EXISTENCE OF SOLUTIONS FOR NONLINEAR NONMONOTONE EVOLUTION EQUATIONS IN BANACH SPACES WITH ANTI-PERIODIC BOUNDARY CONDITIONS

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Abstract. The paper is devoted to the study of the existence of solutions for nonlinear nonmonotone evolution equations in Banach spaces involving anti-periodic boundary conditions. Our approach in this study relies on the theory of monotone and maximal monotone operators combined with the Schaefer fixed-point theorem and the monotonicity method. We apply our abstract results in order to solve a diffusion equation of Kirchhoff type involving the Dirichlet p-Laplace operator.

Keywords: existence of solutions; anti-periodic; monotone operator; maximal monotone operator; Schaefer fixed-point theorem; monotonicity method; diffusion equation

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1. Introduction

In the paper we study the existence of solutions to nonlinear nonmonotone evolution equations of parabolic type in Banach spaces involving anti-periodic boundary conditions.

In the first part of this paper, we are interested in the evolution equation

$$\begin{cases} u'(t) + A(t)u(t) = f(t) \quad \text{for a.e. } t \in (0,T), \\ u(0) = -u(T), \end{cases} \tag{1.1}$$

where $(A(t))_{t\in(0,T)}$ is a family of operators between a Banach space V (which is densely and continuously embedded into a Hilbert space H) and its dual space V'. Setting

$$\mathcal{X} = L^p(0,T;V)$$

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and

$$\mathcal{D} = \{u \in \mathcal{X} \colon\ u' \in \mathcal{X}' \text{ and } u(0) = -u(T)\},$$

formula (1.1) can be written as an algebraic equation from $\mathcal{D}$ into $\mathcal{X}'$ having the form

$$\mathcal{L}u + \mathcal{A}u = f \tag{1.2}$$

with

$$\mathcal{L}u = u',\ \ u \in \mathcal{D}$$

and

$$\mathcal{A}u = A(\cdot)u(\cdot),\ \ u \in \mathcal{X}.$$

Our approach in solving problem (1.2) consists in applying the theory of monotone and maximal monotone operators. More precisely, we have the following typical situation:

(i) $\mathcal{L}$ is hemicontinuous and maximal monotone,
(ii) $\mathcal{A}$ is hemicontinuous and strictly monotone,
(iii) $\mathcal{L} + \mathcal{A}$ is coercive with respect to the norm of $\mathcal{V}$.

We apply a result which asserts that, under assumptions (i)–(iii), problem (1.2) admits a unique solution. Consequently, under some suitable assumptions on the family of operators $(A(t))_{t\in(0,T)}$, we obtain the existence and uniqueness of solutions for problem (1.1).

In the second part of this paper, we consider the evolution equation

$$\begin{cases} u'(t) + (A_1(t, u(t)))(A_2u(t)) = f(t) \quad \text{for a.e. } t \in (0, T), \\ u(0) = -u(T), \end{cases} \tag{1.3}$$

where $A_1\colon\ \mathbb{R} \times H \to \mathcal{L}(V')$ and $A_2\colon\ V \to V'$ are two given operators. The main difficulty of this problem is that the operator $u \mapsto (A_1(t, u))(A_2u)$ is not monotone. We study the existence of solutions to this problem by applying a famous result which is called the Schaefer fixed-point theorem. To this end, we shall consider the modified evolution equation

$$\begin{cases} u'(t) + (A_1(t, v(t)))(A_2u(t)) = f(t) \quad \text{for a.e. } t \in (0, T), \\ u(0) = -u(T) \end{cases}$$

with $v \in L^p(0, T; H)$. Using the previous study, we know that this last problem admits a unique solution $u \in \mathcal{D}$, and this allows us to define an operator $\Lambda\colon\ L^p(0, T; H) \to L^p(0, T; H)$ for which we apply the Schaefer fixed-point theorem. As a consequence, this provides the existence of solutions for problem (1.3).

It is worth pointing out that fixed-point theorems play an important role in the study of the existence of anti-periodic solutions for evolution equations. By applying a fixed-point theorem due to Browder and Patryshyn [5], Okochi has proved the existence of an anti-periodic solution to the subgradient system

$$u'(t) + \partial\varphi(u(t)) \ni f(t), \quad t \in \mathbb{R},$$

when the functional φ is assumed to be proper, l.s.c., convex and even; see [19]. He got also further results for nonlinear parabolic differential equations in noncylindrical domains and for nonlinear evolution equations associated with odd subdifferential operator, see [20], [21]. Furthermore, Haraux [16] has obtained the existence of anti-periodic solutions to a nonmonotone nonlinear subgradient system by means of the Schauder fixed-point theorem. Utilizing yet the Schauder fixed-point theorem, Chen [6] has obtained the well-posedness of solutions for the evolution problem

$$\begin{cases} u'(t) + Au(t) + \nabla G(u(t)) + F(t, u(t)) = 0, & t \in \mathbb{R}, \\ u(t+T) = -u(t), \quad t \in \mathbb{R}, \end{cases}$$

under the assumptions that $A\colon D(A) \to H$ is a self-adjoint operator, ∇G is the gradient of a mapping $G\colon H \to \mathbb{R}$ and $F\colon \mathbb{R} \times H \to H$ is a nonlinear mapping. Recently, by applying the theory of maximal monotone and pseudomonotone operators, Zhenhai [29] has obtained the existence of solutions for the evolution equation

$$\begin{cases} u'(t) + Au(t) + G(u(t)) = f(t) & \text{for a.e. } t \in (0, T), \\ u(0) = -u(T) \end{cases}$$

under the following assumptions:

(i) $A\colon V \to V'$ is monotone and demicontinuous.

