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A MULTILEVEL NEWTON'S METHOD FOR EIGENVALUE PROBLEMS

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Abstract. We propose a new type of multilevel method for solving eigenvalue problems based on Newton's method. With the proposed iteration method, solving an eigenvalue problem on the finest finite element space is replaced by solving a small scale eigenvalue problem in a coarse space and a sequence of augmented linear problems, derived by Newton step in the corresponding sequence of finite element spaces. This iteration scheme improves overall efficiency of the finite element method for solving eigenvalue problems. Finally, some numerical examples are provided to validate the efficiency of the proposed numerical scheme.

Keywords: eigenvalue problem; finite element method; Newton's method; multilevel iteration

MSC 2010: 65N30, 65N25, 65L15, 65B99

1. Introduction

In recent decades, the study of solving large scale eigenvalue problems, arising from the modern science and engineering society, has become one of the major focuses of numerical analysts and engineers. However, it is always a difficult task to solve high-dimensional eigenvalue problems that come from physical and chemical sciences. About the solution of eigenvalue problems, [3], [9], [10], [14], [15], [16], [19], [23], [24] and the references cited therein give some types of multilevel or multigrid schemes.

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Newton's method is one of the most powerful and well-known numerical methods for solving a root-finding problem. With a suitable initial guess, Newton's method is guaranteed to converge and the convergence is quadratic under some assumptions. So Newton's method is an extremely powerful technique in numerical computation and is widely applied to minimization and maximization problems, multiplicative inverses of numbers and power series, solving transcendent equations, complex functions, nonlinear systems of equations.

Note that, recently, Newton-type (Jacobi-type) approach has been successfully developed for solving eigenvalue problems [8], [6], [11], [18], [20], [22], [21], including linear and nonlinear cases. However, most existing researches focus mainly on method designing, understanding, implementing and numerical experiments in algebraic version. The aim of this paper is to present a type of multilevel iteration scheme based on Newton's approach for eigenvalue problems by using finite element discretization. In the multilevel iteration scheme, the coarse meshes provide good initial values for fine meshes, which improves the convergence rate of Newton's iteration. Actually, in order to obtain the optimal accuracy, we only need to do one Newton's iteration step in each level of meshes. The standard Galerkin finite element method for eigenvalue problems has been extensively investigated, e.g. Babuška and Osborn [1], [2], Chatelin [5] and references cited therein. Here we adopt some basic results from these papers for our analysis. The corresponding error and complexity of the proposed iteration scheme for the eigenvalue problem will be analyzed. Based on the analysis, the new method can obtain optimal accuracy with an optimal computational work when we can solve the associated augmented linear problems by the optimal complexity. Although Newton's method is sensitive to initial guess, we use multilevel technique to overcome this difficulty. Since it is easy to find a good approximation in the coarse grid for the fine grid approximation, using Newton type iterative method is reasonable. According to the theory of mixed finite element method [4], we also prove the existence and uniqueness of solutions to the proposed scheme. Namely, results should be helpful to better understand solving eigenvalue problems by Newton's approach and further extension.

This paper is organized as follows. In Section 2, we introduce the finite element method for the eigenvalue problem and give the corresponding basic error estimates. A type of one Newton iteration step is presented and the error estimates of the proposed scheme are analyzed in Section 3. In Section 4, we suggest a type of multilevel iteration scheme for more eigenvalues. In Section 5, three numerical examples are presented to validate our theoretical analysis. Some concluding remarks are provided in the last section.

2. Finite element method for eigenvalue problems

In this section, we introduce some notation and error estimates of the finite element method for the eigenvalue problem. The letter C (with or without subscripts) denotes a generic positive constant which may be different at its different occurrences through the paper. For convenience, the symbols $\lesssim$, $\gtrsim$ and $\approx$ will be used in this paper. Namely, $x_1 \lesssim y_1$, $x_2 \gtrsim y_2$ and $x_3 \approx y_3$, mean that $x_1 \leqslant C_1 y_1$, $x_2 \geqslant c_2 y_2$ and $c_3 x_3 \leqslant y_3 \leqslant C_3 x_3$ respectively for some constants C_1, c_2, c_3, and C_3 that are independent of the mesh size (see e.g. [25]).

