ENTRY-EXIT DECISIONS WITH IMPLEMENTATION DELAY UNDER UNCERTAINTY

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Abstract. We employ a natural method from the perspective of the optimal stopping theory to analyze entry-exit decisions with implementation delay of a project, and provide closed expressions for optimal entry decision times, optimal exit decision times, and the maximal expected present value of the project. The results in conventional research were obtained under the restriction that the sum of the entry cost and exit cost is nonnegative. In practice, we may meet cases when this sum is negative, so it is necessary to remove the restriction. If the sum is negative, there may exist two trigger prices of entry decision, which does not happen when the sum is nonnegative, and it is not optimal to enter and then immediately exit the project even though it is an arbitrage opportunity.

Keywords: entry decision time; exit decision time; implementation delay; optimal stopping problem; viscosity solution

MSC 2010: 60G40, 91B06

1. Introduction

A firm plans to invest in a project which could produce a commodity at some variable cost. To activate the project, the firm has to put a sunk cost, and, in order to get the maximal expected profit from the project, the firm may abandon the project at another sunk cost.

What time is optimal to decide to enter the project and what time is optimal to decide to exit the project? This so-called entry-exit decision problem appeals to many authors, because the theoretical results of it may be used to analyze many concrete problems. Isik et al. [8] employed the results to examine the decisions of a firm whether to invest in an emerging market or abandon the investment. Pradhan

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and Leung [17] showed a behavioral study on the entry, stay and exit decisions of the fishers in Hawaii's longline fishery. Kjærland [10] applied the results to study hydropower investment opportunities within the Norwegian context. Leung [11] used them to investigate the entry and exit decisions of foreign banks in Hong Kong.

Many authors answered the above two questions in the setting that there is no time lag between decision times and corresponding implementation times [5], [18], [6], [13], [21], [19], [20], [12], [3], [22]. In practice, a major characteristic of investments is that there exist lags between decision times and corresponding implementation times. Some authors discussed entry-exit decision problems with implementation delay. For example, see [2], [7], [14], [4].

In [2], Bar-Ilan and Strange embedded lags in the classical model presented by Dixit [5]. They considered entry and exit decisions by employing the real option theory and derived a system of equations (see equations (22)–(25) in [2]), then obtained semi-closed solutions for entry and exit decisions. However, they did not prove the existence and uniqueness of the solution to the system. Gauthier and Morellec [7] provided more explicit solutions through assuming *a priori* the forms of decision times. In [14], Øksendal studied two optimal exit decision problems with implementation delay—an assets selling problem and a resource extraction problem. In [4], Costeniuc et al. applied the probabilistic approach to entry and exit decisions with Parisian implementation delay from the view of real options.

The results in conventional research were obtained under the assumption that the sum of the entry cost and exit cost is nonnegative. In practice, we may meet cases when this sum is negative even though it seems to be rather rare. For example, a large number of illiquid assets are planed to be sold quickly, so the transaction price of these assets is in general lower than the current price. The difference of the current price and the transaction price can be considered a negative cost. If the difference is large enough, the sum of the entry cost and exit cost may be negative. We will remove this assumption and study the case where the sum is negative.

If the sum is nonnegative, there exists no arbitrage opportunity, and there is only one trigger price of entry decision. However, if the sum is negative, it is an arbitrage opportunity to enter and then immediately exit the project, and there may be two trigger prices of entry decision ((vi) of Theorem 5.6). We find that it is not optimal to enter and then immediately exit the project even if the sum is negative (see (iii), (v), and (vi) of Theorem 5.6 and Theorem 5.10).

In this paper, we employ a method from the perspective of the optimal stopping theory, which proves to be natural, to rigorously discuss the entry-exit decision problem with implementation delay. We study this problem in three steps. First, we transform the delayed implementation case into an immediate implementation case. Second, we decompose the immediate implementation case into two standard optimal stopping problems, and then solve these two problems. Finally, we provide explicitly an optimal entry decision time, an optimal exit decision time and an expression of the maximal expected present value of the project.

We outline the structure of this paper. In Section 2, we recall briefly the classical optimal stopping theory. In Section 3, we show that delayed optimal stopping problems involving two stopping times can be transformed to immediate stopping ones. In Section 4, we describe the model in detail. In Section 5, we obtain an optimal entry-exit decision as to when the firm decides to enter the project and when the firm decides to exit the project (Theorem 5.10). Some extensions will be given in Section 6 and conclusions will be drawn in Section 7.

2. Some results concerning classical optimal stopping problems

In this section, we recall briefly some results of classical optimal stopping problems. For details, we refer to [16], Section 5.2.

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a filtered probability space with $\{\mathscr{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions and $\mathscr{F}_0$ being the completion of $\{\emptyset, \Omega\}$. Let $W = (W(t),\ t \geqslant 0)$ be a d-dimensional standard Brownian motion defined on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbb{P})$.

Let $X = (X(t),\ t \geqslant 0)$ be a diffusion in $\mathbb{R}^n$ given by

$$\mathrm{d}X(t) = \alpha(X(t))\,\mathrm{d}t + \beta(X(t))\,\mathrm{d}W(t), \quad X(0) = x,$$

where $\alpha\colon\ \mathbb{R}^n \to \mathbb{R}^n$ and $\beta\colon\ \mathbb{R}^n \to \mathbb{R}^{n\times d}$ are some Lipschitz functions.

Let $\mathcal{T}$ denote the set of all stopping times valued in $[0, \infty]$.

Theorem 2.1. *Consider the optimal stopping problem*

$$V(x) := \sup_{\tau\in\mathcal{T}} \mathbb{E}^x\left[\int_0^\tau \exp(-rt)f(X(t))\,\mathrm{d}t + \exp(-r\tau)g(X(\tau))\right] \tag{2.1}$$

for some Lipschitz functions f and g. Here $\mathbb{E}^x[\cdot] := \mathbb{E}[\cdot \mid X(0) = x]$, and $\exp(-r\tau)g(X(\tau)) \equiv 0$ on $\{\tau = \infty\}$.

Assume that $r > 0$ is large enough. Then the following statements are true.

(i) *The value function V is Lipschitz continuous and is the unique viscosity solution with linear growth of the variational inequality*

$$\min\{rV - \mathcal{L}V - f, V - g\} = 0,$$

where $\mathcal{L}$ is the infinitesimal generator of X.

(ii) *Set* $\mathcal{S} := \{x\colon\ x \in \mathbb{R}^n, V(x) = g(x)\}$, *which is called the exercise region. Then* $\tau^* := \inf\{t\colon\ t > 0,\ X(t) \in \mathcal{S}\}$ *is a maximizer of problem* (2.1).

(iii) *The value function* V *is a viscosity solution of*

$$rV - \mathcal{L}V - f = 0 \quad \text{on } \mathcal{C},$$

where $\mathcal{C} := \{x\colon\ x \in \mathbb{R}^n, V(x) > g(x)\}$ *is the continuation region. Moreover, if* $\mathcal{L}$ *is locally uniformly elliptic,* V *is* C^2 *on* $\mathcal{C}$.

(iv) *Assume that* X *is* 1*-dimensional,* $\mathcal{L}$ *is locally uniformly elliptic, and* g *is* C^1 *on* $\mathcal{S}$. *Then* V *is* C^1 *on* $\partial\mathcal{C}$ *and* C^2 *at the isolated points of* $\mathcal{S}$.

(v) *Define a function* $\widehat{V}$ *by*

$$\widehat{V}(x) := \mathbb{E}\biggl[\int_0^\infty \exp(-rt) f(X(t))\,\mathrm{d}t\biggr].$$

Then $\mathcal{S} = \emptyset$ *implies* $\widehat{V} \geqslant g$ *and* $\widehat{V} \geqslant g$ *implies* $V = \widehat{V}$.

(vi) *If* g *is* C^2 *continuous on some open set* $\mathcal{O}$, *then* $\mathcal{S} \subset \{x\colon\ x \in \mathcal{O},\ rg(x) - \mathcal{L}g(x) - f(x) \geqslant 0\} \cup \mathcal{O}^{\mathrm{c}}$.

(vii) *Assume that* X *is* 1*-dimensional and takes values in* $(0,\infty)$, $X(t,x) \to X(t,0) = 0$ *as* $x \to 0$, $\widehat{V}(x_0) < g(x_0)$ *for some* $x_0 > 0$, *and* g *is* C^2 *continuous. We have the following two facts. If* $\mathcal{D} = [a,\infty)$ *for some* $a > 0$, *where* $\mathcal{D} := \{x\colon\ x > 0,\ rg(x) - \mathcal{L}g(x) - f(x) \geqslant 0\}$, *then* $\mathcal{S} = [x^*,\infty)$ *for some* $x^* \in [a,\infty)$. *If* $g(0) \geqslant f(0)/r$ *and* $\mathcal{D} = (0,a]$ *for some* $a > 0$, *then* $\mathcal{S} = (0,x^*]$ *for some* $x^* \in (0,a]$.

Proof. We refer to [16], Section 5.2 for the proof. □

3. A useful transformation

In this section, we show that delayed optimal stopping problems involving two stopping times can be transformed into immediate stopping ones. The proof is similar to that of [15], p. 38, Theorem 2.11.

