63 (2018) APPLICATIONS OF MATHEMATICS No. 4, 483–498

A WEAK COMPARISON PRINCIPLE FOR SOME QUASILINEAR ELLIPTIC OPERATORS: IT COMPARES FUNCTIONS BELONGING TO DIFFERENT SPACES

Akihito Unai, Tokyo

Received April 20, 2018. Published online July 17, 2018.

Abstract. We shall prove a weak comparison principle for quasilinear elliptic operators $-\operatorname{div}(a(x,\nabla u))$ that includes the negative p-Laplace operator, where $a\colon \Omega\times\mathbb{R}^N\to\mathbb{R}^N$ satisfies certain conditions frequently seen in the research of quasilinear elliptic operators. In our result, it is characteristic that functions which are compared belong to different spaces.

Keywords: weak comparison principle; quasilinear elliptic operator; p-Laplace operator

MSC 2010: 35B51, 35J62, 35J25

1. Introduction and statement of the result

There are many comparison principles (maximum principles) for the second order elliptic differential operators (see [4], [5], [6], [8], [9]). The comparison principle implies the unique solvability and some regularity results of solutions to elliptic differential equations.

In this paper, we shall study a weak comparison principle for some quasilinear elliptic operators. In our case, it is characteristic that functions which are compared belong to different spaces. Let Ω be an open set in $\mathbb{R}^N$(additional restriction will be imposed according to situations in the sequel) and $1<p<\infty$. We consider a Carathéodory map $a\colon \Omega\times\mathbb{R}^N\to\mathbb{R}^N$ which satisfies the following conditions (a-1), (a-2), (a-3):

(a-1) there exists $\alpha>0$ depending on p such that

$$a(x,\ \xi)\cdot\xi\geqslant\alpha|\xi|^p\quad\text{a.e. } x\in\Omega\ \forall\,\xi\in\mathbb{R}^N,$$

a dot denotes here the Euclidean scalar product in $\mathbb{R}^N$,

DOI: 10.21136/AM.2018.0126-18

(a-2) there exists $\beta > 0$ depending on p such that

$$|a(x,\ \xi)| \leqslant \beta |\xi|^{p-1} \quad \text{a.e. } x \in \Omega\ \forall \xi \in \mathbb{R}^N,$$

(a-3) there exists $\gamma > 0$ depending on p such that if $p \geqslant 2$, then
(i) $\{a(x,\xi) - a(x,\ \eta)\} \cdot (\xi - \eta) \geqslant \gamma |\xi - \eta|^p$ a.e. $x \in \Omega$, for all ξ, $\eta \in \mathbb{R}^N$,
and if $1 < p < 2$, then
(ii) $\{a(x,\xi) - a(x,\eta)\} \cdot (\xi - \eta) \geqslant \gamma \{|\xi| + |\eta|\}^{p-2} |\xi - \eta|^2$ a.e. $x \in \Omega$, for all ξ, $\eta \in \mathbb{R}^N$ with $|\xi| + |\eta| > 0$.

The above conditions (a-1), (a-2), (a-3) are frequently seen in the research of quasilinear elliptic operators (see [4]). We consider the operator $-\operatorname{div}(a(x, \nabla u))$ generated by the Carathéodory map a mentioned above. The simple model case is the negative p-Laplace operator. We can now state our theorem:

Theorem 1.1. *Let Ω be an open set in $\mathbb{R}^N$ bounded in one direction and $1 < p < \infty$, $1/p + 1/p' = 1$. Assume the above conditions (a-1), (a-2), (a-3). Let $f \in L^{p'}(\Omega)$ and $g \in L^{p'}_{\mathrm{loc}}(\Omega)$. Furthermore, assume that $u \in W_0^{1,p}(\Omega)$, $w \in W^{1,p}_{\mathrm{loc}}(\Omega)$ with $w \geqslant 0$ a.e. in Ω and f, g satisfy the following conditions (c-1), (c-2), (c-3):*

(c-1) $-\operatorname{div}(a(x, \nabla u)) = f$ *in Ω (in the distributional sense),*
(c-2) $-\operatorname{div}(a(x, \nabla w)) = g$ *in Ω (in the distributional sense),*
(c-3) $f \leqslant g$ *a.e. in Ω.*

Then $u \leqslant w$ a.e. in Ω.

Remark 1.1. (i) For example, (c-1) means that

$$\int_\Omega a(x, \nabla u) \cdot \nabla \varphi \, \mathrm{d}x = \int_\Omega f \varphi \, \mathrm{d}x \quad \forall \varphi \in C_0^\infty(\Omega). \tag{1.1}$$

Since $C_0^\infty(\Omega)$ is dense in the space $W_0^{1,p}(\Omega)$, using condition (a-2) we see that (1.1) holds for any $\varphi \in W_0^{1,p}(\Omega)$.

(ii) When $w \in W^{1,p}_{\mathrm{loc}}(\Omega)$ satisfies the above condition (c-2), $w_1 := w + c$ satisfies the same condition (c-2) for all $c \in \mathbb{R}$ as well. Therefore, it follows from Theorem 1.1 that $u \leqslant w_1$ a.e. in Ω whenever there exists a constant $c \in \mathbb{R}$ such that $w_1 = w + c \geqslant 0$ a.e. in Ω.

In the following, we use the so-called positive part and negative part of a (real valued) function u, defined by

$$u^+ = u(x)^+ = \max\{u(x), 0\}, \quad u^- = u(x)^- = -\min\{u(x), 0\}.$$

As an elementary comparison principle for the operator $-\operatorname{div}(a(x, \nabla u))$, the next one is well-known.

(A) Let Ω be an open set in $\mathbb{R}^N$ and $1 < p < \infty$. Let the Carathéodory map a satisfy conditions (a-2) and (a-3)$'$ instead of (a-3) as follows:

(a-3)$'$ $\quad \{a(x,\xi) - a(x,\eta)\} \cdot (\xi - \eta) > 0$ a.e. $x \in \Omega$ for all $\xi,\ \eta \in \mathbb{R}^N$, $\xi \neq \eta$.

