UNIQUE SOLVABILITY AND STABILITY ANALYSIS OF A GENERALIZED PARTICLE METHOD FOR A POISSON EQUATION IN DISCRETE SOBOLEV NORMS

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Abstract. Unique solvability and stability analysis is conducted for a generalized particle method for a Poisson equation with a source term given in divergence form. The generalized particle method is a numerical method for partial differential equations categorized into meshfree particle methods and generally indicates conventional particle methods such as smoothed particle hydrodynamics and moving particle semi-implicit methods. Unique solvability is derived for the generalized particle method for the Poisson equation by introducing a connectivity condition for particle distributions. Moreover, stability is obtained for the discretized Poisson equation by introducing discrete Sobolev norms and a semi-regularity condition of a family of discrete parameters.

Keywords: generalized particle method; Poisson equation; unique solvability; stability; discrete Sobolev norm

MSC 2010: 65M12

1. Introduction

Numerical analysis is conducted for a generalized particle method introduced in [5]. The generalized particle method is a numerical method for partial differential equations categorized into meshfree particle methods and generally indicates conventional particle methods such as smoothed particle hydrodynamics (SPH) [4], [10] and moving particle semi-implicit (MPS) methods [9]. A few studies of numerical analysis of the generalized particle method and related particle methods have been reported. For example, error estimates of particle methods, which are related to the vortex method, for partial differential equations were established in Raviart [11],

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Ben Moussa–Vila [2], and Ben Moussa [1]. Moreover, a truncation error estimate of an approximate gradient operator of MPS was established in Ishijima-Kimura [8]. Furthermore, for the generalized particle method, we established numerical analysis involving the truncation error estimates of approximate operators [6], [7] and the error estimates for the Poisson and heat equations based on the maximum norm [5]. Therefore, we focus on the numerical analysis of the generalized particle method using discrete Sobolev norms as the next step of this study.

This study considers the Poisson equation with a source term given in divergence form. The formulation is selected, because it has several practical applications. For example, a pressure Poisson equation, which appears in formulations of particle methods for the incompressible Navier–Stokes equations [3], [12], uses a source term including a divergence of a velocity predictor.

A connectivity condition for particle distributions and a semi-regularity condition of a family of discrete parameters are introduced for analyzing the discretized Poisson equation. By virtue of the connectivity condition, a unique solvability of the discretized Poisson equation is derived. Further, by demonstrating certain properties of the discrete Sobolev norms, such as integration by parts, stability of the discretized Poisson equation is obtained with the semi-regularity condition.

2. Formulation

Let $\Omega \subset \mathbb{R}^d$ $(d \in \mathbb{N})$ be a bounded domain with smooth boundary Γ. Let $C(\overline{\Omega})$ be the space of real continuous functions defined in $\overline{\Omega}$. For $k \in \mathbb{N}$, let $C^k(\overline{\Omega})$ be the space of functions in $C(\overline{\Omega})$ with derivatives up to the kth order. We define the function space V as

$$V := \{v \in C(\overline{\Omega});\ v(x) = 0\ (x \in \Gamma)\}.$$

Then, we consider the following Poisson equation with homogeneous boundary condition:

$$\text{Find}\, u \in V \cap C^2(\overline{\Omega}) \text{ s.t. } -\Delta u = \nabla \cdot f,$$

where $f \in (V \cap C^1(\overline{\Omega}))^d$ is given.

We introduce approximate operators in the generalized particle method. Let H be a fixed positive number. For Ω and H we define $\Omega_H \subset \mathbb{R}^d$ and Γ_H by

$$\begin{aligned}
\Omega_H &:= \{x \in \mathbb{R}^d;\ \exists\, y \in \Omega \ \text{ s.t. } |x - y| < H\},\\
\Gamma_H &:= \Omega_H \setminus \Omega.
\end{aligned}$$

For $N \in \mathbb{N}$ we define the particle distribution $\mathcal{X}_N$ as

$$\mathcal{X}_N := \{x_i \in \Omega_H\,;\ i = 1, 2, \ldots, N,\ x_i \neq x_j\ (i \neq j)\}.$$

We refer to $x_i \in \mathcal{X}_N$ as a particle. Figure 1 shows an example of particle distribution in $\mathbb{R}^2$.

