LOCALLY POINTWISE SUPERCONVERGENCE OF THE TENSOR-PRODUCT FINITE ELEMENT IN THREE DIMENSIONS

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Abstract. Consider a second-order elliptic boundary value problem in three dimensions with locally smooth coefficients and solution. Discuss local superconvergence estimates for the tensor-product finite element approximation on a regular family of rectangular meshes. It will be shown that, by the estimates for the discrete Green's function and discrete derivative Green's function, and the relationship of norms in the finite element space such as L^2-norms, $W^{1,\infty}$-norms, and negative-norms in locally smooth subsets of the domain Ω, locally pointwise superconvergence occurs in function values and derivatives.

Keywords: tensor-product finite element; local superconvergence; discrete Green's function

MSC 2010: 65N30

1. Introduction

There have been many studies concerned with superconvergence of finite element methods in three dimensions (see [1]–[6], [8]–[16], [18]–[23], [25]). Most of them focus on the global superconvergent properties. However, to obtain the global superconvergent properties, it is necessary to satisfy two fundamental conditions: C-uniform partition (or piecewise C-uniform partition) and highly smooth solution such as $u \in W^{m+2,p}$ $(2 \leqslant p \leqslant \infty)$. Obviously, it is difficult to possess these two conditions in the whole domain Ω. Nevertheless, the above two conditions are easily satisfied in the interior subset of Ω, which leads us to consider superconvergent properties in interior subsets of Ω (so-called local superconvergent properties). Up to now, in fact,

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there have been some local superconvergence results to be stated (see [11], [22], [24] as well as literatures cited by them). Recently, we began to study local estimates for the three-dimensional finite element, and moreover, have made some progress.

In this paper, we will discuss and only focus on local superconvergence of the block finite element in three dimensions. We shall use the letter C to denote a generic constant which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

Consider then a real-valued second-order elliptic boundary value problem with variable coefficients in a bounded domain Ω in $\mathbb{R}^3$,

$$\mathcal{L}u \equiv -\sum_{i,j=1}^{3} \partial_j(a_{ij}(X)\partial_i u) + a_0(X)u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

It will be assumed that the coefficients of $\mathcal{L}$ are locally smooth and $a_{ij} = a_{ji}$, and also that they satisfy the uniform ellipticity condition

$$\sum_{i,j=1}^{3} a_{ij}(X)\xi_i\xi_j \geqslant \sigma_0 \sum_{i=1}^{3} |\xi_i|^2 \quad \forall\,(\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3$$

with σ_0 positive and locally independent of X.

The weak formulation of the above problem reads

$$\text{(1.1)} \qquad \begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ satisfying} \\ a(u,v) = (f,v) \quad \forall\, v \in H_0^1(\Omega), \end{cases}$$

where

$$a(u,v) \equiv \int_\Omega \Big(\sum_{i,j=1}^{3} a_{ij}\partial_i u \partial_j v + a_0 uv\Big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \quad (f,v) \equiv \int_\Omega fv\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Here $\partial_1 u = \partial u/\partial x$, $\partial_2 u = \partial u/\partial y$ and $\partial_3 u = \partial u/\partial z$, which are usual partial derivatives. For a given direction $l \in \mathbb{R}^3$ and $|l| = 1$, we denote by $\partial_l v(Z)$ the onesided directional derivatives defined by

$$\partial_{Z,l} v(Z) = \lim_{|\Delta Z|\to 0} \frac{v(Z+\Delta Z) - v(Z)}{|\Delta Z|}, \quad \Delta Z = |\Delta Z| l.$$

To discretize problem (1.1), we assume that Ω is partitioned into a regular rectangulation $\mathcal{T}^h$ with mesh size $h \in (0,1)$ such that $\overline{\Omega} = \bigcup_{e\in\mathcal{T}^h} \bar{e}$. Further, we denote by

$S_0^h(\Omega)$ the tensor-product m-degree block finite element space. Thus, the discretization of problem (1.1) is

$$\begin{cases} \text{Find } u_h \in S_0^h(\Omega) \text{ satisfying} \\ a(u_h, v) = (f, v) \quad \forall\, v \in S_0^h(\Omega). \end{cases}$$

To derive our main results, for each $Z \in \Omega$ we need yet to introduce a discrete Green's function $G_Z^h \in S_0^h(\Omega)$ and a discrete derivative Green's function $\partial_{Z,l} G_Z^h \in S_0^h(\Omega)$ defined by

$$a(v, G_Z^h) = v(Z) \quad \forall\, v \in S_0^h(\Omega) \tag{1.2}$$

and

$$a(v, \partial_{Z,l} G_Z^h) = \partial_{Z,l} v(Z) \quad \forall\, v \in S_0^h(\Omega). \tag{1.3}$$

