac{1}{3}\pi < \omega \leqslant \frac{2}{3}\pi$, it is one-dimensional and any $l \in \Lambda_0(L)$ can be represented as*
$$l(u) = c \int_\Omega \Delta^2(u\chi)(r,\theta) r^{2-\pi/\omega} \sin\frac{\pi}{\omega}\theta r \,\mathrm{d}r \,\mathrm{d}\theta,$$
*where $c$ is a constant and $\chi$ a $C^\infty$ function compactly supported in $\overline{\Omega \cap B(0,1)}$ and identically equal to 1 in some neighborhood of $O$.*
- ▷ *When $\frac{2}{3}\pi < \omega < \pi$, it is two-dimensional and any $l \in \Lambda_0(L)$ can be represented as*
$$\begin{aligned} l(u) = c_1 &\int_\Omega \Delta^2(u\chi)(r,\theta) r^{2-\pi/\omega} \sin\frac{\pi}{\omega}\theta r \,\mathrm{d}r \,\mathrm{d}\theta \\ &+ c_2 \int_\Omega \Delta^2(u\chi)(r,\theta) r^{2-2\pi/\omega} \sin\frac{2\pi}{\omega}\theta r \,\mathrm{d}r \,\mathrm{d}\theta, \end{aligned}$$
*where $c_1, c_2$ are two constants and $\chi$ a $C^\infty$ function, compactly supported in $\overline{\Omega \cap B(0,1)}$ and identically equal to 1 in some neighborhood of $O$.*

From this lemma it is then easy to recover the Blum and Rannacher [1] description of the singularities of the functions in the domain in a more accurate version.

**Theorem 2.1.** *Let $u \in D(L)$. Then, depending on $\omega$, one has:*

- ▷ *when $\omega \leqslant \frac{1}{3}\pi$, $u \in H^4(\Omega)$;*
- ▷ *when $\frac{1}{3}\pi < \omega \leqslant \frac{2}{3}\pi$, there exists a ball $B$ centered at the origin, a constant $C$ independent of $\omega$, and for each $\omega$ a linear form $\lambda \in \Lambda_0(L)$ such that any $u \in D(L)$ decomposes on $B$ as*

$$u(r,\theta) = \lambda(u) r^{\pi/\omega} \sin\frac{\pi}{\omega}\theta + u_0(r,\theta) \tag{2.1}$$

*with* $u_0 \in H^4(\Omega)$, *where*

$$|\lambda(u)| + \|u_0\|_{H^4(\Omega)} \leqslant C\|u\|_{D(L)};$$

▷ *when* $\frac{2}{3}\pi < \omega < \pi$, *there exists a ball* $B$ *centered at the origin, a constant* $C$ *independent of* $\omega$, *and for each* $\omega$ *two linear forms* $\lambda, \mu \in \Lambda_0(L)$ *such that any* $u \in D(L)$ *decomposes on* $B$ *as*

$$u(r,\theta) = \lambda(u) r^{\pi/\omega} \sin\frac{\pi}{\omega}\theta + \mu(u) r^{2\pi/\omega} \sin\frac{2\pi}{\omega}\theta + u_0(r,\theta) \tag{2.2}$$

*with* $u_0 \in H^4(\Omega)$, *where*

$$|\lambda(u)| + |\mu(u)| + \|u_0\|_{H^4(\Omega)} \leqslant C\|u\|_{D(L)}.$$

In these statements, the linear forms $\lambda$ and $\mu$ are supported on $\{O\}$, being elements of $\Lambda_0(L)$.

Let us now go into the proofs.

**2.2. Proof of Lemma 2.1.** We begin by choosing $u \in D(L) \cap H^4(\Omega)$ and showing that it belongs to $D_0(L)$. Let $\varepsilon \in \,]0, \frac{1}{2}[$, $\eta \in \,]0, \varepsilon[$, and define

$$h_{\varepsilon,\eta}(r) = \begin{cases} 0 & \text{if } r \leqslant \eta, \\ \mathrm{e}^{-(\ln(r-\eta)/\ln\varepsilon - 1)^5/2} & \text{if } \eta < r \leqslant \eta + \varepsilon, \\ 1 & \text{if } r > \eta + \varepsilon. \end{cases}$$

These functions are at least $C^4$, and for $1 \leqslant k \leqslant 4$ we have

$$|h^{(k)}_{\varepsilon,\eta}(r)| \leqslant \frac{C}{|\ln\varepsilon|}\frac{1}{(r-\eta)^k}\mathrm{e}^{-(\ln(r-\eta)/\ln\varepsilon - 1)^5/2}\,\mathbf{1}_{\eta<r<\eta+\varepsilon}$$

uniformly in $\eta$, $\varepsilon$. We set $u_{\varepsilon,\eta} = u h_{\varepsilon,\eta}$, so that $u_{\varepsilon,\eta} \in D_0(L)$. For any $\alpha > 0$, by the Poincaré inequality it will be sufficient to show that there exist $\varepsilon$ and $\eta$ such that $\|\Delta^2 u - \Delta^2 u_{\varepsilon,\eta}\|_{L^2(\Omega)} \leqslant \alpha$.