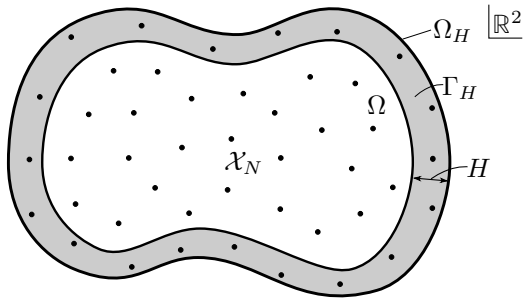

Figure 1. Example of particle distributions $\mathcal{X}_N$ in $\mathbb{R}^2$.

For $N \in \mathbb{N}$ we define the particle volume set $\mathcal{V}_N$ as

$$\mathcal{V}_N := \left\{\omega_i > 0\,;\ i = 1, 2, \ldots, N,\ \sum_{i=1}^{N} \omega_i = |\Omega_H|\right\}.$$

Here $|\Omega_H|$ denotes the volume of Ω_H. We refer to $\omega_i \in \mathcal{V}_N$ as a particle volume. We define the function set $\mathcal{W}$ as

$$\mathcal{W} := \left\{w\colon\ [0, \infty) \to \mathbb{R}\,;\ w(r) > 0\ (0 < r < 1),\ w(r) = 0\ (r \geqslant 1),\ \int_{\mathbb{R}^d} w(|x|)\,\mathrm{d}x = 1\right\}.$$

We refer to $w \in \mathcal{W}$ as a reference weight function. We define the influence radius h as a positive number that satisfies $\min\{|x_i - x_j|\,;\ i \neq j\} < h < H$. For the reference weight function $w \in \mathcal{W}$ and influence radius h, we define the weight function $w_h\colon [0, \infty) \to \mathbb{R}$ as

$$w_h(r) := \frac{1}{h^d} w\Big(\frac{r}{h}\Big).$$

Set discrete parameters $(\mathcal{X}_N, \mathcal{V}_N, h)$ and reference weight function $w \in \mathcal{W}$. Then, we define an approximate divergence operator $\widehat{\nabla}_h \cdot$ for $\psi\colon\ \mathcal{X}_N \to \mathbb{R}^d$ as

$$\widehat{\nabla}_h \cdot \psi(x_i) := d \sum_{j \neq i} \omega_j \frac{\psi(x_j) + \psi(x_i)}{|x_j - x_i|} \cdot \frac{x_j - x_i}{|x_j - x_i|} w_h(|x_j - x_i|) \tag{2.1}$$

and an approximate Laplace operator Δ_h for $\varphi\colon\ \mathcal{X}_N \to \mathbb{R}$ as

$$\Delta_h \varphi(x_i) := 2d \sum_{j \neq i} \omega_i \frac{\varphi(x_j) - \varphi(x_i)}{|x_j - x_i|^2} w_h(|x_j - x_i|). \tag{2.2}$$

We define the index set $\Lambda(S)$ ($S \subset \mathbb{R}^d$) and a function space V_h as

$$\begin{aligned} \Lambda(S) &:= \{i \in \mathbb{N};\ x_i \in \mathcal{X}_N \cap S\}, \\ V_h &:= \{v\colon\ \mathcal{X}_N \to \mathbb{R};\ v(x_i) = 0\ (i \in \Lambda(\Gamma_H))\}. \end{aligned}$$

Then, we consider the following generalized particle method for the Poisson equation:

$$\text{Find}\, \mathrm{u} \in V_h \text{ s.t. } -\Delta_h \mathrm{u}(x_i) = \widehat{\nabla}_h \cdot \widehat{f}(x_i),\ i \in \Lambda(\Omega), \tag{2.3}$$

where $\widehat{f} \in V_h^d$ such that $\widehat{f}(x_i) = f(x_i)$ ($i \in \Lambda(\Omega)$) and $\widehat{f}(x_i) = 0$ ($i \in \Lambda(\Gamma_H)$).