Assume that $u_i \in W^{1,p}(\Omega)$, $i = 1, 2$, satisfy the following:

$$-\operatorname{div}(a(x, \nabla u_1)) \leqslant -\operatorname{div}(a(x, \nabla u_2)) \quad \text{in } \Omega, \tag{1.2}$$

in the sense of distributions, that is,

$$\int_\Omega a(x, \nabla u_1) \cdot \nabla \varphi \, \mathrm{d}x \leqslant \int_\Omega a(x, \nabla u_2) \cdot \nabla \varphi \, \mathrm{d}x \quad \forall\, \varphi \in C_0^\infty(\Omega),\ \varphi \geqslant 0. \tag{1.3}$$

(Then note that (1.3) holds for any $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geqslant 0$ a.e. in Ω by the argument of density and condition (a-2).)

Furthermore, suppose that $u_1 \leqslant u_2$ on $\partial\Omega$ (this means $(u_1 - u_2)^+ \in W_0^{1,p}(\Omega)$ in the definition). Then $u_1 \leqslant u_2$ a.e. in Ω.

On the other hand, it needs various devices to compare functions $u_i \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $i = 1, 2$ (see [5], [6]). Applying [6], Theorem 4.8 to the operator $-\operatorname{div}(a(x, \nabla u))$, for example, we can have the following result:

(B) Let Ω be a bounded open set in $\mathbb{R}^N$ and $1 < p < \infty$. Let the Carathéodory map a satisfy conditions (a-2) and (a-3)$'$. Assume that $u_i \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $i = 1, 2$, satisfy (1.2) in the sense of distributions and $u_1 \leqslant u_2$ on $\partial\Omega$. Then it follows that $u_1 \leqslant u_2$ a.e. in Ω.

Though this is a fine assertion, in this case, the inequality '$u_1 \leqslant u_2$ on $\partial\Omega$' means that for every ε there exists a neighborhood V of $\partial\Omega$ such that for a.e. $x \in V$ we have $u_1(x) \leqslant u_2(x) + \varepsilon$ (see [6], p. 954, Section 4.1). Therefore, to apply this result we need to know the situation of u_i, $i = 1, 2$, in a neighborhood of $\partial\Omega$ in advance. Moreover, it needs the boundedness of Ω.

In our Theorem 1.1, only w belongs to the space $W^{1,p}_{\mathrm{loc}}(\Omega)$ and u belongs to the "good" space $W_0^{1,p}(\Omega)$, however, the open set Ω may be unbounded as long as it is bounded in one direction and there is no difficulty for the corresponding condition to '$u_1 \leqslant u_2$ on $\partial\Omega$'. Needless to say, though u and w belong to the same space $W^{1,p}_{\mathrm{loc}}(\Omega)$ in our case, we use essentially that u belongs to the space $W_0^{1,p}(\Omega)$. In this sense, functions u and w belong to different spaces. Our assertion is different from others in this viewpoint.

Especially, setting $a(x,\xi) = |\xi|^{p-2}\xi$ in Theorem 1.1, we immediately obtain the next corollary for the negative p-Laplace operator $-\Delta_p$:

$$-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

Corollary 1.2. *Let Ω be an open set in $\mathbb{R}^N$ bounded in one direction and $1 < p < \infty$, $1/p + 1/p' = 1$. Let $f \in L^{p'}(\Omega)$ and $g \in L^{p'}_{\mathrm{loc}}(\Omega)$. Assume that $u \in W_0^{1,p}(\Omega)$, $w \in W^{1,p}_{\mathrm{loc}}(\Omega)$ with $w \geqslant 0$ a.e. in Ω and f, g satisfy the following conditions* (i), (ii), (iii)*:*

(i) *$-\Delta_p u = f$ in Ω (in the distributional sense),*

(ii) *$-\Delta_p w = g$ in Ω (in the distributional sense),*

(iii) *$f \leqslant g$ a.e. in Ω.*

Then

$$u \leqslant w \quad \text{a.e. in } \Omega.$$

Remark 1.2. Note that conditions (a-1), (a-2), (a-3) are automatically satisfied for $a(x,\xi) = |\xi|^{p-2}\xi$ with $1 < p < \infty$.

As a simple application to our result, we can show the boundedness of the distributional solution to the p-Laplace equation under the Dirichlet boundary condition. This boundedness result has already been obtained by [3], Theorem 17.7 when Ω is bounded, however, we consider the proof is not applicable when Ω is bounded in only one direction. Therefore, we demonstrate that the proof of [3], Theorem 17.7 is still valid for domains Ω which are bounded in only one direction with our Corollary 1.2.

2. Lemmas

In this section we give three lemmas to prove our theorem. The first one is well-known.

Lemma 2.1. *Let Ω be an open set in $\mathbb{R}^N$ bounded in one direction and $1 < p < \infty$, $1/p + 1/p' = 1$. Assume a Carathéodory map $a\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies conditions* (a-1), (a-2), (a-3)$'$, *which have already been mentioned. Then for every $f \in L^{p'}(\Omega)$ there exists a unique distributional solution $u \in W_0^{1,p}(\Omega)$ such that*

$$-\operatorname{div}(a(x,\nabla u)) = f \quad \text{in } \Omega.$$

The next statement is mentioned in [10], Lemma 2.2.

Lemma 2.2. *Let Ω be an open set in $\mathbb{R}^N$ and $1 \leqslant p < \infty$.*

(i) *Let $v \in W^{1,p}(\Omega)$ and $v^+,\ w \in W_0^{1,p}(\Omega)$. Then we have*

$$(v-w)^+,\ (w-v)^-,\ (w+v)^+,\ (-w-v)^- \in W_0^{1,p}(\Omega).$$

(ii) *Let* $v \in W^{1,p}(\Omega)$ *and* v^-, $w \in W_0^{1,p}(\Omega)$. *Then we have*

$$(-v-w)^+,\ (w+v)^-,\ (w-v)^+,\ (-w+v)^- \in W_0^{1,p}(\Omega).$$

The next statement is concerned with the Sobolev compact embedding.