Theorem 3.1. *Let* δ *be a nonnegative number. Consider the following two optimal stopping problems:*

$$J(x) := \sup_{\substack{\tau_1,\tau_2\in\mathcal{T},\\ \tau_1\leqslant\tau_2}} \mathbb{E}^x\biggl[\int_{\tau_1+\delta}^{\tau_2+\delta} f(X(t))\,\mathrm{d}t + g_1(X(\tau_1+\delta)) + g_2(X(\tau_2+\delta))\biggr], \tag{3.1}$$

where $f, g_1, g_2\colon \mathbb{R}^n \to \mathbb{R}$ *are three functions such that the expectations are finite;*

$$\widetilde{J}(x) := \sup_{\substack{\tau_1,\tau_2\in\mathcal{T},\\ \tau_1\leqslant\tau_2}} \mathbb{E}^x\Big[\int_{\tau_1}^{\tau_2} f(X(t))\,\mathrm{d}t + g_1^\delta(X(\tau_1)) + g_2^\delta(X(\tau_2))\Big], \tag{3.2}$$

where

$$g_1^\delta(x) := \mathbb{E}^x\Big[-\int_0^\delta f(X(t))\,\mathrm{d}t + g_1(X(\delta))\Big]$$

and

$$g_2^\delta(x) := \mathbb{E}^x\Big[\int_0^\delta f(X(t))\,\mathrm{d}t + g_2(X(\delta))\Big].$$

Then $J(x) = \widetilde{J}(x)$*. In addition, if* (τ_1^*, τ_2^*) *is a maximizer of* (3.2)*, it is also a maximizer of* (3.1)*.*

Proof. Note that

$$\begin{aligned}
&\mathbb{E}^x\Big[\int_{\tau_1+\delta}^{\tau_2+\delta} f(X(t))\,\mathrm{d}t + g_1(X(\tau_1+\delta)) + g_2(X(\tau_2+\delta))\Big]\\
&= \mathbb{E}^x\Big[\Big(\int_{\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_1+\delta} + \int_{\tau_2}^{\tau_2+\delta}\Big) f(X(t))\,\mathrm{d}t + g_1(X(\tau_1+\delta)) + g_2(X(\tau_2+\delta))\Big]\\
&= \mathbb{E}^x\Big[\int_{\tau_1}^{\tau_2} f(X(t))\,\mathrm{d}t - \int_{\tau_1}^{\tau_1+\delta} f(X(t))\,\mathrm{d}t + g_1(X(\tau_1+\delta))\\
&\quad + \int_{\tau_2}^{\tau_2+\delta} f(X(t))\,\mathrm{d}t + g_2(X(\tau_2+\delta))\Big].
\end{aligned}$$

Then, by the strong Markov property of the process X, we get

$$\begin{aligned}
&\mathbb{E}^x\Big[\int_{\tau_1+\delta}^{\tau_2+\delta} f(X(t))\,\mathrm{d}t + g_1(X(\tau_1+\delta)) + g_2(X(\tau_2+\delta))\Big]\\
&= \mathbb{E}^x\Big[\int_{\tau_1}^{\tau_2} f(X(t))\,\mathrm{d}t + \mathbb{E}^{X(\tau_1)}\Big[-\int_0^\delta f(X(t))\,\mathrm{d}t + g_1(X(\delta))\Big]\\
&\quad + \mathbb{E}^{X(\tau_2)}\Big[\int_0^\delta f(X(t))\,\mathrm{d}t + g_2(X(\delta))\Big]\Big]\\
&= \mathbb{E}^x\Big[\int_{\tau_1}^{\tau_2} f(X(t))\,\mathrm{d}t + g_1^\delta(X(\tau_1)) + g_2^\delta(X(\tau_2))\Big],
\end{aligned}$$

which completes the proof. □

4. The model

We return to the entry-exit decision problem introduced in Section 1, and assume that the price process P follows

$$\mathrm{d}P(t) = \mu P(t)\,\mathrm{d}t + \sigma P(t)\,\mathrm{d}B(t) \quad \text{and} \quad P(0) = p, \tag{4.1}$$

where $\mu \in \mathbb{R}$, $\sigma, p > 0$, and B is a one-dimensional standard Brownian motion, which models uncertainty, defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\{\mathscr{F}_t^B\}_{t\geqslant 0}$ be the augmentation of the natural filtration generated by the Brownian motion B.

Applying Itô's formula, we deduce that the solution to equation (4.1) is

$$P(t) = P(0)\exp[(\mu - \tfrac{1}{2}\sigma^2)t + \sigma B(t)]. \tag{4.2}$$

To answer the two questions—what time is optimal to make an entry decision and what time is optimal to make an exit decision—we will solve the following optimal problem:

$$\begin{aligned} J(p) := \sup_{\tau_{\rm in}\leqslant \tau_{\rm out}} \mathbb{E}^p\biggl[\int_{\tau_{\rm in}+\delta}^{\tau_{\rm out}+\delta} \exp(-rt)(P(t) - C)\,\mathrm{d}t - \exp(-r(\tau_{\rm in}+\delta))K_{\rm in} \\ - \exp(-r(\tau_{\rm out}+\delta))K_{\rm out}\biggr], \end{aligned} \tag{4.3}$$

where $\tau_{\rm in}$ and $\tau_{\rm out}$ are $\{\mathscr{F}_t^B\}_{t\geqslant 0}$-stopping times, r is the discount rate such that $r > 0$, C is the running cost, $K_{\rm in}$ is the entry cost, $K_{\rm out}$ is the exit cost, and the nonnegative number δ is a time lag between the decision time and the corresponding implementation time. We call stopping times $\tau_{\rm in}$ and $\tau_{\rm out}$ an entry decision time and an exit decision time, respectively, and the function J the maximal expected present value of the project.

Remark 4.1. (1) We do not propose any restriction on the running cost, entry cost and exit cost, except that they are constant.

(2) Note that for any stopping time τ and nonnegative number δ, $\tau + \delta$ is also a stopping time. The maximal expected present value J of the delayed implementation case is no more than that of the corresponding immediate implementation case. We may interpret their difference as the loss due to delayed implementation.

(3) Furthermore, let $0 \leqslant \delta^{(1)} < \delta^{(2)} < \infty$ and $\mathcal{T}$ be the collection of all stopping times. Then $\{\tau + \delta^{(2)} \colon \tau \in \mathcal{T}\} \subset \{\tau + \delta^{(1)} \colon \tau \in \mathcal{T}\}$, and thus the value of J corresponding to $\delta^{(2)}$ is no more than that corresponding to $\delta^{(1)}$. This implies the following principle: *Once one has made a right decision, he/she should activate it as soon as possible.*

In this section, we provide an optimal entry-exit decision and an explicit expression for the function J.

Let us first consider a simple case $r \leqslant \mu$. In this case, noting the expression (4.2) of P, we have

$$\begin{aligned}
\mathbb{E}^p\Big[\int_\delta^\infty \exp(-rt)(P(t)-C)\,\mathrm{d}t\Big] &= \int_\delta^\infty \exp(-rt)(p\,\exp(\mu t)-C)\,\mathrm{d}t \\
&= \begin{cases} \lim\limits_{t\to\infty}\Big(p(t-\delta)+\dfrac{C}{r}(\exp(-rt)-\exp(-r\delta))\Big) & \text{if } r=\mu, \\ \lim\limits_{t\to\infty}\Big(\dfrac{p}{\mu-r}(\exp((\mu-r)t)-\exp((\mu-r)\delta)) & \\ \qquad +\dfrac{C}{r}(\exp(-rt)-\exp(-r\delta))\Big) & \text{if } r<\mu \end{cases} \\
&= \infty,
\end{aligned}$$

where we have used the fact that the process

$$\big(\exp(-\tfrac{1}{2}\sigma^2 t+\sigma B(t)),\ t\geqslant 0\big)$$

is a martingale (see [1], p. 288, Corollary 5.2.2) for the first step.

Thus, we obtain the following result.

Theorem 5.1. *Assume that* $r \leqslant \mu$. *Then* $\tau_{\text{in}}^* := 0$ *a.s. is an optimal entry decision time and* $\tau_{\text{out}}^* := \infty$ *is an optimal exit decision time, i.e., the firm should never exit the project. In addition, the function* J *in* (4.3) *is given by* $J \equiv \infty$.

Now we determine an optimal entry-exit decision for the case $r > \mu$. To this end, we first employ Theorem 3.1 to transform the delayed optimal stopping problem (4.3) to an immediate stopping one.

