ON THE INVERSE EIGENVALUE PROBLEM FOR A SPECIAL KIND OF ACYCLIC MATRICES

Mohammad Heydari, Seyed Abolfazl Shahzadeh Fazeli, Seyed Mehdi Karbassi, Yazd

Received September 6, 2018. Published online April 24, 2019.

Abstract. We study an inverse eigenvalue problem (IEP) of reconstructing a special kind of symmetric acyclic matrices whose graph is a generalized star graph. The problem involves the reconstruction of a matrix by the minimum and maximum eigenvalues of each of its leading principal submatrices. To solve the problem, we use the recurrence relation of characteristic polynomials among leading principal minors. The necessary and sufficient conditions for the solvability of the problem are derived. Finally, a numerical algorithm and some examples are given.

Keywords: inverse eigenvalue problem; leading principal minor; graph of a matrix

MSC 2010: 65F18, 05C50

1. Introduction

An inverse eigenvalue problem (IEP) is the problem of reconstructing a matrix with a special structure from prescribed spectral data. The level of difficulty of the IEP depends on the available spectral data and the structure of the matrices which are to be constructed. In [1], Chu and Golub gave a very perfect characterization of IEPs. IEPs have many practical applications such as control theory, mechanical system simulation, geophysics, molecular spectroscopy, structural analysis, mass spring vibrations, circuit theory and graph theory [1], [5], [6].

An important class of IEPs is that of partially described IEPs and it may occur in computations involving a complicated physical system where it is often difficult or impossible to obtain the entire spectrum. In 2006 Peng et al. [7] discussed two partially described IEPs for a special kind of acyclic matrices. One of the problems is to construct the matrices by the minimal and maximal eigenvalues of its all leading principal submatrices. More precisely, they constructed an $n\times n$ matrix with a special

DOI: 10.21136/AM.2019.0242-18

structure from a given list $\lambda_1^{(n)}, \dots, \lambda_1^{(j)}, \dots, \lambda_1^{(1)}, \dots, \lambda_j^{(j)}, \dots, \lambda_n^{(n)}$ such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are, respectively, the minimum and maximum eigenvalues of all its $j \times j$ leading principal submatrices for any $j = 1, 2, \dots, n$. Sharma and Sen in 2016 [8] and 2018 [9] studied the same problem for symmetric tridiagonal and other kinds of acyclic matrices. In 2017, Xu and Chen [10] solved this problem for two kinds of special banded matrices. In this paper we investigate the same IEP, namely IEPGS, of constructing a special kind of acyclic matrices whose graph is a generalized star graph by the minimal and maximal eigenvalues of its all leading principal submatrices.

The paper is organized as follows: Section 2 describes the preliminary concepts and the notation used in this paper. Section 3 deals with the analysis of the IEPGS and the main results. In Section 4, we present some numerical examples to illustrate the solutions of IEPGS. In Section 5 we conclude the paper.

2. Preliminaries

A graph is an ordered pair $G = (V, E)$ comprising a set V of vertices and a set E of edges which are two-element subsets $\{v_i, v_j\}$, $v_i \neq v_j$ of V. Two vertices in a graph are called *adjacent* if there is an edge between them. A path P_n is a graph G with vertices $v_1, v_2, \dots, v_n$ such that each v_i is adjacent only to v_{i+1}, for $i = 1, \dots, n-1$. Figure 1 shows a path graph with four vertices.

Figure 1. Path graph with four vertices.

Given an $n \times n$ symmetric matrix A, the graph of A, denoted by $G_A(V, E)$, is a graph such that $V = \{v_1, v_2, \dots, v_n\}$ and $E = \bigcup_{a_{ij} \neq 0, i \neq j} v_i v_j$. For a graph $G = (V, E)$ where $|V| = n$, $S(G)$ denotes the set of all $n \times n$ symmetric matrices such that G corresponds to its graph.

The matrix $A^{(n-1)}$ of a P_n is the symmetric matrix

$$A^{(n-1)} = \begin{bmatrix} a_1 & b_1 & 0 & 0 & 0 & 0 & \dots & 0 \\ b_1 & a_2 & b_2 & 0 & 0 & 0 & \dots & 0 \\ 0 & b_2 & a_3 & b_3 & 0 & 0 & \dots & 0 \\ 0 & 0 & b_3 & a_4 & b_4 & 0 & \dots & 0 \\ 0 & 0 & 0 & b_4 & a_5 & b_5 & \dots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & b_{n-2} & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & b_{n-1} & a_n \end{bmatrix}$$

where b_is are nonzero.

Johnson et al. in 2003 [4] introduced the generalized stars graph and fully described the lists of multiplicities of eigenvalues and properties of the matrices of this graph. A generalized star graph is a set of path graphs of arbitrary number of vertices coinciding at a vertex. Figure 2 shows a generalized star graph made by 3 paths of sizes $2, 3, 2$. Each path of a generalized star graph is called an *arm* and the number of vertices of a generalized star graph is the sum of the number of vertices of each arm plus one.

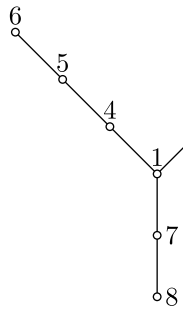

Figure 2. Generalized star graph.

Xu and Chen in 2017 [10] investigated matrices of tridiagonal-and-paw and paw-and-tridiagonal graphs which we generalized to a larger class of graphs, named generalized star graphs. The graph they studied is shown in Figure 3, labeling the vertices of this graph from left to right or from right to left determines whether it is a tridiagonal-and-paw or a paw-and-tridiagonal graph.

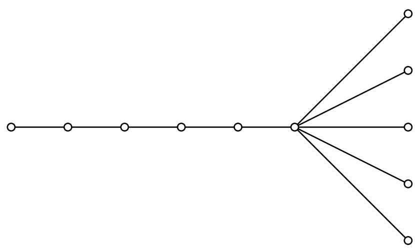

Figure 3. Tridiagonal-and-paw and paw-and-tridiagonal graph.

For a given list of natural numbers $L = (n_1, n_2, \ldots, n_r)$, the generalized star graph of this list with n vertices is named $\mathrm{GS}(n_1, n_2, \ldots, n_r)$, it has r arms, the ith arm has n_i vertices, $n = 1 + \sum_{i=1}^{r} n_i$, and is labeled as follows:

(1) The vertex at the center is labeled by 1.
(2) The first arm is labeled by $2, 3, \ldots, n_1 + 1$ and the center vertex is connected to the vertex of label 2.

(3) The jth arm, $j = 2, 3, \ldots r$, is labeled by $B_j + 1, B_j + 2, \ldots, B_j + n_j$ where $B_j = 1 + \sum_{i=1}^{j-1} n_i$ and the center is connected to the vertex labeled by $B_j + 1$.