Lemma 2.3. *Let* Ω *be an open set in* $\mathbb{R}^N$ *and* $1 \leqslant p < \infty$. *Assume that* $(u_k)_k$ *is a sequence in* $W_0^{1,p}(\Omega)$ *and there exists* $v \in W_0^{1,p}(\Omega)$ *such that*

$$u_k \rightharpoonup v \quad \text{weakly in } W_0^{1,p}(\Omega) \quad \text{as } k \to \infty. \tag{2.1}$$

Then

$$u_k \to v \quad \text{in } L^p_{\mathrm{loc}}(\Omega) \quad \text{as } k \to \infty.$$

Remark 2.1. The conclusion of Lemma 2.3 remains valid if the space $W_0^{1,p}(\Omega)$ is replaced by $W^{1,p}(\Omega)$.

Proof. We use the notation "$\omega \Subset \Omega$" when ω is strongly included in Ω, i.e. $\overline{\omega}$ (the closure of ω in $\mathbb{R}^N$) is compact and $\overline{\omega} \subset \Omega$.

Take any open set $U \Subset \Omega$. Fix a function $\lambda \in C_0^\infty(\Omega)$ such that $\lambda(x) = 1$ in U. Let U_λ be a bounded open set such that

$$\operatorname{supp} \lambda \subset U_\lambda \Subset \Omega,$$

here "$\operatorname{supp} \lambda$" means support of a function λ. First, we easily see that

$$(\lambda u_k)|_{U_\lambda} \in W_0^{1,p}(U_\lambda) \quad \text{and} \quad (\lambda v)|_{U_\lambda} \in W_0^{1,p}(U_\lambda), \tag{2.2}$$

here $f|_{U_\lambda}$ denotes the restriction of the function f to U_λ. Furthermore, it follows from assumption (2.1) that

$$(\lambda u_k)|_{U_\lambda} \rightharpoonup (\lambda v)|_{U_\lambda} \quad \text{weakly in } W_0^{1,p}(U_\lambda) \quad \text{as } k \to \infty. \tag{2.3}$$

Indeed, let $F \in W^{-1,p'}(U_\lambda)$ (the dual space of $W_0^{1,p}(U_\lambda)$), where $1/p + 1/p' = 1$. From the representation theorem of the continuous linear functional on $W_0^{1,p}(U_\lambda)$ (see [2], Prop. 9.20), there exist functions $f_0, f_1, \ldots, f_N \in L^{p'}(U_\lambda)$ such that

$$\begin{aligned} &\langle F, (\lambda u_k)|_{U_\lambda} \rangle_{W^{-1,p'}(U_\lambda), W_0^{1,p}(U_\lambda)} \\ &\quad = \int_{U_\lambda} f_0 (\lambda u_k)\,\mathrm{d}x + \sum_{i=1}^N \int_{U_\lambda} f_i \frac{\partial}{\partial x_i}(\lambda u_k)\,\mathrm{d}x \\ &\quad = \int_{U_\lambda} f_0 (\lambda u_k)\,\mathrm{d}x + \sum_{i=1}^N \int_{U_\lambda} f_i \Big(\frac{\partial \lambda}{\partial x_i} u_k + \lambda \frac{\partial u_k}{\partial x_i}\Big)\,\mathrm{d}x \\ &\quad = \int_{U_\lambda} \Big(f_0 \lambda + \sum_{i=1}^N f_i \frac{\partial \lambda}{\partial x_i}\Big) u_k\,\mathrm{d}x + \sum_{i=1}^N \int_{U_\lambda} (f_i \lambda) \frac{\partial u_k}{\partial x_i}\,\mathrm{d}x. \end{aligned} \tag{2.4}$$

We denote by $\bar{f}_i$, $i = 0, 1, \ldots, N$ its extension by zero outside U_λ, that is,

$$\bar{f}_i(x) = \begin{cases} f_i(x) & \text{if } x \in U_\lambda, \\ 0 & \text{if } x \in \Omega \setminus U_\lambda. \end{cases}$$

Using the representation theorem of the continuous linear functional on $W_0^{1,p}(\Omega)$ (not on $W_0^{1,p}(U_\lambda)$) this time and assumption (2.1), we have from (2.4) that

$$\begin{aligned}
&\langle F, (\lambda u_k)|_{U_\lambda}\rangle_{W^{-1,p'}(U_\lambda), W_0^{1,p}(U_\lambda)} \\
&\quad = \int_\Omega \Big(\bar{f}_0\lambda + \sum_{i=1}^N \bar{f}_i \frac{\partial\lambda}{\partial x_i}\Big) u_k \,\mathrm{d}x + \sum_{i=1}^N \int_\Omega (\bar{f}_i\lambda)\frac{\partial u_k}{\partial x_i}\,\mathrm{d}x \\
&\quad \to \int_\Omega \Big(\bar{f}_0\lambda + \sum_{i=1}^N \bar{f}_i \frac{\partial\lambda}{\partial x_i}\Big) v \,\mathrm{d}x + \sum_{i=1}^N \int_\Omega (\bar{f}_i\lambda)\frac{\partial v}{\partial x_i}\,\mathrm{d}x \\
&\quad = \int_\Omega \bar{f}_0(\lambda v)\,\mathrm{d}x + \sum_{i=1}^N \int_\Omega \bar{f}_i \frac{\partial}{\partial x_i}(\lambda v)\,\mathrm{d}x = \int_{U_\lambda} f_0(\lambda v)\,\mathrm{d}x + \sum_{i=1}^N \int_{U_\lambda} f_i \frac{\partial}{\partial x_i}(\lambda v)\,\mathrm{d}x \\
&\quad = \langle F, (\lambda v)|_{U_\lambda}\rangle_{W^{-1,p'}(U_\lambda), W_0^{1,p}(U_\lambda)}
\end{aligned}$$

as $k \to \infty$. This implies (2.3).

Since U_λ is a bounded open set, by the Sobolev compact embedding $W_0^{1,p}(U_\lambda) \hookrightarrow L^p(U_\lambda)$ we obtain from (2.2) and (2.3) that

$$(\lambda u_k)|_{U_\lambda} \to (\lambda v)|_{U_\lambda} \text{ in } L^p(U_\lambda) \quad \text{as } k \to \infty,$$

without any regularity assumption on U_λ. Considering U instead of U_λ, it follows

$$u_k|_U \to v|_U \quad \text{in } L^p(U) \quad \text{as } k \to \infty.$$

This proves Lemma 2.3. $\square$

3. Proof of our theorem

We give the proof of our Theorem 1.1 in this section.