Theorem 5.2. *The delayed optimal stopping problem* (4.3) *is equivalent to the optimal stopping problem*

$$\begin{aligned}
(5.1)\quad \widetilde{J}(p) := \sup_{\tau_{\text{in}}\leqslant\tau_{\text{out}}} \mathbb{E}^p\Big[&\int_{\tau_{\text{in}}}^{\tau_{\text{out}}} \exp(-rt)(P(t)-C)\,\mathrm{d}t \\
&- \exp(-r\tau_{\text{in}})(k_1P(\tau_{\text{in}})+k_0) - \exp(-r\tau_{\text{out}})(l_1P(\tau_{\text{out}})+l_0)\Big],
\end{aligned}$$

where

$$k_1 := \frac{\exp((\mu-r)\delta)-1}{\mu-r},\quad k_0 := \frac{C}{r}(\exp(-r\delta)-1)+\exp(-r\delta)K_{\text{in}},$$
$$l_1 := -\frac{\exp((\mu-r)\delta)-1}{\mu-r},\quad l_0 := -\frac{C}{r}(\exp(-r\delta)-1)+\exp(-r\delta)K_{\text{out}}.$$

P r o o f. 1. Define the process X by $X(t) := [s+t, P(t)]^{\mathrm{T}}$, where $s \in \mathbb{R}$. Then

$$\mathrm{d}X(t) = \begin{bmatrix} 1 \\ \mu P(t) \end{bmatrix} \mathrm{d}t + \begin{bmatrix} 0 \\ \sigma P(t) \end{bmatrix} \mathrm{d}B(t), \quad X(0) = \begin{bmatrix} s \\ p \end{bmatrix}.$$

2. According to Theorem 3.1, we need to calculate

$$\mathbb{E}^p\Big[-\int_0^\delta \exp(-r(s+t))(P(t)-C)\,\mathrm{d}t - \exp(-r(s+\delta))K_{\mathrm{in}}\Big] \tag{5.2}$$

and

$$\mathbb{E}^p\Big[\int_0^\delta \exp(-r(s+t))(P(t)-C)\,\mathrm{d}t - \exp(-r(s+\delta))K_{\mathrm{out}}\Big]. \tag{5.3}$$

For (5.2) we have

$$\begin{aligned}
&\mathbb{E}^p\Big[-\int_0^\delta \exp(-r(s+t))(P(t)-C)\,\mathrm{d}t - \exp(-r(s+\delta))K_{\mathrm{in}}\Big] \\
&\quad = -\int_0^\delta \exp(-r(s+t))(p\exp(\mu t)-C)\,\mathrm{d}t - \exp(-r(s+\delta))K_{\mathrm{in}} \\
&\quad = -\exp(-rs)\Big(\frac{p}{\mu-r}(\exp((\mu-r)\delta)-1) \\
&\qquad + \frac{C}{r}(\exp(-r\delta)-1) + \exp(-r\delta)K_{\mathrm{in}}\Big),
\end{aligned}$$

where we have used the fact that the process

$$\big(\exp(-\tfrac{1}{2}\sigma^2 t + \sigma B(t)),\ t \geqslant 0\big)$$

is a martingale (see [1], p. 288, Corollary 5.2.2) for the first step.

Similarly, we can calculate (5.3). Therefore, in light of Theorem 3.1, the delayed optimal stopping problem (4.3) is equivalent to the optimal stopping problem (5.1). □

In order to solve the optimal stopping problem (5.1), we will solve the following two optimal stopping problems:

$$G(p) := \sup_{\tau_{\mathrm{out}}} \mathbb{E}^p\Big[\int_0^{\tau_{\mathrm{out}}} \exp(-rt)(P(t)-C)\,\mathrm{d}t - \exp(-r\tau_{\mathrm{out}})(l_1 P(\tau_{\mathrm{out}}) + l_0)\Big] \tag{5.4}$$

and

$$H(p) := \sup_{\tau_{\mathrm{in}}} \mathbb{E}^p[\exp(-r\tau_{\mathrm{in}})(G(P(\tau_{\mathrm{in}})) - k_1 P(\tau_{\mathrm{in}}) - k_0)]. \tag{5.5}$$

Assume that $r > \mu$. Let λ_1 and λ_2 be the solutions of the quadratic equation

$$r - \mu\lambda - \tfrac{1}{2}\sigma^2\lambda(\lambda - 1) = 0$$

with $\lambda_1 < \lambda_2$. Then we have $\lambda_1 < 0$ and $\lambda_2 > 1$.

Theorem 5.3. *For the optimal stopping problem* (5.4), *the following are true.*

(i) *If $r > \mu$ and $C \leqslant rK_{\mathrm{out}}$, then $\tau^*_{\mathrm{out}} := \infty$ a.s. is a maximizer of* (5.4). *In addition, $G(p) = p/(r-\mu) - C/r$.*

(ii) *If $r > \mu$ and $C > rK_{\mathrm{out}}$, then $\tau^*_{\mathrm{out}} := \inf\{t\colon t > 0,\ P(t) \leqslant p_{\mathrm{out}}\}$ a.s. is a maximizer of* (5.4), *where*

$$p_{\mathrm{out}} = \exp(-\mu\delta)\frac{\lambda_1}{\lambda_1 - 1}(r-\mu)\Big(\frac{C}{r} - K_{\mathrm{out}}\Big).$$

In addition,

$$G(p) = \begin{cases} Ap^{\lambda_1} + \dfrac{p}{r-\mu} - \dfrac{C}{r} & \text{if } p > p_{\mathrm{out}}, \\ \dfrac{p}{\mu - r}(\exp((\mu - r)\delta) - 1) & \\ \qquad + \dfrac{C}{r}(\exp(-r\delta) - 1) - \exp(-r\delta)K_{\mathrm{out}} & \text{if } p \leqslant p_{\mathrm{out}}, \end{cases}$$

where $A = \exp((\mu - r)\delta)p_{\mathrm{out}}^{1-\lambda_1}/(\lambda_1(\mu - r))$.

Proof. 1. Assume that $r > \mu$ and $C \leqslant rK_{\mathrm{out}}$.

Noting that

$$\mathbb{E}^p\Big[\int_0^\infty \exp(-rt)(P(t) - C)\,\mathrm{d}t\Big] = \frac{p}{r-\mu} - \frac{C}{r},$$

we have

$$\mathbb{E}^p\Big[\int_0^\infty \exp(-rt)(P(t) - C)\,\mathrm{d}t\Big] \geqslant -l_1 p - l_0.$$

Therefore, by (v) of Theorem 2.1, we obtain (i).

2. Assume that $r > \mu$ and $C > rK_{\mathrm{out}}$.

In this case, we have $\mathcal{D} = (0, \exp(-\mu\delta)(C - rK_{\mathrm{out}})]$. Thus, by (vii) of Theorem 2.1, the exercise region is of the form $(0, p_{\mathrm{out}}]$ for some $p_{\mathrm{out}} \in (0, \infty)$. On the continuation region $(p_{\mathrm{out}}, \infty)$, G satisfies the equation

$$rG - \mu pG' - \tfrac{1}{2}\sigma^2 p^2 G'' - p + C = 0$$

by (iii) of Theorem 2.1. Furthermore, by the Lipschitz property of G, we have

$$G(p) = Ap^{\lambda_1} + \frac{p}{r-\mu} - \frac{C}{r}$$

for some constant A.

Note that G is C^1 continuous at $p_{\rm out}$ by (iv) of Theorem 2.1. We get the system

$$\begin{cases} Ap_{\rm out}^{\lambda_1} + \dfrac{p_{\rm out}}{r-\mu} - \dfrac{C}{r} = \dfrac{p_{\rm out}}{\mu - r}(\exp((\mu-r)\delta) - 1) \\ \qquad\qquad + \dfrac{C}{r}(\exp(-r\delta) - 1) - \exp(-r\delta)K_{\rm out}, \\ \lambda_1 A p_{\rm out}^{\lambda_1 - 1} + \dfrac{1}{r-\mu} = \dfrac{\exp((\mu-r)\delta) - 1}{\mu - r}, \end{cases}$$

from which we obtain

$$p_{\rm out} = \exp(-\mu\delta)\frac{\lambda_1}{\lambda_1 - 1}(r-\mu)\Big(\frac{C}{r} - K_{\rm out}\Big) \tag{5.6}$$

and

$$A = \exp((\mu - r)\delta)\frac{p_{\rm out}^{1-\lambda_1}}{\lambda_1(\mu - r)}.$$

The proof is complete. □

Remark 5.4. We will prove in Theorem 5.10 that $p_{\rm out}$ is the trigger price of exit decision.

Corollary 5.5. *The optimal exit trigger price* $p_{\rm out}$ *in Theorem* 5.3 *satisfies* $p_{\rm out} < \exp(-\mu\delta)(C - rK_{\rm out})$.

Proof. Note that $1/\lambda_1 < \mu/r$. Then thanks to (5.6), the conclusion follows. □

Theorem 5.6. *For the optimal stopping problem* (5.5), *the following are true.*

(i) *If* $r > \mu$, $C - rK_{\rm out} \leqslant 0$ *and* $C + rK_{\rm in} \leqslant 0$, *then* $\tau_{\rm in}^* := 0$ *a.s. is a maximizer of* (5.5). *In addition,*

$$H(p) = \frac{\exp((\mu - r)\delta)}{r - \mu}p - \exp(-r\delta)\Big(\frac{C}{r} + K_{\rm in}\Big).$$

(ii) *If* $r > \mu$, $C - rK_{\rm out} \leqslant 0$ *and* $C + rK_{\rm in} > 0$, *then* $\tau_{\rm in}^* := \inf\{t\colon\ t > 0,\ P(t) \geqslant p_{\rm in}\}$ *a.s. is a maximizer of* (5.5), *where*

$$p_{\rm in} = \exp(-\mu\delta)\frac{\lambda_2}{\lambda_2 - 1}(r-\mu)\Big(\frac{C}{r} + K_{\rm in}\Big).$$

In addition,

$$H(p) = \begin{cases} Bp^{\lambda_2} & \text{if } p < p_{\text{in}}, \\ \dfrac{\exp((\mu - r)\delta)}{r - \mu} p - \exp(-r\delta)\Big(\dfrac{C}{r} + K_{\text{in}}\Big) & \text{if } p \geqslant p_{\text{in}}, \end{cases}$$

where $B = \exp((\mu - r)\delta) p_{\text{in}}^{1-\lambda_2} / (\lambda_2(r - \mu))$.