Let $n_0 = 1$ and $s_j = \sum_{i=0}^{j} n_i$, $j = 0, 1, \ldots, r$. Matrix A^L of a $\mathrm{GS}(n_1, \ldots, n_r)$ is the symmetric matrix

$$\begin{pmatrix} A(1) & B_{1,2} & B_{1,3} & \ldots & B_{1,r-1} & B_{1,r} \\ B_{1,2}^T & A(2) & 0 & \ldots & 0 & 0 \\ B_{1,3}^T & 0 & A(3) & \ddots & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & 0 & 0 \\ B_{1,r-1}^T & 0 & 0 & \ldots & A(r-1) & 0 \\ B_{1,r}^T & 0 & 0 & \ldots & 0 & A(r) \end{pmatrix},$$

where $B_{1,i}$, $i = 2, \ldots, r$ is a $s_1 \times n_i$ matrix with only one nonzero entry $b_{i,s_{i-1}+1}$ at the entry of the first row and column and $A(1)$ and $A(i)$, $i = 2, \ldots, r$, are matrices of a path graph indexed as follow:

$$A(1) = \begin{bmatrix} a_1 & b_{1,2} & 0 & 0 & \ldots & 0 \\ b_{1,2} & a_2 & b_{1,3} & 0 & \ldots & 0 \\ 0 & b_{1,3} & a_3 & b_{1,4} & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & b_{1,s_1-1} & a_{s_1-1} & b_{1,s_1} \\ 0 & 0 & 0 & 0 & b_{1,s_1} & a_{s_1} \end{bmatrix},$$

$$A(i) = \begin{bmatrix} a_{s_{i-1}+1} & b_{i,s_{i-1}+2} & 0 & 0 & \ldots & 0 \\ b_{i,s_{i-1}+2} & a_{s_{i-1}+2} & b_{i,s_{i-1}+3} & 0 & \ldots & 0 \\ 0 & b_{i,s_{i-1}+3} & a_{s_{i-1}+3} & b_{i,s_{i-1}+4} & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & b_{i,s_i-1} & a_{s_i-1} & b_{i,s_i} \\ 0 & 0 & 0 & 0 & b_{i,s_i} & a_{s_i} \end{bmatrix}.$$

For example the matrix of Figure 2 is

$$A^{(2,3,2)} = \begin{bmatrix} a_1 & b_{1,2} & 0 & b_{2,4} & 0 & 0 & b_{3,7} & 0 \\ b_{1,2} & a_2 & b_{1,3} & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{1,3} & a_3 & 0 & 0 & 0 & 0 & 0 \\ b_{2,4} & 0 & 0 & a_4 & b_{2,5} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{2,5} & a_5 & b_{2,6} & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{2,6} & a_6 & 0 & 0 \\ b_{3,7} & 0 & 0 & 0 & 0 & 0 & a_7 & b_{3,8} \\ 0 & 0 & 0 & 0 & 0 & 0 & b_{3,8} & a_8 \end{bmatrix}.$$

A *star* graph is a specific case of the generalized star graph where the size of each arm is exactly 1.

In this paper, we will use the following notation:

(1) A_i^L will be the ith leading principal submatrix of a symmetric matrix A^L.
(2) $\chi_i(\lambda) = \det(\lambda I_i - A_i^L)$, i.e., the characteristic polynomial of the ith leading principal minor of A^L, I_i being the identity matrix of order i.

In the following definition we define the IEPP which will be used in this paper.

Definition 2.1. IEPP: Given $2n-1$ real numbers $\lambda_1^{(j)}$, $\lambda_j^{(j)}$ for $j = 1, 2, \ldots, n$, find an $n \times n$ matrix $A^{(n-1)}$ of a path graph such that $\lambda_1^{(j)}$, $\lambda_j^{(j)}$ are respectively the minimum and maximum eigenvalues of $A_j^{(n-1)}$, $j = 1, 2, \ldots, n$.

The following results will be necessary in this paper:

Lemma 2.1. *For a matrix $A^{(n-1)}$ of a path graph, the sequence $\{\chi_j(\lambda) = \det(\lambda I_j - A_j^{(n-1)})\}$, $j = 1, \ldots, n$, of characteristic polynomials of $A_j^{(n-1)}$ satisfies the following recurrence relations:*

(1) $\chi_1(\lambda) = \lambda - a_1$,
(2) $\chi_j(\lambda) = (\lambda - a_j)\chi_{j-1}(\lambda) - b_{j-1}^2 \chi_{j-2}(\lambda)$;

for the sake of writing the recurrence relations, let $\chi_0(\lambda) = 0$, $b_0 = 0$.

Lemma 2.2 ([8])**.** *The IEPP has a unique solution if and only if*

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_{n-1}^{(n-1)} < \lambda_n^{(n)},$$

and the solution is given by

$$a_1 = \lambda_1^{(1)}, \quad a_2 = \frac{\lambda_1^{(2)}\chi_1(\lambda_1^{(2)}) - \lambda_2^{(2)}\chi_1(\lambda_2^{(2)})}{\chi_1(\lambda_1^{(2)}) - \chi_1(\lambda_2^{(2)})}, \quad b_1^2 = \frac{(\lambda_2^{(2)} - \lambda_1^{(2)})\chi_1(\lambda_1^{(2)})\chi_1(\lambda_2^{(2)})}{\chi_1(\lambda_1^{(2)}) - \chi_1(\lambda_2^{(2)})},$$

and for $j = 3, \ldots, n$ the solution is

$$a_j = \frac{\lambda_1^{(j)}\chi_{j-1}(\lambda_1^{(j)})\chi_{j-2}(\lambda_j^{(j)}) - \lambda_j^{(j)}\chi_{j-1}(\lambda_j^{(j)})\chi_{j-2}(\lambda_1^{(j)})}{\chi_{j-1}(\lambda_1^{(j)})\chi_{j-2}(\lambda_j^{(j)}) - \chi_{j-1}(\lambda_j^{(j)})\chi_{j-2}(\lambda_1^{(j)})},$$
$$b_{j-1}^2 = \frac{(\lambda_j^{(j)} - \lambda_1^{(j)})\chi_{j-1}(\lambda_1^{(j)})\chi_{j-1}(\lambda_j^{(j)})}{\chi_{j-1}(\lambda_1^{(j)})\chi_{j-2}(\lambda_j^{(j)}) - \chi_{j-1}(\lambda_j^{(j)})\chi_{j-2}(\lambda_1^{(j)})}.$$

Lemma 2.3 ([8]). *For any monic polynomial $P(\lambda)$ of order n such that λ_1 and λ_n are its minimal and maximal roots, the following are true*

(1) $\forall\, x > \lambda_n\colon\ P(x) > 0$,

(2) $\forall\, x < \lambda_1\colon\ (-1)^n P(x) > 0$.

Theorem 2.1 (Cauchy's interlacing theorem [3]). *Let $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$ be the eigenvalues of an $n \times n$ real symmetric matrix A and $\mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_{n-1}$ the eigenvalues of an $(n-1) \times (n-1)$ principal submatrix of A. Then*

$$\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \mu_2 \leqslant \ldots \leqslant \lambda_{n-1} \leqslant \mu_{n-1} \leqslant \lambda_n.$$

An immediate consequence of Cauchy's interlacing theorem is the following.