The term $\|\Delta^2 u - (\Delta^2 u) h_{\varepsilon,\eta}\|_{L^2(\Omega)}$ is obvious. The other terms are of the form

$$\|D^k u \; D^{4-k} h_{\varepsilon,\eta}\|_{L^2(\Omega)},$$

where $0 \leqslant k \leqslant 3$ and $D^k$ is a generic notation for any partial derivative of order $k$. For a given $k$ these terms are dominated by the square root of

$$T_k = \frac{C^2}{|\ln\varepsilon|^2}\int_0^\omega\int_{\eta<r<\eta+\varepsilon} D^k u(r,\theta)^2 \frac{1}{(r-\eta)^{8-2k}}\mathrm{e}^{-(\ln(r-\eta)/\ln\varepsilon - 1)^5}\, r\,\mathrm{d}r\,\mathrm{d}\theta.$$

When $2\eta < r < \eta + \varepsilon$, one has

$$\frac{C^2}{|\ln \varepsilon|^2} \frac{1}{(r-\eta)^{8-2k}} \mathrm{e}^{-(\ln(r-\eta)/\ln \varepsilon - 1)^5} \leqslant \frac{C}{r^{8-2k}(\ln r)^2}.$$

We thus have

$$\begin{aligned} (2.3) \qquad T_k \leqslant C \int_0^\omega \int_{r<2\varepsilon} D^k u(r,\theta)^2 \frac{1}{r^{8-2k}(\ln r)^2} r \, \mathrm{d}r \, \mathrm{d}\theta \\ + \frac{C^2}{|\ln \varepsilon|^2} \int_0^\omega \int_{\eta<r<2\eta} D^k u(r,\theta)^2 \frac{1}{(r-\eta)^{8-2k}} \mathrm{e}^{-(\ln(r-\eta)/\ln \varepsilon - 1)^5} r \, \mathrm{d}r \, \mathrm{d}\theta. \end{aligned}$$

The second integral on the r.h.s. of (2.3) above is as small as we want provided $\varepsilon$ is fixed and $\eta$ small enough. The first integral tends to 0 with $\varepsilon$ thanks to the following estimate, pertaining to the class of Hardy inequalities.

**Lemma 2.3.** *For any $u \in D(L) \cap H^4(\Omega)$ and $0 \leqslant k \leqslant 3$, one has*

$$\int_{\Omega \cap B(0,1/2)} D^k u(r,\theta)^2 \frac{1}{r^{8-2k}(\ln r)^2} r \, \mathrm{d}r \, \mathrm{d}\theta \leqslant C \|u\|^2_{H^4(\Omega)}.$$

Proof. Since $u \in H^4(\Omega)$, its trace is at least $C^2$. Therefore, because the two tangent directions defined by the boundary at the origin, along $\Gamma^-$ and $\Gamma^+$, are not collinear ($\omega \neq \pi$!), the Dirichlet boundary condition on $u$ and on $\Delta u$ implies $u(0) = 0$, $\nabla u(0) = 0$, $D^2 u(0) = 0$. That any $u \in H^4(\Omega)$ which vanishes at the origin at the order 2 does satisfy the lemma is shown as follows.

We start by proving the inequality when $k = 3$. It is a direct consequence of the observation that, for all $v \in H^1(\Omega)$,

$$(2.4) \qquad \int_{\Omega \cap B(0,1/2)} v(r,\theta)^2 \frac{1}{r(\ln r)^2} \, \mathrm{d}r \, \mathrm{d}\theta \leqslant C \|v\|^2_{H^1(\Omega)}.$$

Indeed, assuming first that $v \in D(\overline{\Omega})$, integration by part gives

$$\begin{aligned} &\int_{\Omega \cap B(0,1/2)} v(r,\theta)^2 \frac{1}{r(\ln r)^2} \, \mathrm{d}r \, \mathrm{d}\theta \\ &= 2 \int_{\Omega \cap B(0,1/2)} v(r,\theta) \frac{\partial v}{\partial r}(r,\theta) \frac{1}{\ln r} \, \mathrm{d}r \, \mathrm{d}\theta - \int_{\partial(\Omega \cap B(0,\frac{1}{2}))} v(r,\theta)^2 \frac{1}{\ln r} \, \mathrm{d}s \\ &\leqslant 2 \Big( \int_{\Omega \cap B(0,1/2)} v(r,\theta)^2 \frac{1}{r(\ln r)^2} \, \mathrm{d}r \, \mathrm{d}\theta \Big)^{1/2} \Big( \int_{\Omega \cap B(0,1/2)} \frac{\partial v}{\partial r}(r,\theta)^2 r \, \mathrm{d}r \, \mathrm{d}\theta \Big)^{1/2} \\ &\quad + C \|v\|^2_{H^1(\Omega)}, \end{aligned}$$

which implies (2.4) for $v$ in $D(\overline{\Omega})$ . Then the inequality holds in $H^1(\Omega)$ by density.