(iii) *If* $r > \mu$, $C - rK_{\text{out}} > 0$, *and* $C + rK_{\text{in}} \leqslant 0$, *then* $\tau_{\text{in}}^* := 0$ *a.s. is a maximizer of* (5.5)*. In addition,*

$$H(p) = \begin{cases} Ap^{\lambda_1} + \dfrac{\exp((\mu - r)\delta)}{r - \mu} p - \exp(-r\delta)\Big(\dfrac{C}{r} + K_{\text{in}}\Big) & \text{if } p > p_{\text{out}}, \\ -\exp(-r\delta)(K_{\text{in}} + K_{\text{out}}) & \text{if } p \leqslant p_{\text{out}}. \end{cases}$$

(iv) *If* $r > \mu$, $C - rK_{\text{out}} > 0$, $C + rK_{\text{in}} > 0$, *and* $K_{\text{in}} + K_{\text{out}} \geqslant 0$, *then* $\tau_{\text{in}}^* := \inf\{t\colon\ t > 0,\ P(t) \geqslant p_{\text{in}}\}$ *a.s. is a maximizer of* (5.5), *where* p_{in} *is the largest solution of the algebraic equation*

$$A(\lambda_2 - \lambda_1) p_{\text{in}}^{\lambda_1} + \frac{\exp((\mu - r)\delta)}{r - \mu}(\lambda_2 - 1) p_{\text{in}} - \lambda_2 \exp(-r\delta)\Big(\frac{C}{r} + K_{\text{in}}\Big) = 0.$$

In addition,

$$H(p) = \begin{cases} Bp^{\lambda_2} & \text{if } p < p_{\text{in}}, \\ Ap^{\lambda_1} + \dfrac{\exp((\mu - r)\delta)}{r - \mu} p - \exp(-r\delta)\Big(\dfrac{C}{r} + K_{\text{in}}\Big) & \text{if } p \geqslant p_{\text{in}}, \end{cases}$$

where

$$B = \lambda_1 \lambda_2^{-1} A p_{\text{in}}^{\lambda_1 - \lambda_2} + \exp((\mu - r)\delta) \frac{p_{\text{in}}^{1-\lambda_2}}{\lambda_2(r - \mu)}.$$

(v) *If* $r > \mu$, $C - rK_{\text{out}} > 0$, $C + rK_{\text{in}} > 0$, $K_{\text{in}} + K_{\text{out}} < 0$, *and* $p_{\text{out}} \geqslant \exp(-\mu\delta)(C + rK_{\text{in}})$, *then* $\tau_{\text{in}}^* := 0$ *a.s. is a maximizer of* (5.5)*. In addition,*

$$H(p) = \begin{cases} Ap^{\lambda_1} + \dfrac{\exp((\mu - r)\delta)}{r - \mu} p - \exp(-r\delta)\Big(\dfrac{C}{r} + K_{\text{in}}\Big) & \text{if } p > p_{\text{out}}, \\ -\exp(-r\delta)(K_{\text{in}} + K_{\text{out}}) & \text{if } p \leqslant p_{\text{out}}. \end{cases}$$

(vi) *If* $r > \mu$, $C - rK_{\text{out}} > 0$, $C + rK_{\text{in}} > 0$, $K_{\text{in}} + K_{\text{out}} < 0$, *and* $p_{\text{out}} < \exp(-\mu\delta) \times (C + rK_{\text{in}})$, *then* $\tau_{\text{in}}^* := \inf\{t\colon\ t > 0,\ P(t) \leqslant p_{\text{in}}^{(1)} \text{ or } P(t) \geqslant p_{\text{in}}^{(2)}\}$ *a.s. is* *a maximizer of* (5.5), *where* $(p_{\text{in}}^{(1)}, p_{\text{in}}^{(2)})$ *is the solution of the equation*

$$\begin{bmatrix} \lambda_2 {p_{\text{in}}^{(1)}}^{-\lambda_1} & -{p_{\text{in}}^{(1)}}^{-\lambda_1} \\ -\lambda_1 {p_{\text{in}}^{(1)}}^{-\lambda_2} & {p_{\text{in}}^{(1)}}^{-\lambda_2} \end{bmatrix} \begin{bmatrix} -\exp(-r\delta)(K_{\text{in}} + K_{\text{out}}) \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_2 {p_{\text{in}}^{(2)}}^{-\lambda_1} & -{p_{\text{in}}^{(2)}}^{-\lambda_1} \\ -\lambda_1 {p_{\text{in}}^{(2)}}^{-\lambda_2} & {p_{\text{in}}^{(2)}}^{-\lambda_2} \end{bmatrix} \begin{bmatrix} A{p_{\text{in}}^{(2)}}^{\lambda_1} + \frac{\exp((\mu-r)\delta)}{r-\mu} p_{\text{in}}^{(2)} \\ -\exp(-r\delta)\left(\frac{C}{r} + K_{\text{in}}\right) \\ \lambda_1 A{p_{\text{in}}^{(2)}}^{\lambda_1} + \frac{\exp((\mu-r)\delta)}{r-\mu} p_{\text{in}}^{(2)} \end{bmatrix} \tag{5.7}$$

with $p_{\text{in}}^{(1)} < p_{\text{in}}^{(2)}$.

In addition,

$$H(p) = \begin{cases} Ap^{\lambda_1} + \dfrac{\exp((\mu-r)\delta)}{r-\mu} p - \exp(-r\delta)\Big(\dfrac{C}{r} + K_{\text{in}}\Big) & \textit{if } p \geqslant p_{\text{in}}^{(2)}, \\ B_1 p^{\lambda_1} + B_2 p^{\lambda_2} \quad \textit{if } p_{\text{in}}^{(1)} < p < p_{\text{in}}^{(2)}, \\ -\exp(-r\delta)(K_{\text{in}} + K_{\text{out}}) \quad \textit{if } p \leqslant p_{\text{in}}^{(1)}, \end{cases}$$

where

$$B_1 = -\frac{\lambda_2 (p_{\text{in}}^{(1)})^{-\lambda_1} \exp(-r\delta)}{\lambda_2 - \lambda_1}(K_{\text{in}} + K_{\text{out}})$$

and

$$B_2 = \frac{\lambda_1 (p_{\text{in}}^{(1)})^{-\lambda_2} \exp(-r\delta)}{\lambda_2 - \lambda_1}(K_{\text{in}} + K_{\text{out}}).$$

P r o o f. 1. Assume that $r > \mu$, $C - rK_{\text{out}} \leqslant 0$, and $C + rK_{\text{in}} \leqslant 0$. Define a function w by

$$w(p) := \frac{\exp((\mu-r)\delta)}{r-\mu} p - \exp(-r\delta)\Big(\frac{C}{r} + K_{\text{in}}\Big) \quad \text{for } p \in (0, \infty).$$

Then we have

$$rw(p) - \mu p w'(p) - \tfrac{1}{2}\sigma^2 p^2 w''(p) \geqslant 0,$$

which implies w is a viscosity solution of

$$\min\{rV - \mu p V' - \tfrac{1}{2}\sigma^2 p^2 V'', V - w\} = 0 \quad \text{on } (0, \infty).$$

Note that $H(0^+) = w(0^+)$. Thus, by the uniqueness of viscosity solutions (see (i) of Theorem 2.1), we have $H(p) = w(p)$. Consequently, the exercise region is $(0, \infty)$, i.e., $\tau_{\text{in}}^* := 0$ a.s. is a maximizer of (5.5) by (ii) of Theorem 2.1.

2. Assume that $r > \mu$, $C - rK_{\text{out}} \leqslant 0$ and $C + rK_{\text{in}} > 0$.

In this case, we have $\mathcal{D} = [\exp(-\mu\delta)(C+rK_{\text{in}}), \infty)$. Thus, by (vii) of Theorem 2.1, the exercise region is of the form $[p_{\text{in}}, \infty)$ for some $p_{\text{in}} \in (0, \infty)$. On the continuation region $(0, p_{\text{in}})$, H satisfies the equation

$$rH - \mu p H' - \tfrac{1}{2}\sigma^2 p^2 H'' = 0$$

by (iii) of Theorem 2.1. Furthermore, by the Lipschitz property of H, we have $H(p) = Bp^{\lambda_2}$ for some constant B.