As for G_Z^h and $\partial_{Z,l} G_Z^h$, we have (see [15]–[17])

$$|G_Z^h|_{2,1}^h = \mathcal{O}(|\ln h|^{2/3}), \tag{1.4}$$

$$|\partial_{Z,l} G_Z^h|_{2,1}^h = \mathcal{O}(h^{-1}), \tag{1.5}$$

$$|\partial_{Z,l} G_Z^h|_{1,1} = \mathcal{O}(|\ln h|^{4/3}), \tag{1.6}$$

where $|\cdot|_{2,1}^h = \sum\limits_{e \in \mathcal{T}^h} |\cdot|_{2,1,e}$.

2. Several important lemmas

In this section, we will give some lemmas which will be used to derive our main results.

Lemma 2.1. *Suppose $D \subset\subset D' \subset \Omega$ and the integer $k \geqslant 0$. Then we have*

$$\|v\|_{0,D} \leqslant C h^{-k} \|v\|_{-k,D'} \quad \forall\, v \in S_0^h(\Omega). \tag{2.1}$$

Proof. Set $\widehat{D} = \bigcup\limits_e \{e\colon\ e \cap D \neq \emptyset,\ e \in \mathcal{T}^h\}$. For an element $e \subset \widehat{D}$ we define a negative-norm as follows:

$$\|v\|_{-k,e} = \sup_{\varphi \in C_0^\infty(e)} \frac{|(v, \varphi)_e|}{\|\varphi\|_{k,e}}. \tag{2.2}$$

Further, we define an affine transformation by

$$F\colon\ \widetilde{X} \in \tilde{e} \longrightarrow X = \mathbf{B}\widetilde{X} + \mathbf{b} \in e,$$

where $\tilde{e}$ is a standard element and $\mathbf{B} = (b_{ij})$ is a matrix of order 3×3. We write $\widetilde{\varphi}(\widetilde{X}) = \varphi(F(\widetilde{X}))$ and $\tilde{v}(\widetilde{X}) = v(F(\widetilde{X}))$. In addition, we have (see [24])

$$|w|_{k,p,e} \leqslant C\|\mathbf{B}^{-1}\|^k |\det\mathbf{B}|^{1/p} |\widetilde{w}|_{k,p,\tilde{e}} \quad \forall\, \widetilde{w} \in W^{k,p}(\tilde{e}).$$

Thus we get

$$|\varphi|_{k,e} \leqslant Ch_e^{3/2-k} |\widetilde{\varphi}|_{k,\tilde{e}}. \tag{2.3}$$

From (2.3), we obtain

$$\|\varphi\|_{k,e}^2 = \sum_{i=0}^{k} |\varphi|_{i,e}^2 \leqslant Ch_e^{3-2k} \sum_{i=0}^{k} |\widetilde{\varphi}|_{i,\tilde{e}}^2 = Ch_e^{3-2k} \|\widetilde{\varphi}\|_{k,\tilde{e}}^2,$$

namely,

$$\|\varphi\|_{k,e} \leqslant Ch_e^{(3-2k)/2} \|\widetilde{\varphi}\|_{k,\tilde{e}}. \tag{2.4}$$

By (2.4), the definition of the negative norm (2.2), and the equivalence of norms in the finite-dimensional space, we have

$$\begin{aligned} \|v\|_{0,e} &\leqslant Ch_e^{3/2} \|\tilde{v}\|_{0,\tilde{e}} \leqslant Ch_e^{3/2} \|\tilde{v}\|_{-k,\tilde{e}} \leqslant Ch_e^{3/2} \sup_{\widetilde{\varphi} \in C_0^\infty(\tilde{e})} \frac{|(\tilde{v}, \widetilde{\varphi})_{\tilde{e}}|}{\|\widetilde{\varphi}\|_{k,\tilde{e}}} \\ &\leqslant Ch_e^{3/2-3+(3-2k)/2} \sup_{\varphi \in C_0^\infty(e)} \frac{|(v, \varphi)_e|}{\|\varphi\|_{k,e}}, \end{aligned}$$

namely,

$$\|v\|_{0,e} \leqslant Ch_e^{-k} \|v\|_{-k,e}. \tag{2.5}$$

Thus, from (2.5) and $1 \leqslant h/h_e \leqslant C_0$,

$$\|v\|_{0,\widehat{D}}^2 = \sum_e \|v\|_{0,e}^2 \leqslant Ch^{-2k} \sum_e \|v\|_{-k,e}^2. \tag{2.6}$$