P r o o f of Theorem 1.1. We divide our proof into four steps.

Step 1. Take a sequence of open sets Ω_k as $k = 1, 2, \ldots$ such that

$$\Omega = \bigcup_{k=1}^\infty \Omega_k, \quad \Omega_k \Subset \Omega_{k+1} \text{ (see proof of Lemma 2.3)}.$$

Now let f_k be the restriction of the function f to Ω_k:

$$f_k(x) := f|_{\Omega_k}(x), \quad x \in \Omega_k.$$

Then $f_k \in L^{p'}(\Omega_k)$. Using Lemma 2.1 there exists a unique distributional solution $u_k \in W_0^{1,p}(\Omega_k)$ such that

$$-\mathrm{div}(a(x, \nabla u_k)) = f_k \quad \text{in } \Omega_k \tag{3.1}$$

for every $k \in \mathbb{N}$.

Step 2. On the other hand, for every $k \in \mathbb{N}$ it follows that (the restriction of the function w to Ω_k) $w|_{\Omega_k} \in W^{1,p}(\Omega_k)$ satisfies

$$-\mathrm{div}(a(x, \nabla w)) = g \quad \text{in } \Omega_k, \tag{3.2}$$

in the distributional sense. And the assumption '$w \geqslant 0$ a.e. in Ω' leads to $(u_k - w|_{\Omega_k})^+ \in W_0^{1,p}(\Omega_k)$ by Lemma 2.2 (ii), that is, $u_k \leqslant w$ on $\partial\Omega_k$. So we conclude from (3.1) and (3.2) that

$$u_k \leqslant w \quad \text{a.e. in } \Omega_k \quad \text{for } k = 1, 2, \ldots,$$

with the comparison principle of the type (A) of Section 1. Combining this inequality and $w \geqslant 0$ a.e. in Ω again, we have

$$\bar{u}_k \leqslant w \quad \text{a.e. in } \Omega \quad \text{for } k = 1, 2, \ldots, \tag{3.3}$$

where the function $\bar{u}_k$ is defined as:

$$\bar{u}_k(x) := \begin{cases} u_k(x) & \text{if } x \in \Omega_k, \\ 0 & \text{if } x \in \Omega \setminus \Omega_k. \end{cases} \tag{3.4}$$

Step 3. By Remark 1.1 (i), first note that (3.1) means

$$\int_{\Omega_k} a(x, \nabla u_k) \cdot \nabla \varphi \, \mathrm{d}x = \int_{\Omega_k} f_k \varphi \, \mathrm{d}x \quad \forall\, \varphi \in W_0^{1,p}(\Omega_k). \tag{3.5}$$

Substituting $\varphi = u_k \in W_0^{1,p}(\Omega_k)$ in (3.5) and using condition (a-1), it follows

$$\begin{aligned} \alpha \|\nabla \bar{u}_k\|_{L^p(\Omega)}^p &= \alpha \|\nabla u_k\|_{L^p(\Omega_k)}^p \\ &\leqslant \int_{\Omega_k} a(x, \nabla u_k) \cdot \nabla u_k \, \mathrm{d}x = \int_{\Omega_k} f_k u_k \, \mathrm{d}x \\ &\leqslant \|f_k\|_{L^{p'}(\Omega_k)} \|u_k\|_{L^p(\Omega_k)} \leqslant \|f\|_{L^{p'}(\Omega)} \|\bar{u}_k\|_{L^p(\Omega)} \end{aligned}$$

for every $k \in \mathbb{N}$. Note that $\bar{u}_k \in W_0^{1,p}(\Omega)$, thanks to Poincaré's inequality we obtain

$$\alpha \|\nabla \bar{u}_k\|_{L^p(\Omega)}^p \leqslant C\|f\|_{L^{p'}(\Omega)} \|\nabla \bar{u}_k\|_{L^p(\Omega)},$$

that is

$$\|\nabla \bar{u}_k\|_{L^p(\Omega)}^{p-1} \leqslant \frac{C}{\alpha}\|f\|_{L^{p'}(\Omega)} \quad \text{for } k = 1, 2, \dots, \tag{3.6}$$

where C is a constant. Since $W_0^{1,p}(\Omega)$ $(1 < p < \infty)$ is reflexive, there exists $v \in W_0^{1,p}(\Omega)$ and a subsequence of $(\bar{u}_k)_k$, still denoted by $(\bar{u}_k)_k$, such that

$$\bar{u}_k \rightharpoonup v \quad \text{weakly in } W_0^{1,p}(\Omega) \quad \text{as } k \to \infty.$$

Hence, we have that

$$\bar{u}_k \to v \quad \text{in } L^p_{\mathrm{loc}}(\Omega) \quad \text{as } k \to \infty, \tag{3.7}$$

by Lemma 2.3.

Moreover, using the diagonal method there exists a further subsequence of $(\bar{u}_k)_k$, still denoted by $(\bar{u}_k)_k$, such that

$$\bar{u}_k(x) \to v(x) \quad \text{a.e. } x \in \Omega \text{ as } k \to \infty. \tag{3.8}$$

Then passing to the limit in (3.3), we obtain that

$$v(x) \leqslant w(x) \quad \text{a.e. } x \in \Omega. \tag{3.9}$$

Step 4. To establish our Theorem 1.1 it now suffices to prove that $v \in W_0^{1,p}(\Omega)$ in (3.7) satisfies

$$\int_\Omega a(x, \nabla v) \cdot \nabla \varphi \,\mathrm{d}x = \int_\Omega f\varphi \,\mathrm{d}x \quad \forall\, \varphi \in C_0^\infty(\Omega). \tag{3.10}$$

Indeed, since u satisfies condition (c-1), it follows that

$$v(x) = u(x) \quad \text{a.e. in } \Omega, \tag{3.11}$$

by the uniqueness of solutions to (c-1) (see Lemma 2.1). We thus deduce that

$$u(x) = v(x) \leqslant w(x) \quad \text{a.e. in } \Omega,$$

from (3.9) and (3.11). This proves Theorem 1.1.