Note that H is C^1 continuous at p_{in} by (iv) of Theorem 2.1. We get the system

$$\begin{cases} Bp_{\text{in}}^{\lambda_2} = \dfrac{\exp((\mu - r)\delta)}{r - \mu} p_{\text{in}} - \exp(-r\delta)\Big(\dfrac{C}{r} + K_{\text{in}}\Big), \\ \lambda_2 B p_{\text{in}}^{\lambda_2 - 1} = \dfrac{\exp((\mu - r)\delta)}{r - \mu}, \end{cases}$$

from which we obtain

$$p_{\text{in}} = \exp(-\mu\delta)\frac{\lambda_2}{\lambda_2 - 1}(r - \mu)\Big(\frac{C}{r} + K_{\text{in}}\Big)$$

and

$$B = \exp((\mu - r)\delta)\frac{p_{\text{in}}^{1-\lambda_2}}{\lambda_2(r - \mu)}.$$

3. Assume that $r > \mu$, $C - rK_{\text{out}} > 0$ and $C + rK_{\text{in}} \leqslant 0$.

Define a function w by

$$w(p) := \begin{cases} Ap^{\lambda_1} + \dfrac{\exp((\mu - r)\delta)}{r - \mu} p - \exp(-r\delta)\Big(\dfrac{C}{r} + K_{\text{in}}\Big) & \text{if } p > p_{\text{out}}, \\ -\exp(-r\delta)(K_{\text{in}} + K_{\text{out}}) & \text{if } p \leqslant p_{\text{out}}. \end{cases}$$

Then w is a viscosity subsolution of

$$\min\{rV - \mu p V' - \tfrac{1}{2}\sigma^2 p^2 V'', V - w\} = 0 \quad \text{on } (0, \infty). \tag{5.8}$$

We prove that w is a viscosity supersolution of (5.8). To this end, we only need to prove

$$rw(p_{\text{out}}) - \mu p \varphi'(p_{\text{out}}) - \tfrac{1}{2}\sigma^2 p^2 \varphi''(p_{\text{out}}) \geqslant 0 \tag{5.9}$$

for any function $\varphi \in C^2(\mathcal{N}(p_{\text{out}}))$ such that $w(p_{\text{out}}) = \varphi(p_{\text{out}})$ and $w(p) \geqslant \varphi(p)$ on some neighbourhood $\mathcal{N}(p_{\text{out}})$ of p_{out}, since $rw - \mu p w' - \sigma^2 p^2 w''/2 \geqslant 0$ on $(0, p_{\text{out}}) \cup (p_{\text{out}}, \infty)$.

Noting that p_{out} is a minimizer of $w-\varphi$ on $\mathcal{N}(p_{\text{out}})$, we have $w'(p_{\text{out}})-\varphi'(p_{\text{out}})=0$ and $w''_{-}(p_{\text{out}})-\varphi''(p_{\text{out}})\geqslant 0$, i.e. $\varphi'(p_{\text{out}})=0$ and $\varphi''(p_{\text{out}})\leqslant 0$. In addition, thanks to $C-rK_{\text{out}}>0$ and $C+rK_{\text{in}}\leqslant 0$, $K_{\text{in}}+K_{\text{out}}<0$. So (5.9) holds.

In summary, w is a viscosity solution of

$$\min\{rV-\mu pV'-\tfrac{1}{2}\sigma^2p^2V'', V-w\}=0 \quad \text{on } (0,\infty).$$

Note that $H(0^+)=w(0^+)$. Then, by the uniqueness of viscosity solutions (see (i) of Theorem 2.1), we get $H(p)=w(p)$ for $p\in(0,\infty)$. Consequently, the exercise region is $(0,\infty)$, i.e., $\tau^*_{\text{in}}:=0$ a.s. is a maximizer of (5.5) by (ii) of Theorem 2.1.

4. Assume that $r>\mu$, $C-rK_{\text{out}}>0$, $C+rK_{\text{in}}>0$, and $K_{\text{in}}+K_{\text{out}}\geqslant 0$.

First assume $K_{\text{in}}+K_{\text{out}}>0$. Then, in light of (vi) of Theorem 2.1, we have

$$\mathcal{S}\subset[\exp(-\mu\delta)(C+rK_{\text{in}}),\infty)\cup\{p_{\text{out}}\}.$$

Note that $p_{\text{out}}<\exp(-\mu\delta)(C-rK_{\text{out}})$ by Corollary 5.5, and $K_{\text{in}}+K_{\text{out}}>0$. These imply $p_{\text{out}}<\exp(-\mu\delta)(C+rK_{\text{in}})$. Consequently, following the proof of (vii) of Theorem 2.1 (see [16], p. 104) and using (iv) of Theorem 2.1, we can see that there is a point $p_{\text{in}}\in[\exp(-\mu\delta)(C+rK_{\text{in}}),\infty)$ such that

$$H(p)=G(p)-k_1p-k_0 \quad \text{for } p\in[p_{\text{in}},\infty)$$

and

$$rH-\mu pH'-\tfrac{1}{2}\sigma^2p^2H''=0 \quad \text{on } (0,p_{\text{in}}). \tag{5.10}$$

Thus, by the Lipschitz property of H, we have $H(p)=Bp^{\lambda_2}$ for some constant B from (5.10).

Note that H is C^1 continuous at p_{in} by (iv) of Theorem 2.1. We get the system

$$\begin{cases} Bp_{\text{in}}^{\lambda_2}=Ap_{\text{in}}^{\lambda_1}+\dfrac{\exp((\mu-r)\delta)}{r-\mu}p_{\text{in}}-\exp(-r\delta)\Big(\dfrac{C}{r}+K_{\text{in}}\Big),\\ \lambda_2Bp_{\text{in}}^{\lambda_2-1}=\lambda_1Ap_{\text{in}}^{\lambda_1-1}+\dfrac{\exp((\mu-r)\delta)}{r-\mu}, \end{cases} \tag{5.11}$$

from which we obtain

$$A(\lambda_2-\lambda_1)p_{\text{in}}^{\lambda_1}+\frac{\exp((\mu-r)\delta)}{r-\mu}(\lambda_2-1)p_{\text{in}}-\lambda_2\exp(-r\delta)\Big(\frac{C}{r}+K_{\text{in}}\Big)=0.$$

We will show ahead in Lemma 5.9 that the above algebraic equation has only two roots. One is less than p_{out} and the other is greater than p_{out}. Since $p_{\text{in}}\geqslant$

$\exp(-\mu\delta)(C + rK_{\text{in}})$, $p_{\text{out}} < \exp(-\mu\delta)(C - rK_{\text{out}})$ by Corollary 5.5, and $K_{\text{in}} + K_{\text{out}} > 0$, we must choose the greater one. Furthermore, we have

$$B = \lambda_1 \lambda_2^{-1} A p_{\text{in}}^{\lambda_1 - \lambda_2} + \exp((\mu - r)\delta) \frac{p_{\text{in}}^{1-\lambda_2}}{\lambda_2(r - \mu)}.$$

For proving that the exercise region is $[p_{\text{in}}, \infty)$, we only need to show

$$H(p_{\text{out}}) > G(p_{\text{out}}) - k_1 p_{\text{out}} - k_0. \tag{5.12}$$

To see this, consider the function $f(p) := H(p) - G(p) + k_1 p + k_0$ for $p \in [0, p_{\text{out}}]$. Then we have $f(0) = \exp(-r\delta)(K_{\text{in}} + K_{\text{out}}) > 0$. In addition, $f'(p) = \lambda_2 B p^{\lambda_2 - 1} > 0$ for $p \in (0, p_{\text{out}})$, since $B > 0$ by (5.11) and Lemma 5.9. The inequality (5.12) follows.

Now consider the case $K_{\text{in}} + K_{\text{out}} = 0$. We refer to the following Step 6. To solve systems (5.13) and (5.14), we put $B_1 = p_{\text{in}}^{(1)} = 0$, then systems (5.13) and (5.14) are reduced to (5.11). By repeating the proof of the case $K_{\text{in}} + K_{\text{out}} > 0$, we achieve our aim.

5. Assume that $r > \mu$, $C - rK_{\text{out}} > 0$, $C + rK_{\text{in}} > 0$, $K_{\text{in}} + K_{\text{out}} < 0$, and $p_{\text{out}} \geqslant \exp(-\mu\delta)(C + rK_{\text{in}})$. The proof of this case is the same as that of the case (iii).

6. Assume that $r > \mu$, $C - rK_{\text{out}} > 0$, $C + rK_{\text{in}} > 0$, $K_{\text{in}} + K_{\text{out}} < 0$, and $p_{\text{out}} < \exp(-\mu\delta)(C + rK_{\text{in}})$.

In this case, we have

$$\mathcal{S} \subset (0, p_{\text{out}}] \cup [\exp(-\mu\delta)(C + rK_{\text{in}}), \infty).$$

Thus, following the proof of (vii) of Theorem 2.1 (see [16], p. 104), we can see that the exercise region is of the form $(0, p_{\text{in}}^{(1)}] \cup [p_{\text{in}}^{(2)}, \infty)$ for some $p_{\text{in}}^{(1)} \in (0, p_{\text{out}}]$ and $p_{\text{in}}^{(2)} \in [\exp(-\mu\delta)(C + rK_{\text{in}}), \infty)$. On the continuation region $(p_{\text{in}}^{(1)}, p_{\text{in}}^{(2)})$, H satisfies the equation

$$rH - \mu p H' - \tfrac{1}{2}\sigma^2 p^2 H'' = 0$$

by (iii) of Theorem 2.1.

Thus, we have $H(p) = B_1 p^{\lambda_1} + B_2 p^{\lambda_2}$ on $(p_{\text{in}}^{(1)}, p_{\text{in}}^{(2)})$ for some constants B_1 and B_2.

