HOMOGENIZATION OF A LINEAR PARABOLIC PROBLEM WITH A CERTAIN TYPE OF MATCHING BETWEEN THE MICROSCOPIC SCALES

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Received December 15, 2017. Published online September 25, 2018.

Abstract. This paper is devoted to the study of the linear parabolic problem $\varepsilon\partial_t u_\varepsilon(x,t) - \nabla\cdot(a(x/\varepsilon, t/\varepsilon^3)\nabla u_\varepsilon(x,t)) = f(x,t)$ by means of periodic homogenization. Two interesting phenomena arise as a result of the appearance of the coefficient ε in front of the time derivative. First, we have an elliptic homogenized problem although the problem studied is parabolic. Secondly, we get a parabolic local problem even though the problem has a different relation between the spatial and temporal scales than those normally giving rise to parabolic local problems. To be able to establish the homogenization result, adapting to the problem we state and prove compactness results for the evolution setting of multiscale and very weak multiscale convergence. In particular, assumptions on the sequence $\{u_\varepsilon\}$ different from the standard setting are used, which means that these results are also of independent interest.

Keywords: homogenization; parabolic problem; multiscale convergence; very weak multiscale convergence; two-scale convergence

MSC 2010: 35B27, 35K20

1. Introduction

We will study homogenization of a linear parabolic partial differential equation with one microscopic scale in space and in time, respectively. More precisely, we study, as $\varepsilon \to 0$, the equation

$$\begin{aligned}(1.1)\qquad \varepsilon\partial_t u_\varepsilon(x,t) - \nabla\cdot\Big(a\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^3}\Big)\nabla u_\varepsilon(x,t)\Big) &= f(x,t) \quad \text{in } \Omega_T,\\ u_\varepsilon(x,0) &= u_0(x) \quad \text{in } \Omega,\\ u_\varepsilon(x,t) &= 0 \quad \text{on } \partial\Omega\times(0,T),\end{aligned}$$

DOI: 10.21136/AM.2018.0350-17

where $f \in L^2(\Omega_T)$ and $u_0 \in L^2(\Omega)$. Here $\Omega_T = \Omega \times (0,T)$, where Ω is an open bounded subset of $\mathbb{R}^N$ with smooth boundary and $(0,T)$ is an open bounded interval in $\mathbb{R}$. The thermal conductivity, i.e. the coefficient a, is a periodic function with respect to the unit cube $Y = (0,1)^N$ in $\mathbb{R}^N$ in its first variable and to the interval $S = (0,1)$ in its second variable. For a more detailed description of the equation see Section 3.

The fact that the coefficient in front of the time derivative, the volumetric heat capacity, equals ε gives rise to two phenomena. These concern the character of the homogenized and the local problem and will be visible in the homogenization result.

In the homogenization process we need, among other things, the evolution setting of multiscale and very weak multiscale convergence. These concepts of convergence have been studied in quite general settings for sequences bounded in $W^{1,2}(0,T;H_0^1(\Omega),L^2(\Omega))$, meaning that $\{u_\varepsilon\}$ is bounded in $L^2(0,T;H_0^1(\Omega))$ and $\{\partial_t u_\varepsilon\}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$, see e.g. [7], [19] or [9]. Our problem has a sequence of solutions which is bounded in $L^2(0,T;H_0^1(\Omega))$ but there is, up to the authors' knowledge, no existing proof of boundedness in $W^{1,2}(0,T;H_0^1(\Omega),L^2(\Omega))$. Hence, we need convergence results applicable to sequences of our type and in this paper we establish such results where the usual requirement of boundedness of $\{\partial_t u_\varepsilon\}$ is replaced by a certain condition. These convergence results, see Theorem 2.7 and Theorem 2.10, will be applied in the homogenization of (1.1) but they are also of independent interest.

The homogenization result that we state and prove is presented in Theorem 3.2. We show that, when ε tends to zero, the sequence of solutions $\{u_\varepsilon\}$ to (1.1) converges weakly to a limit u in $L^2(0,T;H_0^1(\Omega))$ which is the unique solution to the homogenized problem

$$-\nabla\cdot(b\nabla u(x,t)) = f(x,t) \quad \text{in } \Omega_T,$$
$$u(x,t) = 0 \quad \text{on } \partial\Omega\times(0,T),$$

where

$$b\nabla u(x,t) = \int_{\mathcal{Y}_{1,1}} a(y,s)(\nabla u(x,t) + \nabla_y u_1(x,t,y,s))\,\mathrm{d}y\,\mathrm{d}s.$$

Here $u_1 \in L^2(\Omega_T;\mathcal{W})$ is the unique solution to the local problem

$$\partial_s u_1(x,t,y,s) - \nabla_y\cdot(a(y,s)(\nabla u(x,t) + \nabla_y u_1(x,t,y,s))) = 0.$$

For the used notation, see Notation 1.1.

The homogenization result reveals the two special phenomena announced above. The first phenomenon is that we get an elliptic homogenized problem although the original problem is of parabolic type. The second phenomenon is that we get so-called resonance, by which we mean that the local problem contains a derivative with respect to a local time variable. It was established already in [4] that parabolic problems normally have this property when the temporal microscopic scale is the square of the spatial scale, see also e.g. [10], [15], [8], [21] or [9]. But, in our case we have resonance even though the spatial and the temporal scale do not relate to each other in that way.

There are a number of other articles treating problems related to (1.1) in the sense that the coefficient in front of the time derivative depends on the parameter ε, see e.g. [17], [3], [6], [8], [21], and [5]. A significant difference is that in those articles the coefficient oscillates, while in our case it vanishes, as ε tends to zero. However, none among those of these articles which treat problems with rapid time oscillations exhibit any other kind of resonance than the standard one mentioned above.

The paper is organized as follows. In Section 2 we briefly recall the concepts of two-scale convergence, evolution multiscale convergence and very weak evolution multiscale convergence. Further, we state and prove a characterization of the evolution multiscale limit of $\{\nabla u_\varepsilon\}$ as well as a very weak evolution multiscale convergence result for the sequence $\{\varepsilon^{-1}u_\varepsilon\}$. In Section 3 we apply the convergence results in the homogenization of the parabolic partial differential equation (1.1).