In our methodology description, we are concerned with the following model problem: Find $(\lambda, u) \in \mathbb{R} \times V$ such that $b(u,u) = 1$ and

$$a(u,v) = \lambda b(u,v) \quad \forall\, v \in V, \tag{2.1}$$

where $V := H_0^1(\Omega)$, and $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are bilinear forms defined by

$$a(u,v) = \int_\Omega \nabla u \nabla v \,\mathrm{d}\Omega, \quad b(u,v) = \int_\Omega uv \,\mathrm{d}\Omega.$$

Here $\Omega \subset \mathbb{R}^d$ denotes a bounded domain with Lipschitz boundary.

In this paper, based on these two bilinear forms, we define the norms $\|\cdot\|_a$ and $\|\cdot\|_b$ as follows:

$$\|v\|_a^2 = a(v,v), \quad \|v\|_b^2 = b(v,v).$$

It is well known that $\|\cdot\|_a$ is a norm in the space V and $\|\cdot\|_b$ is a norm in the space $L^2(\Omega)$.

For the eigenvalue λ, there exists the Rayleigh quotient expression (see e.g. [1], [2], [26])

$$\lambda = \frac{a(u,u)}{b(u,u)}.$$

From [2], [5], we know (2.1) has an eigenvalue sequence $\{\lambda_j\}$:

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_k \leqslant \ldots, \quad \lim_{k\to\infty} \lambda_k = \infty,$$

and the associated eigenfunctions

$$u_1, u_2, \ldots, u_k, \ldots,$$

where $b(u_i, u_j) = \delta_{ij}$, and δ_{ij} is the Kronecker's delta. In the sequence $\{\lambda_j\}$, the λ_j are repeated according to their geometric multiplicity. In order to give the error estimates, let $M(\lambda_i)$ denote the eigenfunction space corresponding to the eigenvalue λ_i which is defined by

$$M(\lambda_i) = \{w \in V\colon\ w \text{ is an eigenfunction of (2.1) corresponding to } \lambda_i\}.$$

Now, let us define the finite element approximations of the problem (2.1). First we generate a shape-regular decomposition of the computing domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ into triangles or rectangles for $d = 2$ (tetrahedrons or hexahedrons for $d = 3$). The diameter of a cell $K \in \mathcal{T}_h$ is denoted by h_K. The mesh diameter h describes the maximum diameter of all cells $K \in \mathcal{T}_h$. Based on the mesh $\mathcal{T}_h$, we can construct the linear finite element space denoted by $V_h \subset V$. The finite element space V_h is assumed to satisfy the assumption

$$\lim_{h \to 0} \inf_{v_h \in V_h} \|w - v_h\|_a = 0, \quad \text{for any } w \in V. \tag{2.2}$$

The finite element approximation for (2.1) is defined as follows: Find $(\bar{\lambda}_h, \bar{u}_h) \in \mathbb{R} \times V_h$ such that $b(\bar{u}_h, \bar{u}_h) = 1$ and

$$a(\bar{u}_h, v_h) = \bar{\lambda}_h b(\bar{u}_h, v_h) \quad \forall\, v_h \in V_h. \tag{2.3}$$

From (2.3), we know the following Rayleigh quotient expression for $\bar{\lambda}_h$ holds (see e.g. [1], [2], [26]):

$$\bar{\lambda}_h = \frac{a(\bar{u}_h, \bar{u}_h)}{b(\bar{u}_h, \bar{u}_h)}.$$

Similarly, the eigenvalue problem (2.3) has eigenvalues (see e.g. [2], [5])

$$0 < \bar{\lambda}_{1,h} \leqslant \bar{\lambda}_{2,h} \leqslant \ldots \leqslant \bar{\lambda}_{k,h} \leqslant \ldots \leqslant \bar{\lambda}_{N_h,h},$$

and the corresponding eigenfunctions

$$\bar{u}_{1,h}, \bar{u}_{2,h}, \ldots, \bar{u}_{k,h}, \ldots, \bar{u}_{N_h,h},$$

where $b(\bar{u}_{i,h}, \bar{u}_{j,h}) = \delta_{ij}, 1 \leqslant i, j \leqslant N_h$ (N_h is the dimension of the finite element space V_h).

