A PENALTY APPROACH FOR A BOX CONSTRAINED VARIATIONAL INEQUALITY PROBLEM

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Received November 27, 2017. Published online May 22, 2018.

Abstract. We propose a penalty approach for a box constrained variational inequality problem (BVIP). This problem is replaced by a sequence of nonlinear equations containing a penalty term. We show that if the penalty parameter tends to infinity, the solution of this sequence converges to that of BVIP when the function F involved is continuous and strongly monotone and the box C contains the origin. We develop the algorithmic aspect with theoretical arguments properly established. The numerical results tested on some examples are satisfactory and confirm the theoretical approach.

Keywords: box constrained variational inequality problem; power penalty approach; strongly monotone operator

MSC 2010: 47J20, 65K10, 65J15

1. Introduction

Many real-world phenomena in engineering and economics are governed by a box constrained variational inequality problem BVIP, see [10]. Extensive studies of BVIP have been done in [11], [7] and the references therein. Numerical methods for solving BVIP have been extensively investigated in the literature such as smoothing Newton methods [5], [14], [17], interior point method [2] and nonsmooth equation methods [9], [8]. However, it seems that there are few studies of penalty methods for BVIP.

Recently, a power penalty approach has been proposed for linear, nonlinear, and mixed nonlinear complementarity problems in both the finite-dimensional space $\mathbb{R}^n$ and the infinite-dimensional functional spaces [9], [6], [8], [16]. This approach consists of approximating a box constrained variational inequality problem BVIP by a sequence of nonlinear penalty equations with a penalty term. The penalty method has the merit of not introducing any extra or auxiliary variables. Besides, the re-

DOI: 10.21136/AM.2018.0334-17

sulting algebraic equations are easily solvable by a conventional numerical method such as that of Newton type.

Based on the method presented in [9], [6], [8], [16] for linear, nonlinear, and mixed nonlinear complementarity problems, the present study aims to develop and analyze a penalty approach for the BVIP.

This paper is organized as follows. At the beginning, we present the box constrained variational inequality problem BVIP and its penalized problem. In Section 2, we analyze the convergence of the penalty approach. In Section 3, the corresponding algorithm is constructed and we prove the global convergence results under the hypothesis of continuity and strong-monotonicity of F. Furthermore, some numerical experiments are presented in Section 4. Finally, we give some concluding remarks in Section 5.

Throughout the paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n\,;\ x \geqslant 0\}$ being the positive orthant, $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the Euclidean inner product and norm, respectively.

1.1. The problem BVIP and its penalty formulation. Consider the following box constrained variational inequality problem BVIP.

Find $\bar{x} \in C \subset \mathbb{R}^n$ such that

$$\langle F(\bar{x}), x - \bar{x} \rangle \geqslant 0 \quad \forall\, x \in C, \tag{1.1}$$

where $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is a continuous differentiable mapping from $\mathbb{R}^n$ to itself and $C = \prod_{i=1}^{n} [l_i, u_i]$, $-\infty \leqslant l_i \leqslant u_i \leqslant \infty$, $i = 1, 2, \ldots, n$.

The problem (1.1) is called the box constrained variational inequality problem BVIP.

Now, we present a penalty approach for (1.1). First, we give an important definition.

Definition 1.1. An operator $B\colon \mathbb{R}^n \to \mathbb{R}^n$ is called a penalty operator relative to a closed convex set $C \subseteq \mathbb{R}^n$ if it satisfies

(i) B is a continuous operator on $\mathbb{R}^n$.

(ii) For any $x \in C$:

$$\langle B(y), y - x \rangle \begin{cases} = 0 & \text{if } y \in C, \\ > 0 & \text{if } y \notin C. \end{cases}$$

Many penalty operators have been proposed in the literature (see [1]). Among them, we have the projection operator, which is defined by:

$$B(x) = x - \mathrm{Pr}_C(x),$$

where $\mathrm{Pr}_C(x)$ stands for the Euclidean projection of x onto C.