In what follows, we give the proof that $v \in W_0^{1,p}(\Omega)$ satisfies (3.10). So fix any $\phi \in C_0^\infty(\Omega)$. Let ω be an open set such that

$$\operatorname{supp}\phi \subset \omega \Subset \Omega,$$

and Ω_{k_0} be such that

$$\omega \Subset \Omega_{k_0}.$$

Fix $h \in C_0^\infty(\Omega)$ such that

$$0 \leqslant h(x) \leqslant 1, \quad \operatorname{supp} h \subset \Omega_{k_0}, \quad h(x) = 1 \text{ in a neighborhood of } \omega.$$

First of all, using the extension of functions outside Ω_k by zero like in (3.4), we have from (3.5) that

$$\int_\Omega a(x, \nabla \bar{u}_k) \cdot \nabla \bar{\varphi} \,\mathrm{d}x = \int_\Omega \overline{f}_k \bar{\varphi} \,\mathrm{d}x \quad \forall \varphi \in W_0^{1,p}(\Omega_k) \tag{3.12}$$

for every $k \in \mathbb{N}$. Let $k,\ l \geqslant k_0$. Because of $\operatorname{supp}\{h(\bar{u}_k - \bar{u}_l)\} \subset \Omega_{k_0}$ we have

$$\{h(\bar{u}_k - \bar{u}_l)\}|_{\Omega_k} \in W_0^{1,p}(\Omega_k), \quad \{h(\bar{u}_k - \bar{u}_l)\}|_{\Omega_l} \in W_0^{1,p}(\Omega_l).$$

Hence, we can substitute $\{h(\bar{u}_k - \bar{u}_l)\}|_{\Omega_k}$ for φ in (3.12) and substitute $\{h(\bar{u}_k - \bar{u}_l)\}|_{\Omega_l}$ for φ in (3.12) replacing k with l. Noting that

$$\overline{\{h(\bar{u}_k - \bar{u}_l)\}|_{\Omega_k}} = \overline{\{h(\bar{u}_k - \bar{u}_l)\}|_{\Omega_l}} = h(\bar{u}_k - \bar{u}_l),$$

we then obtain

$$\int_\Omega \{a(x, \nabla \bar{u}_k) - a(x, \nabla \bar{u}_l)\} \cdot \nabla\{h(\bar{u}_k - \bar{u}_l)\} \,\mathrm{d}x = \int_\Omega (\bar{f}_k - \bar{f}_l) h(\bar{u}_k - \bar{u}_l) \,\mathrm{d}x. \tag{3.13}$$

Since $\bar{f}_k(x)h(x) = \bar{f}_l(x)h(x)$ a.e. in Ω, we see

$$(\text{the right-hand side of } (3.13)) = 0. \tag{3.14}$$

Now we deal with the left-hand side of (3.13). According to condition (a-3), (i), (ii), we get the following two cases.

Case 1: $p \geqslant 2$.

By condition (a-3) (i) and $h \geqslant 0$, it follows that

$$\begin{aligned} &(\text{the left-hand side of } (3.13)) \\ &\quad = \int_\Omega \{a(x, \nabla \bar{u}_k) - a(x, \nabla \bar{u}_l)\} \cdot \{(\nabla h)(\bar{u}_k - \bar{u}_l) + h\nabla(\bar{u}_k - \bar{u}_l)\} \,\mathrm{d}x \\ &\quad \geqslant \gamma \int_\Omega |\nabla(\bar{u}_k - \bar{u}_l)|^p h \,\mathrm{d}x + \int_\Omega \{a(x, \nabla \bar{u}_k) - a(x, \nabla \bar{u}_l)\} \cdot (\nabla h)(\bar{u}_k - \bar{u}_l) \,\mathrm{d}x. \end{aligned} \tag{3.15}$$

We deduce from (3.13), (3.14), (3.15) and condition (a-2) that

$$\begin{aligned}
\gamma \int_\omega |\nabla(\bar{u}_k - \bar{u}_l)|^p \,\mathrm{d}x &\leqslant \gamma \int_\Omega |\nabla(\bar{u}_k - \bar{u}_l)|^p h \,\mathrm{d}x \\
&\leqslant -\int_\Omega \{a(x, \nabla\bar{u}_k) - a(x, \nabla\bar{u}_l)\} \cdot (\nabla h)(\bar{u}_k - \bar{u}_l) \,\mathrm{d}x \\
&\leqslant \int_{\Omega_{k_0}} \{|a(x, \nabla\bar{u}_k)| + |a(x, \nabla\bar{u}_l)|\} |\nabla h| |\bar{u}_k - \bar{u}_l| \,\mathrm{d}x \\
&\leqslant \beta \|\nabla h\|_{L^\infty(\Omega)} \int_{\Omega_{k_0}} \{|\nabla\bar{u}_k|^{p-1} + |\nabla\bar{u}_l|^{p-1}\} |\bar{u}_k - \bar{u}_l| \,\mathrm{d}x.
\end{aligned}$$

Using Hölder's inequality and (3.6), we have

$$\begin{aligned}
\|\nabla(\bar{u}_k - \bar{u}_l)\|^p_{L^p(\omega)} &\leqslant \frac{\beta}{\gamma} \|\nabla h\|_{L^\infty(\Omega)} \{\|\nabla\bar{u}_k\|^{p-1}_{L^p(\Omega)} + \|\nabla\bar{u}_l\|^{p-1}_{L^p(\Omega)}\} \|\bar{u}_k - \bar{u}_l\|_{L^p(\Omega_{k_0})} \\
&\leqslant 2\frac{C\beta}{\alpha\gamma} \|\nabla h\|_{L^\infty(\Omega)} \|f\|_{L^{p'}(\Omega)} \|\bar{u}_k - \bar{u}_l\|_{L^p(\Omega_{k_0})}.
\end{aligned}$$

The above inequality and (3.7) implies that $(\bar{u}_k|_\omega)_k$ is a Cauchy sequence in $W^{1,p}(\omega)$. By the completeness of $W^{1,p}(\omega)$ and (3.7) again, we consequently obtain

$$\bar{u}_k|_\omega \to v|_\omega \quad \text{in } W^{1,p}(\omega) \quad \text{as } k \to \infty. \tag{3.16}$$

Case 2: $1 < p < 2$.