Notation 1.1. *We let* $\mathcal{Y}_{n,m} = Y^n \times S^m$ *with* $Y^n = Y_1 \times Y_2 \times \ldots \times Y_n$ *and* $S^m = S_1 \times S_2 \times \ldots \times S_m$*, where* $Y_1 = Y_2 = \ldots = Y_n = Y = (0,1)^N$ *and* $S_1 = S_2 = \ldots = S_m = S = (0,1)$*. We denote* $y^n = y_1, y_2, \ldots, y_n$*,* $\mathrm{d}y^n = \mathrm{d}y_1\,\mathrm{d}y_2 \ldots \mathrm{d}y_n$*,* $s^m = s_1, s_2, \ldots, s_m$ *and* $\mathrm{d}s^m = \mathrm{d}s_1\,\mathrm{d}s_2 \ldots \mathrm{d}s_m$*. Moreover, for* $k = 1, \ldots, n$ *and* $j = 1, \ldots, m$*, the scales* $\varepsilon_k(\varepsilon)$ *and* $\varepsilon'_j(\varepsilon)$ *are strictly positive functions such that they tend to zero when* ε *does. Further, we let* $\{\varepsilon_1, \ldots, \varepsilon_n\}$ *and* $\{\varepsilon'_1, \ldots, \varepsilon'_m\}$ *be lists of spatial and temporal scales, respectively. Lastly, we define the space* $\mathcal{W} = \{z \in L^2_\sharp(S; H^1_\sharp(Y)/\mathbb{R})\colon\ \partial_s z \in L^2_\sharp(S; (H^1_\sharp(Y)/\mathbb{R})')\}$ *together with the norm* $\|z\|_{\mathcal{W}} = \|z\|_{L^2_\sharp(S;H^1_\sharp(Y)/\mathbb{R})} + \|\partial_s z\|_{L^2_\sharp(S;(H^1_\sharp(Y)/\mathbb{R})')}$*. The subscript* $\sharp$ *denotes periodicity of the functions involved with respect to the domain in question.*

2. Preliminaries

The main tools in this paper are variants or generalizations of the classical concept of two-scale convergence, which was first introduced by Nguetseng in [13] and [14]. Nguetseng applied the technique to a linear elliptic problem with one spatial microscopic scale. In [1], Allaire provided a proof of compactness for some alternative classes of admissible test functions. He also treated nonlinear elliptic problems and problems defined on perforated domains.

Definition 2.1. A sequence $\{u_\varepsilon\}$ in $L^2(\Omega)$ is said to two-scale converge to $u_0 \in L^2(\Omega \times Y)$ if

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x) v\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \int_\Omega \int_Y u_0(x,y) v(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

for all $v \in L^2(\Omega; C_\sharp(Y))$. We write

$$u_\varepsilon(x) \overset{2}{\rightharpoonup} u_0(x,y).$$

In [2], Allaire and Briane generalized the concept of two-scale convergence to include multiple scales in space and named it multiscale convergence. A compactness result involving an arbitrary number of scales in both space and time was presented in [18] (see also the appendix of [9]). We give the definition of the so-called evolution multiscale convergence.

Definition 2.2. A sequence $\{u_\varepsilon\}$ in $L^2(\Omega_T)$ is said to $(n+1, m+1)$-scale converge to $u_0 \in L^2(\Omega_T \times \mathcal{Y}_{n,m})$ if

$$\begin{aligned}\lim_{\varepsilon\to 0} &\int_{\Omega_T} u_\varepsilon(x,t) v\Big(x, t, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}, \frac{t}{\varepsilon'_1}, \dots, \frac{t}{\varepsilon'_m}\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} u_0(x,t,y^n,s^m) v(x,t,y^n,s^m)\,\mathrm{d}y^n\,\mathrm{d}s^m\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for all $v \in L^2(\Omega_T; C_\sharp(\mathcal{Y}_{n,m}))$. This is denoted by

$$u_\varepsilon(x,t) \xrightharpoonup{n+1,m+1} u_0(x,t,y^n,s^m).$$

We proceed by making some assumptions on the scales. Following [2], we say that the scales in a list are separated if

$$\lim_{\varepsilon\to 0} \frac{\varepsilon_{k+1}}{\varepsilon_k} = 0$$

and well-separated if there exists a positive integer l such that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon_k}\Big(\frac{\varepsilon_{k+1}}{\varepsilon_k}\Big)^l = 0,$$

where $k = 1, \dots, n-1$. The generalization from one to two lists is called jointly separated and jointly well-separated lists of scales and was first presented by Persson, see e.g. [19]. We give the definition.

Definition 2.3. Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ and $\{\varepsilon'_1, \ldots, \varepsilon'_m\}$ be lists of (well-)separated scales. Collect all elements from both the lists in one common list. If from possible duplicates, where by duplicates we mean scales which tend to zero equally fast, one member of each pair is removed and the list in order of magnitude of all the remaining elements is (well-)separated, the lists $\{\varepsilon_1, \ldots, \varepsilon_n\}$ and $\{\varepsilon'_1, \ldots, \varepsilon'_m\}$ are said to be jointly (well-)separated.

Here a compactness result for evolution multiscale convergence follows.

Theorem 2.4. *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^2(\Omega_T)$ and suppose that the lists $\{\varepsilon_1, \ldots, \varepsilon_n\}$ and $\{\varepsilon'_1, \ldots, \varepsilon'_m\}$ are jointly separated. Then, up to a subsequence,*

$$u_\varepsilon(x,t) \xrightarrow{n+1,m+1} u_0(x,t,y^n,s^m)$$

where $u_0 \in L^2(\Omega_T \times \mathcal{Y}_{n,m})$.

Proof. See Theorem A.1 in [9]. □

As the next theorem states, the evolution multiscale limit is unique.

Theorem 2.5. *The $(n+1, m+1)$-scale limit is unique.*

Proof. A proof of uniqueness of the two-scale limit can be found in the discussion below Definition 1 in [12]. The proof for the $(n+1, m+1)$-scale limit can be done in a similar way. □

Since (1.1) has two spatial and two temporal scales we will apply the evolution multiscale convergence with $n = m = 1$, i.e. we will use $(2,2)$-scale convergence. We proceed by stating and proving the $(2,2)$-scale convergence result for the gradient under certain assumptions, suitable for our problem. First, we give the following lemma, which will be used in the orthogonal reasoning in the proof of the convergence result.

Lemma 2.6. *Let H be the space of generalized divergence-free functions in $L^2(\Omega; L^2_\sharp(Y))^N$ defined by*

$$H = \{v \in L^2(\Omega; L^2_\sharp(Y))^N;\ \nabla_y \cdot v = 0\}.$$

The space H has the following properties:

(i) *$D(\Omega; C^\infty_\sharp(Y))^N \cap H$ is dense in H,*

(ii) *the orthogonal complement of H is*

$$H^\perp = \{\nabla_y u_1(x,y);\ u_1 \in L^2(\Omega; H^1_\sharp(Y))\}.$$

P r o o f. See Lemma 3.7 in [2] with $n = 1$. $\square$

We are now ready to give the convergence result.