Note that H is C^1 continuous at $p_{\text{in}}^{(1)}$ and $p_{\text{in}}^{(2)}$ by (iv) of Theorem 2.1. We get the following systems:

$$\begin{cases} B_1 {p_{\text{in}}^{(1)}}^{\lambda_1} + B_2 {p_{\text{in}}^{(1)}}^{\lambda_2} = -\exp(-r\delta)(K_{\text{in}} + K_{\text{out}}), \\ \lambda_1 B_1 {p_{\text{in}}^{(1)}}^{\lambda_1 - 1} + \lambda_2 B_2 {p_{\text{in}}^{(1)}}^{\lambda_2 - 1} = 0 \end{cases} \tag{5.13}$$

and

$$
(5.14) \quad \begin{cases} B_1 p_{\text{in}}^{(2)\lambda_1} + B_2 p_{\text{in}}^{(2)\lambda_2} = A p_{\text{in}}^{(2)\lambda_1} + \dfrac{\exp((\mu - r)\delta)}{r-\mu} p_{\text{in}}^{(2)} - \exp(-r\delta)\Big(\dfrac{C}{r} + K_{\text{in}}\Big), \\ \lambda_1 B_1 p_{\text{in}}^{(2)\lambda_1 - 1} + \lambda_2 B_2 p_{\text{in}}^{(2)\lambda_2 - 1} = \lambda_1 A p_{\text{in}}^{(2)\lambda_1 - 1} + \dfrac{\exp((\mu - r)\delta)}{r-\mu}, \end{cases}
$$

from which, by solving B_1 and B_2, respectively, we obtain (5.7). □

Remark 5.7. We will prove in Theorem 5.10 that p_{in}, $p_{\text{in}}^{(1)}$, and $p_{\text{in}}^{(2)}$ are the trigger prices of entry decision.

Corollary 5.8. *The optimal entry trigger prices* p_{in} *in* (ii) *and* (iv) *of Theorem* 5.6 *satisfy* $p_{\text{in}} > \exp(-\mu\delta)(C + rK_{\text{in}})$; *the optimal entry trigger prices* $p_{\text{in}}^{(1)}$ *and* $p_{\text{in}}^{(2)}$ *in* (vi) *of Theorem* 5.6 *satisfy* $p_{\text{in}}^{(1)} < \exp(-\mu\delta)(C - rK_{\text{out}})$ *and* $p_{\text{in}}^{(2)} > \exp(-\mu\delta)(C + rK_{\text{in}})$.

Proof. 1. For the case (ii) of Theorem 5.6, we have

$$
p_{\text{in}} = \exp(-\mu\delta) \frac{\lambda_2}{\lambda_2 - 1} (r - \mu) \Big(\frac{C}{r} + K_{\text{in}}\Big).
$$

In addition, note that $1/\lambda_2 > \mu/r$. The inequality $p_{\text{in}} > \exp(-\mu\delta)(C + rK_{\text{in}})$ follows.

2. Consider the case (iv) of Theorem 5.6.

Define a function U by

$$
U(p) := Bp^{\lambda_2} - G(p) + k_1 p + k_0 \quad \text{for } p \in [p_{\text{out}}, \infty).
$$

Then we have $U(p_{\text{out}}) \geqslant 0$ and $U(p_{\text{in}}) = 0$.

We prove that the equation $U''(p) = 0$ has a solution in $(p_{\text{out}}, p_{\text{in}})$. To this end, suppose that the equation $U''(p) = 0$ has no solution in $(p_{\text{out}}, p_{\text{in}})$. Then the function $U'(\cdot)$ is strictly monotonous on $[p_{\text{out}}, p_{\text{in}}]$. In addition, note that $U'(p_{\text{out}}) = \lambda_2 B p_{\text{out}}^{\lambda_2 - 1} > 0$ and $U'(p_{\text{in}}) = 0$. We get $U'(p) > 0$ for $p_{\text{out}} < p < p_{\text{in}}$. Consequently, $0 \leqslant U(p_{\text{out}}) < U(p_{\text{in}}) = 0$, which is a contradiction.

On the other hand, by noting $U''(p) = \lambda_2(\lambda_2 - 1)Bp^{\lambda_2 - 2} - \lambda_1(\lambda_1 - 1)Ap^{\lambda_1 - 2}$, the equation $U''(p) = 0$ has at most one solution in (p_{out}, ∞).

Therefore, $U''(p_{\text{in}}) > 0$ and then

$$
\begin{aligned}
r(G(p_{\text{in}}) - k_1 p_{\text{in}} - k_0) - \mu p_{\text{in}}(G'(p_{\text{in}}) - k_1) - \tfrac{1}{2}\sigma^2 p_{\text{in}}^2 G''(p_{\text{in}}) \\
> rH(p_{\text{in}}) - \mu p_{\text{in}} H'(p_{\text{in}}) - \tfrac{1}{2}\sigma^2 p_{\text{in}}^2 H''_-(p_{\text{in}}) = 0,
\end{aligned}
$$

i.e.

$$
p_{\text{in}} > \exp(-\mu\delta)(C + rK_{\text{in}}),
$$

where we have used $rG(p_{\text{in}}) - \mu p_{\text{in}} G'(p_{\text{in}}) - \sigma^2 p_{\text{in}}^2 G''(p_{\text{in}})/2 = p_{\text{in}} - C$.

3. The inequality $p_{\text{in}}^{(1)} < \exp(-\mu\delta)(C - rK_{\text{out}})$ follows from $p_{\text{in}}^{(1)} \leqslant p_{\text{out}}$ and Corollary 5.5. The proof of the inequality $p_{\text{in}}^{(2)} > \exp(-\mu\delta)(C + rK_{\text{in}})$ is similar to Step 2. □

Lemma 5.9. *Assume that* $r > \mu$, $C - rK_{\rm out} > 0$, $C + rK_{\rm in} > 0$, *and* $K_{\rm in} + K_{\rm out} \geqslant 0$. *Then the equation*

$$A(\lambda_2 - \lambda_1)p^{\lambda_1} + \frac{\exp((\mu - r)\delta)}{r - \mu}(\lambda_2 - 1)p - \lambda_2 \exp(-r\delta)\Big(\frac{C}{r} + K_{\rm in}\Big) = 0$$

has only two solutions p_1 *and* p_2 *in* $(0, \infty)$ *satisfying* $p_1 \leqslant p_{\rm out}$ *and* $p_2 > p_{\rm out}$. *Furthermore,* $\lambda_1 A p_2^{\lambda_1} + \exp((\mu - r)\delta)/(r - \mu) > 0$.

Proof. The proof is similar to that of [22], Lemma 5.5.

1. Define a function E by

$$E(p) := A(\lambda_2 - \lambda_1)p^{\lambda_1} + \frac{\exp((\mu - r)\delta)}{r - \mu}(\lambda_2 - 1)p - \lambda_2 \exp(-r\delta)\Big(\frac{C}{r} + K_{\rm in}\Big).$$

Suppose that the equation $E(p) = 0$ has three solutions in $(0, \infty)$. Then by Rolle's mean value theorem, there is a positive number ξ such that $E''(\xi) = 0$, i.e.

$$A(\lambda_2 - \lambda_1)\lambda_1(\lambda_1 - 1)\xi^{\lambda_1 - 2} = 0,$$

which is impossible. Thus, the equation $E(p) = 0$ has at most two solutions in $(0, \infty)$.

2. In this step, we will estimate $E(p_{\rm out})$ and $E'(p_{\rm out})$.

We first estimate $E(p_{\rm out})$ as

$$\begin{aligned} E(p_{\rm out}) = {} & (\lambda_2 - \lambda_1)\Big(\frac{\exp((\mu - r)\delta)}{\mu - r}p_{\rm out} + \exp(-r\delta)\Big(\frac{C}{r} - K_{\rm out}\Big)\Big) \\ & + \frac{\exp((\mu - r)\delta)}{r - \mu}(\lambda_2 - 1)p_{\rm out} - \lambda_2 \exp(-r\delta)\Big(\frac{C}{r} + K_{\rm in}\Big) \\ = {} & -\lambda_2 \exp(-r\delta)(K_{\rm in} + K_{\rm out}) \leqslant 0, \end{aligned}$$

where we have used the continuity of the function G at $p_{\rm out}$ for the first equality.

Now we estimate $E'(p_{\rm out})$:

$$\begin{aligned} E'(p_{\rm out}) &= (\lambda_1 - \lambda_2)\frac{\exp((\mu - r)\delta)}{r - \mu} + (\lambda_2 - 1)\frac{\exp((\mu - r)\delta)}{r - \mu} \\ &= (\lambda_1 - 1)\frac{\exp((\mu - r)\delta)}{r - \mu} < 0, \end{aligned}$$

where we have used the C^1 continuity of the function G at $p_{\rm out}$ for the first equality.

3. Note that $\lim\limits_{p \to 0^+} E(p) = \infty$, $\lim\limits_{p \to \infty} E(p) = \infty$, $E(p_{\rm out}) \leqslant 0$, and $E'(p_{\rm out}) < 0$. We find that the equation $E(p) = 0$ has only two solutions p_1 and p_2 in $(0, \infty)$ satisfying $p_1 \leqslant p_{\rm out}$ and $p_2 > p_{\rm out}$.