Write

$$A(u_k, u_l) := \{x \in \Omega;\ |\nabla\bar{u}_k(x)| + |\nabla\bar{u}_l(x)| > 0\},$$

and set for every $0 < \varepsilon < 1$ that

$$\begin{aligned}
B^{\varepsilon-}(u_k, u_l) &:= \Big\{x \in \Omega;\ \frac{|\nabla(\bar{u}_k - \bar{u}_l)(x)|}{|\nabla\bar{u}_k(x)| + |\nabla\bar{u}_l(x)|} > \varepsilon\Big\} \cap A(u_k, u_l), \\
B^{\varepsilon+}(u_k, u_l) &:= \Big\{x \in \Omega;\ \frac{|\nabla(\bar{u}_k - \bar{u}_l)(x)|}{|\nabla\bar{u}_k(x)| + |\nabla\bar{u}_l(x)|} \leqslant \varepsilon\Big\} \cap A(u_k, u_l).
\end{aligned}$$

First we have

$$\begin{aligned}
\int_\omega |\nabla(\bar{u}_k - \bar{u}_l)|^p \,\mathrm{d}x &= \int_{\omega \cap A(u_k, u_l)} |\nabla(\bar{u}_k - \bar{u}_l)|^p \,\mathrm{d}x \\
&= \int_{\omega \cap B^{\varepsilon-}(u_k, u_l)} |\nabla(\bar{u}_k - \bar{u}_l)|^p \,\mathrm{d}x + \int_{\omega \cap B^{\varepsilon+}(u_k, u_l)} |\nabla(\bar{u}_k - \bar{u}_l)|^p \,\mathrm{d}x.
\end{aligned} \tag{3.17}$$

We estimate two terms of (3.17).

The first term:

By condition (a-3) (ii), it follows that

$$
\begin{aligned}
(3.18)\qquad & (\text{the left-hand side of } (3.13)) \\
&= \int_{\Omega_{k_0}} \{a(x,\nabla \bar{u}_k) - a(x,\nabla \bar{u}_l)\} \cdot \{(\nabla h)(\bar{u}_k - \bar{u}_l) + h\nabla(\bar{u}_k - \bar{u}_l)\}\,\mathrm{d}x \\
&\geqslant \gamma \int_{\Omega_{k_0}\cap A(u_k,u_l)} h\{|\nabla \bar{u}_k| + |\nabla \bar{u}_l|\}^{p-2} |\nabla(\bar{u}_k - \bar{u}_l)|^2\,\mathrm{d}x \\
&\quad + \int_{\Omega_{k_0}} \{a(x,\nabla \bar{u}_k) - a(x,\nabla \bar{u}_l)\} \cdot (\nabla h)(\bar{u}_k - \bar{u}_l)\,\mathrm{d}x.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
(3.19)\qquad & \gamma\varepsilon^{2-p} \int_{\omega\cap B^{\varepsilon-}(u_k,u_l)} |\nabla(\bar{u}_k - \bar{u}_l)|^p\,\mathrm{d}x \\
&\leqslant \gamma\varepsilon^{2-p} \int_{\Omega_{k_0}\cap B^{\varepsilon-}(u_k,u_l)} h|\nabla(\bar{u}_k - \bar{u}_l)|^p\,\mathrm{d}x \\
&\leqslant \gamma \int_{\Omega_{k_0}\cap B^{\varepsilon-}(u_k,u_l)} h\Big(\frac{|\nabla(\bar{u}_k - \bar{u}_l)(x)|}{|\nabla \bar{u}_k(x)| + |\nabla \bar{u}_l(x)|}\Big)^{2-p} |\nabla(\bar{u}_k - \bar{u}_l)|^p\,\mathrm{d}x \\
&= \gamma \int_{\Omega_{k_0}\cap B^{\varepsilon-}(u_k,u_l)} h\{|\nabla \bar{u}_k| + |\nabla \bar{u}_l|\}^{p-2} |\nabla(\bar{u}_k - \bar{u}_l)|^2\,\mathrm{d}x \\
&\leqslant \gamma \int_{\Omega_{k_0}\cap B^{\varepsilon-}(u_k,u_l)} h\{|\nabla \bar{u}_k| + |\nabla \bar{u}_l|\}^{p-2} |\nabla(\bar{u}_k - \bar{u}_l)|^2\,\mathrm{d}x \\
&\qquad + \int_{\Omega_{k_0}\cap B^{\varepsilon+}(u_k,u_l)} h\{|\nabla \bar{u}_k| + |\nabla \bar{u}_l|\}^{p-2} |\nabla(\bar{u}_k - \bar{u}_l)|^2\,\mathrm{d}x \\
&= \gamma \int_{\Omega_{k_0}\cap A(u_k,u_l)} h\{|\nabla \bar{u}_k| + |\nabla \bar{u}_l|\}^{p-2} |\nabla(\bar{u}_k - \bar{u}_l)|^2\,\mathrm{d}x \\
&\leqslant - \int_{\Omega_{k_0}} \{a(x,\nabla \bar{u}_k) - a(x,\nabla \bar{u}_l)\} \cdot (\nabla h)(\bar{u}_k - \bar{u}_l)\,\mathrm{d}x \\
&\qquad \text{(here we used (3.13), (3.14), and (3.18))} \\
&\leqslant \beta\|\nabla h\|_{L^\infty(\Omega)} \int_{\Omega_{k_0}} \{|\nabla \bar{u}_k|^{p-1} + |\nabla \bar{u}_l|^{p-1}\}|\bar{u}_k - \bar{u}_l|\,\mathrm{d}x \\
&\qquad \text{(here we used condition (a-2))} \\
&\leqslant \beta\|\nabla h\|_{L^\infty(\Omega)} \{\|\nabla \bar{u}_k\|^{p-1}_{L^p(\Omega)} + \|\nabla \bar{u}_l\|^{p-1}_{L^p(\Omega)}\} \|\bar{u}_k - \bar{u}_l\|_{L^p(\Omega_{k_0})} \\
&\leqslant 2\beta\|\nabla h\|_{L^\infty(\Omega)} \Big(\frac{C}{\alpha}\|f\|_{L^{p'}(\Omega)}\Big) \|\bar{u}_k - \bar{u}_l\|_{L^p(\Omega_{k_0})} \\
&\qquad \text{(here we used (3.6)).}
\end{aligned}
$$