Let $C = \prod_{i=1}^{n} [l_i, u_i]$. Then the projection operator $B(x) = x - \mathrm{Pr}_C(x)$ is defined by $B(x) = B_1(x) + \ldots + B_n(x)$, where

$$B_i(x) = \begin{cases} (0, \ldots, x_i - l_i, \ldots, 0)^{\mathrm{T}} & \text{if } x_i < l_i, \\ (0, \ldots, 0, \ldots, 0)^{\mathrm{T}} & \text{if } l_i \leqslant x_i \leqslant u_i, \\ (0, \ldots, x_i - u_i, \ldots, 0)^{\mathrm{T}} & \text{if } x_i > u_i. \end{cases}$$

1.1.1. Penalized problem of BVIP. Consider the following sequence of nonlinear equations BVEP_r:

Find a vector $x_r \in \mathbb{R}^n$ such that:

$$E(x_r) = F(x_r) + rB(x_r) = 0, \tag{1.2}$$

where $r > 0$ is the penalty parameter and B is the previous projection operator.

The formula (1.2) is an approximate problem of the initial problem (1.1).

When $r \to \infty$, we expect that the solution x_r of problem (1.2) converges to that of problem (1.1). A detailed convergence analysis of the solution of (1.2) will be developed in the next section under some mild assumptions on the operator F and the box C.

2. Theoretical aspect of the problem BVEP_r

In this section, we establish some upper bounds for the distance between the solutions of problems (1.1) and (1.2). Before, we first make the following assumptions on F and C.

(A1) F is continuous on $\mathbb{R}^n$.

(A2) F is strongly monotone, i.e., there exists a constant $\alpha > 0$ such that

$$\langle F(x) - F(y), x - y \rangle \geqslant \alpha \|x - y\|^2 \quad \forall\, x, y \in \mathbb{R}^n.$$

(A3) $0 \in C$.

Lemma 2.1 ([10]). *If F is strongly monotone, then F is strongly coercive, i.e., there exists $x^0 \in C$ such that*

$$\lim_{\|x\| \to \infty} \frac{\langle F(x), x - x^0 \rangle}{\|x - x^0\|} = \infty.$$

It can be easily shown that, under assumptions (A1) and (A2), the problem (1.1) has a unique solution (cf., for example, Theorem 2.3.3 of [1]). Furthermore, since $F(\cdot) + rB(\cdot)$ is also strongly monotone for any $r > 0$, the variational inequality problem (with $C = \mathbb{R}^n$) corresponding to problem (1.2) has a unique solution.

2.1. Convergence of BVEP$_r$ to BVIP. We start our convergence analysis with the following lemma.

Lemma 2.2. *Let x_r be a solution of* (1.2) *for any $r > 0$ and let assumptions* (A1), (A2), *and* (A3) *be satisfied. Then there exists a positive constant M independent of x_r such that*

$$\|x_r\| \leqslant M \quad \forall\, r > 0. \tag{2.1}$$

Proof. For $r > 0$, let x_r be a solution of (1.2). Left-multiplying both sides of (1.2) by x_r^{T}, we obtain

$$\langle x_r, E(x_r)\rangle = \langle x_r, F(x_r)\rangle + r\langle x_r, B(x_r)\rangle = 0.$$

From assumption (A3), we have $0 \in C$. Then,

$$\langle B(x_r), x_r - 0\rangle \geqslant 0,$$

which gives according to Definition 1.1

$$\langle x_r, F(x_r)\rangle = -r\langle x_r, B(x_r)\rangle = -r\langle B(x_r), x_r - 0\rangle \leqslant 0 \quad \forall\, r > 0.$$

Moreover, we have

$$\langle x_r, F(x_r)\rangle = \langle x_r, F(x_r)\rangle - \langle x_r, F(0)\rangle + \langle x_r, F(0)\rangle$$

which is equivalent to

$$\langle x_r, F(x_r) - F(0)\rangle \leqslant -\langle x_r, F(0)\rangle.$$

Using the above estimate and the Cauchy-Schwarz inequality, we get

$$\langle x_r, F(x_r) - F(0)\rangle \leqslant -\langle x_r, F(0)\rangle \leqslant \|x_r\|\|F(0)\|.$$

From assumption (A2) and the above estimate, we have

$$\alpha\|x_r\|^2 \leqslant \langle x_r, F(x_r) - F(0)\rangle \leqslant \|x_r\|\|F(0)\|,$$

and so,

$$\|x_r\| \leqslant \frac{\|F(0)\|}{\alpha}. \tag{2.2}$$

This completes the proof with $M = \|F(0)\|/\alpha$. □

R e m a r k 2.3. Lemma 2.2 shows that for any non-negative r, the solution of (1.2) always belongs to a bounded closed set $D = \{y \in \mathbb{R}^n;\ \|y\| \leqslant M\}$. Due to assumption (A1), this guarantees that there exists a positive constant L independent of x_r and r such that

$$\|F(x_r)\| \leqslant L \quad \forall\, r > 0. \tag{2.3}$$

This result will serve us in the proof of the following lemma.