(ii) $G\colon V \to V'$ is continuous, weakly continuous and for any sequence (u_n) in V which converges weakly to u in V we have

$$\limsup \langle Gu_n, u_n - u\rangle_{V',V} \geqslant 0.$$

(iii) There exist positive constants c_1, c_2, c_3, and c_4 such that for every $u \in V$

$$\|Au\|_{V'} \leqslant c_1(\|u\|_V^{p-1} + 1), \ \|Gu\|_{V'} \leqslant c_2(\|u\|_V^{p-1} + 1),$$
$$\langle Au + Gu, u\rangle_{V',V} \geqslant c_3\|u\|_V^p - c_4.$$

By applying the Schaefer fixed-point theorem combined with the continuity method, Boussandel [4] solved, in Hilbert spaces, the evolution equation

$$\begin{cases} u'(t) + Q(t, u(t))^{-1}\nabla E(u(t)) = f(t) & \text{for a.e. } t \in (0, T), \\ u(0) = -u(T), \end{cases}$$

where $Q\colon [0,T]\times H\to\mathcal{L}(H)$ is an operator such that $Q(t,u)$ is invertible for every $(t,u)\in[0,T]\times H$ and ∇E denotes the gradient of a quadratic form E on V with respect to the fixed inner product $\langle\cdot,\cdot\rangle_H$ on H. As a consequence of this result, we obtain the existence of solutions of the quasilinear diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - m(t,\cdot,u)\Delta u = f & \text{in} \quad (0,T)\times\Omega, \\ u = 0 & \text{on} \quad (0,T)\times\partial\Omega, \\ u(0,\cdot) = -u(T,\cdot) & \text{in} \quad \Omega, \end{cases} \tag{1.4}$$

where $m\colon [0,T]\times\Omega\times\mathbb{R}\to[\varepsilon,1/\varepsilon]$ is a measurable function such that $m(t,x,\cdot)$ is continuous for every $(t,x)\in[0,T]\times\Omega$.

The existence of anti-periodic solutions to first order evolution problems has been considered by many mathematicians under several assumptions on nonlinear terms; we invite the readers to refer to [2], [12], [7], [8], [10], [9], [11], [23], [24] for further reading. Second order problems involving anti-periodic boundary conditions are considered in [1], [2], [3], [17], [18], [25], [26], [27].

The rest of the paper is organized as follows. In the next section, we recall some necessary definitions and results on monotone and maximal monotone operators. Then we state the main results with their proofs. Finally, in Section 3, we apply our abstract results of Section 2 to a diffusion equation of Kirchhoff type involving the Dirichlet p-Laplace operator.

2. Functional setting and main results

We start by presenting the functional setting and introducing some definitions, notions and preliminary facts which will be useful in the sequel. Let V be a real reflexive and separable Banach space with norm $\|\cdot\|_V$, and let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and induced norm $\|\cdot\|_H$ such that V is densely and continuously embedded into H. The duality bracket between the dual space V' and V is denoted by $\langle\cdot,\cdot\rangle_{V,V'}$. Identifying H to its dual H', we have $V\subset H\subset V'$. Let $T>0$, $p\geqslant 2$ and let

$$\mathcal{X} = L^p(0,T;V)$$

endowed with its natural norm, i.e.

$$\|u\|_{\mathcal{X}} = \left(\int_0^T \|u(t)\|_V^p \,\mathrm{d}t\right)^{1/p}.$$

The dual space $\mathcal{X}'$ of $\mathcal{X}$ is given by

$$\mathcal{X}' = L^q(0,T;V')$$

with $1/p+1/q=1$. Let further

$$\mathcal{D}=\{u\in\mathcal{X}\colon\ u'\in\mathcal{X}' \text{ and } u(0)=-u(T)\},$$

which is a Banach space for the norm

$$\|u\|_{\mathcal{D}}=\|u\|_{\mathcal{X}}+\|u'\|_{\mathcal{X}'}.$$

Remark 2.1. We note that $\mathcal{D}$ is continuously embedded into $C([0,T];H)$ (see [28], Proposition 23.23), so that if $u\in\mathcal{D}$, then the anti-periodic condition $u(0)=-u(T)$ makes sense.

Let in addition $(A(t))_{t\in(0,T)}$ be a family of operators from V into V'.

Definition 2.1. Let $\mathcal{V}$ be a Banach space and let $\mathcal{A}\colon\ \mathcal{V}\to\mathcal{V}'$ be an operator from $\mathcal{V}$ into its dual space $\mathcal{V}'$.