Theorem 2.7. *Assume that $\{u_\varepsilon\}$ is bounded in $L^2(0,T;H_0^1(\Omega))$ and, for any $v_1 \in D(\Omega)$, $c_1 \in D(0,T)$, $c_2 \in C_\sharp^\infty(S)$ and $r > 0$,*

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} u_\varepsilon(x,t)v_1(x)\partial_t\Big(\varepsilon^r c_1(t)c_2\Big(\frac{t}{\varepsilon^r}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{2.1}$$

Then, with $\varepsilon_1 = \varepsilon$ and $\varepsilon_1' = \varepsilon^r$, up to a subsequence,

$$u_\varepsilon(x,t) \rightharpoonup u(x,t) \quad \text{in } L^2(0,T;H_0^1(\Omega)) \tag{2.2}$$

and

$$\nabla u_\varepsilon(x,t) \overset{2,2}{\rightharpoonup} \nabla u(x,t) + \nabla_y u_1(x,t,y,s), \tag{2.3}$$

where $u \in L^2(0,T;H_0^1(\Omega))$ and $u_1 \in L^2(\Omega_T \times S; H_\sharp^1(Y)/\mathbb{R})$.

P r o o f. The weak convergence (2.2) follows immediately from the boundedness of $\{u_\varepsilon\}$ in $L^2(0,T;H_0^1(\Omega))$. From the same boundedness we have that $\{\nabla u_\varepsilon\}$ is bounded in $L^2(\Omega_T)^N$. Theorems 2.4 and 2.5 give that there exist unique functions $u_0 \in L^2(\Omega_T \times \mathcal{Y}_{1,1})$ and $\tau_0 \in L^2(\Omega_T \times \mathcal{Y}_{1,1})^N$ such that, up to a subsequence,

$$u_\varepsilon(x,t) \overset{2,2}{\rightharpoonup} u_0(x,t,y,s) \tag{2.4}$$

and

$$\nabla u_\varepsilon(x,t) \overset{2,2}{\rightharpoonup} \tau_0(x,t,y,s). \tag{2.5}$$

We continue by showing that the $(2,2)$-scale limit u_0 depends neither on y nor on s, meaning that $u_0 \in L^2(\Omega_T)$. On the left-hand side of (2.3), we choose the test function

$$v(x)c(t) = \varepsilon v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big)c_1(t)c_2\Big(\frac{t}{\varepsilon^r}\Big),$$

where $v_1 \in D(\Omega)$, $v_2 \in C_\sharp^\infty(Y)^N$, $c_1 \in D(0,T)$ and $c_2 \in C_\sharp^\infty(S)$. By integration by parts and after differentiations we have that

$$\begin{aligned}\int_{\Omega_T} &\nabla u_\varepsilon(x,t)\varepsilon v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big)c_1(t)c_2\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_{\Omega_T} -u_\varepsilon(x,t)\Big(\varepsilon\nabla v_1(x)\cdot v_2\Big(\frac{x}{\varepsilon}\Big) + v_1(x)\nabla_y\cdot v_2\Big(\frac{x}{\varepsilon}\Big)\Big)c_1(t)c_2\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

and as $\varepsilon \to 0$, due to (2.5) the sequence $\{\varepsilon \nabla u_\varepsilon\}$ tends to zero and we obtain

$$\lim_{\varepsilon\to 0} \int_{\Omega_T} -u_\varepsilon(x,t)\Big(\varepsilon\nabla v_1(x)\cdot v_2\Big(\frac{x}{\varepsilon}\Big) + v_1(x)\nabla_y \cdot v_2\Big(\frac{x}{\varepsilon}\Big)\Big)c_1(t)c_2\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

The first term vanishes and from (2.4) we get

$$\int_{\Omega_T}\int_{\mathcal{Y}_{1,1}} -u_0(x,t,y,s)v_1(x)\nabla_y\cdot v_2(y)c_1(t)c_2(s)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t = 0$$

and by the Variational Lemma

$$-\int_Y u_0(x,t,y,s)\nabla_y\cdot v_2(y)\,\mathrm{d}y = 0$$

a.e. in $\Omega_T \times S$. Thus, u_0 is independent of y.

To show independence of s we carry out the differentiations in (2.1) and obtain

$$\lim_{\varepsilon\to 0}\Big(\int_{\Omega_T}\varepsilon^r u_\varepsilon(x,t)v_1(x)\partial_t c_1(t)c_2\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} u_\varepsilon(x,t)v_1(x)c_1(t)\partial_s c_2\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t\Big) = 0.$$

Passing to the limit, using (2.4), we arrive at

$$\int_{\Omega_T}\int_{\mathcal{Y}_{1,1}} u_0(x,t,s)v_1(x)c_1(t)\partial_s c_2(s)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t = 0$$

and the Variational Lemma gives

$$\int_S u_0(x,t,s)\partial_s c_2(s)\,\mathrm{d}s = 0$$

a.e. in Ω_T, hence u_0 does not depend on the local time variable s. Thus, the independences yield

$$u_\varepsilon(x,t) \overset{2,2}{\rightharpoonup} u_0(x,t), \tag{2.6}$$

where $u_0 \in L^2(\Omega_T)$.

Now we will show that $u_0 \in L^2(0,T;H_0^1(\Omega))$. Since (2.2) holds, we also have

$$u_\varepsilon(x,t) \rightharpoonup u(x,t) \;\text{ in } L^2(\Omega_T) \tag{2.7}$$

for the same $u \in L^2(0,T;H_0^1(\Omega))$. If the $(2,2)$-scale limit, u_0, in (2.6) is the same as u in (2.7) we have that $u_0 \in L^2(0,T;H_0^1(\Omega))$. Note that (2.6) means

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} u_\varepsilon(x,t)v\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T}\int_{\mathcal{Y}_{1,1}} u_0(x,t)v(x,t,y,s)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t$$

for all $v \in L^2(\Omega_T;C_\sharp(\mathcal{Y}_{1,1}))$. Since $L^2(\Omega_T) \subset L^2(\Omega_T;C_\sharp(\mathcal{Y}_{1,1}))$ this convergence implies, for all $v \in L^2(\Omega_T)$,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega_T} u_\varepsilon(x,t)v(x,t)\,\mathrm{d}x\,\mathrm{d}t &= \int_{\Omega_T}\int_{\mathcal{Y}_{1,1}} u_0(x,t)v(x,t)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_{\Omega_T} u_0(x,t)v(x,t)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where in the last step we integrated over y and s. Hence, we see that the $(2,2)$-scale limit u_0 coincides with the weak $L^2(0,T;H_0^1(\Omega))$ limit u.