For every $\varepsilon > 0$, choosing $\varepsilon_e > 0$ such that $\sum\limits_e \varepsilon_e = \varepsilon$, we have

$$\|v\|_{-k,e}^2 - \varepsilon_e \leqslant |(v, \varphi_e)_e|^2, \quad \varphi_e \in C_0^\infty(e) \text{ and } \|\varphi_e\|_{k,e} = 1. \tag{2.7}$$

We write $\omega = \sum\limits_e (v, \varphi_e)_e \varphi_e \in C_0^\infty(D')$ and then

$$(v, \omega)_{D'} = \sum_e |(v, \varphi_e)_e|^2. \tag{2.8}$$

Combining (2.6)–(2.8) yields

$$\|v\|^2_{0,\widehat{D}} \leqslant Ch^{-2k}((v, \omega)_{D'} + \varepsilon) \leqslant Ch^{-2k}(\|v\|_{-k,D'}\|\omega\|_{k,D'} + \varepsilon). \tag{2.9}$$

In addition,

$$\begin{aligned}
\|\omega\|^2_{k,D'} &= \int_{D'} \sum_{0\leqslant s\leqslant k} \Big|\sum_e (v, \varphi_e)_e \nabla^s \varphi_e\Big|^2 \,\mathrm{d}X \\
&= \sum_{0\leqslant s\leqslant k} \int_{D'} \Big|\sum_e (v, \varphi_e)_e \nabla^s \varphi_e\Big|^2 \,\mathrm{d}X \\
&= \sum_{0\leqslant s\leqslant k} \sum_e |(v, \varphi_e)_e|^2 \int_e |\nabla^s \varphi_e|^2 \,\mathrm{d}X \\
&= \sum_e |(v, \varphi_e)_e|^2 = (v, \omega)_{D'} \leqslant \|v\|_{-k,D'}\|\omega\|_{k,D'}.
\end{aligned}$$

Thus,

$$\|\omega\|_{k,D'} \leqslant \|v\|_{-k,D'}. \tag{2.10}$$

When $\varepsilon \to 0$, we have by (2.9) and (2.10)

$$\|v\|_{0,\widehat{D}} \leqslant Ch^{-k}\|v\|_{-k,D'}.$$

Obviously, $D \subset \widehat{D}$, thus $\|v\|_{0,D} \leqslant \|v\|_{0,\widehat{D}} \leqslant Ch^{-k}\|v\|_{-k,D'}$. The proof of Lemma 2.1 is completed. □

Lemma 2.2. *Suppose $D \subset\subset D' \subset \Omega$, $d \equiv \operatorname{dist}(\partial D, \partial D')$, and the boundary $\partial D'$ is smooth enough. Let the integer $k \geqslant 0$, $a_{ij} \in W^{k+2,\infty}(D')$, and $\chi \in S_0^h(\Omega)$ satisfy $a(\chi, v) = 0$ for all $v \in S_0^h(D')$. Then we have*

$$\|\chi\|_{-k,D} \leqslant C(d)h\|\chi\|_{1,D'} + C(d)\|\chi\|_{-k-1,D'}. \tag{2.11}$$

P r o o f. Choosing D_1 such that $D \subset\subset D_1 \subset\subset D'$, $\operatorname{dist}(\partial D_1, \partial D') = \operatorname{dist}(\partial D_1, \partial D) = d/2$, and $\mu \in C^\infty(\Omega)$ satisfying $\operatorname{supp}\mu \subset\subset D'$ and $\mu|_{D_1} = 1$, and setting $\widehat{\chi} = \mu\chi$, we have by the a priori estimate

$$\|\chi\|_{-k,D} \leqslant \|\widehat{\chi}\|_{-k,D'} = \sup_{\varphi \in C_0^\infty(D')} \frac{|(\varphi, \widehat{\chi})_{D'}|}{\|\varphi\|_{k,D'}} \leqslant C \sup_{w \in \mathcal{H}} \frac{|a(w, \widehat{\chi})_{D'}|}{\|w\|_{k+2,D'}}, \tag{2.12}$$

where $\mathcal{L}w = \varphi$ and $w \in \mathcal{H} \equiv H^{k+2}(D') \cap H_0^1(D')$. Similarly to the arguments of Theorem 5.6 in [24], with the conditions of this lemma we get