Remark 2.1. We can derive approximate operators (2.1) and (2.2) using the weighted averages of approximations based on the finite volume method, as shown in Appendix A.

3. Connectivity and semi-regularity conditions

We introduce a connectivity condition for particle distributions $\mathcal{X}_N$ and a semi-regularity condition for families of discrete parameters.

Definition 3.1. For influence radius h we say that a particle distribution $\mathcal{X}_N$ satisfies the *h-connectivity condition* if for all $i \in \Lambda(\Omega)$ there exists an integer ζ ($1 \leqslant \zeta \leqslant N$) and a sequence $\{i_k\}_{k=1}^{\zeta} \subset \{1, 2, \ldots, N\}$ such that

$$i_1 = i, \quad |x_{i_k} - x_{i_{k+1}}| < h\ (1 \leqslant k < \zeta), \quad i_k \in \Lambda(\Omega)\ (1 \leqslant k < \zeta), \quad i_\zeta \in \Lambda(\Gamma_H). \tag{3.1}$$

Definition 3.2. A family $\{(\mathcal{X}_N, \mathcal{V}_N, h)\}$ satisfies the *semi-regularity condition* if there exists a positive constant c_0 such that for all elements of the family,

$$\max_{i=1,2,\ldots,N} \sum_{j \neq i} \omega_j w_h(|x_j - x_i|) \leqslant c_0. \tag{3.2}$$

Constant c_0 is called the semi-regularity constant.

Remark 3.3. Consider the graph G whose vertex set is the particle distribution $\mathcal{X}_N$ and whose edges are the pairs (x_i, x_j) that satisfy $0 < |x_i - x_j| < h$; see Figure 2. By Definition 3.1, the particle distribution $\mathcal{X}_N$ satisfies the h-connectivity condition if and only if all vertices of G in Ω have a path to a vertex of G in Γ_H.

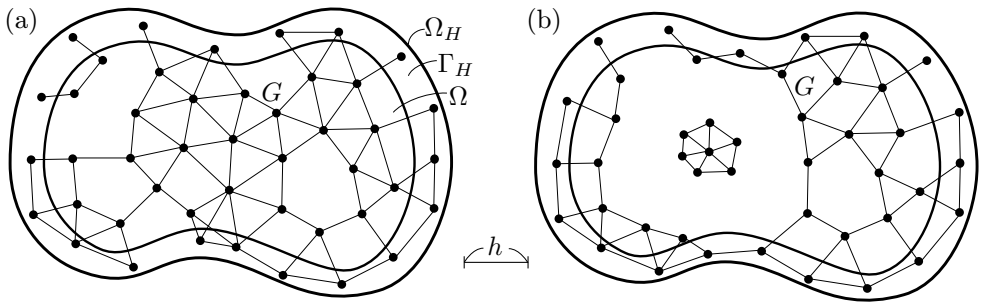

Figure 2. Particle distributions $\mathcal{X}_N$ and their graphs G: the left and right figures show a particle distribution that (a) satisfies, and that (b) does not satisfy the h-connectivity condition, respectively.

4. Unique solvability and stability analysis

First, we show the unique solvability for the discrete Poisson equation (2.3).

Theorem 4.1. *If the particle distribution $\mathcal{X}_N$ satisfies the h-connectivity condition, then the discrete Poisson equation* (2.3) *has a unique solution.*

Proof. Let N_Ω be the number of particles included in Ω. We renumber the index of particles so that $x_i \in \Omega$ $(i = 1, 2, \ldots, N_\Omega)$. Let a_{ij} $(i, j = 1, 2, \ldots, N)$ be

$$a_{ij} := \begin{cases} 0, & i = j, \\ 2d\dfrac{w_h(|x_j - x_i|)}{|x_j - x_i|^2}, & i \neq j. \end{cases}$$

We define matrix $A \in \mathbb{R}^{N_\Omega \times N_\Omega}$ as

$$A_{ij} := \begin{cases} \displaystyle\sum_{k=1}^{N} \frac{\omega_k}{\omega_i} a_{ik}, & i = j, \\ -a_{ij}, & i \neq j. \end{cases}$$