Lemma 2.4. *Let x_r be a solution of* (1.2) *and let the assumptions* (A1) *and* (A2) *be satisfied. Then there exists a positive constant L independent of x_r and r such that*

$$\|B(x_r)\| \leqslant \frac{L}{r} \tag{2.4}$$

for all $r > 0$.

P r o o f. Left-multiplying both sides of (1.2) by $[B(x_r)]^{\mathrm{T}}$, we obtain

$$[B(x_r)]^{\mathrm{T}} F(x_r) + r[B(x_r)]^{\mathrm{T}} B(x_r) = 0.$$

Now using the Cauchy-Schwarz inequality, we have from the above equation

$$\|B(x_r)\|^2 = \frac{-1}{r}[B(x_r)]^{\mathrm{T}} F(x_r) \leqslant \frac{1}{r}\|B(x_r)\|\|F(x_r)\|,$$

and so

$$\|B(x_r)\| \leqslant \frac{\|F(x_r)\|}{r}.$$

Finally, (2.4) holds due to the above inequality and (2.3). □

Now, we present and prove our main convergence result.

Theorem 2.5. *Let $\bar{x}$ and x_r be the solutions of problems* (1.1) *and* (1.2), *respectively and let the assumptions* (A1), (A2), *and* (A3) *be satisfied. Then there exists a constant $c_1 > 0$ independent of x_r and r such that*

$$\|\bar{x} - x_r\| \leqslant \frac{c_1}{\sqrt{r}}. \tag{2.5}$$

P r o o f. Let $\bar{x}$ be a solution of problem (1.1) and x_r a solution of problem (1.2). We decompose $\bar{x} - x_r$ as follows

$$\bar{x} - x_r = (\bar{x} - \mathrm{Pr}_C(x_r)) - (x_r - \mathrm{Pr}_C(x_r)) = s_r - (x_r - \mathrm{Pr}_C(x_r)), \tag{2.6}$$

where $s_r = (\bar{x} - \mathrm{Pr}_C(x_r))$. Then

$$\bar{x} - s_r = \mathrm{Pr}_C(x_r) \in C. \tag{2.7}$$

From (1.1), for $x = \bar{x} - s_r \in C$, we obtain

$$\langle F(\bar{x}), \bar{x} - s_r - \bar{x}\rangle = \langle F(\bar{x}), -s_r\rangle \geqslant 0. \tag{2.8}$$

Multiplying (1.2) by s_r^{T}, we have

$$\langle F(x_r), s_r\rangle + r\langle B(x_r), s_r\rangle = 0. \tag{2.9}$$

Adding up (2.8) and (2.9), we deduce

$$\langle F(x_r) - F(\bar{x}), s_r\rangle + r\langle B(x_r), s_r\rangle \geqslant 0. \tag{2.10}$$

Note that

$$\langle B(x_r), s_r\rangle = \langle x_r - \mathrm{Pr}_C(x_r), \bar{x} - \mathrm{Pr}_C(x_r)\rangle \leqslant 0.$$

Thus (2.10) leads to

$$\langle F(\bar{x}) - F(x_r), \bar{x} - \mathrm{Pr}_C(x_r)\rangle = \langle F(\bar{x}) - F(x_r), s_r\rangle \leqslant 0.$$

Then

$$\langle F(\bar{x}) - F(x_r), (\bar{x} - x_r) + (x_r - \mathrm{Pr}_C(x_r))\rangle \leqslant 0$$

which gives

$$\langle F(\bar{x}) - F(x_r), (\bar{x} - x_r)\rangle \leqslant \langle F(x_r) - F(\bar{x}), x_r - \mathrm{Pr}_C(x_r)\rangle.$$