$$a(w, \widehat{\chi})_{D'} = a(\widehat{w}, \chi)_{D'} + I_{D'} = a(\widehat{w} - \Pi\widehat{w}, \chi)_{D'} + I_{D'}, \tag{2.13}$$

where $\widehat{w} = \mu w$ and

$$I_{D'} = \int_{D'} \sum_{i,j=1}^{3} (-\partial_j(\chi w a_{ij} \partial_i \mu) + \chi \partial_j(w a_{ij} \partial_i \mu) + \chi a_{ij} \partial_i w \partial_j \mu) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z.$$

Since $w \in \mathcal{H}$, we have

$$\begin{aligned} |I_{D'}| &= \left| \int_{D'} \sum_{i,j=1}^{3} (\chi \partial_j(w a_{ij} \partial_i \mu) + \chi a_{ij} \partial_i w \partial_j \mu) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \right| \\ &\leqslant C(d) \|\chi\|_{-k-1,D'} \|w\|_{k+2,D'}. \end{aligned} \tag{2.14}$$

From (2.13) and (2.14), we obtain

$$\begin{aligned} |a(w, \widehat{\chi})_{D'}| &\leqslant C\|\chi\|_{1,D'} \|\widehat{w} - \Pi\widehat{w}\|_{1,D'} + C(d)\|\chi\|_{-k-1,D'} \|w\|_{k+2,D'} \\ &\leqslant C(d)h\|\chi\|_{1,D'} \|w\|_{k+2,D'} + C(d)\|\chi\|_{-k-1,D'} \|w\|_{k+2,D'}. \end{aligned} \tag{2.15}$$

Combining (2.12) and (2.15) yields the result (2.11). The proof of Lemma 2.2 is completed. □

Lemma 2.3. *Suppose $D' \subset \Omega$ and the boundary $\partial D'$ is smooth enough. Let the integer $k \geqslant 0$, $a_{ij} \in W^{k+2,\infty}(D')$, and $\chi \in S_0^h(\Omega)$ satisfy $a(\chi, v) = 0$ for all $v \in S_0^h(D')$. For each D^* and D^{**} satisfying $D^* \subset\subset D^{**} \subset\subset D'$, we then have*

$$\|\chi\|_{1,\infty,D^*} + \|\chi\|_{-k,D^*} \leqslant C(d)\|\chi\|_{-k-1,D^{**}}, \tag{2.16}$$

where $d \equiv \operatorname{dist}(\partial D^*, \partial D^{**})$.

P r o o f. When $k=0$, choosing $\widetilde{D}$ such that $D^* \subset\subset \widetilde{D} \subset\subset D^{**}$ and $\operatorname{dist}(\partial\widetilde{D}, \partial D^{**}) = \operatorname{dist}(\partial\widetilde{D}, \partial D^*) = d/2$, we have by similar arguments as for Theorem 5.6 in [24]

$$\|\chi\|_{1,\infty,D^*} \leqslant C(d)h\|\chi\|_{0,\widetilde{D}} + C(d)\|\chi\|_{-1,\widetilde{D}}. \tag{2.17}$$

Combining (2.1) and (2.17) yields

$$\|\chi\|_{0,D^*} \leqslant \|\chi\|_{1,\infty,D^*} \leqslant C(d)\|\chi\|_{-1,D^{**}}, \tag{2.18}$$

which indicates, when $k=0$, the result (2.16) holds. Next, when $k=t$, we suppose the result (2.16) holds, namely,

$$\|\chi\|_{1,\infty,D^*} + \|\chi\|_{-t,D^*} \leqslant C(d)\|\chi\|_{-t-1,D^{**}}. \tag{2.19}$$

We consider the case of $k=t+1$. Choosing $\{D_i\}_{i=0}^{t+2}$ such that $D^* \subset\subset \widetilde{D} \subset\subset D_0 \subset\subset D_1 \subset\subset D_2 \subset\subset \ldots \subset\subset D_{t+2} \subset\subset D^{**}$, and $\operatorname{dist}(\partial\widetilde{D}, \partial D_0) = \operatorname{dist}(\partial D_i, \partial D_{i+1}) = d/(2(t+4))$, $i = 0, \ldots, t+1$, we have by (2.11) and (2.19)

$$\begin{aligned} \|\chi\|_{-t-1,\widetilde{D}} &\leqslant C(d)h\|\chi\|_{1,D_0} + C(d)\|\chi\|_{-t-2,D_0} \\ &\leqslant C(d)h\|\chi\|_{1,\infty,D_0} + C(d)\|\chi\|_{-t-2,D_0} \\ &\leqslant C(d)h\|\chi\|_{-t-1,D_1} + C(d)\|\chi\|_{-t-2,D_1}. \end{aligned} \tag{2.20}$$