Then the discrete Poisson equation (2.3) is equivalent to:

$$\text{Find}\, y \in \mathbb{R}^{N_\Omega} \quad \text{s.t. } ADy = b,$$

where $D := \mathrm{diag}(\omega_i)$, $b_i := \widehat{\nabla}_h \cdot \widehat{f}(x_i)$ $(i = 1, 2, \ldots, N_\Omega)$, and $y_i := \mathrm{u}(x_i)$ $(i = 1, 2, \ldots, N_\Omega)$. As $\omega_i > 0$ $(i = 1, 2, \ldots, N)$, the diagonal matrix D is a non-singular matrix. Therefore, it is sufficient to prove that A is a non-singular matrix. As A

is symmetric, it is sufficient to prove that A is a positive definite matrix. For all $\alpha \in \mathbb{R}^{N_\Omega} \setminus \{0\}$ we have

$$
\begin{aligned}
(4.1) \qquad \sum_{i,j=1}^{N_\Omega} \alpha_i \alpha_j A_{ij} &= 2 \sum_{1 \leqslant i < j \leqslant N_\Omega} \alpha_i \alpha_j A_{ij} + \sum_{i=1}^{N_\Omega} \alpha_i^2 A_{ii} \\
&= -2 \sum_{1 \leqslant i < j \leqslant N_\Omega} \alpha_i \alpha_j a_{ij} + \sum_{i=1}^{N_\Omega} \alpha_i^2 \sum_{k=1}^{N} \frac{\omega_k}{\omega_i} a_{ik} \\
&= \sum_{1 \leqslant i < j \leqslant N_\Omega} \frac{(\omega_j \alpha_i - \omega_i \alpha_j)^2}{\omega_i \omega_j} a_{ij} + \sum_{i=1}^{N_\Omega} \alpha_i^2 \sum_{k=N_\Omega+1}^{N} \frac{\omega_k}{\omega_i} a_{ik}.
\end{aligned}
$$

As a_{ij} is nonnegative, (4.1) is nonnegative. For $\alpha \in \mathbb{R}^{N_\Omega} \setminus \{0\}$ we set i such that $\alpha_i \neq 0$. Because the particle distribution $\mathcal{X}_N$ satisfies the h-connectivity condition, we can consider a sequence $\{i_k\}_{k=1}^{\zeta}$ such that (3.1) holds. As all terms of the last equation in (4.1) are nonnegative, we have

$$
(4.2) \qquad \sum_{i,j=1}^{N_\Omega} \alpha_i \alpha_j A_{ij} \geqslant \sum_{k=1}^{\zeta-1} \frac{\left(\omega_{i_{k+1}} \alpha_{i_k} - \omega_{i_k} \alpha_{i_{k+1}}\right)^2}{\omega_{i_k} \omega_{i_{k+1}}} a_{i_k i_{k+1}} + \frac{\omega_{i_\zeta}}{\omega_{i_{\zeta-1}}} \alpha_{i_{\zeta-1}}^2 a_{i_{\zeta-1} i_\zeta}.
$$

As $|x_{i_k} - x_{i_{k+1}}| < h$, the value of $a_{i_k i_{k+1}}$ $(k = 1, 2, \ldots, \zeta - 1)$ is positive. Thus, if the right-hand side of (4.2) equals zero, then $\alpha_{i_k} = 0$ $(k = 1, 2, \ldots, \zeta)$. As this is inconsistent with α_i $(= \alpha_{i_1}) \neq 0$, the right-hand side of (4.2) is positive. Therefore, matrix A is a positive definite matrix. □