1. We say that $\mathcal{A}$ is monotone if

$$\langle\mathcal{A}u-\mathcal{A}v,u-v\rangle_{\mathcal{V}',\mathcal{V}}\geqslant 0\quad\forall\,u,v\in\mathcal{V}. \tag{2.1}$$

The operator $\mathcal{A}$ is called strictly monotone if for $u\neq v$ the strict inequality holds in (2.1).

2. We say that $\mathcal{A}$ is hemicontinuous if the function

$$t\mapsto\langle\mathcal{A}(u+tv),w\rangle_{\mathcal{V}',\mathcal{V}}$$

is continuous on $[0,1]$ for all u, v, $w\in\mathcal{V}$.

3. We say that $\mathcal{A}$ is coercive if

$$\lim_{\|u\|_{\mathcal{V}}\to\infty}\frac{\|\mathcal{A}u\|_{\mathcal{V}'}}{\|u\|_{\mathcal{V}}}=\infty.$$

Definition 2.2. Let $\mathcal{V}$ be a Banach space and let $\mathcal{C}$ be a linear dense subspace of $\mathcal{V}$. Let $\mathcal{L}\colon\ \mathcal{C}\to\mathcal{V}'$ be a linear operator. The operator $\mathcal{L}$ is called maximal monotone if for every $(u,f)\in\mathcal{V}\times\mathcal{V}'$ the implication

$$(\langle Lv-f,v-u\rangle_{\mathcal{V}',\mathcal{V}}\geqslant 0\quad\forall\,v\in\mathcal{C})\Rightarrow u\in\mathcal{C}\text{ and }f=\mathcal{L}u$$

holds true.

We assume that the following assumptions hold:

(H1) For each $u \in V$, the function $A(\cdot)u\colon (0,T) \to V'$ is measurable.
(H2) For each $t \in (0,T)$, the operator $A(t)$ is hemicontinuous.
(H3) For each $t \in (0,T)$, the operator $A(t)$ is strictly monotone.
(H4) There exists $c_1 > 0$ such that for every $t \in (0,T)$ and every $u \in V$

$$\|A(t)u\|_{V'} \leqslant c_1 \|u\|_V^{p-1}.$$

(H5) There exist $c_2 > 0$ and $c_3 \geqslant 0$ such that for every $t \in (0,T)$ and every $u \in V$

$$\langle A(t)u, u\rangle_{V',V} \geqslant c_2 \|u\|_V^p - c_3 \|u\|_V.$$

We consider the first order differential equation

$$\begin{cases} u'(t) + A(t)u(t) = f(t) & \text{for a.e. } t \in (0,T), \\ u(0) = -u(T), \end{cases} \tag{2.2}$$

where $f\colon (0,T) \to V'$ is a given function. We have the following result.

Theorem 2.1. *Under assumptions* (H1)–(H5), *for every* $f \in \mathcal{X}'$, *problem* (2.2) *admits a unique solution* $u \in \mathcal{D}$.

The proof of Theorem 2.1 will be based on the following result, see [14], Chapter 3, Section 2.2.

Lemma 2.1. *Let $\mathcal{V}$ be a real separable and reflexive Banach space and let $\mathcal{C}$ be a linear dense subspace of $\mathcal{V}$. Let $\mathcal{A}\colon \mathcal{V} \to \mathcal{V}'$ be a hemicontinuous and monotone operator. Let $\mathcal{L}\colon \mathcal{C} \to \mathcal{V}'$ be a linear hemicontinuous and maximal monotone operator. We assume that $\mathcal{L} + \mathcal{A}$ is coercive with respect to the norm of $\mathcal{V}$. Then $\mathcal{L} + \mathcal{A}$ is surjective. Moreover, if $\mathcal{A}$ is strictly monotone, then $\mathcal{L} + \mathcal{A}$ is invertible.*

Proof of Theorem 2.1. Let $\mathcal{L}\colon \mathcal{D} \to \mathcal{X}'$ be the linear operator defined for every $u \in \mathcal{D}$ by $\mathcal{L}u = u'$. Let further $\mathcal{A}\colon \mathcal{X} \to \mathcal{X}'$ be the operator defined for every $u \in \mathcal{X}$ by

$$\mathcal{A}u = A(\cdot)u(\cdot).$$

Step 1. ($\mathcal{A}$ is well defined.) From assumption (H4) we have for every $u \in \mathcal{X}$

$$\int_0^T \|A(t)u(t)\|_{V'}^q \,\mathrm{d}t \leqslant c_1 \int_0^T \|u(t)\|_V^{q(p-1)} \,\mathrm{d}t = c_1 \int_0^T \|u(t)\|_V^p \,\mathrm{d}t,$$

which shows that $\mathcal{A}$ is well defined.