Similarly,

$$\|\chi\|_{-t-1,D_i} \leqslant C(d)h\|\chi\|_{-t-1,D_{i+1}} + C(d)\|\chi\|_{-t-2,D_{i+1}}, \quad i = 1, 2, \ldots, t+1. \tag{2.21}$$

From (2.1), (2.20), and (2.21),

$$\begin{aligned} \|\chi\|_{-t-1,\widetilde{D}} &\leqslant C(d)h^{t+2}\|\chi\|_{-t-1,D_{t+2}} + C(d)\|\chi\|_{-t-2,D_{t+2}} \\ &\leqslant C(d)h^{t+2}\|\chi\|_{0,D_{t+2}} + C(d)\|\chi\|_{-t-2,D_{t+2}} \\ &\leqslant C(d)\|\chi\|_{-t-2,D^{**}}. \end{aligned} \tag{2.22}$$

In addition, from (2.19) and (2.22),

$$\|\chi\|_{1,\infty,D^*} \leqslant C(d)\|\chi\|_{-t-1,\widetilde{D}} \leqslant C(d)\|\chi\|_{-t-2,D^{**}}. \tag{2.23}$$

Thus, from (2.22) and (2.23),

$$\|\chi\|_{1,\infty,D^*} + \|\chi\|_{-t-1,D^*} \leqslant C(d)\|\chi\|_{-t-2,D^{**}},$$

which shows, when $k=t+1$, the result (2.16) holds. The proof of Lemma 2.3 is completed. $\square$

Lemma 2.4. *Suppose $D \subset\subset D' \subset \Omega$, $d \equiv \operatorname{dist}(\partial D, \partial D')$, and the boundary $\partial D'$ is smooth enough. Let the integer $k \geqslant 0$, $a_{ij} \in W^{k+2,\infty}(D')$, and $\chi \in S_0^h(\Omega)$ satisfy $a(\chi, v) = 0$ for all $v \in S_0^h(D')$. Then we have*

$$\|\chi\|_{0,D} \leqslant C(d)\|\chi\|_{-k-1,D'}, \tag{2.24}$$

$$\|\chi\|_{1,\infty,D} \leqslant C(d)\|\chi\|_{-k-1,D'}. \tag{2.25}$$

Proof. Choosing $\{D_i\}_{i=1}^{k+1}$ such that $D \subset\subset D_1 \subset\subset D_2 \subset\subset \ldots \subset\subset D_k \subset\subset D_{k+1} = D'$, and $\operatorname{dist}(\partial D, \partial D_1) = \operatorname{dist}(\partial D_i, \partial D_{i+1}) = d/(k+1)$, $i = 1, \ldots, k$, we have by (2.16)

$$\begin{aligned}\|\chi\|_{1,\infty,D} \leqslant C(d)\|\chi\|_{-1,D_1} \leqslant C(d)\|\chi\|_{-2,D_2} \leqslant C(d)\|\chi\|_{-3,D_3}\\ \leqslant \ldots \leqslant C(d)\|\chi\|_{-k-1,D'},\end{aligned}$$

which is the result (2.25). Obviously,

$$\|\chi\|_{0,D} \leqslant \|\chi\|_{1,\infty,D}.$$

Combined with (2.25), we immediately obtain the result (2.24). The proof of Lemma 2.4 is completed. □

3. Locally pointwise superconvergence

In this section, we will give our main results on the locally pointwise superconvergence.

Theorem 3.1. *Suppose $D \subset\subset D' \subset \Omega$ and let u_h be the tensor-product m-degree block finite element approximation to the solution u of (1.1), and Π_m the corresponding interpolation operator. When $u \in W^{m+2,\infty}(D') \cap H_0^1(\Omega)$ and $m \geqslant 1$, we have*

$$|u_h - \Pi_m u|_{1,\infty,D} \leqslant C(h^{m+1}|\ln h|^{4/3}\|u\|_{m+2,\infty,D'} + \|u - u_h\|_{-1,D'}). \tag{3.1}$$