Next, if $k \leqslant 2$, we write

$$D^k u(r,\theta) = \int_0^r \frac{\partial}{\partial \varrho} D^k u(\varrho,\theta)\,\mathrm{d}\varrho,$$

using that $D^k u$ vanishes at the origin. Hence, one can readily verify that this allows to deduce the inequality of the lemma for such a $k$ from the same inequality for $k+1$. This ends the proof. □

Therefore, we have obtained that $u \in D_0(L)$ as desired.

Let us conversely show why any $u \in D_0(L)$ belongs to $H^4(\Omega)$. We may assume that $u$ is compactly supported in the ball $B(0,\frac{1}{2})$. We will prove that, in this case,

$$\int_\Omega |D^4(u)|^2 = \int_\Omega (\Delta^2 u)^2, \tag{2.5}$$

where we have defined

$$|D^4(u)|^2 = \Big(\frac{\partial^4 u}{\partial x^4}\Big)^2 + 4\Big(\frac{\partial^4 u}{\partial x^3 \partial y}\Big)^2 + 6\Big(\frac{\partial^4 u}{\partial x^2 \partial y^2}\Big)^2 + 4\Big(\frac{\partial^4 u}{\partial x \partial y^3}\Big)^2 + \Big(\frac{\partial^4 u}{\partial y^4}\Big)^2.$$

By density it is enough to prove this when, moreover, $u$ is vanishing in some neighborhood of $O$. But then, using an appropriate partition of unity, we can write $u = u_1 + u_2$, where $u_1$ is supported in the half-plane $P_1 = \{y > 0\}$, $u_2$ is supported in the half-plane $P_2 = \{x \sin\omega - y\cos\omega > 0\}$, both are in $H^4(P_i)$, satisfy the boundary condition $u_i = \Delta u_i = 0$ on $\partial P_i$, and $\operatorname{supp} u_1 \cap \operatorname{supp} u_2$ is a compact subset of $\Omega$. In fact, equality (2.5) is satisfied by each $u_i$ separately: To see this, by density one can consider first $u_i \in C_c^\infty(P_i)$ integration by parts and commutativity of partial derivatives for smooth functions give immediately (2.5) for each $u_i$. We thus have

$$\begin{aligned}\int_\Omega |D^4(u)|^2 &= \int_\Omega |D^4(u_1)|^2 + 2\int_\Omega D^4(u_1)\cdot D^4(u_2) + \int_\Omega |D^4(u_2)|^2 \\ &= \int_\Omega (\Delta^2 u_1)^2 + 2\int_\Omega D^4(u_1)\cdot D^4(u_2) + \int_\Omega (\Delta^2 u_2)^2.\end{aligned} \tag{2.6}$$

Choosing $h \in C_c^\infty(\Omega)$ with $h = 1$ on $\operatorname{supp} u_1 \cap \operatorname{supp} u_2$, we also have

$$\begin{aligned}\int_\Omega D^4(u_1)\cdot D^4(u_2) &= \int_{\mathbb{R}^2} D^4(u_1 h)\cdot D^4(u_2 h) \\ &= \int_{\mathbb{R}^2} \Delta^2(u_1 h)\Delta^2(u_2 h) = \int_\Omega \Delta^2(u_1)\Delta^2(u_2),\end{aligned}$$

the second equality relying on the same arguments of commutativity of partial derivatives as for $u_1$ and $u_2$ separately. Inserting this identity into (2.6) gives inequality (2.5). This completes the proof of Lemma 2.1.

Remark 2.1. It follows from the argument above that there exists a constant $C$ independent of $\omega$ such that

$$\forall u \in D_0(L) \|u\|_{H^4(\Omega)} \leqslant C\|u\|_{D(L)}.$$

It should be pointed out that one can easily check that, for all $u \in D_0(L)$, the estimate $u(r,\theta) = o(r^3|\ln r|)$ holds true near the origin (see [15]).

**2.3. Proof of Lemma 2.2.** Since, $\|\Delta^2 u\|_{L^2(\Omega)}$ is a norm on $D(L)$, any linear form $l$ on the domain of $L$ can be written as

$$l(u) = \int_\Omega \Delta^2 u z$$

for some $z \in L^2(\Omega)$. If now $l$ vanishes on $D_0(L)$, we have $\int_\Omega \Delta^2 \varphi z = 0$ for any test function $\varphi$ on $\Omega$, which means that

$$\Delta^2 z = 0 \quad \text{in } D'(\Omega).$$

Let $\chi \in C_c^\infty(]0,1[), k \geqslant 1$ and $u(r,\theta) = \chi(r)\sin(k\pi\theta/\omega)$. Then $u \in D_0(L)$, so that $l(u) = 0$. This gives in polar coordinates followed by integration w.r.t. $\theta$ the equation

$$\int_0^1 \Big( \Big[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r\frac{\mathrm{d}}{\mathrm{d}r}\Big) - \Big(\frac{k\pi}{\omega}\Big)^2 \frac{1}{r^2}\Big]^2 \chi(r) r\Big) z_k(r)\,\mathrm{d}r = 0,$$

where

$$z_k(r) = \frac{2}{\omega}\int_0^\omega z(r,\theta)\sin\frac{k\pi}{\omega}\,\mathrm{d}\theta.$$