On the other hand, since it follows that

$$
(3.20)\qquad |\nabla(\bar{u}_k - \bar{u}_l)| \leqslant \varepsilon(|\nabla \bar{u}_k| + |\nabla \bar{u}_l|) \quad \text{a.e. in } \omega\cap B^{\varepsilon+}(u_k,u_l),
$$

we obtain from (3.20) that

$$\begin{aligned}
(3.21)\qquad \|\nabla(\bar u_k-\bar u_l)\|^p_{L^p(\omega\cap B^{\varepsilon+}(u_k,u_l))} &= \int_{\omega\cap B^{\varepsilon+}(u_k,u_l)} |\nabla(\bar u_k-\bar u_l)|^p\,\mathrm{d}x\\
&\leqslant \varepsilon^p \int_{\omega\cap B^{\varepsilon+}(u_k,u_l)} (|\nabla\bar u_k|+|\nabla\bar u_l|)^p\,\mathrm{d}x \leqslant \varepsilon^p \|\,|\nabla\bar u_k|+|\nabla\bar u_l|\,\|^p_{L^p(\Omega)}\\
&\leqslant 2^p\varepsilon^p\Big(\frac{C}{\alpha}\|f\|_{L^{p'}(\Omega)}\Big)^{p/(p-1)}.
\end{aligned}$$

Consequently, we deduce from (3.19) and (3.21) that

$$\begin{aligned}
(3.22)\qquad &\int_\omega |\nabla(\bar u_k-\bar u_l)|^p\,\mathrm{d}x\\
&= \int_{\omega\cap B^{\varepsilon-}(u_k,u_l)} |\nabla(\bar u_k-\bar u_l)|^p\,\mathrm{d}x + \int_{\omega\cap B^{\varepsilon+}(u_k,u_l)} |\nabla(\bar u_k-\bar u_l)|^p\,\mathrm{d}x\\
&\leqslant \frac{2\beta}{\gamma\varepsilon^{2-p}}\|\nabla h\|_{L^\infty(\Omega)}\Big(\frac{C}{\alpha}\|f\|_{L^{p'}(\Omega)}\Big)\|\bar u_k-\bar u_l\|_{L^p(\Omega_{k_0})}\\
&\quad + 2^p\varepsilon^p\Big(\frac{C}{\alpha}\|f\|_{L^{p'}(\Omega)}\Big)^{p/(p-1)}.
\end{aligned}$$

From (3.7) it follows that

$$(3.23)\qquad \text{(the right-hand side of (3.22))} \to 2^p\varepsilon^p\Big(\frac{C}{\alpha}\|f\|_{L^{p'}(\Omega)}\Big)^{p/(p-1)} \quad \text{as } k,l\to\infty,$$

and since $0<\varepsilon<1$ is arbitrary, we can see from (3.22) and (3.23) that $(\bar u_k|_\omega)_{k\geqslant k_0}$ is a Cauchy sequence in $W^{1,p}(\omega)$. By the completeness of $W^{1,p}(\omega)$ and (3.7) again, we consequently obtain

$$(3.24)\qquad \bar u_k|_\omega \to v|_\omega \quad \text{in } W^{1,p}(\omega) \quad \text{as } k\to\infty.$$

Thus, we have for $1<p<\infty$

$$(3.25)\qquad \bar u_k|_\omega \to v|_\omega \quad \text{in } W^{1,p}(\omega) \quad \text{as } k\to\infty,$$

from (3.16) and (3.24). Now remember that $\bar u_k$ as $k\in\mathbb{N}$ satisfy (3.12) and $\phi\in C_0^\infty(\Omega)$, $\operatorname{supp}\phi\subset\omega\Subset\Omega_k$ as $k\geqslant k_0$. Hence, we have from (3.12) that

$$(3.26)\qquad \int_\Omega a(x,\nabla\bar u_k)\cdot\nabla\phi\,\mathrm{d}x = \int_\Omega \bar f_k\phi\,\mathrm{d}x \quad \forall\, k\geqslant k_0.$$

Passing to the limit for $k\to\infty$

$$(3.27)\qquad \text{(the right-hand side of (3.26))} \to \int_\Omega f\phi\,\mathrm{d}x,$$

by the definition of $\bar{f}_k$, on the other hand,

$$\text{(the left-hand side of (3.26))} \to \int_\Omega a(x, \nabla v) \cdot \nabla \phi \,\mathrm{d}x, \tag{3.28}$$

as follows.

P r o o f of (3.28). We can prove (3.28) as in [10], step 3 in the proof of Lemma 2.3. Indeed, set

$$(N_a \xi)(x) = a(x, \xi(x)) \quad (= (a_1(x, \xi(x)), \dots, a_N(x, \xi(x)))),$$

for any $\xi = (\xi_1, \dots, \xi_N) \in (L^p(\Omega))^N$. Then we can use *Nemitski's composition theorem* ([1], Theorem 3.6, [7], Section 3.6, Corollary 3, note that Nemitski's composition theorem is valid for any open set Ω) from condition (a-2) and obtain that the operator $N_a\colon (L^p(\Omega))^N \to (L^{p'}(\Omega))^N$ is continuous. Using this with ω and $\nabla \bar{u}_k$ instead of Ω and ξ, respectively, it follows

$$N_a(\nabla \bar{u}_k) \to N_a(\nabla v) \quad \text{in } (L^{p'}(\omega))^N \quad \text{as } k \to \infty$$

from (3.25). This is equivalent to

$$\|a(\cdot, (\nabla \bar{u}_k)(\cdot)) - a(\cdot, (\nabla v)(\cdot))\|_{L^{p'}(\omega)} \to 0 \quad \text{as } k \to \infty.$$

Accordingly, noting that $\operatorname{supp} \phi \subset \omega$, it follows that

$$\begin{aligned}
&\left| \int_\Omega a(x, \nabla \bar{u}_k(x)) \cdot \nabla \phi(x) \,\mathrm{d}x - \int_\Omega a(x, \nabla v(x)) \cdot \nabla \phi(x) \,\mathrm{d}x \right| \\
&\qquad \leqslant \int_\omega |a(x, \nabla \bar{u}_k(x)) - a(x, \nabla v(x))| |\nabla \phi(x)| \,\mathrm{d}x \\
&\qquad \leqslant \|a(\cdot, (\nabla \bar{u}_k)(\cdot)) - a(\cdot, (\nabla v)(\cdot))\|_{L^{p'}(\omega)} \|\nabla \phi\|_{L^p(\omega)} \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

Thus we arrive at (3.28).