Using the Cauchy-Schwarz inequality, assumption (A2) and the above inequality, it follows that

$$\begin{aligned}
\alpha\|\bar{x} - x_r\|^2 &\leqslant \langle F(\bar{x}) - F(x_r), (\bar{x} - x_r)\rangle \\
&\leqslant \langle F(x_r) - F(\bar{x}), (x_r - \mathrm{Pr}_C(x_r))\rangle \\
&\leqslant \|F(\bar{x}) - F(x_r)\|\|x_r - \mathrm{Pr}_C(x_r)\| \\
&= \|F(\bar{x}) - F(x_r)\|\|B(x_r)\|.
\end{aligned}$$

Finally, from (2.3) and Lemma 2.4, we obtain

$$\alpha\|\bar{x} - x_r\|^2 \leqslant \frac{2L^2}{r}$$

which implies

$$\|\bar{x} - x_r\| \leqslant \frac{c_1}{\sqrt{r}}.$$

This completes the proof with $c_1 = \sqrt{2}L/\sqrt{\alpha}$. □

3. The algorithm and its convergence

In this section, the corresponding algorithm is constructed for solving the problem BVIP.

3.1. Algorithm.

Begin algorithm

1. Initialization
   Let $\varepsilon > 0$ be a given precision and $\theta > 1$.
   Let $x^0 \in \mathbb{R}^n$, $r_0 = 1$ and $k = 0$.
2. Iteration
   Find x^{k+1} solution of nonlinear equation

$$E(x) = F(x) + r_k B(x) = 0.$$

3. **If** $\|B(x^k)\| \leqslant \varepsilon$ or $\|x^{k+1} - x^k\| \leqslant \varepsilon$, **then** stop: x^{k+1} is an approximate solution of BVIP.
   If not take
   - ▷ $r_{k+1} = \theta r_k$,
   - ▷ $x^k = x^{k+1}$,
   - ▷ $k = k + 1$, and go back to 2.

End algorithm

3.2. Comments on the algorithm.

- ▷ Clearly, the analysis and the properties of the algorithm depend largely on the treatment of the step 2.
- ▷ To solve the equation of step 2, we can consider any conventional method such as fixed point, Newton, etc.
- ▷ The choice of the method depends on the properties of the operator B and the function F of problem (1.1).

3.3. Convergence of the algorithm. The result given in the following theorem is the same as that of Theorem 4.4 from [1], established with a detailed proof.

Theorem 3.1. *Let assumption* (A1) *hold and F be strongly-coercive. Then the sequence $\{x^k\}$ generated by the Algorithm* 3.1 *converges to the unique adherent value $\bar{x}$ of the solution of the problem* (1.1).

P r o o f. The proof of this theorem is done in three steps:

Recall that since F is strongly-coercive, the step 2 of Algorithm 3.1 is realized, because the operator $E = F + r_k B$ is also strongly-coercive on $\mathbb{R}^n$.

First, we prove that the sequence $\{x^k\}$ of Algorithm 3.1 is bounded. Suppose the contrary. Then there exists a subsequence $\{x^{k_j}\}$ of $\{x^k\}$ such that $\|x^{k_j}\| \to \infty$ as $j \to \infty$. From step 2 of Algorithm 3.1, we have:

$$\frac{\langle E(x^{k_j}), x^{k_j} - x^0\rangle}{\|x^{k_j} - x^0\|} = \frac{\langle F(x^{k_j}), x^{k_j} - x^0\rangle}{\|x^{k_j} - x^0\|} + r_k \frac{\langle B(x^{k_j}), x^{k_j} - x^0\rangle}{\|x^{k_j} - x^0\|} = 0$$
$$\Longrightarrow \frac{\langle F(x^{k_j}), x^{k_j} - x^0\rangle}{\|x^{k_j} - x^0\|} = -r_k \frac{\langle B(x^{k_j}), x^{k_j} - x^0\rangle}{\|x^{k_j} - x^0\|}.$$

According to the property (ii) in Definition 1.1 of B, we have:

$$\frac{\langle F(x^{k_j}), x^{k_j} - x^0\rangle}{\|x^{k_j} - x^0\|} \leqslant 0.$$

On the other hand, the strong coercivity of F gives

$$\lim_{\|x^{k_j}\| \to \infty} \frac{\langle F(x^{k_j}), x^{k_j} - x^0\rangle}{\|x^{k_j} - x^0\|} = \infty > 0$$

which is a contradiction. Hence, the sequence $\{x^k\}$ is bounded.