Thus, we have

$$r\Big[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r\frac{\mathrm{d}}{\mathrm{d}r}\Big) - \Big(\frac{k\pi}{\omega}\Big)^2 \frac{1}{r^2}\Big]^2 z_k(r) = 0 \quad \text{in } D'(]0,1[)$$

or equivalently, since $r = 0$ is not in $]0,1[$, the associated Dirac mass at 0 will vanish for test functions in $C_c^\infty(]0,1[)$ so that one can omit the factor $r$ from the left-hand side of this last equation and obtain in $D'(]0,1[)$ the system

$$\begin{cases} \Big[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r\frac{\mathrm{d}}{\mathrm{d}r}\Big) - \Big(\frac{k\pi}{\omega}\Big)^2 \frac{1}{r^2}\Big] z_k(r) = \varphi_k, \\ \Big[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r\frac{\mathrm{d}}{\mathrm{d}r}\Big) - \Big(\frac{k\pi}{\omega}\Big)^2 \frac{1}{r^2}\Big] \varphi_k = 0, \end{cases} \tag{2.7}$$

where the second equation has straightforward analytical solutions,

$$\varphi_k(r) = \alpha_k r^{k\pi/\omega} + \beta_k r^{-k\pi/\omega},$$

which are in $L^2_{\text{loc}}(]0,1[)$ for all constants $\alpha_k, \beta_k \in \mathbb{R}$. It follows by standard arguments of interior elliptic regularity from the second equation of (2.7) that $\varphi_k \in H^2_{\text{loc}}(]0,1[)$,[1] and consequently the first equation of (2.7) implies under the same arguments that $z_k \in H^4_{\text{loc}}(]0,1[)$. As a result, $z_k$ satisfies in $L^2_{\text{loc}}(]0,1[)$ the differential equation

$$\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right) - \left(\frac{k\pi}{\omega}\right)^2\frac{1}{r^2}\right)^2 z_k(r) = 0.$$

It follows that

$$z_k = a_k r^{2+k\pi/\omega} + b_k r^{k\pi/\omega} + c_k r^{2-k\pi/\omega} + d_k r^{-k\pi/\omega}$$

for some constants $a_k$, $b_k$, $c_k$, $d_k$. Moreover, we immediately get rid of most of the constants $c_k$, $d_k$. Indeed, we have $z_k \in L^2(]0,1[, r\,\mathrm{d}r)$, since $z$ is square-integrable on $\Omega$, so that $d_k$ must vanish for all $k$ and $c_k$ vanishes as soon as $k \geqslant 3\omega/\pi$, too.

Define

$$z_0(r,\theta) = \sum_{k=1}^{\infty} a_k r^{2+k\pi/\omega} \sin\frac{k\pi}{\omega}\theta + \sum_{k=1}^{\infty} b_k r^{k\pi/\omega} \sin\frac{k\pi}{\omega}\theta,$$

so that, on $\Omega \cap B(0,1)$, we have $z = z_0$ when $\omega \leqslant \frac{1}{3}\pi$, $z = z_0 + c_1 r^{2-\pi/\omega}$ when $\frac{1}{3}\pi < \omega \leqslant \frac{2}{3}\pi$, and $z = z_0 + c_1 r^{2-\pi/\omega} + c_2 r^{2-2\pi/\omega}$ when $\frac{2}{3}\pi < \omega < \pi$. Notice that these series have their radii of convergence equal to at least 1, since

$$\sum_{k=1}^{\infty} \frac{1}{k}(a_k^2 + b_k^2) \leqslant C \int_{\{r\leqslant 1\}} z(r,\theta)^2 r\,\mathrm{d}r\,\mathrm{d}\theta < \infty.$$

Choose $\chi$ a $C^\infty$ function, compactly supported in $\overline{\Omega \cap B(0,\frac{1}{2})}$ and identically equal to 1 in some neighborhood of $O$. Lemma 2.2 directly follows from the fact that

$$\forall u \in D(L) \int_\Omega \Delta^2(u\chi) z_0 = 0. \tag{2.8}$$

To prove it, we forget $\chi$ and assume that $u$ itself is compactly supported in $\overline{\Omega \cap B(0,\frac{1}{2})}$. Let $\varepsilon > 0$ and $\Omega_\varepsilon = \Omega \cap \overline{B(0,\varepsilon)}^c$. We have

$$\int_\Omega \Delta^2(u) z_0 = \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \Delta^2(u) z_0.$$

[1] For any integer $m \geqslant 0$, $H^m_{\text{loc}}(]0,1[) \overset{\text{def}}{=} \{v \in D'(]0,1[),\ \chi v \in H^m(]0,1[) \text{ for all } \chi \in C_c^\infty(]0,1[)\}$.