Step 2. ($\mathcal{A}$ is strictly monotone.) Using assumption (H3) we have for every $u, v \in \mathcal{X}$ such that $u \neq v$

$$\langle \mathcal{A}u - \mathcal{A}v, u - v \rangle_{\mathcal{X}', \mathcal{X}} = \int_0^T \langle A(t)(u(t)) - A(t)(v(t)), u(t) - v(t) \rangle_{V', V} \, \mathrm{d}t > 0,$$

which means that $\mathcal{A}$ is strictly monotone.

Step 3. ($\mathcal{A}$ is hemicontinuous.) Let $u, v, w \in \mathcal{X}$ be fixed and let $\alpha \in [0,1]$ and (t_n) be any sequence in $[0,1]$ which converges to α. We have from the definition of the operator $\mathcal{A}$

$$\langle \mathcal{A}(u + t_n v), w \rangle_{\mathcal{X}', \mathcal{X}} = \int_0^T \langle A(t)(u(t) + t_n v(t)), w(t) \rangle_{V', V} \, \mathrm{d}t.$$

From assumption (H2) we can derive the convergence

$$\langle A(t)(u(t) + t_n v(t)), w(t) \rangle_{V', V} \to \langle A(t)(u(t) + \alpha v(t)), w(t) \rangle_{V', V} \quad \text{for a.e. } t \in (0, T).$$

Moreover, from assumption (H4) we obtain the estimate

$$|\langle A(t)(u(t) + t_n v(t)), w(t) \rangle_{V', V}| \leqslant c'(\|u(t)\|_V^{p-1} + \|v(t)\|_V)^{p-1} \|w(t)\|_V$$

with $c' > 0$ being independent of n. By Hölder's inequality, it follows that we have the following two estimates:

$$\int_0^T \|u(t)\|_V^{p-1} \|w(t)\|_V \, \mathrm{d}s \leqslant \left(\int_0^T \|u(t)\|_V^p \, \mathrm{d}s \right)^{1/p'} \left(\int_0^T \|w(t)\|_V^p \, \mathrm{d}s \right)^{1/p}$$

and

$$\int_0^T \|v(t)\|_V^{p-1} \|w(t)\|_V \, \mathrm{d}t \leqslant \left(\int_0^T \|v(t)\|_V^p \, \mathrm{d}s \right)^{1/p'} \left(\int_0^T \|w(t)\|_V^p \, \mathrm{d}s \right)^{1/p}.$$

This yields that the function

$$t \mapsto (\|u(t)\|_V^{p-1} + \|v(t)\|_V^{p-1}) \|w(t)\|_V$$

belongs to $L^1(0,T)$. The desired result follows from the dominated convergence theorem.

Step 4. ($\mathcal{L}$ is maximal monotone.) For the proof of this fact, we invite the reader to consult [29].

Step 5. ($\mathcal{L}$ is hemicontinuous.) This is an immediate consequence of the fact that $\mathcal{L}$ is a linear bounded operator.

Step 6. ($\mathcal{L}+\mathcal{A}$ is coercive.) Let $R>0$ and let $u \in \mathcal{X}$, $u \neq 0$, such that

$$\frac{\langle \mathcal{L}u + \mathcal{A}u, u\rangle_{\mathcal{X}',\mathcal{X}}}{\|u\|_{\mathcal{X}}} \leqslant R,$$

that is

$$\langle \mathcal{L}u + \mathcal{A}u, u\rangle_{\mathcal{X}',\mathcal{X}} \leqslant R\|u\|_{\mathcal{X}}. \tag{2.3}$$

We prove that there exists $C(R)>0$ such that

$$\|u\|_{\mathcal{X}} \leqslant C(R).$$

From the definition of the operators $\mathcal{L}$ and $\mathcal{A}$ we have the following identity:

$$\langle \mathcal{L}u + \mathcal{A}u, u\rangle_{\mathcal{X}',\mathcal{X}} = \int_0^T \langle u', u\rangle_{V',V}\,\mathrm{d}t + \int_0^T \langle A(t)u, u\rangle_{V',V}\,\mathrm{d}t.$$

Using the fact that $u(0) = -u(T)$, we get from [28], Proposition 23.23,

$$\int_0^T \langle u', u\rangle_{V',V}\,\mathrm{d}t = \int_0^T \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_H^2\,\mathrm{d}t = \frac{1}{2}\|u(T)\|_H^2 - \frac{1}{2}\|u(0)\|_H^2 = 0.$$

It follows from assumption (H5) that

$$c_2\|u\|_{\mathcal{X}}^p - c_3\int_0^T \|u\|_V\,\mathrm{d}t \leqslant \langle \mathcal{L}u + \mathcal{A}u, u\rangle_{\mathcal{X}',\mathcal{X}},$$

which implies by Hölder's inequality that

$$c_2\|u\|_{\mathcal{X}}^p - c_3T\|u\|_{\mathcal{X}} \leqslant \langle \mathcal{L}u + \mathcal{A}u, u\rangle_{\mathcal{X}',\mathcal{X}}.$$