Corollary 2.1. *Let A be an $n \times n$ real symmetric matrix with minimum and maximum eigenvalues λ_1, λ_n, respectively, and let $\hat{A}$ be the $(n-2)\times(n-2)$ principal submatrix of rows and columns $2, \ldots, n-1$ of A with minimum and maximum eigenvalues μ_1, μ_{n-2}, respectively. Then*

$$\mu_1 \geqslant \lambda_1 \quad \text{and} \quad \mu_{n-2} \leqslant \lambda_n.$$

3. Solution of IEPGS

Now we investigate the eigenvalue problem for the generalized star graph as follows:

IEPGS: Given $2n-1$ real numbers $\lambda_1^{(j)}, \lambda_j^{(j)}$, for $1 \leqslant j \leqslant n$, and a list of natural numbers $L = (n_1, n_2, \ldots, n_r)$, where $1 + \sum\limits_{i=1}^{r} n_i = n$, find an $n \times n$ matrix $A^L \in \mathrm{GS}(n_1, n_2, \ldots, n_r)$ such that for $j = 1, 2, \ldots, n$, $\lambda_1^{(j)}, \lambda_j^{(j)}$ are the minimum and the maximum eigenvalues of A_j^L, respectively.

In the following lemma, we investigate the relation between the successive leading principal minors of $I_n\lambda - A^L$.

Lemma 3.1. *The sequence $\{\chi_j(\lambda) = \det(I_j\lambda - A_j^L)\}_{j=1}^n$ of characteristic polynomials of A_j^L satisfies the following recurrence relations:*

(1) $\chi_1(\lambda) = \lambda - a_1$,

(2) $\chi_2(\lambda) = (\lambda - a_2)\chi_1(\lambda) - b_{1,2}^2$,

(3) $\chi_j(\lambda) = (\lambda - a_j)\chi_{j-1}(\lambda) - b_{i,j}^2 \det(R_j^\lambda)$ *for* $j = s_{i-1} + 1$, $i = 2, \ldots, r$,

(4) $\chi_j(\lambda) = (\lambda - a_j)\chi_{j-1}(\lambda) - b_{i,j}^2 \chi_{j-2}(\lambda)$ *for* $s_{i-1} + 2 \leqslant j \leqslant s_i$, $i = 1, \ldots, r$,

where $\chi_0(\lambda) = 1$ and R_j^λ is the submatrix of rows and columns $2, 3, \ldots, j-1$ of the matrix $\lambda I_n - A^L$.

P r o o f. It is easy to verify by Lemma 2.1 and expanding the determinant. $\square$

It is worthwhile to mention that by [2], the time complexity of computing the determinant of R_j^{λ} is $O(n\log n)$, because it is a symmetric tridiagonal matrix.

By Cauchy's interlacing theorem (Theorem 2.1), the eigenvalues of a symmetric matrix and those of any of its principal submatrix interlace each other. If we apply this theorem recursively to each principal leading submatrix of A^L, the following inequalities are satisfied:

$$\lambda_1^{(n)} \leqslant \lambda_1^{(n-1)} \leqslant \ldots \leqslant \lambda_1^{(2)} \leqslant \lambda_1^{(1)} \leqslant \lambda_2^{(2)} \leqslant \ldots \leqslant \lambda_{n-1}^{(n-1)} \leqslant \lambda_n^{(n)}.$$

Since $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are eigenvalues of A_j^L, hence solving the IEPGS equals solving the following system of equations of Lemma 3.1, for $j = 1, 2, \ldots, n$:

$$\begin{cases} \chi_j(\lambda_1^{(j)}) = 0, \\ \chi_j(\lambda_j^{(j)}) = 0. \end{cases} \tag{3.1}$$

For the first equation we have $\chi_1(\lambda_1^{(1)}) = 0$ and from Lemma 3.1 we get $a_1 = \lambda_1^{(1)}$. The solution and the necessary and sufficient conditions of equations (2) and (4) of Lemma 3.1 are given in Lemma 2.2 so we only need to investigate the equation (3) of Lemma 3.1. In the next Lemma, we investigate necessary and sufficient conditions of equation (3) of Lemma 3.1 for A_j^L.

Lemma 3.2. *If $\lambda_1^{(j)}, \lambda_j^{(j)}$ are the minimum and the maximum eigenvalues of A_j^L, $j = s_{i-1}+1$, $i = 2, \ldots, r$, then the system of equations* (3.1) *for the recurrence relation*

$$\chi_j(\lambda) = (\lambda - a_j)\chi_{j-1}(\lambda) - b_{i,j}^2 \det(R_j^{\lambda})$$

has a unique solution for a_j and $b_{i,j}^2$ if and only if $\lambda_1^{(j)} < \lambda_1^{(j-1)}$ and $\lambda_{j-1}^{(j-1)} < \lambda_j^{(j)}$.

P r o o f. The system of equations (3.1) for the given recurrence relation equals the following system of equations which are linear in a_j and $b_{i,j}^2$:

$$\begin{cases} (\lambda_1^{(j)} - a_j)\chi_{j-1}(\lambda_1^{(j)}) - b_{i,j}^2 \det\bigl(R_j^{\lambda_1^{(j)}}\bigr) = 0 = \chi_j(\lambda_1^{(j)}), \\ (\lambda_j^{(j)} - a_j)\chi_{j-1}(\lambda_j^{(j)}) - b_{i,j}^2 \det\bigl(R_j^{\lambda_j^{(j)}}\bigr) = 0 = \chi_j(\lambda_j^{(j)}). \end{cases} \tag{3.2}$$

This can be written as

$$\begin{bmatrix} \chi_{j-1}(\lambda_1^{(j)}) & \det\bigl(R_j^{\lambda_1^{(j)}}\bigr) \\ \chi_{j-1}(\lambda_j^{(j)}) & \det\bigl(R_j^{\lambda_j^{(j)}}\bigr) \end{bmatrix} \begin{bmatrix} a_j \\ b_{i,j}^2 \end{bmatrix} = \begin{bmatrix} \lambda_1^{(j)}\chi_{j-1}(\lambda_1^{(j)}) \\ \lambda_j^{(j)}\chi_{j-1}(\lambda_j^{(j)}) \end{bmatrix}.$$

This system of linear equations has a unique solution if and only if D_j, the determinant of the matrix of coefficients, is nonzero. Thus we should have

$$D_j = \chi_{j-1}(\lambda_1^{(j)}) \det\big(R_j^{\lambda_j^{(j)}}\big) - \chi_{j-1}(\lambda_j^{(j)}) \det\big(R_j^{\lambda_1^{(j)}}\big) \neq 0. \tag{3.3}$$