Our functions are regular enough to allow the use of Green's formula on $\Omega_\varepsilon$. Since $u$ vanishes on the boundary of $\Omega$ and $z_0$ is biharmonic on $\operatorname{supp} u$, we obtain

$$\begin{aligned}\int_{\Omega_\varepsilon} \Delta^2 u z_0 &= \int_{\gamma_\varepsilon} \Big(\frac{\partial \Delta u}{\partial n} z_0 - \Delta u \frac{\partial z_0}{\partial n}\Big)\,\mathrm{d}s + \int_{\Omega_\varepsilon} \Delta u \Delta z_0 \\ &= \int_{\gamma_\varepsilon} \Big(\frac{\partial \Delta u}{\partial n} z_0 - \Delta u \frac{\partial z_0}{\partial n}\Big)\,\mathrm{d}s + \int_{\gamma_\varepsilon} \Big(\frac{\partial u}{\partial n} z_0 - u \frac{\partial z_0}{\partial n}\Big)\,\mathrm{d}s,\end{aligned}$$

where $\gamma_\varepsilon$ is the arc $\{(\varepsilon, \theta);\ 0 < \theta < \omega\}$ and $\mathrm{d}s$ its arclength measure. We estimate the r.h.s. above using standard results on the traces:

$$\begin{aligned}\Big|\int_{\Omega_\varepsilon} \Delta^2 u z_0\Big| &\leqslant \Big\|\frac{\partial \Delta u}{\partial n}\Big\|_{H^{-3/2}(\gamma_\varepsilon)} \|z_0\|_{H^{3/2}(\gamma_\varepsilon)} + \|\Delta u\|_{H^{-1/2}(\gamma_\varepsilon)} \Big\|\frac{\partial z_0}{\partial n}\Big\|_{H^{1/2}(\gamma_\varepsilon)} \\ &\quad + \Big\|\frac{\partial u}{\partial n}\Big\|_{H^{-1/2}(\gamma_\varepsilon)} \|z_0\|_{H^{1/2}(\gamma_\varepsilon)} + \|u\|_{H^{1/2}(\gamma_\varepsilon)} \Big\|\frac{\partial z_0}{\partial n}\Big\|_{H^{-1/2}(\gamma_\varepsilon)} \\ &\leqslant C\|u\|_{D(L)} \|z_0 h_\varepsilon\|_{H^2(\Omega_\varepsilon)} + C\|u\|_{H^1(\Omega_\varepsilon)} \|z_0 h_\varepsilon\|_{H^1(\Omega_\varepsilon)}\end{aligned}$$

where $h_\varepsilon$ is a smooth cut-off function identically equal to 1 on $\gamma_\varepsilon$ and supported on $B(0, 3\varepsilon)$. One can check that $\|z_0 h_\varepsilon\|_{H^2} \leqslant C\varepsilon^{\pi/\omega - 1}$ and $\|z_0 h_\varepsilon\|_{H^1} \leqslant C\varepsilon^{\pi/\omega}$. This concludes the proof of (2.8) and of the Lemma 2.2.

**2.4. Proof of Theorem 2.1.** When $\omega \leqslant \frac{1}{3}\pi$, we have $D(L) = D_0(L)$ by Lemma 2.2, hence the $H^4$-regularity of the solutions to problem $(P_\omega)$ whatever $f \in L^2(\Omega)$.

When $\frac{1}{3}\pi < \omega \leqslant \frac{2}{3}\pi$ and $u \in D(L)$, define $\lambda \in \Lambda_0(L)$ by

$$\lambda(u) = \frac{1}{4\pi(\pi/\omega - 1)} \int_\Omega \Delta^2(u\chi)(r, \theta) r^{2-\pi/\omega} \sin\frac{\pi}{\omega}\theta r\,\mathrm{d}r\,\mathrm{d}\theta$$

for a fixed $C^\infty$ radial $\chi$, compactly supported in $\overline{\Omega \cap B(0, \frac{1}{2})}$ and identically equal to 1 in some neighborhood of $O$, picked on once and for all. Note that, using the proof of (2.8), we also have

$$\lambda(u) = \frac{1}{4\pi(\pi/\omega - 1)} \int_\Omega \Delta^2(u\chi)(r, \theta)(r^{2-\pi/\omega} - r^{\pi/\omega}) \sin\frac{\pi}{\omega}\theta r\,\mathrm{d}r\,\mathrm{d}\theta. \tag{2.9}$$

This identity implies that $\|\lambda\|$ is bounded uniformly with respect to $\omega$. Indeed, the proof results from the Cauchy-Schwarz inequality and the uniform estimate

$$|\delta(r, \theta)| \leqslant \frac{1}{2\pi} \quad \forall\, r \in [0, 1],\ \theta \in [0, \pi],\ \text{and } \omega \in \Big[\frac{\pi}{3}, \pi\Big],$$

where

$$\delta(r,\theta)=\begin{cases}\dfrac{r^{\pi/\omega}}{4\pi(\pi/\omega-1)}(r^{2-2\pi/\omega}-1)\sin\dfrac{\pi}{\omega}\theta & \text{if } \dfrac{\pi}{3}\leqslant\omega<\pi,\\ \dfrac{r}{4\pi}\ln r^2\sin\theta & \text{if } \omega=\pi,\end{cases}\tag{2.10}$$

so that $\delta(r,\theta)$ is continuous with respect to $\omega$ at $\omega=\pi$.