Especially, if the boundary $\partial D'$ is smooth enough and the integer $k \geqslant 0$, we then have

$$|u_h - \Pi_m u|_{1,\infty,D} \leqslant C(h^{m+1}|\ln h|^{4/3}\|u\|_{m+2,\infty,D'} + \|u - u_h\|_{-k-1,D'}). \tag{3.2}$$

P r o o f. Choose D'' such that $D \subset\subset D'' \subset\subset D'$, and take $\mu \in C^\infty(\Omega)$ such that $\operatorname{supp}\mu \subset\subset D'$, and $\mu|_{D''} = 1$. Set $\tilde{u} = \mu u$ and $\bar{u} = u - \tilde{u}$. Thus, for each $Z \in D$ we have by the weak estimate of the first type (see [19]) and (1.6)

$$
\begin{aligned}
|\partial_l(\tilde{u}_h - \Pi_m \tilde{u})(Z)| &= a(\tilde{u}_h - \Pi_m \tilde{u}, \partial_{Z,l} G_Z^h) = a(\tilde{u} - \Pi_m \tilde{u}, \partial_{Z,l} G_Z^h) \\
&\leqslant C h^{m+1} \|\tilde{u}\|_{m+2,\infty,D'} |\partial_{Z,l} G_Z^h|_{1,1} \\
&\leqslant C h^{m+1} |\ln h|^{4/3} \|u\|_{m+2,\infty,D'}.
\end{aligned} \tag{3.3}
$$

Obviously, $\bar{u} = 0$ in D''. For every $v \in S_0^h(D'')$ we have $a(\bar{u}_h - \bar{u}, v) = 0$. Thus,

$$
a(\bar{u}_h, v) = a(\bar{u}, v) = 0. \tag{3.4}
$$

In addition, similarly to (2.17) we get

$$
\|\bar{u}_h\|_{1,\infty,D} \leqslant C(\varrho) h \|\bar{u}_h\|_{0,D''} + C(\varrho) \|\bar{u}_h\|_{-1,D''}, \tag{3.5}
$$

where $\varrho \equiv \operatorname{dist}(\partial D, \partial D'')$. From (2.1), (3.4), and (3.5),

$$
\begin{aligned}
|\bar{u}_h - \Pi_m \bar{u}|_{1,\infty,D} &= |\bar{u}_h|_{1,\infty,D} \leqslant C h \|\bar{u}_h\|_{0,D''} + C \|\bar{u}_h\|_{-1,D''} \\
&\leqslant C \|\bar{u}_h\|_{-1,D''} = C \|\bar{u} - \bar{u}_h\|_{-1,D''} \\
&\leqslant C \|u - u_h\|_{-1,D''} + C \|\tilde{u} - \tilde{u}_h\|_{-1,D''}.
\end{aligned} \tag{3.6}
$$

In addition,

$$
\|\tilde{u} - \tilde{u}_h\|_{-1,D''} \leqslant \|\tilde{u} - \tilde{u}_h\|_{0,D''} \leqslant C h^{m+1} \|\tilde{u}\|_{m+1,D'} \leqslant C h^{m+1} \|u\|_{m+1,D'}. \tag{3.7}
$$

Combining (3.3), (3.6), and (3.7) yields the result (3.1). If the boundary $\partial D'$ is smooth enough and the integer $k \geqslant 0$, similarly to the arguments of (3.1), we may get by Lemma 2.4 the result (3.2). The proof of Theorem 3.1 is completed. □

Theorem 3.1 is concerning the case of $m \geqslant 1$. In fact, when $m \geqslant 2$, we have yet the following superconvergent estimates.

Theorem 3.2. *Suppose $D \subset\subset D' \subset \Omega$ and let u_h be the tensor-product m-degree block finite element approximation to the solution u of (1.1), and Π_m the corresponding interpolation operator. When $u \in W^{m+2,\infty}(D') \cap H_0^1(\Omega)$ and $m \geqslant 2$, we have*

$$
|u_h - \Pi_m u|_{1,\infty,D} \leqslant C(h^{m+1} \|u\|_{m+2,\infty,D'} + \|u - u_h\|_{-1,D'}). \tag{3.8}
$$

Especially, if the boundary $\partial D'$ is smooth enough and the integer $k \geqslant 0$, we then have

$$
|u_h - \Pi_m u|_{1,\infty,D} \leqslant C(h^{m+1} \|u\|_{m+2,\infty,D'} + \|u - u_h\|_{-k-1,D'}). \tag{3.9}
$$