Since the sequence $\{x^k\}_{k\in\mathbb{N}}$ is bounded, we can extract a subsequence converging to the adhesion value $\bar{x}$ of the sequence $\{x^k\}$.

Note that for all $x \in C$

$$\liminf_{k\to\infty} r_k \langle B(x^k), x^k - x\rangle \begin{cases} \geqslant 0 & \text{if } x^k \in C, \\ \infty & \text{if } x^k \notin C. \end{cases}$$

Therefore,

$$\liminf_{k\to\infty} \langle F(x^k), x - x^k\rangle = \liminf_{k\to\infty} r_k \langle B(x^k), x^k - x\rangle.$$

On the other hand, we have F continuous over $\mathbb{R}^n$ (assumption (A1)), hence

$$\liminf_{k\to\infty} \langle F(x^k), x - x^k\rangle = \langle F(\bar{x}), x - \bar{x}\rangle \geqslant 0.$$

We deduce that

$$\bar{x} \in C \text{ and } \langle F(\bar{x}), x - \bar{x} \rangle \leqslant 0 \quad \forall\, x \in C,$$

which gives that $\bar{x}$ is a solution of the problem (1.1).

This completes the proof. □

4. Numerical experiments

To give some insight into the behavior of our Algorithm 3.1, we implemented it on Matlab and have run it on a set of problems which are described below.

We denote by x^0 the starting point of the algorithm. The examples are tested for different values of θ with $\theta > 1$ and the tolerance considered is taken $\varepsilon = 10^{-6}$.

In the tables of results, (Iter) represents the number of iterations needed to obtain the solution of BVIP, and CPU(s) represents the time of computation.

Finally, we note that the nonlinear equation (1.2) is solved thanks to the f-solver procedure from the Matlab optimization toolbox.

4.1. Examples with fixed sizes.

Example 1 ([11]). The operator $F\colon \mathbb{R}^4 \to \mathbb{R}^4$ is

$$F(x) = \begin{pmatrix} 400x_1^3 + 2x_1 - 400x_1x_2 - 2 \\ -200x_1^2 + 200.2x_2 + 19.8x_4 - 40 \\ 360x_1^3 + 2x_2 - 360x_3x_4 - 2 \\ 19.8x_2 - 180x_3^2 + 220.2x_4^2 - 40 \end{pmatrix}$$

and $C = [-10, 10]^4$.

Using the starting point $x^0 = (3, 3, 3, 3)^{\mathrm{T}}$, the numerical results are given in Table 1.

θ	Iter	CPU(s)
10	1	0.100601
10^2	1	0.071838
10^3	1	0.059665
10^4	1	0.041732
10^5	1	0.024124

Table 1.

E x a m p l e 2 ([15]). Kojima-Shindo problem (MCPLIB file kojsshin.gms) where the operator is defined by

$$F\colon \mathbb{R}^4 \to \mathbb{R}^4, \quad F(x) = \begin{pmatrix} 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\ 2x_1^2 + x_1 + 2x_2^2 + 2x_3 + 2x_4 - 2 \\ 3x_1^2 + x_1x_2 + 2x_2^2 + 9x_4 - 9 \\ x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3 \end{pmatrix},$$

and the feasible set is $C = [-0.5, 0.5]^4$.

In this problem, we consider $\theta = 10$, and we present the computational results in Table 2 for different choices of the starting point x^0.

x^0	Iter	CPU(s)
$(5,-1,1,1)^{\mathrm{T}}$	8	0.128521
$(1,7,1,1)^{\mathrm{T}}$	8	0.177527
$(2,7,-2,-1)^{\mathrm{T}}$	8	0.137462
$(-1,-5,0,-3)^{\mathrm{T}}$	8	0.116928
$(0.6,4,0,8)^{\mathrm{T}}$	8	0.147629
$(1,-2,0.7,1)^{\mathrm{T}}$	8	0.1103788
$(1,-6,5,3)^{\mathrm{T}}$	8	0.135970
$(-1,-1,-1,-1)^{\mathrm{T}}$	8	0.151304

Table 2.