Next we will identify τ_0. Using the product of $v \in D(\Omega;C_\sharp^\infty(Y))^N \cap H$ defined in Lemma 2.6, $c_1 \in D(0,T)$ and $c_2 \in C_\sharp^\infty(S)$ as test functions in (2.5) we get, up to a subsequence,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\Omega_T} \nabla u_\varepsilon(x,t)\cdot v\Big(x,\frac{x}{\varepsilon}\Big)c_1(t)c_2\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad= \int_{\Omega_T}\int_{\mathcal{Y}_{1,1}} \tau_0(x,t,y,s)\cdot v(x,y)c_1(t)c_2(s)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t,\end{aligned}$$

for some $\tau_0 \in L^2(\Omega_T \times \mathcal{Y}_{1,1})^N$. Integration by parts on the left-hand side leads to

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\Omega_T} -u_\varepsilon(x,t)\nabla\cdot v\Big(x,\frac{x}{\varepsilon}\Big)c_1(t)c_2\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad= \lim_{\varepsilon\to 0}\int_{\Omega_T} -u_\varepsilon(x,t)\Big(\nabla_x\cdot v\Big(x,\frac{x}{\varepsilon}\Big)+\varepsilon^{-1}\nabla_y\cdot v\Big(x,\frac{x}{\varepsilon}\Big)\Big)c_1(t)c_2\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad= \lim_{\varepsilon\to 0}\int_{\Omega_T} -u_\varepsilon(x,t)\nabla_x\cdot v\Big(x,\frac{x}{\varepsilon}\Big)c_1(t)c_2\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where, in the last step, the second term has vanished due to the fact that $\nabla_y \cdot v = 0$. Passing to the limit yields

$$\begin{aligned}&\int_{\Omega_T}\int_{\mathcal{Y}_{1,1}} -u(x,t)\nabla_x\cdot v(x,y)c_1(t)c_2(s)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\\ &\quad= \int_{\Omega_T}\int_{\mathcal{Y}_{1,1}} \nabla u(x,t)\cdot v(x,y)c_1(t)c_2(s)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

and hence we have

$$\int_{\Omega_T}\int_{\mathcal{Y}_{1,1}} \tau_0(x,t,y,s)\cdot v(x,y)c_1(t)c_2(s)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t \\ = \int_{\Omega_T}\int_{\mathcal{Y}_{1,1}} \nabla u(x,t)\cdot v(x,y)c_1(t)c_2(s)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t.$$

From this we deduce that

$$\int_{\Omega_T}\int_{\mathcal{Y}_{1,1}} (\tau_0(x,t,y,s)-\nabla u(x,t))\cdot v(x,y)c_1(t)c_2(s)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t = 0,$$

which means, according to the Variational Lemma, that

$$\int_{\Omega}\int_{Y} (\tau_0(x,t,y,s)-\nabla u(x,t))\cdot v(x,y)\,\mathrm{d}y\,\mathrm{d}x = 0$$

a.e. in $(0,T)\times S$. As we can see $\tau_0(x,t,y,s)-\nabla u(x,t)$ is orthogonal to $v\in D(\Omega; C_\sharp^\infty(Y))^N\cap H$ and by property (i) in Lemma 2.6 to the whole space H. Hence, we have that

$$\tau_0(x,t,y,s)-\nabla u(x,t)\in H^\perp.$$

By property (ii) in Lemma 2.6 we conclude that there exists a function u_1 in $L^2(\Omega_T\times S; H_\sharp^1(Y)/\mathbb{R})$ such that

$$\tau_0(x,t,y,s)-\nabla u(x,t) = \nabla_y u_1(x,t,y,s),$$

which proves (2.3). □

In the homogenization procedure of (1.1) the product $\varepsilon^{-1}u_\varepsilon$ will appear. Since $\{\varepsilon^{-1}u_\varepsilon\}$ is not guaranteed to be bounded in $L^2(\Omega_T)$ it may lack a multiscale limit and hence we need another type of convergence. The idea was originally presented in Corollary 3.3 in [10], where the convergence of $\{\varepsilon^{-1}(u_\varepsilon - u)\}$ was established. Nguetseng published, in [15], a closely related result for a somewhat different class of test functions, which led to the abbreviation of $\{\varepsilon^{-1}(u_\varepsilon - u)\}$ to $\{\varepsilon^{-1}u_\varepsilon\}$. The convergence, in its present form, is called very weak multiscale convergence and its definition was given for an arbitrary number of spatial scales in [7], where also the name was first introduced. Later it was generalized to include arbitrarily many temporal scales as well, see e.g. [20] or [9]. We give the definition of very weak evolution multiscale convergence.

Definition 2.8. A sequence $\{w_\varepsilon\}$ in $L^1(\Omega_T)$ is said to $(n+1, m+1)$-scale converge very weakly to $w_0 \in L^1(\Omega_T \times \mathcal{Y}_{n,m})$ if

$$\lim_{\varepsilon\to 0} \int_{\Omega_T} w_\varepsilon(x,t) v_1\Big(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_{n-1}}\Big) v_2\Big(\frac{x}{\varepsilon_n}\Big) c\Big(t, \frac{t}{\varepsilon'_1}, \ldots, \frac{t}{\varepsilon'_m}\Big) \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} w_0(x,t,y^n,s^m) v_1(x, y^{n-1}) v_2(y_n) c(t, s^m) \,\mathrm{d}y^n\,\mathrm{d}s^m\,\mathrm{d}x\,\mathrm{d}t$$

for any $v_1 \in D(\Omega; C^\infty_\sharp(Y^{n-1}))$, $v_2 \in C^\infty_\sharp(Y_n)/\mathbb{R}$ and $c \in D(0,T; C^\infty_\sharp(S^m))$, where

$$\int_{Y_n} w_0(x,t,y^n,s^m) \,\mathrm{d}y_n = 0. \tag{2.8}$$

We write

$$w_\varepsilon(x,t) \xrightarrow[vw]{n+1,m+1} w_0(x,t,y^n,s^m).$$

R e m a r k 2.9. Due to (2.8) the very weak evolution multiscale limit is unique.

A compactness result for very weak evolution multiscale convergence was proved for sequences bounded in $W^{1,2}(0,T; H^1_0(\Omega), L^2(\Omega))$, see e.g. [20] or [9]. Here, we apply somewhat different assumptions to suit e.g. our problem (1.1). As we did in Theorem 2.7 we let $n = m = 1$. Note that (2.9) is the same as (2.1) in Theorem 2.7.

Theorem 2.10. *Assume that $\{u_\varepsilon\}$ is bounded in $L^2(0,T; H^1_0(\Omega))$ and, for any $v_1 \in D(\Omega)$, $c_1 \in D(0,T)$, $c_2 \in C^\infty_\sharp(S)$ and $r > 0$,*

$$\lim_{\varepsilon\to 0} \int_{\Omega_T} u_\varepsilon(x,t) v_1(x) \partial_t\Big(\varepsilon^r c_1(t) c_2\Big(\frac{t}{\varepsilon^r}\Big)\Big) \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{2.9}$$

Then, with $\varepsilon_1 = \varepsilon$ and $\varepsilon'_1 = \varepsilon^r$, up to a subsequence,

$$\varepsilon^{-1} u_\varepsilon(x,t) \xrightarrow[vw]{2,2} u_1(x,t,y,s), \tag{2.10}$$

where $u_1 \in L^2(\Omega_T \times S; H^1_\sharp(Y)/\mathbb{R})$ is the same as in (2.3) in Theorem 2.7.