Combining this last estimate with (2.3), we obtain

$$c_2\|u\|_{\mathcal{X}}^p - (c_3T + R)\|u\|_{\mathcal{X}} \leqslant 0.$$

Therefore,

$$\|u\|_{\mathcal{X}} \leqslant \Big(\frac{c_3T+R}{c_2}\Big)^{1/(p-1)}.$$

This proves that $\mathcal{L}+\mathcal{A}$ is a coercive operator. The claim follows from Lemma 2.1. $\square$

In the sequel, we shall consider two operators

$$A_1\colon\ \mathbb{R}\times H\to\mathcal{L}(V'),\quad (t,u)\mapsto A_1(t,u),$$

and

$$A_2\colon\ V\to V',\quad u\mapsto A_2u.$$

We assume that the following assumptions hold:

(A1) For each $u\in V$ and for each $v\in H$, the function $(A_1(\cdot,v))(A_2u)\colon\ \mathbb{R}\to V'$ is measurable.

(A2) For each $(t,v)\in\mathbb{R}\times H$, the operator $A_1(t,v)A_2\colon\ V\to V'$ is hemicontinuous.

(A3) For each $(t,v)\in\mathbb{R}\times H$, the operator $A_1(t,v)A_2\colon\ V\to V'$ is strictly monotone.

(A4) There exist $c_4>0$ and $c_5\geqslant 0$ such that for every $(t,v)\in\mathbb{R}\times H$ and every $u\in V$

$$\langle (A_1(t,v))(A_2u),u\rangle_{V',V}\geqslant c_4\|u\|_V^p-c_5\|u\|_V.$$

(A5) There exists $c_6>0$ such that for every $(t,v)\in\mathbb{R}\times H$

$$\|A_1(t,v)\|_{\mathcal{L}(V')}\leqslant c_6.$$

(A6) There exists $c_7>0$ such that for every $u\in V$

$$\|A_2u\|_{V'}\leqslant c_7\|u\|_V^{p-1}.$$

(A7) If $(v_n)\subset H$ is such that

$$v_n\to v \text{ in } H,$$

then

$$A_1(t,v_n)\to A_1(t,v) \text{ in } \mathcal{L}(V') \text{ a.e. } t\in(0,T).$$

(A8) The embedding $V\hookrightarrow H$ is compact.

We consider the following first order differential equation:

$$\begin{cases} u'(t)+(A_1(t,u(t)))(A_2u(t))=f(t) & \text{for a.e. } t\in(0,T),\\ u(0)=-u(T), \end{cases} \tag{2.4}$$

where $f\colon\ (0,T)\to V'$ is a given function. As a consequence of Theorem 2.1, we obtain the following result.

Theorem 2.2. *Under assumptions* (A1)–(A8), *for every* $f\in\mathcal{X}'$, *problem* (2.4) *admits a solution* $u\in\mathcal{D}$.

P r o o f. To prove this theorem, we shall use the Schaefer fixed-point theorem (see [15]). Set $\mathcal{Y} = L^p(0,T;H)$. Let $f \in \mathcal{X}'$, $v \in \mathcal{Y}$ be fixed and consider the first order differential equation

$$\begin{cases} u'(t) + (A_1(t,v(t)))(A_2u(t)) = f(t) \quad \text{for a.e. } t \in (0,T), \\ u(0) = -u(T). \end{cases} \tag{2.5}$$

Let

$$B(t)\colon V \to V',$$
$$u \mapsto (A_1(t,v(t)))(A_2u).$$

We check that the family of operators $(B(t))_{t\in(0,T)}$ satisfies all assumptions (H1)–(H5). It follows from Theorem 2.1 that problem (2.4) admits a unique solution $u \in \mathcal{D}$. Therefore, we can define the solution mapping

$$\Lambda\colon \mathcal{Y} \to \mathcal{Y},$$
$$v \mapsto \Lambda v = u,$$

where $u \in \mathcal{D}$ is the unique solution of problem (2.5).

Step 1. (Λ is continuous from $\mathcal{Y}$ into $\mathcal{Y}$.) Let $(v_n) \subset \mathcal{Y}$ be such that

$$v_n \to \overline{v} \text{ in } \mathcal{Y}, \tag{2.6}$$

and let $u_n = \Lambda v_n$ and $\overline{u} = \Lambda\overline{v}$. Prove that $u_n \to \overline{u}$ in $\mathcal{Y}$. It suffices to prove that $u_n \to \overline{u}$ in $\mathcal{Y}$ for a subsequence. By convergence (2.6), we can extract from (v_n) a subsequence denoted again by (v_n) such that

$$v_n(t) \to \overline{v}(t) \text{ in } H \text{ a.e. } t \in (0,T). \tag{2.7}$$

By the definition of the mapping Λ, we have

$$\begin{cases} u_n'(t) + (A_1(t,v_n(t)))(A_2u_n(t)) = f(t) \quad \text{a.e. } t \in (0,T), \\ u_n(0) = -u_n(T). \end{cases} \tag{2.8}$$