From Lemma 3.1 we have

$$\begin{aligned}
\det\big(R_j^{\lambda_1^{(j)}}\big) &= \frac{1}{b_{i,j}^2}((\lambda_1^{(j)} - a_j)\chi_{j-1}(\lambda_1^{(j)}) - \chi_j(\lambda_1^{(j)})),\\
\det\big(R_j^{\lambda_j^{(j)}}\big) &= \frac{1}{b_{i,j}^2}((\lambda_j^{(j)} - a_j)\chi_{j-1}(\lambda_j^{(j)}) - \chi_j(\lambda_j^{(j)})),
\end{aligned}$$

and by replacing them in the equation (3.3), we get

$$\begin{aligned}
D_j = \chi_{j-1}(\lambda_1^{(j)})&\frac{1}{b_{i,j}^2}((\lambda_j^{(j)} - a_j)\chi_{j-1}(\lambda_j^{(j)}) - \chi_j(\lambda_j^{(j)}))\\
&- \chi_{j-1}(\lambda_j^{(j)})\frac{1}{b_{i,j}^2}((\lambda_1^{(j)} - a_j)\chi_{j-1}(\lambda_1^{(j)}) - \chi_j(\lambda_1^{(j)})).
\end{aligned}$$

The following inequality must hold in order to have a nonzero determinant:

$$\begin{aligned}
\chi_{j-1}(\lambda_1^{(j)})&\frac{1}{b_{i,j}^2}((\lambda_j^{(j)} - a_j)\chi_{j-1}(\lambda_j^{(j)}) - \chi_j(\lambda_j^{(j)}))\\
&\neq \chi_{j-1}(\lambda_j^{(j)})\frac{1}{b_{i,j}^2}((\lambda_1^{(j)} - a_j)\chi_{j-1}(\lambda_1^{(j)}) - \chi_j(\lambda_1^{(j)})).
\end{aligned}$$

We assumed $b_{i,j}$s are nonzero, so we can omit them from both sides to get

$$\begin{aligned}
\chi_{j-1}(\lambda_1^{(j)})&((\lambda_j^{(j)} - a_j)\chi_{j-1}(\lambda_j^{(j)}) - \chi_j(\lambda_j^{(j)}))\\
&\neq \chi_{j-1}(\lambda_j^{(j)})((\lambda_1^{(j)} - a_j)\chi_{j-1}(\lambda_1^{(j)}) - \chi_j(\lambda_1^{(j)})).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\chi_{j-1}(\lambda_1^{(j)})(\lambda_j^{(j)}\chi_{j-1}(\lambda_j^{(j)}) - a_j\chi_{j-1}(\lambda_j^{(j)}) - \chi_j(\lambda_j^{(j)}))\\
&\qquad \neq \chi_{j-1}(\lambda_j^{(j)})(\lambda_1^{(j)}\chi_{j-1}(\lambda_1^{(j)}) - a_j\chi_{j-1}(\lambda_1^{(j)}) - \chi_j(\lambda_1^{(j)}))\\
&\quad \Rightarrow \lambda_j^{(j)}\chi_{j-1}(\lambda_1^{(j)})\chi_{j-1}(\lambda_j^{(j)}) - a_j\chi_{j-1}(\lambda_1^{(j)})\chi_{j-1}(\lambda_j^{(j)}) - \chi_{j-1}(\lambda_1^{(j)})\chi_j(\lambda_j^{(j)})\\
&\qquad \neq \lambda_1^{(j)}\chi_{j-1}(\lambda_1^{(j)})\chi_{j-1}(\lambda_j^{(j)}) - a_j\chi_{j-1}(\lambda_1^{(j)})\chi_{j-1}(\lambda_j^{(j)}) - \chi_{j-1}(\lambda_j^{(j)})\chi_j(\lambda_1^{(j)})\\
&\quad \Rightarrow (\lambda_j^{(j)} - \lambda_1^{(j)})\chi_{j-1}(\lambda_1^{(j)})\chi_{j-1}(\lambda_j^{(j)}) \neq \chi_{j-1}(\lambda_1^{(j)})\chi_j(\lambda_j^{(j)}) - \chi_{j-1}(\lambda_j^{(j)})\chi_j(\lambda_1^{(j)}).
\end{aligned}$$

The right-hand side of the last statement is zero, because $\lambda_1^{(j)}$, $\lambda_j^{(j)}$ are roots of the polynomial $\chi_j(\lambda)$, so the left-hand side must be nonzero. It is nonzero if $\lambda_j^{(j)} \neq \lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $\lambda_1^{(j)}$ are not roots of the polynomial $\chi_{j-1}(\lambda)$. Hence, by the interlacing theorem must be $\lambda_j^{(j)} > \lambda_{j-1}^{(j-1)}$ and $\lambda_1^{(j)} < \lambda_1^{(j-1)}$.

Conversely, the system of equations (3.2) will have a unique solution for a_j and $b_{i,j}^2$ if and only if $D_j \neq 0$. From the above, $D_j \neq 0$ if and only if $\chi_{j-1}(\lambda_1^{(j)}) \neq 0$ and $\chi_{j-1}(\lambda_j^{(j)}) \neq 0$, i.e. $\lambda_1^{(j)}$, $\lambda_j^{(j)}$ are not zeros of $\chi_{j-1}(\lambda)$, and $\lambda_j^{(j)} - \lambda_1^{(j)} \neq 0$. Hence, by the interlacing theorem $\lambda_1^{(j)} < \lambda_j^{(j)}$ and $\lambda_{j-1}^{(j-1)} < \lambda_j^{(j)}$ and $\lambda_1^{(j)} < \lambda_1^{(j-1)}$ and this completes the proof. □

In the following theorem, we give the necessary and sufficient conditions for existence of a solution to IEPGS.

Theorem 3.1. *The IEPGS has a unique solution for a_j and $b_{i,j}^2$ if and only if*

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_{n-1}^{(n-1)} < \lambda_n^{(n)}.$$

P r o o f. Immediate from Lemmas 3.2 and 2.2. □

Now since $D_j \neq 0$, $j = s_{i-1} + 1$, $i = 2, \ldots, r$, there is a unique solution to the system of equations (3.2). In the next theorem we give the solution to the system of equations (3.1) for equation (3) of Lemma 3.1.

Theorem 3.2. *If $\lambda_1^{(j)}$, $\lambda_j^{(j)}$ are the minimum and the maximum eigenvalues of A_j^L, $j = s_{i-1} + 1$, $i = 2, \ldots, r$, then the unique solution to the system of equations* (3.1) *for the recurrence relation*

$$\chi_j(\lambda) = (\lambda - a_j)\chi_{j-1}(\lambda) - b_{i,j}^2 \det(R_j^\lambda)$$

is given by the relations

$$a_j = \frac{\lambda_1^{(j)}\chi_{j-1}(\lambda_1^{(j)}) \det\big(R_j^{\lambda_j^{(j)}}\big) - \lambda_j^{(j)}\chi_{j-1}(\lambda_j^{(j)}) \det\big(R_j^{\lambda_1^{(j)}}\big)}{\chi_{j-1}(\lambda_1^{(j)}) \det\big(R_j^{\lambda_j^{(j)}}\big) - \chi_{j-1}(\lambda_j^{(j)}) \det\big(R_j^{\lambda_1^{(j)}}\big)},$$

$$b_{i,j}^2 = \frac{\lambda_j^{(j)}\chi_{j-1}(\lambda_j^{(j)})\chi_{j-1}(\lambda_1^{(j)}) - \lambda_1^{(j)}\chi_{j-1}(\lambda_1^{(j)})\chi_{j-1}(\lambda_j^{(j)})}{\chi_{j-1}(\lambda_1^{(j)}) \det\big(R_j^{\lambda_j^{(j)}}\big) - \chi_{j-1}(\lambda_j^{(j)}) \det\big(R_j^{\lambda_1^{(j)}}\big)},$$

if and only if $\lambda_1^{(j)} < \lambda_1^{(j-1)}$ and $\lambda_{j-1}^{(j-1)} < \lambda_j^{(j)}$.