P r o o f. We start by pointing out that (2.10) means

$$\lim_{\varepsilon\to 0} \int_{\Omega_T} \varepsilon^{-1} u_\varepsilon(x,t) v_1(x) v_2\Big(\frac{x}{\varepsilon}\Big) c_1(t) c_2\Big(\frac{t}{\varepsilon^r}\Big) \,\mathrm{d}x\,\mathrm{d}t \tag{2.11}$$
$$= \int_{\Omega_T} \int_{\mathcal{Y}_{1,1}} u_1(x,t,y,s) v_1(x) v_2(y) c_1(t) c_2(s) \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t,$$

where $v_1 \in D(\Omega)$, $v_2 \in C^\infty_\sharp(Y)/\mathbb{R}$, $c_1 \in D(0,T)$ and $c_2 \in C^\infty_\sharp(S)$. We note that any $v_2 \in C^\infty_\sharp(Y)/\mathbb{R}$ can be expressed as

$$v_2(y) = \Delta_y \varrho(y) = \nabla_y \cdot (\nabla_y \varrho(y)) \tag{2.12}$$

for some $\varrho \in C^\infty_\sharp(Y)/\mathbb{R}$. By (2.12), the left-hand side of (2.11) becomes

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_{\Omega_T} \varepsilon^{-1} u_\varepsilon(x,t) v_1(x) c_1(t) c_2\Big(\frac{t}{\varepsilon^r}\Big) \nabla_y \cdot \Big(\nabla_y \varrho\Big(\frac{x}{\varepsilon}\Big)\Big) \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \lim_{\varepsilon\to 0} \int_{\Omega_T} u_\varepsilon(x,t) v_1(x) c_1(t) c_2\Big(\frac{t}{\varepsilon^r}\Big) \nabla \cdot \Big(\nabla_y \varrho\Big(\frac{x}{\varepsilon}\Big)\Big) \,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

and by integration by parts we obtain

$$\begin{aligned} \lim_{\varepsilon\to 0} \Big(&\int_{\Omega_T} -\nabla u_\varepsilon(x,t) v_1(x) c_1(t) c_2\Big(\frac{t}{\varepsilon^r}\Big) \cdot \nabla_y \varrho\Big(\frac{x}{\varepsilon}\Big) \,\mathrm{d}x\,\mathrm{d}t \\ &- \int_{\Omega_T} u_\varepsilon(x,t) \nabla v_1(x) c_1(t) c_2\Big(\frac{t}{\varepsilon^r}\Big) \cdot \nabla_y \varrho\Big(\frac{x}{\varepsilon}\Big) \,\mathrm{d}x\,\mathrm{d}t \Big). \end{aligned}$$

Passing to the limit, using Theorem 2.7, we get, up to a subsequence,

$$\begin{aligned} &\int_{\Omega_T}\int_{\mathcal{Y}_{1,1}} - (\nabla u(x,t) + \nabla_y u_1(x,t,y,s)) v_1(x) c_1(t) c_2(s) \cdot \nabla_y \varrho(y) \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad - \int_{\Omega_T}\int_{\mathcal{Y}_{1,1}} u(x,t) \nabla v_1(x) c_1(t) c_2(s) \cdot \nabla_y \varrho(y) \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

After integration by parts in the last term with respect to x we arrive at

$$-\int_{\Omega_T}\int_{\mathcal{Y}_{1,1}} \nabla_y u_1(x,t,y,s) v_1(x) c_1(t) c_2(s) \cdot \nabla_y \varrho(y) \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t$$

and by integration by parts with respect to y we obtain

$$\begin{aligned} &\int_{\Omega_T}\int_{\mathcal{Y}_{1,1}} u_1(x,t,y,s) v_1(x) c_1(t) c_2(s) \nabla_y \cdot (\nabla_y \varrho(y)) \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad = \int_{\Omega_T}\int_{\mathcal{Y}_{1,1}} u_1(x,t,y,s) v_1(x) v_2(y) c_1(t) c_2(s) \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Hence, the proof is complete. □

Remark 2.11. The assumption (2.1) used in Theorem 2.7 and Theorem 2.10, to overcome the lack of boundedness of $\{\partial_t u_\varepsilon\}$ in $L^2(0,T;H^{-1}(\Omega))$, was to the authors' knowledge first introduced in [11]. This can be seen as a compactness assumption on the distributional derivative of $\{u_\varepsilon\}$ in a certain weak sense.

3. Homogenization

In this section we will give the homogenization result for the partial differential equation (1.1) presented in the introduction of this paper. We consider

$$
\begin{aligned}
\varepsilon\partial_t u_\varepsilon(x,t) - \nabla\cdot\Big(a\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^3}\Big)\nabla u_\varepsilon(x,t)\Big) &= f(x,t) \quad \text{in } \Omega_T,\\
u_\varepsilon(x,0) &= u_0(x) \quad \text{in } \Omega,\\
u_\varepsilon(x,t) &= 0 \quad \text{on } \partial\Omega\times(0,T),
\end{aligned} \tag{3.1}
$$

where $a \in C_\sharp(\mathcal{Y}_{1,1})^{N\times N}$, $f \in L^2(\Omega_T)$, and $u_0 \in L^2(\Omega)$. We assume that the coefficient a satisfies the coercivity condition

$$
a(y,s)\xi\cdot\xi \geqslant C_0|\xi|^2 \tag{3.2}
$$

for a.e. $(y,s) \in \mathcal{Y}_{1,1}$, for every $\xi \in \mathbb{R}^N$ and for some $C_0 > 0$. The problem possesses a unique solution $u_\varepsilon \in W^{1,2}(0,T;H_0^1(\Omega),L^2(\Omega))$, see Section 23.7 in [22]. Note that a sequence $\{u_\varepsilon\}$ that lies in $W^{1,2}(0,T;H_0^1(\Omega),L^2(\Omega))$ is not necessarily uniformly bounded in that space.

We will show that our problem satisfies the conditions required for Theorem 2.7 and Theorem 2.10. For this and for the homogenization procedure we need the weak form of (3.1), which is

$$
\begin{aligned}
\int_{\Omega_T} -\varepsilon u_\varepsilon(x,t)v(x)\partial_t c(t) + a\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^3}\Big)\nabla u_\varepsilon(x,t)\cdot\nabla v(x)c(t)\,\mathrm{d}x\,\mathrm{d}t&\\
= \int_{\Omega_T} f(x,t)v(x)c(t)\,\mathrm{d}x\,\mathrm{d}t&,
\end{aligned} \tag{3.3}
$$

where $v \in H_0^1(\Omega)$ and $c \in D(0,T)$.