Let $\psi\in D(L)$ be such that $\psi(r,\theta)=r^{\pi/\omega}\sin(\pi\theta/\omega)$ when $(r,\theta)\in\operatorname{supp}\chi$ (that such a $\psi$ does exist is straightforward). Then, setting $\chi_1(r)=2\pi\chi'(r)/\omega+(r\chi'(r))'$, we have

$$\begin{aligned}\lambda(\psi)&=\frac{1}{4\pi(\pi/\omega-1)}\int_\Omega\Big(\Big(1-2\frac{\pi}{\omega}\Big)\chi_1(r)\\&\qquad+2\Big(\frac{\pi}{\omega}-1\Big)r\chi_1'(r)+r(r\chi_1'(r))'\Big)\sin^2\frac{\pi}{\omega}\theta\,\mathrm{d}r\,\mathrm{d}\theta\\&=\frac{1}{4\pi(\pi/\omega-1)}\int_0^\omega\sin^2\frac{\pi}{\omega}\theta\,\mathrm{d}\theta\int_0^1\Big(\Big(1-2\frac{\pi}{\omega}\Big)\chi_1(r)\\&\qquad+2\Big(\frac{\pi}{\omega}-1\Big)r\chi_1'(r)+r(r\chi_1'(r))'\Big)\,\mathrm{d}r.\end{aligned}$$

Note that the expression of $\chi_1$ contains only derivatives of $\chi$ so that all integrals w.r.t. $r$ in the previous identity are independent of the interior values of $\chi$ given that they involve only its boundary values at $r=0$ or $r=1$ and that $\chi(0)=1$ and $\chi(1)=0$. So, integration by parts leads after calculus to $\lambda(\psi)=\chi(0)=1$.

Therefore, $u_0=u-\lambda(u)\psi$ is in the kernel of the form $\lambda$, that is to say, in $D_0(L)$, since $\Lambda_0(L)$ is generated by $\lambda$. This means, by Lemma 2.1, that $u_0$ is in $H^4$, as desired. Moreover, the estimate given in Remark 2.1 shows that

$$\|u_0\|_{H^4(\Omega)}\leqslant C(\|u\|_{D(L)}+|\lambda(u)|)\leqslant C\|u\|_{D(L)},$$

which completes the case $\frac{1}{3}\pi<\omega\leqslant\frac{2}{3}\pi$.

When $\frac{2}{3}\pi<\omega<\pi$, we keep on the definition of the form $\lambda$ above, and define another form $\mu\in\Lambda_0(L)$ by

$$\mu(u)=\frac{1}{8\pi(2\pi/\omega-1)}\int_\Omega\Delta^2(u\chi)(r,\theta)r^{2-2\pi/\omega}\sin\frac{2\pi}{\omega}\theta r\,\mathrm{d}r\,\mathrm{d}\theta.$$

Note that $\|\mu\|$ is bounded uniformly with respect to $\omega$. Then, if we choose $\zeta\in D(L)$ such that $\zeta(r,\theta)=r^{2\pi/\omega}\sin(2\pi\theta/\omega)$ when $(r,\theta)\in\operatorname{supp}\chi$, an analogous calculation gives $\mu(\zeta)=1$. Note that $\lambda(\zeta)=\mu(\psi)=0$. Therefore, $u_0=u-\lambda(u)\psi-\mu(u)\zeta$ is in the kernel of the forms $\lambda,\mu$, that is to say, in $D_0(L)$, since $\Lambda_0(L)$ is generated by

$\lambda$ and $\mu$. This means, by Lemma 2.1, that $u_0$ is in $H^4$, as desired. Moreover, the estimate given in Remark 2.1 shows that

$$\|u_0\|_{H^4(\Omega)} \leqslant C(\|u\|_{D(L)} + |\lambda(u)| + |\mu(u)|) \leqslant C\|u\|_{D(L)},$$

which ends the proof of Theorem 2.1.

**2.5. Study of the singularity of solutions in the neighborhood of $\pi$.** We want now to describe more precisely how the functions $u_\omega$ converge toward $u_\pi$, emphasizing the resolution of their singularities. Recall that, for reason of simplicity, we restrict ourselves to the case $\omega \leqslant \pi$.

We know from Theorem 2.1 and from (2.9) that, when $\frac{2}{3}\pi < \omega < \pi$, the solution $u_\omega$ of the problem $(P_\omega)$ has near the origin the decomposition

$$u_\omega(r,\theta) = \lambda_\omega(u_\omega(r,\theta))r^{\pi/\omega} \sin\frac{\pi}{\omega}\theta + \mu_\omega(u_\omega(r,\theta))r^{2\pi/\omega} \sin\frac{2\pi}{\omega}\theta + u_{\omega,0}(r,\theta), \tag{2.11}$$

where

$$\lambda_\omega(u_\omega) = \frac{1}{4\pi(\pi/\omega - 1)} \int_{\Omega_\omega} \Delta^2(u_\omega\chi)(r,\theta)(r^{2-\pi/\omega} - r^{\pi/\omega}) \sin\frac{\pi}{\omega}\theta r \,\mathrm{d}r\,\mathrm{d}\theta, \tag{2.12}$$

$$\mu_\omega(u_\omega) = \frac{1}{8\pi(2\pi/\omega - 1)} \int_{\Omega_\omega} \Delta^2(u_\omega\chi)(r,\theta) r^{2-2\pi/\omega} \sin\frac{2\pi}{\omega}\theta r \,\mathrm{d}r\,\mathrm{d}\theta, \tag{2.13}$$

and $u_{\omega,0}$ is the regular part, of regularity $H^4$. Here, $\chi$ is a $C^\infty$ function, compactly supported in $\overline{\Omega \cap B(0,1)}$ and identically equal to 1 in some neighborhood of $O$. In the sequel, we forget $\chi$ and assume that $u_\omega$ and $u_\pi$ are compactly supported in $\overline{\Omega \cap B(0,1)}$. This simplification is harmless, as the reader will easily check.