Furthermore, $E'(p_2) > 0$. It follows that

$$\lambda_1 A p_2^{\lambda_1} + \frac{\exp((\mu - r)\delta)}{r - \mu} > \lambda_1 A p_2^{\lambda_1} + \frac{\lambda_2 - 1}{\lambda_2 - \lambda_1} \frac{\exp((\mu - r)\delta)}{r - \mu} = \frac{1}{\lambda_2 - \lambda_1} E'(p_2) > 0.$$

The proof is complete. □

Recall problem (4.3). The following Theorem 5.10 provides a solution of entry and exit decisions and an explicit expression of the maximal expected present value of the project.

Theorem 5.10. *In each case of Theorem* 5.6, $\tau_{\rm in}^*$ *is an optimal entry decision time, and* $\tau_{\rm out}^*$ *is an optimal exit decision time, where* $\tau_{\rm out}^* := \infty$ *if* $C \leqslant rK_{\rm out}$ *and* $\tau_{\rm out}^* := \inf\{t \colon t > \tau_{\rm in}^*$ *and* $P(t) \leqslant p_{\rm out}\}$ *if* $C > rK_{\rm out}$*, respectively. In addition, we have* $J(p) = H(p)$*, where the functions* J *and* H *are given by* (4.3) *and* (5.5)*, respectively.*

Proof. 1. For an entry decision time $\tau_{\rm in}$, define a process $Q := (Q(t), t \geqslant 0)$ by $Q(t) := P(\tau_{\rm in} + t)$ and a filtration $\{\mathscr{G}_t\}_{t \geqslant 0}$ by $\mathscr{G}_t := \mathscr{F}^B_{\tau_{\rm in}+t}$. Then Q is a geometric Brownian motion on the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{G}_t\}_{t\geqslant 0}, \mathbb{P})$ with the same drift and volatility as P.

2. For any random time $\tau \geqslant \tau_{\rm in}$ we have:

$\tau - \tau_{\rm in}$ is a $\{\mathscr{G}_t\}_{t\geqslant 0}$-stopping time if and only if τ is a $\{\mathscr{F}^B_t\}_{t\geqslant 0}$-stopping time.

Suppose that $\tau' := \tau - \tau_{\rm in}$ is a $\{\mathscr{G}_t\}_{t\geqslant 0}$-stopping time. Note that $\{\tau < t\} = \{\tau_{\rm in} + \tau' < t\} = \bigcup_{q\in[0,t)\cap\mathbb{Q}} \{\tau_{\rm in} \leqslant q,\ \tau' < t - q\} = \bigcup_{q\in[0,t)\cap\mathbb{Q}} \{\tau_{\rm in} + t - q \leqslant t,\ \tau' < t - q\}$. Since τ' is a $\{\mathscr{G}_t\}_{t\geqslant 0}$-stopping time, thanks to [9], p. 6, Proposition 2.3, we have $\{\tau' < t - q\} \in \mathscr{G}_{t-q} = \mathscr{F}^B_{\tau_{\rm in}+t-q}$. By the definition of $\mathscr{F}^B_{\tau_{\rm in}+t-q}$ (cf. [9], p. 8, Definition 2.12), it holds that $\{\tau_{\rm in} + t - q \leqslant s, \tau' < t - q\} \in \mathscr{F}^B_s$ for any $s \geqslant 0$. In particular, via taking $s = t$, $\{\tau_{\rm in} + t - q \leqslant t, \tau' < t - q\} \in \mathscr{F}^B_t$. Thus $\{\tau < t\} \in \mathscr{F}^B_t$, and then from the right continuity of $\{\mathscr{F}^B_t\}_{t\geqslant 0}$ it follows that τ is a $\{\mathscr{F}^B_t\}_{t\geqslant 0}$-stopping time. Conversely, if τ is a $\{\mathscr{F}^B_t\}_{t\geqslant 0}$-stopping time, then $\{\tau - \tau_{\rm in} \leqslant t\} = \{\tau \leqslant \tau_{\rm in} + t\} \in \mathscr{F}^B_{\tau_{\rm in}+t}$ by [9], p. 8, Lemma 2.16, i.e. $\tau - \tau_{\rm in}$ is a $\{\mathscr{G}_t\}_{t\geqslant 0}$-stopping time.

3. Noting

$$\begin{aligned}
&\int_{\tau_{\rm in}}^{\tau_{\rm out}} \exp(-rt)(P(t) - C)\,\mathrm{d}t \\
&\qquad - \exp(-r\tau_{\rm in})(k_1 P(\tau_{\rm in}) + k_0) - \exp(-r\tau_{\rm out})(l_1 P(\tau_{\rm out}) + l_0) \\
&\quad = \exp(-r\tau_{\rm in}) \int_0^{\tau_{\rm out}-\tau_{\rm in}} \exp(-rt)(Q(t) - C)\,\mathrm{d}t - \exp(-r\tau_{\rm in})(k_1 P(\tau_{\rm in}) + k_0) \\
&\qquad - \exp(-r\tau_{\rm in}) \exp(-r(\tau_{\rm out} - \tau_{\rm in}))(l_1 Q(\tau_{\rm out} - \tau_{\rm in}) + l_0),
\end{aligned}$$

we have

$$\sup_{\tau_{\rm in}\leqslant\tau_{\rm out}} \mathbb{E}^p\bigg[\int_{\tau_{\rm in}}^{\tau_{\rm out}} \exp(-rt)(P(t)-C)\,\mathrm{d}t - \exp(-r\tau_{\rm in})(k_1P(\tau_{\rm in})+k_0)$$

$$- \exp(-r\tau_{\rm out})(l_1P(\tau_{\rm out})+l_0)\bigg]$$

$$= \sup_{\tau_{\rm in}}\sup_{\tau'_{\rm out}} \mathbb{E}^p\bigg[\exp(-r\tau_{\rm in})\int_0^{\tau'_{\rm out}} \exp(-rt)(Q(t)-C)\,\mathrm{d}t$$

$$- \exp(-r\tau_{\rm in})(k_1P(\tau_{\rm in})+k_0) - \exp(-r\tau_{\rm in})\exp(-r(\tau'_{\rm out}))(l_1Q(\tau'_{\rm out})+l_0)\bigg],$$

where the random times $\tau'_{\rm out}$ are $\{\mathscr{G}_t\}_{t\geqslant 0}$-stopping times.

Therefore, together with (5.1), (5.4), and (5.5) it follows that $\widetilde{J}(p) = H(p)$ and $(\tau^*_{\rm in}, \tau^*_{\rm out})$ is a solution of problem (5.1). Consequently, these and Theorem 5.2 complete the proof. □

6. Extensions

We here discuss some extensions in the following directions.

(a) Instead of the common lag δ, one may prefer that the entry lag δ_1 and exit lag δ_2 are different, but still deterministic.

(b) The common lag δ is random, and independent of the price process P from (4.1).

(c) Combining (a) and (b), the different entry lag δ_1 and exit lag δ_2 are both random, and $(\delta_1,\ \delta_2)$ and the price process given by P (4.1) are of mutual independence.

In all of these extensions, if $r \leqslant \mu$, we always have the same result as Theorem 5.1. Thus in the following, we discuss entry-exit decisions for $r > \mu$. We are aware that transforming delayed optimal stopping problems into stopping problems without delay is the point, and complete it in Propositions 6.1, 6.3, and 6.4.

For the extension (a), we replace naturally (4.3) with

$$J^{\delta_1,\delta_2}(p) := \sup_{\substack{\tau_{\rm in}\leqslant\tau_{\rm out},\\ \tau_{\rm in}+\delta_1\leqslant\tau_{\rm out}+\delta_2}} \mathbb{E}^p\bigg[\int_{\tau_{\rm in}+\delta_1}^{\tau_{\rm out}+\delta_2} \exp(-rt)(P(t)-C)\,\mathrm{d}t$$

$$- \exp(-r(\tau_{\rm in}+\delta_1))K_{\rm in} - \exp(-r(\tau_{\rm out}+\delta_2))K_{\rm out}\bigg]. \tag{6.1}$$

Proposition 6.1. (i) *Assume that $r > \mu$ and $\delta_1 < \delta_2$. Consider the problem*

$$\widetilde{J}^{\delta_1<\delta_2}(p) := \sup_{\tau_{\rm in}\leqslant\tau_{\rm out}} \mathbb{E}^p\bigg[\int_{\tau_{\rm in}}^{\tau_{\rm out}} \exp(-rt)(P(t)-C)\,{\rm d}t - \exp(-r\tau_{\rm in})(k_1P(\tau_{\rm in})+k_0) - \exp(-r\tau_{\rm out})(l_1P(\tau_{\rm out})+l_0)\bigg], \tag{6.2}$$

where

$$k_1 := \frac{\exp((\mu-r)\delta_1)-1}{\mu-r}, \quad k_0 := \frac{C}{r}(\exp(-r\delta_1)-1)+\exp(-r\delta_1)K_{\rm in},$$
$$l_1 := -\frac{\exp((\mu-r)\delta_2)-1}{\mu-r}, \quad l_0 := -\frac{C}{r}(\exp(-r\delta_2)-1)+\exp(-r\delta_2)K_{\rm out}.$$

Then $J^{\delta_1,\delta_2}(p) = \widetilde{J}^{\delta_1<\delta_2}(p)$. In addition, if $(\tau_{\rm in}^*, \tau_{\rm out}^*)$ is a maximizer of (6.2), *it is also a maximizer of* (6.1).