Proposition 3.1. *Let $\{u_\varepsilon\}$ be a sequence of solutions to (3.1) in $W^{1,2}(0,T;H_0^1(\Omega),L^2(\Omega))$. Then the following properties hold.*

(i) *The sequence $\{u_\varepsilon\}$ is bounded in $L^2(0,T;H_0^1(\Omega))$, i.e. it satisfies the a priori estimate*

$$
\|u_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \leqslant C, \tag{3.4}
$$

where $C > 0$ is a constant independent of ε.

(ii)

$$
\lim_{\varepsilon\to 0}\int_{\Omega_T} u_\varepsilon(x,t)v_1(x)\partial_t\Big(\varepsilon^3 c_1(t)c_2\Big(\frac{t}{\varepsilon^3}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t = 0,
$$

where $v_1 \in D(\Omega)$, $c_1 \in D(0,T)$ and $c_2 \in C_\sharp^\infty(S)$.

P r o o f. We start by proving (i). Using $u_\varepsilon \in W^{1,2}(0,T;H_0^1(\Omega),L^2(\Omega))$ as a test function in the operator form of (3.1), we obtain

$$\int_0^T \varepsilon\langle \partial_t u_\varepsilon, u_\varepsilon\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,\mathrm{d}t + \int_{\Omega_T} a\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^3}\Big)\nabla u_\varepsilon(x,t)\cdot\nabla u_\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f(x,t)u_\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t,$$

see Section 30.3 in [23]. Multiplying by 2 and using formula (25) in Section 23.6 in [22], we get

$$\int_\Omega \varepsilon((u_\varepsilon(x,T))^2-(u_0(x))^2)\,\mathrm{d}x + 2\int_{\Omega_T} a\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^3}\Big)\nabla u_\varepsilon(x,t)\cdot\nabla u_\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t = 2\int_{\Omega_T} f(x,t)u_\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t$$

and rewriting gives

$$\varepsilon\|u_\varepsilon(\cdot,T)\|^2_{L^2(\Omega)} + 2\int_{\Omega_T} a\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^3}\Big)\nabla u_\varepsilon(x,t)\cdot\nabla u_\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t = \varepsilon\|u_0\|^2_{L^2(\Omega)} + 2\int_{\Omega_T} f(x,t)u_\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

The coercivity condition (3.2) now yields

$$\varepsilon\|u_\varepsilon(\cdot,T)\|^2_{L^2(\Omega)} + 2C_0\int_{\Omega_T} |\nabla u_\varepsilon(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant \varepsilon\|u_0\|^2_{L^2(\Omega)} + 2\int_{\Omega_T} f(x,t)u_\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t$$

or equivalently

$$\varepsilon\|u_\varepsilon(\cdot,T)\|^2_{L^2(\Omega)} + 2C_0\|u_\varepsilon\|^2_{L^2(0,T;H_0^1(\Omega))} \leqslant \varepsilon\|u_0\|^2_{L^2(\Omega)} + 2\int_{\Omega_T} f(x,t)u_\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Further, using the property

$$\int_{\Omega_T} f(x,t)u_\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t \leqslant C_1\|u_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))},$$

where $C_1>0$ is independent of ε, and applying the elementary inequality

$$2xy \leqslant C_0^{-1}x^2 + C_0y^2,$$

we obtain

$$\begin{aligned}\varepsilon\|u_\varepsilon(\cdot,T)\|^2_{L^2(\Omega)} &+ 2C_0\|u_\varepsilon\|^2_{L^2(0,T;H^1_0(\Omega))} \\ &\leqslant \varepsilon\|u_0\|^2_{L^2(\Omega)} + C_0^{-1}C_1^2 + C_0\|u_\varepsilon\|^2_{L^2(0,T;H^1_0(\Omega))}.\end{aligned}$$

Rewriting and noting that

$$\varepsilon\|u_\varepsilon(\cdot,T)\|^2_{L^2(\Omega)} \geqslant 0,$$

we arrive at

$$\|u_\varepsilon\|^2_{L^2(0,T;H^1_0(\Omega))} \leqslant \varepsilon C_0^{-1}\|u_0\|^2_{L^2(\Omega)} + C_0^{-2}C_1^2.$$

Since $u_0 \in L^2(\Omega)$ are known, the left-hand side will stay bounded while $\varepsilon \to 0$. This implies the a priori estimate (3.4).

We continue by proving (ii). Using the weak form (3.3) with

$$v(x)c(t) = \varepsilon^2 v_1(x)c_1(t)c_2\Big(\frac{t}{\varepsilon^3}\Big),$$

where $v_1 \in D(\Omega)$, $c_1 \in D(0,T)$ and $c_2 \in C^\infty_\sharp(S)$, we get

$$\begin{aligned}&\int_{\Omega_T} -\varepsilon u_\varepsilon(x,t)\varepsilon^2 v_1(x)\partial_t\Big(c_1(t)c_2\Big(\frac{t}{\varepsilon^3}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_{\Omega_T} a\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^3}\Big)\nabla u_\varepsilon(x,t)\cdot\varepsilon^2\nabla v_1(x)c_1(t)c_2\Big(\frac{t}{\varepsilon^3}\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{\Omega_T} f(x,t)\varepsilon^2 v_1(x)c_1(t)c_2\Big(\frac{t}{\varepsilon^3}\Big)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

and by rearranging we obtain

$$\begin{aligned}&\int_{\Omega_T} u_\varepsilon(x,t)v_1(x)\partial_t\Big(\varepsilon^3 c_1(t)c_2\Big(\frac{t}{\varepsilon^3}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{\Omega_T} \varepsilon^2 a\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^3}\Big)\nabla u_\varepsilon(x,t)\cdot\nabla v_1(x)c_1(t)c_2\Big(\frac{t}{\varepsilon^3}\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad - \int_{\Omega_T} \varepsilon^2 f(x,t)v_1(x)c_1(t)c_2\Big(\frac{t}{\varepsilon^3}\Big)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

By (i) we know that $\{u_\varepsilon\}$ is bounded in $L^2(0,T;H^1_0(\Omega))$ and therefore $\{\nabla u_\varepsilon\}$ is bounded in $L^2(\Omega_T)^N$. Hence,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\Omega_T} u_\varepsilon(x,t)v_1(x)\partial_t\Big(\varepsilon^3 c_1(t)c_2\Big(\frac{t}{\varepsilon^3}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \lim_{\varepsilon\to 0}\Big(\int_{\Omega_T} \varepsilon^2 a\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^3}\Big)\nabla u_\varepsilon(x,t)\cdot\nabla v_1(x)c_1(t)c_2\Big(\frac{t}{\varepsilon^3}\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad - \int_{\Omega_T} \varepsilon^2 f(x,t)v_1(x)c_1(t)c_2\Big(\frac{t}{\varepsilon^3}\Big)\,\mathrm{d}x\,\mathrm{d}t\Big) = 0\end{aligned}$$

and the proof is complete. □

Finally, we are ready to give the homogenization result. Here we see that the coefficient ε, indeed, gives rise to the phenomena stated in the introduction.