P r o o f. By the arguments of Theorem 3.1, we find that it is only needed to prove

$$|\tilde{u}_h - \Pi_m \tilde{u}|_{1,\infty,D} \leqslant Ch^{m+1}\|u\|_{m+2,\infty,D'}. \tag{3.10}$$

In fact, for each $Z \in D$ we have by the weak estimate of the second type (see [19]) and (1.5)

$$\begin{aligned}|\partial_l(\tilde{u}_h - \Pi_m\tilde{u})(Z)| &= a(\tilde{u}_h - \Pi_m\tilde{u}, \partial_{Z,l}G_Z^h) = a(\tilde{u} - \Pi_m\tilde{u}, \partial_{Z,l}G_Z^h)\\ &\leqslant Ch^{m+2}\|\tilde{u}\|_{m+2,\infty,D'}|\partial_{Z,l}G_Z^h|_{2,1}\\ &\leqslant Ch^{m+1}\|u\|_{m+2,\infty,D'},\end{aligned}$$

which leads to (3.10). Thus, the result (3.8) holds. If the boundary $\partial D'$ is smooth enough and the integer $k \geqslant 0$, similarly to the arguments of (3.8), we may get by Lemma 2.4 the result (3.9). The proof of Theorem 3.2 is completed. □

The above two theorems are about local superconvergence of derivatives. As for function values, we have also the similar results.

Theorem 3.3. *Suppose $D \subset\subset D' \subset \Omega$ and let u_h be the tensor-product m-degree block finite element approximation to the solution u of* (1.1), *and Π_m the corresponding interpolation operator. When $u \in W^{m+2,\infty}(D') \cap H_0^1(\Omega)$ and $m \geqslant 2$, we have*

$$|u_h - \Pi_m u|_{0,\infty,D} \leqslant C(h^{m+2}|\ln h|^{2/3}\|u\|_{m+2,\infty,D'} + \|u - u_h\|_{-1,D'}). \tag{3.11}$$

Especially, if the boundary $\partial D'$ is smooth enough and the integer $k \geqslant 0$, we then have

$$|u_h - \Pi_m u|_{0,\infty,D} \leqslant C(h^{m+2}|\ln h|^{2/3}\|u\|_{m+2,\infty,D'} + \|u - u_h\|_{-k-1,D'}). \tag{3.12}$$

R e m a r k 3.1. Similarly to the arguments of Theorems 3.1 and 3.2, together with (1.4), the results (3.11) and (3.12) will be proved.

R e m a r k 3.2. As for the negative norms in the above theorems, we now give their bounds. From the definition of the negative norm, we have $\|u - u_h\|_{-k-1,D'} \leqslant \|u - u_h\|_{-k-1,\Omega}$. Thus, for each $\varphi \in H^{k+1}(\Omega)$ there exists an $\widetilde{\varphi} \in H^{k+3}(\Omega) \cap H_0^1(\Omega)$ such that

$$|(u - u_h, \varphi)| = |a(u - u_h, \widetilde{\varphi} - \Pi_m\widetilde{\varphi})| \leqslant C\|u - u_h\|_1\,\|\widetilde{\varphi} - \Pi_m\widetilde{\varphi}\|_1.$$

Further, when $m \geqslant 2$ and $0 \leqslant k \leqslant m-2$, we have by the interpolation error estimate the optimal approximation estimate and the a priori estimate

$$|(u-u_h,\varphi)| \leqslant Ch^{m+k+2}\|u\|_{m+1}\|\widetilde{\varphi}\|_{k+3} \leqslant Ch^{m+k+2}\|u\|_{m+1}\|\varphi\|_{k+1}.$$

Thus we have

$$\|u-u_h\|_{-k-1,D'} \leqslant \|u-u_h\|_{-k-1,\Omega} \leqslant Ch^{m+k+2}\|u\|_{m+1}, \quad 0 \leqslant k \leqslant m-2.$$

When $m=1$, we have

$$\|u-u_h\|_{-1,D'} \leqslant \|u-u_h\|_{-1,\Omega} \leqslant C\|u-u_h\|_{0,\Omega} \leqslant Ch^2\|u\|_2.$$

The above results show that the negative norms do not spoil the order of superconvergence.

Using the results in Theorems 3.1–3.3 and the estimates for negative norms, we easily obtain the corresponding superconvergence points and their estimates.

Let $X^* \in D$ be an interpolation point of the operator Π_m. Then we have the function value superconvergence estimate

$$|(u-u_h)(X^*)| \leqslant Ch^{m+2}|\ln h|^{2/3}\|u\|_{m+2,\infty,D'}, \quad m \geqslant 2. \tag{3.13}$$

In fact, when $X^* \in D$ is a zero-point of the antiderivative of the *Legendre polynomial* of degree m, the superconvergent result (3.13) holds too.