P r o o f. By Lemma 3.2, the system of equations (3.1) for the given recurrence relation has a unique solution and the determinant

$$D_j = \chi_{j-1}(\lambda_1^{(j)}) \det\big(R_j^{\lambda_j^{(j)}}\big) - \chi_{j-1}(\lambda_j^{(j)}) \det\big(R_j^{\lambda_1^{(j)}}\big)$$

of the following matrix of coefficients is nonzero if and only if the conditions of this theorem are satisfied:

$$\begin{bmatrix} \chi_{j-1}(\lambda_1^{(j)}) & \det\big(R_j^{\lambda_1^{(j)}}\big) \\ \chi_{j-1}(\lambda_j^{(j)}) & \det\big(R_j^{\lambda_j^{(j)}}\big) \end{bmatrix} \begin{bmatrix} a_j \\ b_{i,j}^2 \end{bmatrix} = \begin{bmatrix} \lambda_1^{(j)} \chi_{j-1}(\lambda_1^{(j)}) \\ \lambda_j^{(j)} \chi_{j-1}(\lambda_j^{(j)}) \end{bmatrix}.$$

The solution to a_j and $b_{i,j}^2$ is given by

$$\begin{bmatrix} a_j \\ b_{i,j}^2 \end{bmatrix} = \frac{1}{D_j} \begin{bmatrix} \det\big(R_j^{\lambda_j^{(j)}}\big) & -\det\big(R_j^{\lambda_1^{(j)}}\big) \\ -\chi_{j-1}(\lambda_j^{(j)}) & \chi_{j-1}(\lambda_1^{(j)}) \end{bmatrix} \begin{bmatrix} \lambda_1^{(j)} \chi_{j-1}(\lambda_1^{(j)}) \\ \lambda_j^{(j)} \chi_{j-1}(\lambda_j^{(j)}) \end{bmatrix},$$

by multiplying the matrices on the right-hand side, we get

$$\begin{bmatrix} a_j \\ b_{i,j}^2 \end{bmatrix} = \frac{1}{D_j} \begin{bmatrix} \lambda_1^{(j)} \chi_{j-1}(\lambda_1^{(j)}) \det\big(R_j^{\lambda_j^{(j)}}\big) - \lambda_j^{(j)} \chi_{j-1}(\lambda_j^{(j)}) \det\big(R_j^{\lambda_1^{(j)}}\big) \\ \lambda_j^{(j)} \chi_{j-1}(\lambda_j^{(j)}) \chi_{j-1}(\lambda_1^{(j)}) - \lambda_1^{(j)} \chi_{j-1}(\lambda_1^{(j)}) \chi_{j-1}(\lambda_j^{(j)}) \end{bmatrix} = \frac{1}{D_j} \begin{bmatrix} \Psi \\ \Omega \end{bmatrix},$$

thus

$$a_j = \frac{\Psi}{D_j}, \quad b_{i,j}^2 = \frac{\Omega}{D_j}, \tag{3.4}$$

where

$$\begin{aligned} \Psi &= \lambda_1^{(j)} \chi_{j-1}(\lambda_1^{(j)}) \det\big(R_j^{\lambda_j^{(j)}}\big) - \lambda_j^{(j)} \chi_{j-1}(\lambda_j^{(j)}) \det\big(R_j^{\lambda_1^{(j)}}\big), \\ \Omega &= \lambda_j^{(j)} \chi_{j-1}(\lambda_j^{(j)}) \chi_{j-1}(\lambda_1^{(j)}) - \lambda_1^{(j)} \chi_{j-1}(\lambda_1^{(j)}) \chi_{j-1}(\lambda_j^{(j)}). \end{aligned}$$

By equation (3.4), the unique solution to a_j is given by

$$a_j = \frac{\lambda_1^{(j)} \chi_{j-1}(\lambda_1^{(j)}) \det\big(R_j^{\lambda_j^{(j)}}\big) - \lambda_j^{(j)} \chi_{j-1}(\lambda_j^{(j)}) \det\big(R_j^{\lambda_1^{(j)}}\big)}{\chi_{j-1}(\lambda_1^{(j)}) \det\big(R_j^{\lambda_j^{(j)}}\big) - \chi_{j-1}(\lambda_j^{(j)}) \det\big(R_j^{\lambda_1^{(j)}}\big)}.$$

We need to prove $\Omega/D_j > 0$ to get a real value for $b_{i,j}$. We have $\Omega/D_j > 0$ if and only if $(-1)^{j-1}\Omega/((-1)^{j-1}D_j) > 0$. We prove both the terms $(-1)^{j-1}\Omega$ and $(-1)^{j-1}D_j$ are positive. We start by the numerator,

$$\begin{aligned} (-1)^{j-1}\Omega &= (-1)^{j-1}(\lambda_j^{(j)} \chi_{j-1}(\lambda_j^{(j)}) \chi_{j-1}(\lambda_1^{(j)}) - \lambda_1^{(j)} \chi_{j-1}(\lambda_1^{(j)}) \chi_{j-1}(\lambda_j^{(j)})) \\ &= (-1)^{j-1}(\lambda_j^{(j)} - \lambda_1^{(j)}) \chi_{j-1}(\lambda_1^{(j)}) \chi_{j-1}(\lambda_j^{(j)}). \end{aligned}$$

By the assumption of the theorem, $\lambda_j^{(j)} > \lambda_1^{(j)}$, $\lambda_j^{(j)} > \lambda_{j-1}^{(j-1)}$ and $\lambda_1^{(j)} < \lambda_1^{(j-1)}$, thus by Lemma 2.3 the terms $(-1)^{j-1}\chi_{j-1}(\lambda_1^{(j)}) > 0$ and $\chi_{j-1}(\lambda_j^{(j)}) > 0$ and the term $(\lambda_j^{(j)} - \lambda_1^{(j)})$ is clearly positive, so the numerator is positive.