Theorem 3.2. *Let* $\{u_\varepsilon\}$ *be a sequence of solutions to* (3.1) *in* $W^{1,2}(0,T;H_0^1(\Omega), L^2(\Omega))$. *Then*

$$u_\varepsilon(x,t) \rightharpoonup u(x,t) \quad \text{in } L^2(0,T;H_0^1(\Omega))$$

and

$$\nabla u_\varepsilon(x,t) \overset{2,2}{\rightharpoonup} \nabla u(x,t) + \nabla_y u_1(x,t,y,s),$$

where $u \in L^2(0,T;H_0^1(\Omega))$ *is the unique solution to*

$$\begin{aligned} -\nabla\cdot(b\nabla u(x,t)) &= f(x,t) \quad \text{in } \Omega_T,\\ u(x,t) &= 0 \quad \text{on } \partial\Omega\times(0,T) \end{aligned} \tag{3.5}$$

with

$$b\nabla u(x,t) = \int_{\mathcal{Y}_{1,1}} a(y,s)(\nabla u(x,t) + \nabla_y u_1(x,t,y,s))\,\mathrm{d}y\,\mathrm{d}s.$$

Here, $u_1 \in L^2(\Omega_T;\mathcal{W})$ *is the unique solution to the local problem*

$$\partial_s u_1(x,t,y,s) - \nabla_y\cdot(a(y,s)(\nabla u(x,t) + \nabla_y u_1(x,t,y,s))) = 0. \tag{3.6}$$

Letting

$$u_1(x,t,y,s) = \nabla u(x,t)\cdot z(y,s),$$

where $z_j \in \mathcal{W}^N$, *the local problem can be expressed as*

$$\partial_s z_j(y,s) - \nabla_y\cdot(a(y,s)(e_j + \nabla_y z_j(y,s))) = 0,$$

where $j = 1,\ldots,N$, *and the coefficient* b *in the homogenized problem as*

$$b_{ij} = \int_{\mathcal{Y}_{1,1}} a_{ij}(y,s) + \sum_{k=1}^{N} a_{ik}(y,s)\partial_{y_k} z_j(y,s)\,\mathrm{d}y\,\mathrm{d}s.$$

Proof. Proposition 3.1 holds, hence Theorem 2.7 guarantees, up to a subsequence, that

$$u_\varepsilon(x,t) \rightharpoonup u(x,t) \quad \text{in } L^2(0,T;H_0^1(\Omega)) \tag{3.7}$$

and

$$\nabla u_\varepsilon(x,t) \overset{2,2}{\rightharpoonup} \nabla u(x,t) + \nabla_y u_1(x,t,y,s), \tag{3.8}$$

where $u \in L^2(0,T;H_0^1(\Omega))$ and $u_1 \in L^2(\Omega_T\times S;H_\sharp^1(Y)/\mathbb{R})$.

To obtain the homogenized problem we choose test functions in (3.3) without microscopic oscillations. More precisely, by choosing

$$v(x)c(t) = v_1(x)c_1(t),$$

where $v_1 \in H_0^1(\Omega)$ and $c_1 \in D(0,T)$, we get

$$\int_{\Omega_T} -\varepsilon u_\varepsilon(x,t)v_1(x)\partial_t c_1(t) + a\Big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^3}\Big)\nabla u_\varepsilon(x,t)\cdot\nabla v_1(x)c_1(t)\,\mathrm{d}x\,\mathrm{d}t \\ = \int_{\Omega_T} f(x,t)v_1(x)c_1(t)\,\mathrm{d}x\,\mathrm{d}t.$$

When ε tends to zero the first term vanishes due to (3.7) and by (3.8) we have

$$\int_{\Omega_T}\int_{\mathcal{Y}_{1,1}} a(y,s)(\nabla u(x,t) + \nabla_y u_1(x,t,y,s))\cdot\nabla v_1(x)c_1(t)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\ = \int_{\Omega_T} f(x,t)v_1(x)c_1(t)\,\mathrm{d}x\,\mathrm{d}t.$$

By the Variational Lemma one has

$$\int_\Omega\bigg(\int_{\mathcal{Y}_{1,1}} a(y,s)(\nabla u(x,t) + \nabla_y u_1(x,t,y,s))\,\mathrm{d}y\,\mathrm{d}s\bigg)\cdot\nabla v_1(x)\,\mathrm{d}x \\ = \int_\Omega f(x,t)v_1(x)\,\mathrm{d}x$$

for a.e. $t \in (0,T)$, which is the weak form of (3.5).

To find the local problem we choose test functions in (3.3) that capture the microscopic oscillations, i.e. we choose

$$v(x)c(t) = \varepsilon v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big)c_1(t)c_2\Big(\frac{t}{\varepsilon^3}\Big),$$

where $v_1 \in D(\Omega)$, $v_2 \in C_\sharp^\infty(Y)/\mathbb{R}$, $c_1 \in D(0,T)$ and $c_2 \in C_\sharp^\infty(S)$. After differentiation we get

$$\int_{\Omega_T} -u_\varepsilon(x,t)v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big)\Big(\varepsilon^2\partial_t c_1(t)c_2\Big(\frac{t}{\varepsilon^3}\Big) + \varepsilon^{-1}c_1(t)\partial_s c_2\Big(\frac{t}{\varepsilon^3}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t \\ + \int_{\Omega_T} a\Big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^3}\Big)\nabla u_\varepsilon(x,t)\cdot\Big(\varepsilon\nabla v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big) + v_1(x)\nabla_y v_2\Big(\frac{x}{\varepsilon}\Big)\Big)c_1(t)c_2\Big(\frac{t}{\varepsilon^3}\Big)\,\mathrm{d}x\,\mathrm{d}t \\ = \int_{\Omega_T} f(x,t)\varepsilon v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big)c_1(t)c_2\Big(\frac{t}{\varepsilon^3}\Big)\,\mathrm{d}x\,\mathrm{d}t$$

and passing to the limit, omitting terms that equal zero, leaves us with

$$\lim_{\varepsilon\to 0}\biggl(\int_{\Omega_T} -\varepsilon^{-1}u_\varepsilon(x,t)v_1(x)v_2\Bigl(\frac{x}{\varepsilon}\Bigr)c_1(t)\partial_s c_2\Bigl(\frac{t}{\varepsilon^3}\Bigr)\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} a\Bigl(\frac{x}{\varepsilon},\frac{t}{\varepsilon^3}\Bigr)\nabla u_\varepsilon(x,t)\cdot v_1(x)\nabla_y v_2\Bigl(\frac{x}{\varepsilon}\Bigr)c_1(t)c_2\Bigl(\frac{t}{\varepsilon^3}\Bigr)\,\mathrm{d}x\,\mathrm{d}t\biggr) = 0.$$