Multiplying (2.8) by $u_n(t)$ with respect to the duality bracket $\langle\cdot,\cdot\rangle_{V',V}$, then integrating over the interval $(0,T)$ and using the fact that $u_n(0) = -u_n(T)$, we obtain the integral identity

$$\int_0^T \langle (A_1(t,v_n))(A_2u_n), u_n\rangle_{V',V}\,\mathrm{d}t = \int_0^T \langle f, u_n\rangle_{V',V}\,\mathrm{d}t. \tag{2.9}$$

Using assumption (A4), this yields the estimate

$$c_4 \int_0^T \|u_n\|_V^p \,\mathrm{d}t \leqslant \int_0^T (c_5 + \|f\|_{V'})\|u_n\|_V \,\mathrm{d}t.$$

It follows from Hölder's inequality that

$$\|u_n\|_{\mathcal{X}} \leqslant \left[\frac{1}{c_4}\left(\int_0^T (c_5 + \|f\|_{V'})^{p'} \,\mathrm{d}t\right)^{1/p'}\right]^{1/(p-1)}, \tag{2.10}$$

that is

$$(u_n) \text{ is bounded in } \mathcal{X}. \tag{2.11}$$

Moreover, by using assumptions (A5)–(A6), Hölder's inequality and estimate (2.10), we have for every $v \in \mathcal{X}$

$$\begin{aligned}
\left|\int_0^T \langle (A_1(t, v_n))(A_2 u_n), v\rangle_{V',V} \,\mathrm{d}t\right| &\leqslant c_6 c_7 \int_0^T \|u_n\|_V^{p-1} \|v\|_V \,\mathrm{d}t \\
&\leqslant c_6 c_7 \left(\int_0^T \|u_n\|_V^p \,\mathrm{d}t\right)^{(p-1)/p} \left(\int_0^T \|v\|_V^p \,\mathrm{d}t\right)^{1/p} \\
&\leqslant \frac{c_6 c_7}{c_4}\left(\int_0^T c_5 + \|f\|_{V'}^{p'} \,\mathrm{d}t\right)^{1/p'} \left(\int_0^T \|v\|_V^p \,\mathrm{d}t\right)^{1/p},
\end{aligned}$$

which means that

$$((A_1(\cdot, v_n))(A_2 u_n)) \text{ is bounded in } \mathcal{X}'. \tag{2.12}$$

From (2.8) we get that

$$(u_n') \text{ is bounded in } \mathcal{X}'. \tag{2.13}$$

The boundednesses (2.13), (2.12), and (2.11) yield that we can extract from (u_n) a subsequence denoted again by (u_n) such that

$$u_n \rightharpoonup \overline{w} \quad \text{in } \mathcal{X}, \tag{2.14}$$

$$u_n' \rightharpoonup \overline{w}' \quad \text{in } \mathcal{X}', \tag{2.15}$$

$$(A_1(\cdot, v_n))(A_2 u_n) \rightharpoonup \xi \quad \text{in } \mathcal{X}'. \tag{2.16}$$

Applying weak limit (2.16) to (2.8), we get

$$u_n' \rightharpoonup f - \xi \quad \text{in } \mathcal{X}'.$$

Combining this last limit with (2.15), we obtain the identity

$$\overline{w}' + \xi = f. \tag{2.17}$$

Moreover, since $u_n(0) = -u_n(T)$ and since the space $\{u \in \mathcal{X} \colon u' \in \mathcal{X}'\}$ is continuously embedded into $C([0,T];H)$, we deduce from (2.14) and (2.15) that

$$\overline{w}(0) = -\overline{w}(T). \tag{2.18}$$

Combining (2.18), (2.17), and (2.14), identity (2.9) yields that

$$\begin{aligned}\lim_{n\to\infty} \int_0^T \langle (A_1(t,v_n))(A_2u_n)), u_n\rangle_{V',V} &= \int_0^T \langle f(t), \overline{w}\rangle_{V',V}\,\mathrm{d}t \\ = \int_0^T \langle f(t), \overline{w}\rangle_{V',V}\,\mathrm{d}t - \int_0^T \langle \overline{w}', \overline{w}\rangle_{V',V}\,\mathrm{d}t &= \int_0^T \langle \xi, \overline{w}\rangle_{V',V}\,\mathrm{d}t.\end{aligned} \tag{2.19}$$

Let $v \in \mathcal{X}$ and set $w_\lambda = (1-\lambda)\overline{w} + \lambda v$, $\lambda \in (0,1)$. Then, by assumption (A3) we have

$$\int_0^T \langle (A_1(t,v_n))(A_2u_n) - (A_1(t,v_n))(A_2w_\lambda), u_n - w_\lambda\rangle_{V',V}\,\mathrm{d}t \geqslant 0,$$

which can be rewritten as

$$\begin{aligned}\lambda \int_0^T &\langle (A_1(t,v_n))(A_2u_n), \overline{w} - v\rangle_{V',V}\,\mathrm{d}t \\ &\geqslant \lambda \int_0^T \langle (A_1(t,v_n))(A_2w_\lambda), \overline{w} - v\rangle_{V',V}\,\mathrm{d}t \\ &\quad + \int_0^T \langle (A_1(t,v_n))(A_2w_\lambda), u_n - \overline{w}\rangle_{V',V}\,\mathrm{d}t \\ &\quad - \int_0^T \langle (A_1(t,v_n))(A_2u_n), u_n - \overline{w}\rangle_{V',V}\,\mathrm{d}t.\end{aligned} \tag{2.20}$$