For the denominator we obtain

$$(-1)^{j-1}D_j = (-1)^{j-1}\chi_{j-1}(\lambda_1^{(j)}) \det\big(R_j^{\lambda_j^{(j)}}\big) - (-1)^{j-1}\chi_{j-1}(\lambda_j^{(j)}) \det\big(R_j^{\lambda_1^{(j)}}\big). \tag{3.5}$$

The matrix R_j^λ is a symmetric tridiagonal matrix of rows and columns $2, \dots, j-1$ of $\lambda I_n - A^L$. The determinant $\det(R_j^\lambda)$ is a monic polynomial of order $j-2$ and let $\mu_1^{(j-2)}$, $\mu_{j-2}^{(j-2)}$ be the minimum and the maximum roots of this polynomial, respectively. Because $\mu_1^{(j-2)}$, $\mu_{j-2}^{(j-2)}$ are the minimum and the maximum roots of $\det(R_j^\lambda)$, hence $\mu_1^{(j-2)}$, $\mu_{j-2}^{(j-2)}$ are minimum and the maximum eigenvalues of the submatrix of rows and columns $2, 3, \dots, j-1$ of A^L. By Corollary 2.1, $\mu_1^{(j-2)} > \lambda_1^{(j)}$ and $\mu_{j-2}^{(j-2)} < \lambda_j^{(j)}$. From equation (3.5) we get

$$(-1)^{j-1}D_j = (-1)^{j-1}\chi_{j-1}(\lambda_1^{(j)}) \det\big(R_j^{\lambda_j^{(j)}}\big) + (-1)^{j-2}\chi_{j-1}(\lambda_j^{(j)}) \det\big(R_j^{\lambda_1^{(j)}}\big).$$

Because $\det(R_j^\lambda)$ is a monic polynomial of order $j-2$, hence by Lemma 2.3 $(-1)^{j-2}\det\big(R_j^{\lambda_1^{(j)}}\big) > 0$, $\det\big(R_j^{\lambda_j^{(j)}}\big) > 0$, $(-1)^{j-1}\chi_{j-1}(\lambda_1^{(j)}) > 0$ and $\chi_{j-1}(\lambda_j^{(j)}) > 0$, consequently $(-1)^{j-1}D_j > 0$ and from equation (3.4) we obtain

$$b_{i,j}^2 = \frac{\Omega}{D_j} = \frac{\lambda_j^{(j)}\chi_{j-1}(\lambda_j^{(j)})\chi_{j-1}(\lambda_1^{(j)}) - \lambda_1^{(j)}\chi_{j-1}(\lambda_1^{(j)})\chi_{j-1}(\lambda_j^{(j)})}{\chi_{j-1}(\lambda_1^{(j)}) \det\big(R_j^{\lambda_j^{(j)}}\big) - \chi_{j-1}(\lambda_j^{(j)}) \det\big(R_j^{\lambda_1^{(j)}}\big)}.$$

Conversely, suppose IEPGS has a solution with unique values of the entries a_j and $b_{i,j}^2$ for indices given in this theorem, then the system of equations (3.2) will have a unique solution. Hence, from Lemma 3.2, $\lambda_1^{(j)} < \lambda_1^{(j-1)}$ and $\lambda_{j-1}^{(j-1)} < \lambda_j^{(j)}$ and this completes the proof. □

Theorem 3.3. *The unique solution to IEPGS for a_j and $b_{i,j}^2$ is given by*

$$a_1 = \lambda_1^{(1)}, \; a_2 = \frac{\lambda_1^{(2)}\chi_1(\lambda_1^{(2)}) - \lambda_2^{(2)}\chi_1(\lambda_2^{(2)})}{\chi_1(\lambda_1^{(2)}) - \chi_1(\lambda_2^{(2)})}, \; b_{1,2}^2 = \frac{(\lambda_2^{(2)} - \lambda_1^{(2)})\chi_1(\lambda_1^{(2)})\chi_1(\lambda_2^{(2)})}{\chi_1(\lambda_1^{(2)}) - \chi_1(\lambda_2^{(2)})},$$

for $j = s_{i-1}+1, i = 2, \dots, r$,

$$a_j = \frac{\lambda_1^{(j)}\chi_{j-1}(\lambda_1^{(j)}) \det\big(R_j^{\lambda_j^{(j)}}\big) - \lambda_j^{(j)}\chi_{j-1}(\lambda_j^{(j)}) \det\big(R_j^{\lambda_1^{(j)}}\big)}{\chi_{j-1}(\lambda_1^{(j)}) \det\big(R_j^{\lambda_j^{(j)}}\big) - \chi_{j-1}(\lambda_j^{(j)}) \det\big(R_j^{\lambda_1^{(j)}}\big)},$$

$$b_{i,j}^2 = \frac{\lambda_j^{(j)}\chi_{j-1}(\lambda_j^{(j)})\chi_{j-1}(\lambda_1^{(j)}) - \lambda_1^{(j)}\chi_{j-1}(\lambda_1^{(j)})\chi_{j-1}(\lambda_j^{(j)})}{\chi_{j-1}(\lambda_1^{(j)}) \det\big(R_j^{\lambda_j^{(j)}}\big) - \chi_{j-1}(\lambda_j^{(j)}) \det\big(R_j^{\lambda_1^{(j)}}\big)},$$

and for $s_{i-1}+2 \leqslant j \leqslant s_i$, $i=1,\dots,r$,

$$a_j = \frac{\lambda_1^{(j)}\chi_{j-1}(\lambda_1^{(j)})\chi_{j-2}(\lambda_j^{(j)}) - \lambda_j^{(j)}\chi_{j-1}(\lambda_j^{(j)})\chi_{j-2}(\lambda_1^{(j)})}{\chi_{j-1}(\lambda_1^{(j)})\chi_{j-2}(\lambda_j^{(j)}) - \chi_{j-1}(\lambda_j^{(j)})\chi_{j-2}(\lambda_1^{(j)})},$$

$$b_{i,j}^2 = \frac{(\lambda_j^{(j)} - \lambda_1^{(j)})\chi_{j-1}(\lambda_1^{(j)})\chi_{j-1}(\lambda_j^{(j)})}{\chi_{j-1}(\lambda_1^{(j)})\chi_{j-2}(\lambda_j^{(j)}) - \chi_{j-1}(\lambda_j^{(j)})\chi_{j-2}(\lambda_1^{(j)})},$$

if and only if

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_{n-1}^{(n-1)} < \lambda_n^{(n)}.$$

P r o o f. Immediate from Lemma 2.2 and Theorem 3.2. □

The above results lead to Algorithm 1 on inputs

$$\Lambda = \{\lambda_1^{(n)}, \lambda_1^{(n-1)}, \dots, \lambda_1^{(1)}, \dots, \lambda_{n-1}^{(n-1)}, \lambda_n^{(n)}\}$$

and a list $L=(n_1,n_2,\dots,n_r)$ of natural numbers such that $n = 1 + \sum\limits_{i=1}^{r} n_i$. The $\chi_a(b)$ denotes the procedure call $\chi(a,b)$.

R e m a r k. The expressions for entries $b_{i,j}$ are quadratic so the positive value and the negative value of the square of the term $b_{i,j}^2$ can be chosen as $b_{i,j}$. Thus at each step of the algorithm there is a choice in selecting either one and each choice gives a new solution. But the solutions differ only in the sign of nonzero off-diagonal entries.