Applying, for $r=3$, Theorem 2.10 to the first term and Theorem 2.7 to the second, we obtain

$$\int_{\Omega_T}\int_{\mathcal{Y}_{1,1}} - u_1(x,t,y,s)v_1(x)v_2(y)c_1(t)\partial_s c_2(s)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T}\int_{\mathcal{Y}_{1,1}} a(y,s)(\nabla u(x,t) + \nabla_y u_1(x,t,y,s))\cdot v_1(x)\nabla_y v_2(y)c_1(t)c_2(s)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Using the Variational Lemma, we arrive at

$$\int_{\mathcal{Y}_{1,1}} - u_1(x,t,y,s)v_2(y)\partial_s c_2(s)\,\mathrm{d}y\,\mathrm{d}s + \int_{\mathcal{Y}_{1,1}} a(y,s)(\nabla u(x,t) + \nabla_y u_1(x,t,y,s))\cdot\nabla_y v_2(y)c_2(s)\,\mathrm{d}y\,\mathrm{d}s = 0$$

a.e. in Ω_T, which is the weak form of (3.6). □

R e m a r k 3.3. The well-posedness of the homogenized equation (3.5), including both the homogenized coefficient and the local problem, has been studied in earlier works. Already in [4], a well-posed local problem of the same type as (3.6) is formulated and it is shown that the thereby obtained homogenized coefficient generates an elliptic operator, thus (3.5) has a unique solution for every fixed t. Regarding the uniqueness and regularity of the solution to the local problem, a detailed study of the weak form of a monotone parabolic local problem, obtained by methods of two-scale convergence type, is found in [21]. The authors also formulate the specialization to the linear case. See also [3]. Equation (3.6) appears as a special case of the respective local problems obtained in [3] and [21] and the existence of a unique solution in $L^2(\Omega_T;\mathcal{W})$ of the weak form of (3.6) follows. For more studies on the well-posedness of parabolic local problems, see e.g. [16].

References

[1] *G. Allaire*: Homogenization and two-scale convergence. SIAM J. Math. Anal. *23* (1992), 1482–1518. zbl MR doi

[2] *G. Allaire, M. Briane*: Multiscale convergence and reiterated homogenisation. Proc. R. Soc. Edinb., Sect. A. *126* (1996), 297–342. zbl MR doi

[3] *G. Allaire, A. Piatnitski*: Homogenization of nonlinear reaction-diffusion equation with a large reaction term. Ann. Univ. Ferrara, Sez. VII, Sci. Mat. *56* (2010), 141–161. zbl MR doi

[4] *A. Bensoussan, J.-L. Lions, G. Papanicolaou*: Asymptotic Analysis for Periodic Structures. Studies in Mathematics and Its Applications 5, North-Holland Publishing, Amsterdam, 1978. zbl MR doi

[5] *H. Douanla, J. L. Woukeng*: Homogenization of reaction-diffusion equations in fractured porous media. Electron. J. Differ. Equ. *2015* (2015), Paper No. 253, 23 pages. zbl MR

[6] *L. Flodén, A. Holmbom, M. Olsson Lindberg*: A strange term in the homogenization of parabolic equations with two spatial and two temporal scales. J. Funct. Spaces Appl. *2012* (2012), Article ID 643458, 9 pages. zbl MR doi

[7] *L. Flodén, A. Holmbom, M. Olsson, J. Persson*: Very weak multiscale convergence. Appl. Math. Lett. *23* (2010), 1170–1173. zbl MR doi

[8] *L. Flodén, A. Holmbom, M. Olsson Lindberg, J. Persson*: A note on parabolic homogenization with a mismatch between the spatial scales. Abstr. Appl. Anal. *2013* (2013), Article ID 329704, 6 pages. zbl MR doi

[9] *L. Flodén, A. Holmbom, M. Olsson Lindberg, J. Persson*: Homogenization of parabolic equations with an arbitrary number of scales in both space and time. J. Appl. Math. *2014* (2014), Article ID 101685, 16 pages. MR doi

[10] *A. Holmbom*: Homogenization of parabolic equations an alternative approach and some corrector-type results. Appl. Math., Praha *42* (1997), 321–343. zbl MR doi

[11] *T. Lobkova*: Homogenization of linear parabolic equations with a certain resonant matching between rapid spatial and temporal oscillations in periodically perforated domains. Available at `https://arxiv.org/abs/1704.01483` (2017).

[12] *D. Lukkassen, G. Nguetseng, P. Wall*: Two-scale convergence. Int. J. Pure Appl. Math. *2* (2002), 35–86. zbl MR

[13] *G. Nguetseng*: A general convergence result for a functional related to the theory of homogenization. SIAM J. Math. Anal. *20* (1989), 608–623. zbl MR doi

[14] *G. Nguetseng*: Asymptotic analysis for a stiff variational problem arising in mechanics. SIAM J. Math. Anal. *21* (1990), 1394–1414. zbl MR doi

[15] *G. Nguetseng, J. L. Woukeng*: Σ-convergence of nonlinear parabolic operators. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *66* (2007), 968–1004. zbl MR doi

[16] *A. Pankov*: *G*-Convergence and Homogenization of Nonlinear Partial Differential Operators. Mathematics and Its Applications 422, Kluwer Academic Publishers, Dordrecht, 1997. zbl MR doi

[17] *F. Paronetto*: Homogenization of degenerate elliptic-parabolic equations. Asymptotic Anal. *37* (2004), 21–56. zbl MR

[18] *J. Persson*: Homogenization of Some Selected Elliptic and Parabolic Problems Employing Suitable Generalized Modes of Two-Scale Convergence. Mid Sweden University Licentiate Thesis 45, Department of Engineering and Sustainable Development, Mid Sweden University, 2010.

[19] *J. Persson*: Homogenization of monotone parabolic problems with several temporal scales. Appl. Math., Praha *57* (2012), 191–214. zbl MR doi

[20] *J. Persson*: Selected Topics in Homogenization. Mid Sweden University Doctoral Thesis 127, Department of Engineering and Sustainable Development, Mid Sweden University, 2012.

[21] *N. Svanstedt, J. L. Woukeng*: Periodic homogenization of strongly nonlinear reaction-diffusion equations with large reaction terms. Appl. Anal. *92* (2013), 1357–1378. zbl MR doi
[22] *E. Zeidler*: Nonlinear Functional Analysis and Its Applications. II/A: Linear Monotone Operators. Springer, New York, 1990. zbl MR doi
[23] *E. Zeidler*: Nonlinear Functional Analysis and Its Applications. II/B: Nonlinear Monotone Operators. Springer, New York, 1990. zbl MR doi

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