(ii) *Assume that $r > \mu$ and $\delta_1 > \delta_2$. Consider the problem*

$$\widetilde{J}^{\delta_1>\delta_2}(p) := \sup_{\tau_{\rm in}+\delta_1\leqslant\tau_{\rm out}+\delta_2} \mathbb{E}^p\bigg[\int_{\tau_{\rm in}}^{\tau_{\rm out}} \exp(-rt)(P(t)-C)\,{\rm d}t - \exp(-r\tau_{\rm in})(k_1P(\tau_{\rm in})+k_0) - \exp(-r\tau_{\rm out})(l_1P(\tau_{\rm out})+l_0)\bigg], \tag{6.3}$$

where

$$k_1 := \frac{\exp((\mu-r)\delta_1)-1}{\mu-r}, \quad k_0 := \frac{C}{r}(\exp(-r\delta_1)-1)+\exp(-r\delta_1)K_{\rm in},$$
$$l_1 := -\frac{\exp((\mu-r)\delta_2)-1}{\mu-r}, \quad l_0 := -\frac{C}{r}(\exp(-r\delta_2)-1)+\exp(-r\delta_2)K_{\rm out}.$$

Then $J^{\delta_1,\delta_2}(p) = \widetilde{J}^{\delta_1>\delta_2}(p)$. In addition, if $(\tau_{\rm in}^*, \tau_{\rm out}^*)$ is a maximizer of (6.3), *it is also a maximizer of* (6.1).

Proof. For case (i), noting that $\tau_{\rm in} \leqslant \tau_{\rm out}$ and $\delta_1 < \delta_2$ imply $\tau_{\rm in}+\delta_1 \leqslant \tau_{\rm out}+\delta_2$, we follow the proofs of Theorems 3.1 and 5.2 to get the conclusion of case (i). Case (ii) is similar. □

Remark 6.2. For case (ii), in light of the proof of Theorem 5.10, we should first solve the problem

$$G^{\delta_1>\delta_2}(p) := \sup_{\tau_{\rm out}\geqslant\delta_1-\delta_2} \mathbb{E}^p\bigg[\int_0^{\tau_{\rm out}} \exp(-rt)(P(t)-C)\,{\rm d}t - \exp(-r\tau_{\rm out})(l_1P(\tau_{\rm out})+l_0)\bigg], \tag{6.4}$$

and then solve the problem

$$H^{\delta_1>\delta_2}(p) := \sup_{\tau_{\rm in}} \mathbb{E}^p[\exp(-r\tau_{\rm in})(G^{\delta_1>\delta_2}(P(\tau_{\rm in})) - k_1P(\tau_{\rm in}) - k_0)]. \tag{6.5}$$

Solving problem (6.5) is standard. To solve problem (6.4), we may adopt the following procedure.

Step 1. Rewrite (6.4) as

$$G^{\delta_1>\delta_2}(p) = \sup_{\tau_{\rm out}\geqslant\delta_1-\delta_2} \mathbb{E}^p\left[\int_{\delta_1-\delta_2}^{\tau_{\rm out}} \exp(-rt)(P(t)-C)\,{\rm d}t - \exp(-r\tau_{\rm out})(l_1P(\tau_{\rm out})+l_0)\right]$$
$$+\mathbb{E}^p\left[\int_0^{\delta_1-\delta_2} \exp(-rt)(P(t)-C)\,{\rm d}t\right].$$

Step 2. Solve the variational inequality

$$\min\{r\phi - \mu p\phi' - \tfrac{1}{2}\sigma^2p^2\phi'' - p + C,\ \phi + l_1p + l_0\} = 0 \quad \text{on } (0,\infty).$$

It has been done in the proof of Theorem 5.3. Let $\mathcal{S}_{\rm exit}$ be the exercise region of the above variational inequality. Then by the standard verification argument, we find that $\tau_{\rm exit} := \inf\{t\colon\ t > \delta_1-\delta_2, P(t)\in\mathcal{S}_{\rm exit}\}$ is a maximizer of (6.4) and

$$G^{\delta_1>\delta_2}(p) = \mathbb{E}^p[\exp(-r(\delta_1-\delta_2))\phi(P(\delta_1-\delta_2))] + \mathbb{E}^p\left[\int_0^{\delta_1-\delta_2} \exp(-rt)(P(t)-C)\,{\rm d}t\right].$$

Step 3. To obtain the explicit expression of $G^{\delta_1>\delta_2}$, by the Feynman–Kac formula we solve the parabolic equation

$$\frac{\partial u}{\partial t} = \mu p\frac{\partial u}{\partial p} + \frac{1}{2}\sigma^2p^2\frac{\partial^2 u}{\partial p} - ru + p - C$$

subject to the initial condition $u(0,p) = \phi(p)$.

Then it follows that $G^{\delta_1>\delta_2}(p) = u(\delta_1-\delta_2, p)$.

Turning to the extension (b), we have the following proposition.

Proposition 6.3. *Assume that* δ *in* (4.3) *is a random variable with moment generating function* ϕ_δ*. Consider the problem*

$$\widetilde{J}^{\rm rdm}(p) := \sup_{\tau_{\rm in}\leqslant\tau_{\rm out}} \mathbb{E}^p\bigg[\int_{\tau_{\rm in}}^{\tau_{\rm out}} \exp(-rt)(P(t)-C)\,{\rm d}t - \exp(-r\tau_{\rm in})(k_1P(\tau_{\rm in})+k_0)$$
$$- \exp(-r\tau_{\rm out})(l_1P(\tau_{\rm out})+l_0)\bigg], \tag{6.6}$$

where

$$k_1 := \frac{\phi_\delta(\mu - r) - 1}{\mu - r}, \quad k_0 := \frac{C}{r}(\phi_\delta(-r) - 1) + \phi_\delta(-r)K_{\rm in},$$
$$l_1 := -\frac{\phi_\delta(\mu - r) - 1}{\mu - r}, \quad l_0 := -\frac{C}{r}(\phi_\delta(-r) - 1) + \phi_\delta(-r)K_{\rm out}.$$

Then $J(p) = \widetilde{J}^{\rm rdm}(p)$*. In addition, if* $(\tau_{\rm in}^*, \tau_{\rm out}^*)$ *is a maximizer of* (6.6)*, it is also a maximizer of* (4.3)*.*

P r o o f. The proof is similar to the proofs of Theorems 3.1 and 5.2, but noting the independence of δ and P. □

Finally, we consider the extension (c).

Proposition 6.4. *Assume that* δ_1 *and* δ_2 *in* (6.1) *are random variables with moment generating functions* ϕ_{δ_1} *and* ϕ_{δ_2}*, respectively, and* $\Delta := (\delta_1 - \delta_2)^+$ *being a* $\{\mathscr{F}_t^B\}_{t\geqslant 0}$*-stopping time. Consider the problem*

$$\begin{aligned}\widetilde{J}^{\rm rdm12}(p) := \sup_{\tau_{\rm in}+\Delta\leqslant\tau_{\rm out}} \mathbb{E}^p\bigg[\int_{\tau_{\rm in}}^{\tau_{\rm out}} &\exp(-rt)(P(t) - C)\,{\rm d}t \\ &- \exp(-r\tau_{\rm in})(k_1P(\tau_{\rm in}) + k_0) \\ &- \exp(-r\tau_{\rm out})(l_1P(\tau_{\rm out}) + l_0)\bigg],\end{aligned} \tag{6.7}$$

where

$$k_1 := \frac{\phi_{\delta_1}(\mu - r) - 1}{\mu - r}, \quad k_0 := \frac{C}{r}(\phi_{\delta_1}(-r) - 1) + \phi_{\delta_1}(-r)K_{\rm in},$$
$$l_1 := -\frac{\phi_{\delta_2}(\mu - r) - 1}{\mu - r}, \quad l_0 := -\frac{C}{r}(\phi_{\delta_2}(-r) - 1) + \phi_{\delta_2}(-r)K_{\rm out}.$$

Then $J^{\delta_1,\delta_2}(p) = \widetilde{J}^{\rm rdm12}(p)$*. In addition, if* $(\tau_{\rm in}^*, \tau_{\rm out}^*)$ *is a maximizer of* (6.7)*, it is also a maximizer of* (6.1)*.*

P r o o f. The proof is similar to the proofs of Theorems 3.1 and 5.2, but noting the independence of (δ_1, δ_2) and P. □

R e m a r k 6.5. Remark 6.2 also works for Proposition 6.4.

7. Conclusions

We have to face the fact that activating decisions is usually postponed. In this paper, we study an entry-exit decision problem with delayed implementation. We first clarify it from the view of optimal stopping in Section 4, then discuss the trivial case in Theorem 5.1, and finally obtain an explicit solution in Theorem 5.10 for the nontrivial case. Some extensions are discussed in Section 6.

The results in literature were obtained under the assumption that the sum of the entry cost and exit cost is nonnegative, which limits the application of the model (4.3). We here abandon the assumption. If the sum is negative, two trigger prices of optimal entry decisions may exist, which does not happen if the sum is nonnegative. Even though it is an arbitrage opportunity to enter a project and then exit it immediately if the sum is negative, it is still not advisable.

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