E x a m p l e 3 ([15]). We take again the Kojima-Shindo problem (MCPLIB file kojsshin.gms) with $C = [0,3]^4$. This problem has two degenerate solutions $\bar{x}_1 = (1,0,3,0)^{\mathrm{T}}$, $F(\bar{x}_1) = (0,31,0,4)^{\mathrm{T}}$ and $\bar{x}_2 = (\frac{\sqrt{6}}{2},0,0,\frac{1}{2})^{\mathrm{T}}$, $F(\bar{x}_2) = (0, 2+\frac{\sqrt{6}}{2},0,0)^{\mathrm{T}}$.

Using the starting point $x^0 = (-1,-1,-1,-1)^{\mathrm{T}}$, the numerical results are given in Table 3.

Similarly, taking another starting point $x^0 = (9,0,-1,1)^{\mathrm{T}}$, we obtain the numerical results given in Table 4.

E x a m p l e 4 ([15]). The operator is

$$F(x) = \begin{pmatrix} x_1^3 - 8 \\ x_2 - x_3 + x_2^3 + 3 \\ x_2 - x_3 + 2x_3^3 - 3 \\ x_4 - 2x_4^3 \end{pmatrix}$$

and $C = [0,5]^4$. This problem has one degenerate solution $\bar{x} = (2,0,1,0)^{\mathrm{T}}$, $F(\bar{x}) = (0,2,0,0)^{\mathrm{T}}$.

θ	Iter	CPU(s)	x^k
6	10	0.197873	$\begin{pmatrix} 1.224745 \\ 0.000003 \\ 0.000000 \\ 0.499999 \end{pmatrix}$
7	9	0.172599	$\begin{pmatrix} 1.224752 \\ 0.000006 \\ 0.000000 \\ 0.499999 \end{pmatrix}$
10	8	0.152000	$\begin{pmatrix} 1.224745 \\ 0.000003 \\ 0.000000 \\ 0.499999 \end{pmatrix}$
15	7	0.133142	$\begin{pmatrix} 1.224745 \\ 0.000003 \\ 0.000000 \\ 0.499999 \end{pmatrix}$
22	6	0.130594	$\begin{pmatrix} 1.224752 \\ 0.000003 \\ 0.000000 \\ 0.499997 \end{pmatrix}$
25	$*$	$*$	$*$

Table 3. The star $*$ indicates that the algorithm does not provide any solution when θ takes a value greater than or equal to the last value displayed. Therefore, we consider that the method does not converge.

θ	Iter	CPU(s)	x^k
10	9	0.056870	$\begin{pmatrix} 1.000000 \\ 0.000003 \\ 2.999999 \\ 0.000000 \end{pmatrix}$
25	7	0.050231	$\begin{pmatrix} 1.000000 \\ 0.000002 \\ 2.999999 \\ 0.000000 \end{pmatrix}$
50	6	0.0491033	$\begin{pmatrix} 1.000000 \\ 0.000001 \\ 2.999999 \\ 0.000000 \end{pmatrix}$
100	5	0.049065	$\begin{pmatrix} 1.224745 \\ 0.000000 \\ 0.000000 \\ 0.500000 \end{pmatrix}$
200	4	0.044569	$\begin{pmatrix} 1.224745 \\ 0.000000 \\ 0.000000 \\ 0.499999 \end{pmatrix}$
415	$*$	$*$	$*$

Table 4.

Using the starting point $x^0 = (-6, -6, -10, -1)^{\mathrm{T}}$, the numerical results are given in Table 5.

θ	Iter	CPU(s)
5	11	0.066163
10	8	0.049526
20	6	0.039950
100	5	0.039827
200	4	0.033758
1500	3	0.026853
4600	*	*

Table 5.

Example 5 ([15]). The operator is the same as that defined in Example 4 and $C = [-1, 1]^4$.

Using the starting point $x^0 = (6, -6, 10, 3)^{\mathrm{T}}$, the numerical results are given in Table 6.

θ	Iter	CPU(s)
5	11	0.147629
10	8	0.217512
10^2	5	0.095258
10^3	4	0.046632
10^4	3	0.033456

Table 6.