Algorithm 1 IEPGS Algorithm

1: Set

$$a_1 = \lambda_1^{(1)}, \quad a_2 = \frac{\lambda_1^{(2)}\chi_1(\lambda_1^{(2)}) - \lambda_2^{(2)}\chi_1(\lambda_2^{(2)})}{\chi_1(\lambda_1^{(2)}) - \chi_1(\lambda_2^{(2)})},$$

$$b_{1,2} = \sqrt{\frac{(\lambda_2^{(2)} - \lambda_1^{(2)})\chi_1(\lambda_1^{(2)})\chi_1(\lambda_2^{(2)})}{\chi_1(\lambda_1^{(2)}) - \chi_1(\lambda_2^{(2)})}}.$$

2: $n_0 = 1$, $s_j = \sum\limits_{i=0}^{j} n_i$, $j = 1,2,\dots,r-1$.

3: **For** $j=3$ **to** n

4: **If** $j = s_{i-1} + 1$ *for any* $i = 2, 3, \ldots, r$

Then

$$a_j = \frac{\lambda_1^{(j)} \chi_{j-1}(\lambda_1^{(j)}) \det\bigl(R_j^{\lambda_j^{(j)}}\bigr) - \lambda_j^{(j)} \chi_{j-1}(\lambda_j^{(j)}) \det\bigl(R_j^{\lambda_1^{(j)}}\bigr)}{\chi_{j-1}(\lambda_1^{(j)}) \det\bigl(R_j^{\lambda_j^{(j)}}\bigr) - \chi_{j-1}(\lambda_j^{(j)}) \det\bigl(R_j^{\lambda_1^{(j)}}\bigr)},$$

$$b_{i,j} = \sqrt{\frac{\lambda_j^{(j)} \chi_{j-1}(\lambda_j^{(j)}) \chi_{j-1}(\lambda_1^{(j)}) - \lambda_1^{(j)} \chi_{j-1}(\lambda_1^{(j)}) \chi_{j-1}(\lambda_j^{(j)})}{\chi_{j-1}(\lambda_1^{(j)}) \det\bigl(R_j^{\lambda_j^{(j)}}\bigr) - \chi_{j-1}(\lambda_j^{(j)}) \det\bigl(R_j^{\lambda_1^{(j)}}\bigr)}}.$$

5: **Else** for any i such that $s_{i-1} + 2 \leqslant j \leqslant s_i$, $i = 1, \ldots, r$

$$a_j = \frac{\lambda_1^{(j)} \chi_{j-1}(\lambda_1^{(j)}) \chi_{j-2}(\lambda_j^{(j)}) - \lambda_j^{(j)} \chi_{j-1}(\lambda_j^{(j)}) \chi_{j-2}(\lambda_1^{(j)})}{\chi_{j-1}(\lambda_1^{(j)}) \chi_{j-2}(\lambda_j^{(j)}) - \chi_{j-1}(\lambda_j^{(j)}) \chi_{j-2}(\lambda_1^{(j)})},$$

$$b_{i,j} = \sqrt{\frac{(\lambda_j^{(j)} - \lambda_1^{(j)}) \chi_{j-1}(\lambda_1^{(j)}) \chi_{j-1}(\lambda_j^{(j)})}{\chi_{j-1}(\lambda_1^{(j)}) \chi_{j-2}(\lambda_j^{(j)}) - \chi_{j-1}(\lambda_j^{(j)}) \chi_{j-2}(\lambda_1^{(j)})}}.$$

6: **EndIf**

7: **EndFor**

8: **procedure** $\chi(j, t)$

9: **If** $j = 0$ **Then Return** 1

10: **ElseIf** $j = 1$ **Then Return** $t - a_1$

11: **ElseIf** $j = 2$ **Then Return** $(t - a_2)(t - a_1) - b_{1,2}^2$

12: **ElseIf** $j = s_{i-1} + 1$ **for any** $i = 2, 3, \ldots, r$

Then Return $(t - a_j)\chi_{j-1}(t) - b_{i,j}^2 \det(R_j^t)$

13: **Else** for any i such that $s_{i-1} + 2 \leqslant j \leqslant s_i$, $i = 1, \ldots, r$

Return $(t - a_j)\chi_{j-1}(t) - b_{i,j}^2 \chi_{j-2}(t)$.

14: **end procedure**

4. Numerical examples

We apply results obtained in the previous section to solve the following problem:

Example 4.1. Given 17 real numbers $\lambda_1 = 2$, $\lambda_2 = 4$, $\lambda_3 = 5.3$, $\lambda_4 = -2.7$, $\lambda_5 = 11.43$, $\lambda_6 = 0.23$, $\lambda_7 = 3.6$, $\lambda_8 = 10$, $\lambda_9 = 14.5$, $\lambda_{10} = -13$, $\lambda_{11} = -7.43$, $\lambda_{12} = 15.64$, $\lambda_{13} = -8.83$, $\lambda_{14} = 21$, $\lambda_{15} = 12$, $\lambda_{16} = 45$, $\lambda_{17} = -60$, and a list $L = (3, 2, 3)$, rearrange them to regard the interlacing theorem and then find a matrix $A^L \in \mathrm{GS}(3, 2, 3)$ such that $\lambda_1^{(j)}$, $\lambda_j^{(j)}$ are the minimum and the maximum eigenvalues of A_j^L.

Solution: We rearrange the numbers as follows:

$$\lambda_1^{(9)} < \lambda_1^{(8)} < \lambda_1^{(7)} < \lambda_1^{(6)} < \lambda_1^{(5)} < \lambda_1^{(4)} < \lambda_1^{(3)} < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \lambda_3^{(3)} \\ < \lambda_4^{(4)} < \lambda_5^{(5)} < \lambda_6^{(6)} < \lambda_7^{(7)} < \lambda_8^{(8)} < \lambda_9^{(9)},$$

i.e.,

$$-60 < -13 < -8.83 < -7.43 < -2.7 < 0.23 < 2 < 3.6 < 4 < 5.3 < 10 \\ < 11.43 < 12 < 14.5 < 15.64 < 21 < 45.$$

Using the expressions for a_j, $b_{i,j}$, the A^L is the matrix

$$\begin{bmatrix} 4.0000 & 0.7211 & 0 & 0 & 7.2460 & 0 & 7.2428 & 0 & 0 \\ 0.7211 & 4.9000 & 3.7195 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.7195 & 7.2404 & 3.8879 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.8879 & 4.0492 & 0 & 0 & 0 & 0 & 0 \\ 7.2460 & 0 & 0 & 0 & 5.2426 & 7.7953 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7.7953 & 0.1087 & 0 & 0 & 0 \\ 7.2428 & 0 & 0 & 0 & 0 & 0 & 0.5848 & 14.3316 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 14.3316 & 8.4315 & 47.3511 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 47.3511 & -25.5027 \end{bmatrix}.$$