Consequently, we deduce from (3.26), (3.27), and (3.28) that $v \in W_0^{1,p}(\Omega)$ satisfies

$$\int_\Omega a(x, \nabla v) \cdot \nabla \phi \,\mathrm{d}x = \int_\Omega f \phi \,\mathrm{d}x.$$

This concludes the proof of Theorem 1.1. □

4. Application

In this section we give an application to our result. As mentioned in Section 1, we follow [3], Theorem 17.7.

Let S_ν be a strip-like domain such that

$$S_\nu = \{x \in \mathbb{R}^N;\ (x - x_0) \cdot \nu \ \in (-a,\ a)\}$$

for some $a > 0$ and $x_0 \in \mathbb{R}^N$, where ν is a unit vector and a dot denotes the scalar product in $\mathbb{R}^N$. Let $\Omega \subset S_\nu$ be an open set. We assume that $u \in W_0^{1,p}(\Omega)$, $1 < p < \infty$, satisfies

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f \quad \text{in } \Omega, \tag{4.1}$$

in the distributional sense with $f \in L^\infty(\Omega) \cap L^{p'}(\Omega)$, $1/p + 1/p' = 1$. Then we conclude that $u \in L^\infty(\Omega)$ and

$$\|u\|_{L^\infty(\Omega)} \leqslant \frac{p-1}{p} a^{p/(p-1)} (\|f\|_{L^\infty(\Omega)})^{1/(p-1)}. \tag{4.2}$$

Indeed, first we set $\alpha := 1 + 1/(p-1)$ and

$$\widetilde{w}(x) := a^\alpha - |(x - x_0) \cdot \nu|^\alpha \ (\geqslant 0) \quad \text{in } \Omega.$$

Then we see $\widetilde{w} \in W_{\mathrm{loc}}^{1,p}(\Omega)$ (note that $\widetilde{w}$ does not belong to $W^{1,p}(\Omega)$ in general when the open set Ω $(\subset S_\nu)$ is unbounded). A simple computing leads us to

$$\nabla \widetilde{w} = -\alpha |(x - x_0) \cdot \nu|^{\alpha - 1} \{\operatorname{sgn}((x - x_0) \cdot \nu)\} \nu,$$

where sgn function is defined by

$$\operatorname{sgn}(t) := \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < 0. \end{cases}$$

So we have

$$|\nabla \widetilde{w}|^{p-2} = \alpha^{p-2} |(x - x_0) \cdot \nu|^{(\alpha-1)(p-2)}.$$

Therefore, noting $(\alpha - 1) + (\alpha - 1)(p - 2) = (\alpha - 1)(p - 1) = 1$, it follows that

$$\begin{aligned} |\nabla \widetilde{w}|^{p-2} \nabla \widetilde{w} &= -\alpha^{p-1} |(x - x_0) \cdot \nu|^{(\alpha-1)+(\alpha-1)(p-2)} \{\operatorname{sgn}((x - x_0) \cdot \nu)\} \nu \\ &= -\alpha^{p-1} |(x - x_0) \cdot \nu| \{\operatorname{sgn}((x - x_0) \cdot \nu)\} \nu \\ &= -\alpha^{p-1} ((x - x_0) \cdot \nu) \nu, \end{aligned}$$

and after a simple computation we obtain

$$-\operatorname{div}(|\nabla\widetilde{w}|^{p-2}\nabla\widetilde{w}) = \alpha^{p-1}. \tag{4.3}$$

Therefore, setting $w := \alpha^{-1}\|f\|_{L^\infty(\Omega)}^{1/(p-1)}\widetilde{w}$ $(\in W^{1,p}_{\mathrm{loc}}(\Omega))$, then noting that

$$|\nabla w|^{p-2}\nabla w = \frac{1}{\alpha^{p-1}}\|f\|_{L^\infty(\Omega)}|\nabla\widetilde{w}|^{p-2}\nabla\widetilde{w},$$

we derive from (4.3) that

$$-\operatorname{div}(|\nabla w|^{p-2}\nabla w) = \|f\|_{L^\infty(\Omega)}. \tag{4.4}$$

Since $f \leqslant \|f\|_{L^\infty(\Omega)}$ as a matter of course, combining it with (4.1), (4.4), and $w \geqslant 0$ in Ω, we can apply Corollary 1.2. Hence, we conclude that

$$u \leqslant w = \frac{1}{\alpha}\|f\|_{L^\infty(\Omega)}^{1/(p-1)}\widetilde{w}.$$

Since $-u$ is a solution to (4.1) corresponding to $-f$, we have

$$|u| \leqslant w = \frac{1}{\alpha}\|f\|_{L^\infty(\Omega)}^{1/(p-1)}\widetilde{w}.$$

Finally, noting

$$\frac{1}{\alpha} = \frac{p-1}{p} \quad \text{and} \quad 0 \leqslant \widetilde{w} \leqslant a^\alpha = a^{p/(p-1)},$$

we obtain (4.2).

Remark 4.1. In the above consideration, $u \in W_0^{1,p}(\Omega) \subset W^{1,p}(\Omega)$, however, w does not belong to $W^{1,p}(\Omega)$ in general when the open set Ω $(\subset S_\nu)$ is unbounded. Therefore, the elementary comparison principle of the type (A) of Section 1 cannot be applied to the above inference.

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Author's address: *Akihito Unai*, Department of Applied Mathematics, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-0825, Japan, e-mail: `unai@rs.kagu.tus.ac.jp`.