Next, we introduce a few notations and present certain lemmas. Hereafter, assume that the particle distribution $\mathcal{X}_N$ satisfies the h-connectivity condition. For $S \subset \mathbb{R}^d$ and $n \in \mathbb{N}$ we define the discrete inner product $(\cdot, \cdot)_{h(S)} \colon V_h^n \times V_h^n \to \mathbb{R}$, a discrete L^2-norm $\|\cdot\|_{l^2(S)} : V_h^n \to \mathbb{R}$, and a discrete H_0^1 norm $\|\cdot\|_{h_0^1(S)} : V_h^n \to \mathbb{R}$ as

$$
\begin{aligned}
(\varphi, \psi)_{h(S)} &:= \sum_{i \in \Lambda(S)} \omega_i \, \varphi_i \cdot \psi_i, \\
\|\varphi\|_{l^2(S)} &:= (\varphi, \varphi)_{h(S)}^{1/2} = \left(\sum_{i \in \Lambda(S)} \omega_i \, \varphi_i^2 \right)^{1/2}, \\
\|\varphi\|_{h_0^1(S)} &:= \left(d \sum_{i \in \Lambda(S)} \omega_i \sum_{j \neq i} \omega_j \frac{|\varphi_j - \varphi_i|^2}{|x_j - x_i|^2} w_h(|x_j - x_i|) \right)^{1/2},
\end{aligned}
$$

where $\psi_i := \psi(x_i)$ and $\varphi_i := \varphi(x_i)$. For $\varphi \colon \mathcal{X}_N \to \mathbb{R}$ we define an approximate gradient operator ∇_h by

$$
\nabla_h \varphi(x_i) := d \sum_{j \neq i} \omega_j \frac{\varphi(x_j) - \varphi(x_i)}{|x_j - x_i|} \frac{x_j - x_i}{|x_j - x_i|} w_h(|x_j - x_i|).
$$

R e m a r k 4.2. For function space V_h^n $(n \in \mathbb{N})$, the discrete L^2-norm $\|\cdot\|_{l^2(\Omega)}$ satisfies the conditions of norm. Moreover, the discrete H_0^1-norm $\|\cdot\|_{h_0^1(\Omega)}$ satisfies the conditions of norm if and only if the particle distribution $\mathcal{X}_N$ satisfies the h-connectivity condition.

Then we obtain the following lemma:

Lemma 4.3. *For $\varphi \in V_h$ and $\psi \in V_h^d$ we have*

$$(\widehat{\nabla}_h \cdot \psi, \varphi)_{h(\Omega)} = -(\psi, \nabla_h \varphi)_{h(\Omega)}, \tag{4.3}$$

$$-(\Delta_h \varphi, \varphi)_{h(\Omega)} = \|\varphi\|^2_{h_0^1(\Omega_H)} \geqslant \|\varphi\|^2_{h_0^1(\Omega)}. \tag{4.4}$$

P r o o f. First, we prove (4.3). Let I_{ij} be

$$I_{ij} := \begin{cases} 0, & i = j, \\ d\dfrac{x_j - x_i}{|x_j - x_i|^2} w_h(|x_j - x_i|), & i \neq j. \end{cases}$$

As $\varphi \in V_h$, $\psi \in V_h^d$, and $I_{ij} = -I_{ji}$, we have

$$\begin{aligned}
(\widehat{\nabla}_h \cdot \psi, \varphi)_{h(\Omega)} &= \sum_{i \in \Lambda(\Omega)} \omega_i \varphi_i \sum_{j=1}^{N} \omega_j \left(\psi_j + \psi_i\right) \cdot I_{ij} \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_i \omega_j \varphi_i \left(\psi_j + \psi_i\right) \cdot I_{ij} \\
&= \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_i \omega_j (\varphi_i - \varphi_j) \left(\psi_j + \psi_i\right) \cdot I_{ij} \\
&= \sum_{i=1}^{N} \omega_i \psi_i \cdot \sum_{j=1}^{N} \omega_j (\varphi_i - \varphi_j) I_{ij} \\
&= -(\psi, \nabla_h \varphi)_{h(\Omega)}.
\end{aligned}$$

Next, we prove (4.4). Let J_{ij} $(\geqslant 0)$ be

$$J_{ij} := \begin{cases} 0, & i = j, \\ d\dfrac{w_h(|x_j - x_i|)}{|x_j - x_i|^2}, & i \neq j. \end{cases}$$