We compute the spectra of all of the leading principal submatrices of A^L to verify that the conditions of the IEPGS are satisfied. The minimum and maximum eigenvalues of each principal submatrix are shown in bold.

$$\begin{aligned} \sigma(A^L) &= \{\mathbf{-60.0000}, -9.1000, -4.0725, 0.2597, 4.4405, 5.7479, 11.4140, 15.3651, \\ &\qquad \mathbf{45.0000}\}, \\ \sigma(A_8^L) &= \{\mathbf{-13.0000}, -6.6108, 0.2418, 2.9487, 4.5965, 11.4139, 13.9673, \mathbf{21.0000}\}, \\ \sigma(A_7^L) &= \{\mathbf{-8.8300}, -3.3752, 0.2644, 4.4796, 6.5331, 11.4140, \mathbf{15.6400}\}, \\ \sigma(A_6^L) &= \{\mathbf{-7.4300}, 0.2290, 2.2494, 4.5787, 11.4139, \mathbf{14.5000}\}, \\ \sigma(A_5^L) &= \{\mathbf{-2.7000}, 0.2815, 4.5144, 11.3365, \mathbf{12.0000}\}, \\ \sigma(A_4^L) &= \{\mathbf{0.2300}, 3.6751, 4.8546, \mathbf{11.4300}\}, \\ \sigma(A_3^L) &= \{\mathbf{2.0000}, 4.1404, \mathbf{10.0000}\}, \\ \sigma(A_2^L) &= \{\mathbf{3.6000}, \mathbf{5.3000}\}, \\ \sigma(A_1^L) &= \{\mathbf{4}\}. \end{aligned}$$

E x a m p l e 4.2. The solution to Example 4.1 for $L = (1,1,1,1,1,1,1,1,1)$ (a star graph) is

$$\begin{bmatrix} 4.0000 & 0.7211 & 3.3361 & 4.0132 & 4.0659 & 8.2405 & 5.5476 & 11.8861 & 47.6926 \\ 0.7211 & 4.9000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3.3361 & 0 & 8.1130 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4.0132 & 0 & 0 & 7.3985 & 0 & 0 & 0 & 0 & 0 \\ 4.0659 & 0 & 0 & 0 & 1.4253 & 0 & 0 & 0 & 0 \\ 8.2405 & 0 & 0 & 0 & 0 & 1.3676 & 0 & 0 & 0 \\ 5.5476 & 0 & 0 & 0 & 0 & 0 & 1.8871 & 0 & 0 \\ 11.8861 & 0 & 0 & 0 & 0 & 0 & 0 & 5.3133 & 0 \\ 47.6926 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -21.8061 \end{bmatrix}.$$

The spectra of the leading principal submatrices of A^L are

$$\begin{aligned}
\sigma(A^L) &= \{-\mathbf{60.0000}, 0.0078, 1.4139, 1.7728, 4.1627, 4.9022, 7.2931, 8.0461, \mathbf{45.0000}\},\\
\sigma(A^L_8) &= \{-\mathbf{13.0000}, 1.4133, 1.7319, 3.1661, 4.9016, 7.1976, 7.9942, \mathbf{21.0000}\},\\
\sigma(A^L_7) &= \{-\mathbf{8.8300}, 1.4135, 1.7507, 4.8781, 6.3588, 7.8803, \mathbf{15.6400}\},\\
\sigma(A^L_6) &= \{-\mathbf{7.4300}, 1.4140, 4.8618, 5.9976, 7.8610, \mathbf{14.5000}\},\\
\sigma(A^L_5) &= \{-\mathbf{2.7000}, 3.7410, 4.9795, 7.8162, \mathbf{12.0000}\},\\
\sigma(A^L_4) &= \{\mathbf{0.2300}, 4.9471, 7.8043, \mathbf{11.4300}\},\\
\sigma(A^L_3) &= \{\mathbf{2.0000}, 5.0130, \mathbf{10.0000}\},\\
\sigma(A^L_2) &= \{\mathbf{3.6000}, \mathbf{5.3000}\},\\
\sigma(A^L_1) &= \{\mathbf{4}\}.
\end{aligned}$$

5. Conclusions

In this paper, we solved a partially described inverse eigenvalue problem for a special kind of acyclic matrices whose graph is a generalized star graph by the minimum and maximum eigenvalue of the leading principal submatrices of the required matrix. Such partially described problems are an important class of inverse eigenvalue problems [1] and may occur in computations involving a complicated physical system where it is often difficult or impossible to obtain the entire spectrum. As a future work it would be interesting to solve the problem for other matrix structures.

References

[1] *M. T. Chu, G. H. Golub*: Inverse Eigenvalue Problems: Theory, Algorithms, and Applications. Numerical Mathematics and Scientific Computation, Oxford University Press, Oxford, 2005. zbl MR doi

[2] *E. S. Coakley, V. Rokhlin*: A fast divide-and-conquer algorithm for computing the spectra of real symmetric tridiagonal matrices. Appl. Comput. Harmon. Anal. *34* (2013), 379–414. zbl MR doi

[3] *L. Hogben*: Spectral graph theory and the inverse eigenvalue problem of a graph. Electron. J. Linear Algebra *14* (2005), 12–31. zbl MR doi

[4] *C. R. Johnson, A. L. Duarte, C. M. Saiago*: Inverse eigenvalue problems and lists of multiplicities of eigengvalues for matrices whose graph is a tree: the case of generalized stars and double generalized stars. Linear Algebra Appl. *373* (2003), 311–330. zbl MR doi

[5] *K. H. Monfared, B. L. Shader*: Construction of matrices with a given graph and prescribed interlaced spectral data. Linear Algebra Appl. *438* (2013), 4348–4358. zbl MR doi

[6] *P. Nylen, F. Uhlig*: Inverse eigenvalue problems associated with spring-mass systems. Linear Algebra Appl. *254* (1997), 409–425. zbl MR doi

[7] *J. Peng, X.-Y. Hu, L. Zhang*: Two inverse eigenvalue problems for a special kind of matrices. Linear Algebra Appl. *416* (2006), 336–347. zbl MR doi

[8] *D. Sharma, M. Sen*: Inverse eigenvalue problems for two special acyclic matrices. Mathematics *4* (2016), Article ID 12, 11 pages. zbl doi

[9] *D. Sharma, M. Sen*: Inverse eigenvalue problems for acyclic matrices whose graph is a dense centipede. Spec. Matrices *6* (2018), 77–92. zbl MR doi

[10] *W.-R. Xu, G.-L. Chen*: On inverse eigenvalue problems for two kinds of special banded matrices. Filomat *31* (2017), 371–385. MR doi

Authors' address: *Mohammad Heydari*, *Seyed Abolfazl Shahzadeh Fazeli* (corresponding author), *Seyed Mehdi Karbassi*, Yazd University, Safayieh, Yazd, Yazd Province, Iran, e-mail: `m.heydari@stu.yazd.ac.ir`, `fazeli@yazd.ac.ir`, `smkarbassi@yazd.ac.ir`.