In addition, let $Y^* \in D$ be a stress good point of the operator Π_m (usually taking zero-point of the Legendre polynomial of degree m), thus we have the gradient superconvergence estimates

$$|\overline{\nabla}(u-u_h)(Y^*)| \leqslant Ch^{m+1}|\ln h|^{4/3}\|u\|_{m+2,\infty,D'}, \quad m \geqslant 1, \tag{3.14}$$

and

$$|\overline{\nabla}(u-u_h)(Y^*)| \leqslant Ch^{m+1}\|u\|_{m+2,\infty,D'}, \quad m \geqslant 2. \tag{3.15}$$

Remark 3.3. The study of superconvergence at special points started with the work by Douglas, Dupont, and Wheeler [7]. Later, many works on superconvergence points have been given. For example, Schatz, Sloan, and Wahlbin [22] gave superconvergence estimates on locally symmetric points. Lin and Zhang [14] pointed out where are the natural superconvergence points in three-dimensional setting. However, our results in the present paper differ from the ones in [14] and [22]. The details are as follows.

(1) For function value superconvergence, [22] requires superconvergence points being locally symmetric points, and moreover, the degree m of the finite element is even. However, our results do not need these conditions. When the negative norm $\|u-u_h\|_{-k-1,D'}$ is of the highest order $O(h^{2m})$ with $k=m-2$, the function value convergence order at superconvergence points in [22] is $O(h^{m+1+(m-1)/(m+1.5)}|\ln h|)$ ($m\geqslant 2$ even) and $O(h^{m+2}|\ln h|^{2/3})$ ($m\geqslant 2$ any integer) in the present paper.

(2) For derivative superconvergence, [22] still requires superconvergence points being locally symmetric points, and the degree m of the finite element is odd, which are unnecessary in our paper. When the negative norm $\|u-u_h\|_{-k-1,D'}$ is of the highest order $O(h^{2m})$ with $k=m-2$, the derivative convergence order at superconvergence points in [22] is $O(h^{m+m/(m+2.5)}|\ln h|)$ ($m>2$ odd) and $O(h^{m+1})$ ($m\geqslant 2$ any integer) in the present paper. For $m=1$, the former is $O(h^{1+2/7}|\ln h|)$ and the latter is $O(h^2|\ln h|^{4/3})$.

In summary, our results are better than the ones in [22]. In addition, although [14] also gave the same superconvergence points as the present paper, it did not show the convergence order at these points.

Example 3.1. Consider the following Poisson equation:

$$\begin{cases} -\Delta u = f & \text{in } \Omega=(0,1)\times(0,1)\times(0,1),\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where

$$\begin{aligned} f = (&-\mathrm{e}^x(\mathrm{e}^y-(\mathrm{e}-1)y-1)-\mathrm{e}^y(\mathrm{e}^x-(\mathrm{e}-1)x-1)\\ &+\pi^2(\mathrm{e}^x-(\mathrm{e}-1)x-1)(\mathrm{e}^y-(\mathrm{e}-1)y-1))\sin(\pi z).\end{aligned}$$

The true solution is

$$u=(\mathrm{e}^x-(\mathrm{e}-1)x-1)(\mathrm{e}^y-(\mathrm{e}-1)y-1)\sin(\pi z).$$

Let u_h be the tensor-product two-degree finite element approximation to u. Set $X^*=(0.5,0.5,0.5)$ and

$$Y^*=\Big(\frac{3-\sqrt{3}}{6}h,\frac{3-\sqrt{3}}{6}h,\frac{3-\sqrt{3}}{6}h\Big).$$

Obviously, X^* is an interpolation point and Y^* is a zero-point of the *Legendre polynomial* of degree 2 in Ω. Both of them are the function value superconvergence point and the derivative superconvergence point, respectively. For simplicity, we consider only the numerical results at X^* and Y^*. We solve Example 3.1 and obtain the following numerical results:

h	$\lvert(u-u_h)(X^*)\rvert$	$\lvert\partial_x(u-u_h)(Y^*)\rvert$
0.5	2.4184e − 004	6.4341e − 003
0.25	1.3204e − 005	4.9586e − 004
0.125	7.8025e − 007	5.3748e − 005

Table 3.1. Numerical results at superconvergence points on uniform meshes.

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