As $\varphi \in V_h$ and $J_{ij} = J_{ji}$, we have

$$
\begin{aligned}
-(\Delta_h \varphi, \varphi)_{h(\Omega)} &= 2 \sum_{i \in \Lambda(\Omega)} \omega_i \varphi_i \sum_{j=1}^{N} \omega_j \left(\varphi_i - \varphi_j\right) J_{ij} \\
&= 2 \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_i \omega_j \varphi_i \left(\varphi_i - \varphi_j\right) J_{ij} \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_i \omega_j \left(\varphi_i - \varphi_j\right)^2 J_{ij} = \|\varphi\|^2_{h_0^1(\Omega_H)} \\
&= \|\varphi\|^2_{h_0^1(\Omega)} + \sum_{i \in \Lambda(\Gamma_H)} \omega_i \sum_{j=1}^{N} \omega_j \varphi_j^2 J_{ij} \geqslant \|\varphi\|^2_{h_0^1(\Omega)} .
\end{aligned}
$$

□

Lemma 4.4. *Assume that a family $\{(\mathcal{X}_N, \mathcal{V}_N, h)\}$ satisfies the semi-regularity condition. Then we have*

$$
\|\nabla_h \varphi\|^2_{l^2(\Omega)} \leqslant d\, c_0 \, \|\varphi\|^2_{h_0^1(\Omega)} .
$$

Here, c_0 is the semi-regularity constant in (3.2).

P r o o f. By the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\|\nabla_h \varphi\|^2_{l^2(\Omega)} &= \sum_{i \in \Lambda(\Omega)} \omega_i |\nabla_h \varphi_i|^2 \\
&\leqslant d^2 \sum_{i \in \Lambda(\Omega)} \omega_i \left(\sum_{j \neq i} \omega_j \frac{|\varphi_j - \varphi_i|}{|x_j - x_i|} w_h(|x_j - x_i|) \right)^2 \\
&\leqslant d^2 \sum_{i \in \Lambda(\Omega)} \omega_i \left(\sum_{j \neq i} \omega_j \frac{|\varphi_j - \varphi_i|^2}{|x_j - x_i|^2} w_h(|x_j - x_i|) \right) \\
&\quad \times \max_{i \in \Lambda(\Omega)} \sum_{j \neq i} \omega_j w_h(|x_j - x_i|).
\end{aligned}
$$

As the family $\{(\mathcal{X}_N, \mathcal{V}_N, h)\}$ fulfills the semi-regularity condition, we obtain

$$
\|\nabla_h \varphi\|^2_{l^2(\Omega)} \leqslant d\, c_0 \, \|\varphi\|^2_{h_0^1(\Omega)} .
$$

□

Now, we can establish the following stability of the generalized particle method for the Poisson equation (2.3).

Theorem 4.5. *Assume that the family $\{(\mathcal{X}_N, \mathcal{V}_N, h)\}$ satisfies the semi-regularity condition and that its particle distribution $\mathcal{X}_N$ satisfies the h-connectivity condition. Then there exists a constant c depending only on c_0 and d such that*

$$\|\mathrm{u}\|_{h_0^1(\Omega)} \leqslant c \, \|f\|_{l^2(\Omega)} \, . \tag{4.5}$$

P r o o f. By the Cauchy-Schwarz inequality, (2.3), and Lemmas 4.3 and 4.4, we have

$$\begin{aligned} \|\mathrm{u}\|^2_{h_0^1(\Omega)} &\leqslant |-(\Delta_h \mathrm{u}, \mathrm{u})_{h(\Omega)}| = |(\widehat{\nabla}_h \cdot \widehat{f}, \mathrm{u})_{h(\Omega)}| = |-(\widehat{f}, \nabla_h \mathrm{u})_{h(\Omega)}| \\ &\leqslant \|f\|_{l^2(\Omega)} \, \|\nabla_h \mathrm{u}\|_{l^2(\Omega)} \leqslant \sqrt{d\, c_0} \, \|f\|_{l^2(\Omega)} \, \|\mathrm{u}\|_{h_0^1(\Omega)} \, . \end{aligned}$$

Consequently, we obtain (4.5). □

5. Concluding remarks

We analyzed the stability of a generalized particle method for a Poisson equation with a source term given in divergence form. We obtained the unique solvability of the discretized Poisson equation by introducing a connectivity condition for particle distributions, which is referred to as the h-connectivity condition. Moreover, we established the stability of the discretized Poisson equation based on the semi-regularity condition of a family of discrete parameters and discrete Sobolev norms with properties such as integration by parts.