4.2. Examples with variable sizes.

Example 6 ([13]). We consider the following linear complementarity problem, where the operator F is defined by:

$$F(x) = Mx + q,$$

while the matrix $M \in \mathbb{R}^{n\times n}$ and vector $q \in \mathbb{R}^n$ are of the following forms

$$M = \begin{pmatrix} 1 & 2 & 2 & . & . & . & 2 \\ 0 & 1 & 2 & . & . & . & 2 \\ 0 & 0 & 1 & . & . & . & 2 \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ 0 & 0 & 0 & . & . & . & 1 \end{pmatrix}, \quad q = \begin{pmatrix} -1 \\ -1 \\ . \\ . \\ . \\ . \\ -1 \end{pmatrix}.$$

This example has one degenerate solution $\bar{x} = (0, 0, \ldots, 0, 1)^{\mathrm{T}}$.

The numerical results are presented in the following tables for different sizes of n using the starting point $x^0 = (2, 2, \ldots, 2)^{\mathrm{T}}$ and different values of θ.

For $\theta = 10$

Size (n)	Iter	CPU(s)
10	8	0.47
50	8	5.26
100	8	85.50
150	8	213.88
200	8	561.56

Table 7.

For $\theta = 20$

Size (n)	Iter	CPU(s)
10	7	0.37
50	7	4.78
100	7	54.34
150	7	143.75
200	7	476.12

Table 8.

For $\theta = 30$

Size (n)	Iter	CPU(s)
10	6	0.34
50	6	4.74
100	6	51.70
150	6	91.19
200	6	127.88

Table 9.

Example 7 ([13]), ([12]). Let $F(x) = Mx + q$, where M is the $(n \times n)$ nonsymmetric matrix

$$M = \begin{pmatrix} 4 & -1 & & & & & \\ -1 & 4 & -1 & & & & \\ & & 4 & -1 & & & \\ & & \ddots & \ddots & & & \\ & & & 4 & -1 & & \\ & & & & 4 & -1 & \\ & & & & & 4 & \end{pmatrix}, \quad q = (-1, -1, \ldots, -1)^{\mathrm{T}} \in \mathbb{R}^n,$$

and $C = [0, 1]^n$.

The computational results are summarized in Tables 10 and 11 using two starting points for different sizes of n.

For $x^0 = (-1, -1, \ldots, -1)^{\mathrm{T}}$

Size (n)	10	100	1000	2000	3000
Iter	1	1	1	1	1
CPU(s)	0.09	0.18	32.40	65.84	203.76

Table 10.

For $x^0 = (0, 0, \ldots, 0)^{\mathrm{T}}$

Size (n)	10	100	1000	2000	3000
Iter	1	1	1	1	1
CPU(s)	0.03	0.11	30.95	59.73	171.88

Table 11.

4.3. Comments. Through the examples tested, we note that the number of iterations depends inversely on the value of θ. However, the quality of the solution risks to be deteriorated when θ is too large. On the other hand, the choice of the starting point x^0 does not influence the behavior of the algorithm (see Examples 2 and 7).

We note also that if the solution is in the interior of the admissible set C, then the algorithm converges after one iteration (see Examples 1 and 7). This phenomenon is justified, because if the solution is an interior point of C, then the resolution of BVIP is reduced to the resolution of $F(x) = 0$ (see [3], [4]).

The used selection ($r_0 = 1$ and $r_{k+1} = 10r_k$) of the penalty parameter does not work if the solution is nondegenerate (see Examples 3, 4 and 6). We must then choose appropriate values. It is also noticeable that this algorithm can be considered to solve BVIP of large dimensions as shown in the last two examples.

5. Conclusion

In this paper, we have proposed and analyzed a penalty method for solving the box constrained variational inequality problem. The method consists of formulating the variational inequality as a sequence of penalized nonlinear equations. We have shown that the sequence of solutions of the sequence converges to that of the box constrained variational inequality problem.

To highlight the details of our contribution, we have presented numerical simulations on some examples not necessarily strongly monotone. These simulations illustrate clearly the effectiveness of our approach and consolidate our theoretical results.

A c k n o w l e d g m e n t s. The authors thank the referee and the editor for their valuable comments which greatly improve the presentation of the paper. The authors also thank professor Keraghel Abdelkrim for his invaluable help in carrying out this work.

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