Using (2.19) and (2.16), we obtain the convergence

$$\int_0^T \langle (A_1(t,v_n))(A_2u_n), u_n - \overline{w}\rangle_{V',V}\,\mathrm{d}t \to 0. \tag{2.21}$$

We have the following identity:

$$\begin{aligned}\int_0^T &\langle (A_1(t,v_n))(A_2w_\lambda), u_n - \overline{w}\rangle_{V',V}\,\mathrm{d}t \\ &= \int_0^T \langle (A_1(t,v_n) - A_1(t,\overline{v}))(A_2w_\lambda), u_n - \overline{w}\rangle_{V',V}\,\mathrm{d}t \\ &\quad + \int_0^T \langle (A_1(t,\overline{v}))(A_2w_\lambda), u_n - \overline{w}\rangle_{V',V}\,\mathrm{d}t.\end{aligned} \tag{2.22}$$

The convergence (2.14) yields that

$$\int_0^T \langle (A_1(t,\overline{v}))(A_2 w_\lambda), u_n - \overline{w}\rangle_{V',V}\,\mathrm{d}t \to 0. \tag{2.23}$$

In addition, using the fact that (u_n) is bounded in $\mathcal{X}$, we get the following estimate:

$$\begin{aligned}\left|\int_0^T \langle (A_1(t,v_n) - A_1(t,\overline{v}))(A_2 w_\lambda), u_n - \overline{w}\rangle_{V',V}\,\mathrm{d}t\right| \\ \leqslant C\int_0^T \|A_1(t,v_n) - A_1(t,\overline{v})\|^q_{\mathcal{L}(V')}\|A_2 w_\lambda\|^q_V\,\mathrm{d}t\end{aligned}$$

with $C > 0$ being independent of n. Employing assumptions (A5)–(A7) and limit (2.7), it follows from the dominated convergence theorem that

$$\int_0^T \|(A_1(t,v_n))(A_2 w_\lambda) - (A_1(t,\overline{v}))\|^q_{\mathcal{L}(V')}\|A_2 w_\lambda\|^q_V\,\mathrm{d}t \to 0.$$

Therefore,

$$\int_0^T \langle (A_1(t,v_n) - A_1(t,\overline{v}))(A_2 w_\lambda), u_n - \overline{w}\rangle_{V',V}\,\mathrm{d}t \to 0. \tag{2.24}$$

Hence, combining (2.24), (2.23), and (2.22), we obtain

$$\int_0^T \langle (A_1(t,v_n))(A_2 w_\lambda), u_n - \overline{w}\rangle_{V',V}\,\mathrm{d}t \to 0. \tag{2.25}$$

Letting $n \to \infty$ in (2.20) and using (2.25) and (2.21), we get the estimate

$$\int_0^T \langle \xi, \overline{w} - v\rangle_{V',V}\,\mathrm{d}t \geqslant \int_0^T \langle (A_1(t,\overline{v}))(A_2 w_\lambda), \overline{w} - v\rangle_{V',V}\,\mathrm{d}t.$$

From assumptions (A2), (A5), (A6), and the dominated convergence theorem, we deduce by letting $\lambda \to 0$ in this last estimate that

$$\int_0^T \langle \xi, \overline{w} - v\rangle_{V',V}\,\mathrm{d}t \geqslant \int_0^T \langle (A_1(t,\overline{v}))(A_2 \overline{w}), \overline{w} - v\rangle_{V',V}\,\mathrm{d}t.$$

Since $v \in \mathcal{X}$ is arbitrary, it follows from this last estimate that

$$(A_1(\cdot,\overline{v}))(A_2\overline{w}) = \xi.$$

Thus, $\overline{w} \in \mathcal{D}$ and $\overline{w}$ is a solution of the problem

$$\begin{cases} \overline{w}' + (A_1(\cdot, \overline{v}))(A_2\overline{w}) = f & \text{for a.e. } t \in (0,T), \\ \overline{w}(0) = -\overline{w}(T). \end{cases}$$

Since $\overline{u}$ is the unique solution of this last problem, we conclude that $\overline{w} = \overline{u}$. This proves that Λ is a continuous mapping.

Step 2. (Λ is relatively compact.) It suffices to show that for any bounded sequence (u_n) in $\mathcal{X}$ we can extract a subsequence (denoted again by (u_n)) such that (Λu_n) converges strongly in $\mathcal{X}$. This fact can be handled in a similar way as in the first step. We note that in this step we use the fact that the space $\{u \in \mathcal{X}\colon\ u' \in \mathcal{X}'\}$ is compactly embedded into $L^p(0,T;H)$, since V is compactly embedded into H by assumption (A8) (see [28], Problem 23.13).