The objective of this subsection is to prove the more precise result:

**Theorem 2.2.** *With the previous notation we have*

$$\lim_{\omega\to\pi} \lambda_\omega(u_\omega) = \lambda_\pi(u_\pi) = \frac{\partial u_\pi}{\partial y}(0,0), \tag{2.14}$$

$$\lim_{\omega\to\pi} \mu_\omega(u_\omega) = \mu_\pi(u_\pi) = \frac{1}{2}\frac{\partial^2 u_\pi}{\partial x \partial y}(0,0). \tag{2.15}$$

The Taylor expansion of $u_\pi$ near 0 being of the form

$$u_\pi(x,y) = \frac{\partial u_\pi}{\partial y}(0,0)y + \frac{1}{2}\frac{\partial^2 u_\pi}{\partial x\partial y}(0,0)xy + (x^2+y^2)\varepsilon(x,y),$$

where $\varepsilon(x,y) \to 0$ as $(x,y) \to (0,0)$, Theorem 2.2 shows that, in the decomposition (2.11) of $u_\omega$, each term converges to one of the terms of this development.

P r o o f. We prove the equality (2.14) starting from (2.12), which straightforwardly implies

$$\lim_{\omega\to\pi} \lambda_\omega(u_\omega) = \lambda_\pi(u_\pi) \stackrel{\text{def}}{=} -\frac{1}{2\pi}\int_{\Omega_\pi} \Delta^2(u_\pi)(r,\theta)\, r\ln r \sin\theta r \,\mathrm{d}r\,\mathrm{d}\theta.$$

To prove the equality $\lambda(u_\pi) = \partial u_\pi/\partial y(0,0)$, we apply Green's theorem and obtain

$$\lambda_\pi(u_\pi) = \frac{1}{\pi}\int_{\Omega_\pi} \Delta u_\pi(r,\theta) G^1_\pi(r,\theta)\, r\,\mathrm{d}r\,\mathrm{d}\theta,$$

where $G^1_\pi(r,\theta) = -(\sin\theta)/r$. Note that this integral exists, since $\Delta u_\pi \in L^\infty(\Omega_\pi)$, by Sobolev embeddings.

To go further, we classically put $\Omega_{\pi,\varepsilon} = \Omega_\pi \setminus B(0,\varepsilon)$ and

$$\lambda_{\pi,\varepsilon}(u_\pi) = \frac{1}{\pi}\int_{\Omega_{\pi,\varepsilon}} \Delta u_\pi(r,\theta) G^1_\pi((r,\theta) r\,\mathrm{d}r\,\mathrm{d}\theta$$

so that $\lim\limits_{\varepsilon\to 0} \lambda_{\pi,\varepsilon}(u_\pi) = \lambda_\pi(u_\pi)$. Applying Green's theorem, we find:

$$\begin{aligned}\lambda_{\pi,\varepsilon}(u_\pi) &= \frac{1}{\pi}\int_{\Omega_{\pi,\varepsilon}} \Delta u_\pi(r,\theta) G^1_\pi(r,\theta) r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= -\frac{1}{\pi}\int_{\Omega_{\pi,\varepsilon}} \nabla u_\pi(r,\theta)\cdot\nabla G^1_\pi(r,\theta) r\,\mathrm{d}r d\theta \\ &\quad + \frac{1}{\pi}\int_0^\pi -\frac{\partial u_\pi}{\partial r}(\varepsilon,\theta) G^1_\pi(\varepsilon,\theta)\varepsilon\,\mathrm{d}\theta.\end{aligned} \tag{2.16}$$

The boundary term is equal to

$$\frac{1}{\pi}\int_0^\pi \frac{\partial u_\pi}{\partial r}(\varepsilon,\theta)\sin\theta\,\mathrm{d}\theta.$$

Since $\nabla u_\pi$ is $C^1$, it converges towards

$$\frac{1}{\pi}\int_0^\pi \frac{\partial u_\pi}{\partial r}(0,\theta)\sin\theta\,\mathrm{d}\theta = \frac{1}{2}\frac{\partial u_\pi}{\partial y}(0,0).$$

Applying once more Green's theorem, we obtain that the other term in (2.16) is equal to

$$\begin{aligned}\frac{1}{\pi}\int_0^\pi u_\pi(\varepsilon,\theta)\frac{\partial G^1_\pi}{\partial r}(\varepsilon,\theta)\varepsilon\,\mathrm{d}\theta &= \frac{1}{\pi}\int_0^\pi \Big(\frac{\partial u_\pi}{\partial y}(0,0)\varepsilon\sin\theta + O(\varepsilon^2)\Big)\frac{1}{\varepsilon^2}\sin\theta\varepsilon\,\mathrm{d}\theta \\ &= \frac{1}{2}\frac{\partial u_\pi}{\partial y}(0,0) + O(\varepsilon).\end{aligned}$$

This concludes the proof of (2.14).