In future, we will analyze the error estimates of the discretized Poisson equation by showing properties such as the discrete Poincaré inequality. Moreover, we will extend the analysis to, for example, the Poisson equation with the mixed boundary conditions, namely the Dirichlet and Neumann boundary conditions, or time-dependent problems such as the heat and convection-diffusion equations.

Appendix A. Derivation of approximate operators

Assume a two-dimensional or three-dimensional space, $d = 2, 3$. Assume a particle distribution on a square lattice with spacing Δx. For $i, j = 1, 2, \ldots, N$, let $\sigma_i = (x_i - \Delta x/2, x_i + \Delta x/2)^d$, $\gamma_{ij} := \overline{\sigma}_i \cap \overline{\sigma}_j$, and $\lambda_i := \{k = 1, 2, \ldots, N;\ |\gamma_{ik}| \neq 0,\ k \neq i\}$. As $|\sigma_i| = \Delta x^d$ and $|\gamma_{ij}| = \Delta x^{d-1}$ $(j \in \lambda_i)$, by the divergence theorem, we

can approximate the divergence of $\psi\colon\ \Omega_H \to \mathbb{R}^d$ at $x_i \in \mathcal{X}_N$ as

$$\begin{aligned}
\nabla \cdot \psi_i &\approx \frac{1}{|\sigma_i|}\int_{\sigma_i} \nabla \cdot \psi(x)\,\mathrm{d}x = \frac{1}{\Delta x^d}\int_{\partial\sigma_i} \psi(x)\cdot n\,\mathrm{d}s \\
&\approx \frac{1}{\Delta x^d}\sum_{j\in\lambda_i} |\gamma_{ij}|\psi\Big(\frac{x_i+x_j}{2}\Big)\cdot\frac{x_j-x_i}{|x_j-x_i|} \\
&\approx \frac{1}{\Delta x^d}\sum_{j\in\lambda_i} |\gamma_{ij}|\frac{\psi_j+\psi_i}{2}\cdot\frac{x_j-x_i}{|x_j-x_i|} \\
&= \frac{1}{2}\sum_{j\in\lambda_i} \frac{\psi_j+\psi_i}{|x_j-x_i|}\cdot\frac{x_j-x_i}{|x_j-x_i|},
\end{aligned} \tag{A.1}$$

where n is the outward normal vector on the boundary $\partial\sigma_i$. Further, using the central difference, we can approximate the Laplacian of $\varphi\colon\ \Omega_H \to \mathbb{R}$ at $x_i \in \mathcal{X}_N$ as

$$\begin{aligned}
\Delta\varphi_i &\approx \frac{1}{|\sigma_i|}\int_{\sigma_i} \Delta\varphi(x)\,\mathrm{d}x = \frac{1}{\Delta x^d}\int_{\partial\sigma_i} \nabla\varphi(x)\cdot n\,\mathrm{d}s \\
&\approx \frac{1}{\Delta x^d}\sum_{j\in\lambda_i} |\gamma_{ij}|\nabla\varphi\Big(\frac{x_i+x_j}{2}\Big)\cdot\frac{x_j-x_i}{|x_j-x_i|} \\
&\approx \frac{1}{\Delta x^d}\sum_{j\in\lambda_i} |\gamma_{ij}|\frac{\varphi_j-\varphi_i}{|x_j-x_i|} = \sum_{j\in\lambda_i}\frac{\varphi_j-\varphi_i}{|x_j-x_i|^2}.
\end{aligned} \tag{A.2}$$

By noting that the number of elements of λ_i is $2d$, we can derive approximate operators (2.1) and (2.2) as the weighted averages of (A.1) and (A.2), respectively. As the approximation procedures in (A.1) and (A.2) are the same as those of the finite volume method based on Voronoi decomposition, we can regard approximate operators (2.1) and (2.2) as approximations based on the finite volume method.

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