Step 3. (The Schaefer set $\mathcal{C} := \{u \in \mathcal{X}\colon\ u = \lambda\Lambda u$ for some $\lambda \in [0,1]\}$ is bounded.) Let $u \in \mathcal{C}$. Then there exists $\lambda \in [0,1]$ such that $u = \lambda\Lambda u$, i.e.

$$\begin{cases} u'(t) + (A_1(t,u))(A_2u(t)) = \lambda f(t) & \text{for a.e. } t \in (0,T), \\ u(0) = -u(T). \end{cases}$$

As in the first step, we can see that there exists $C > 0$ which is independent of u such that $\|u\|_{\mathcal{D}} \leqslant C$, which implies that $\mathcal{C}$ is a bounded set. Applying the Schaefer fixed-point theorem, we conclude that there exists $u \in \mathcal{X}$ which is a fixed-point of Λ, that is $u \in \mathcal{D}$ is a solution of problem (2.4). This completes the proof of Theorem 2.2. $\square$

3. Application

Let $\Omega \subset \mathbb{R}^N$ be open and bounded and let

$$\beta\colon\ \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$
$$(t,u) \mapsto \beta(t,u)$$

be a measurable function which satisfies the following conditions.

(β_1) For every $u \in \mathbb{R}$, the function $t \mapsto \beta(t,u)$ is measurable and for every $t \in \mathbb{R}$, the function $u \mapsto \beta(t,u)$ is continuous.

(β_2) There exist $c_8, c_9 > 0$ such that for every $(t,u) \in \mathbb{R} \times \mathbb{R}$

$$c_8 \leqslant \beta(t,u) \leqslant c_9.$$

We consider the diffusion equation of Kirchhoff type

$$(3.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \beta(t, \|u\|_{L^2})\Delta_p u = f & \text{in } (0,T)\times\Omega,\\ u = 0 & \text{on } (0,T)\times\partial\Omega,\\ u(0,\cdot) = -u(T,\cdot) & \text{in } \Omega, \end{cases}$$

where Δ_p is the p-Laplace operator

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

We put

$$V = W_0^{1,p}(\Omega)$$

endowed with the usual norm

$$\|u\|_{1,p} = \|\nabla u\|_{L^p(\Omega)^N}$$

and

$$H = L^2(\Omega)$$

endowed with the usual inner product and norm denoted by $\langle\cdot,\cdot\rangle_{L^2}$ and $\|\cdot\|_{L^2}$, respectively. We introduce the functional $J\colon V \to \mathbb{R}$ defined for every $u \in V$ by

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x.$$

It is well known that J is continuously differentiable on V and we have for every $u, v \in V$

$$J'(u)v = \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,\mathrm{d}x = \langle -\Delta_p u, v\rangle_{V',V}.$$

We set

$$A_1\colon\ \mathbb{R}\times H \to \mathcal{L}(V'),$$
$$(t,u) \to \beta(t, \|u\|_{L^2})I_{V'}$$

and

$$A_2\colon\ V \to V',$$
$$u \mapsto -\Delta_p u,$$

where $I_{V'}$ denotes the identity operator of V'. Assumption (A2) follows from the continuity of the derivative operator J'. It is well known (see [13], Exercise 7.6.13) that the operator A_2 is strongly monotone in the sense that there exists $c > 0$ such that for every $u, v \in V$ we have the estimate

$$\langle A_2u - A_2v, u - v\rangle_{V',V} \geqslant c\|u - v\|_V^p.$$

Hence, combining this fact with assumption (β_2), we get that assumption (A3) is fulfilled. For every $(t,v)\in\mathbb{R}\times H$ and every $u\in V$ one has from assumption (β_2)

$$\langle (A_1(t,v))(A_2u),u\rangle_{V',V}=\beta(t,\|v\|_{L^2})J'(u)u=p\beta(t,\|v\|_{L^2})\|u\|_V^p\geqslant pc_8\|u\|_V^p,$$

which yields that assumption (A4) is satisfied. Assumption (A5) follows immediately from assumption (β_2). In addition, by using Hölder's inequality, we get for every $u\in V$

$$\begin{aligned}|\langle A_2u,v\rangle_{V',V}|&=\left|\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla v\,\mathrm{d}x\right|\leqslant\int_\Omega|\nabla u|^{p-1}|\nabla v|\,\mathrm{d}x\\&\leqslant\|\nabla u\|_{L^p(\Omega)^N}^{p-1}\|\nabla v\|_{L^p(\Omega)^N}=\|u\|_V^{p-1}\|v\|_V,\end{aligned}$$

which implies that

$$\|A_2u\|_{V'}\leqslant\|u\|_V^{p-1}.$$

This proves that assumption (A6) is satisfied. Assumption (A7) is straightforward. Finally, assumption (A8) can be deduced from [22], Corollary 8. As a consequence of Theorem 2.2, we obtain the following result.

Corollary 3.1. *For any $f\in L^q(0,T;W^{-1,q}(\Omega))$ there exists $u\in L^p(0,T;W_0^{1,p}(\Omega))$ such that $u'\in L^q(0,T;W^{-1,q}(\Omega))$, which is the solution of problem* (3.1).

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