We prove similarly the equality (2.15) starting from:

$$\lim_{\omega\to\pi} \mu_\omega(u_\omega) = \mu_\pi(u_\pi) \overset{\text{def}}{=} \frac{1}{8\pi}\int_{\Omega_\pi} \Delta^2(u_\pi)(r,\theta)\sin 2\theta r \,\mathrm{d}r\,\mathrm{d}\theta.$$

We obtain as in the former case

$$\mu_\pi(u_\pi) = \frac{1}{2\pi}\int_{\Omega_\pi} \Delta u_\pi(r,\theta) G_\pi^2(r,\theta) r\,\mathrm{d}r\,\mathrm{d}\theta,$$

where $G_\pi^2(x,y) = -(\sin 2\theta)/r^2$ in polar coordinates. That this integral converges comes from the $H^2$ regularity of $\Delta u_\pi$ together with the Dirichlet boundary condition it fulfils, which implies that $\Delta u_\pi(r,\theta) = o(r^\alpha)$ for all $\alpha < 1$. We set $\Omega_{\pi,\varepsilon} = \Omega_\pi \backslash B(0,\varepsilon)$ and

$$\mu_{\pi,\varepsilon}(u_\pi) = \frac{1}{2\pi}\int_{\Omega_{\pi,\varepsilon}} \Delta u_\pi(r,\theta) G_\pi^2(r,\theta) r\,\mathrm{d}r\,\mathrm{d}\theta,$$

so that $\lim_{\varepsilon\to 0}\mu_{\pi,\varepsilon}(u_\pi) = \mu_\pi(u_\pi)$. Applying Green's theorem gives us

$$\begin{aligned} \mu_{\pi,\varepsilon}(u_\pi) = & -\frac{1}{2\pi}\int_{\Omega_{\pi,\varepsilon}} \nabla u_\pi(r,\theta)\cdot\nabla G_\pi^2(r,\theta) r\,\mathrm{d}r\,\mathrm{d}\theta \\ & + \frac{1}{2\pi}\int_0^\pi -\frac{\partial u_\pi}{\partial r}(r,\theta) G_\pi^2(\varepsilon,\theta)\varepsilon\,\mathrm{d}\theta. \end{aligned} \tag{2.17}$$

The boundary term reads

$$\frac{1}{2\pi}\int_0^\pi \frac{\partial u_\pi}{\partial r}(\varepsilon,\theta)\frac{1}{\varepsilon}\sin 2\theta\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^\pi \Big[\frac{\partial u_\pi}{\partial r}(\varepsilon,\theta) - \frac{\partial u_\pi}{\partial r}(0,\theta)\Big]\frac{1}{\varepsilon}\sin 2\theta\,\mathrm{d}\theta.$$

We thus have

$$\begin{aligned} \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_0^\pi \frac{\partial u_\pi}{\partial r}(\varepsilon,\theta)\frac{1}{\varepsilon}\sin 2\theta\,\mathrm{d}\theta &= \frac{1}{2\pi}\int_0^\pi \frac{\partial^2 u_\pi}{\partial r^2}(0,\theta)\sin 2\theta\,\mathrm{d}\theta \\ &= \frac{1}{4}\frac{\partial^2 u_\pi}{\partial x\partial y}(0,0). \end{aligned} \tag{2.18}$$

The other term in (2.17) is equal, by Green's formula again, to

$$\begin{aligned} &\frac{1}{2\pi}\int_0^\pi u_\pi(\varepsilon,\theta)\frac{\partial G_\pi^2}{\partial r}(\varepsilon,\theta)\varepsilon\,\mathrm{d}\theta \\ &\qquad = \frac{1}{\pi}\int_0^\pi \Big(\frac{\partial u_\pi}{\partial y}(0,0)\varepsilon\sin\theta + \frac{1}{2}\frac{\partial^2 u_\pi}{\partial x\partial y}(0,0)\varepsilon^2\sin 2\theta + o(\varepsilon^2)\Big)\frac{1}{\varepsilon^2}\sin 2\theta\,\mathrm{d}\theta. \end{aligned}$$

Passing to the limit, we obtain

$$\lim_{\varepsilon\to 0} -\frac{1}{2\pi}\int_{\Omega_{\pi,\varepsilon}} \nabla u_\pi(r,\theta)\cdot\nabla G^2_\pi(r,\theta) r\,\mathrm{d}r\,\mathrm{d}\theta = \frac{1}{4}\frac{\partial^2 u_\pi}{\partial x\partial y}(0,0). \tag{2.19}$$

We deduce from (2.19) and (2.18) that

$$\mu_\pi(u_\pi) = \frac{1}{2}\frac{\partial^2 u_\pi}{\partial x\partial y}(0,0).$$

This completes the proof of Theorem 2.2. □

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