Due to the minimum-maximum principle (see e.g. [1], [2]), the following upper bound result holds

$$\lambda_i \leqslant \bar{\lambda}_{i,h}, \quad i = 1, 2, \ldots, N_h.$$

Similarly, let $M_h(\lambda_i)$ denote the approximate eigenfunction space corresponding to the eigenvalue λ_i which is defined by

$$M_h(\lambda_i) = \{w_h \in V_h \colon\ w_h \text{ is an eigenfunction of (2.3) corresponding to } \lambda_i\}.$$

According to [1], [2], each eigenvalue $\bar{\lambda}_{i,h}$ can be defined as follows

$$\bar{\lambda}_{i,h} = \inf_{\substack{v_h \in V_h \\ v_h \perp M_h(\lambda_j) \text{ for } \lambda_j < \lambda_i}} \frac{a(v_h, v_h)}{b(v_h, v_h)}. \tag{2.4}$$

In order to give error estimate results for the eigenvalue problems by the finite element method, we define

$$\delta_h(\lambda_i) = \sup_{\substack{w \in M(\lambda_i) \\ \|w\|_a = 1}} \inf_{v_h \in V_h} \|w - v_h\|_a, \tag{2.5}$$

and

$$\eta_a(h) = \sup_{\substack{f \in V \\ \|f\|_b = 1}} \inf_{v_h \in V_h} \|Tf - v_h\|_a, \tag{2.6}$$

where the operator $T\colon\ V' \to V$ is defined as

$$a(Tf, v) = b(f, v) \quad \forall\, f \in V' \text{ and } \forall\, v \in V.$$

The following error estimates for the eigenpair approximations by finite element method are known:

Proposition 2.1 ([1], Lemma 3.7, (3.29b), [2], p. 699 and [5])**.** (i) *For any eigenfunction approximation* $\bar{u}_{i,h}$ *of* (2.3) $(i = 1, 2, \ldots, N_h)$*, there is an eigenfunction* u_i *of* (2.1) *corresponding to* λ_i *such that* $\|u_i\|_b = 1$ *and*

$$\|u_i - \bar{u}_{i,h}\|_a \leqslant C\delta_h(\lambda_i). \tag{2.7}$$

Furthermore,

$$\|u_i - \bar{u}_{i,h}\|_b \leqslant C\eta_a(h)\|u_i - \bar{u}_{i,h}\|_a. \tag{2.8}$$

(ii) *For each eigenvalue we have*

$$\lambda_i \leqslant \bar{\lambda}_{i,h} \leqslant \lambda_i + C\delta_h^2(\lambda_i). \tag{2.9}$$

Here and hereafter C *is a constant depending on* λ_i *but independent of the mesh size* h*.*

3. Newton's method for eigenvalue problem

The aim of this section is to present a type of one Newton iteration step to improve the accuracy of the given eigenpair approximations. This iteration method only requires solving augmented linear problems in a finer finite element space. Here we only state the numerical method for the first and simple eigenvalue. In the next section, we will show the case of more eigenvalues.

For the analysis in this paper, we introduce the error expansion of the eigenvalue by the Rayleigh quotient formula which comes from [1], [2], [17], [26].

Lemma 3.1 ([1], Lemma 3.1)**.** *Assume $(\bar{\lambda}_h, \bar{u}_h)$ is a true solution of the eigenvalue problem* (2.3) *and $0 \neq \psi_h \in V_h$. Let us define*

$$\widehat{\lambda}_h = \frac{a(\psi_h, \psi_h)}{b(\psi_h, \psi_h)}.$$

Then we have

$$\widehat{\lambda}_h - \bar{\lambda}_h = \frac{a(\bar{u}_h - \psi_h, \bar{u}_h - \psi_h)}{b(\psi_h, \psi_h)} - \bar{\lambda}_h \frac{b(\bar{u}_h - \psi_h, \bar{u}_h - \psi_h)}{b(\psi_h, \psi_h)}.$$

3.1. Existence and uniqueness of solutions. This subsection introduces the main idea that deduces our numerical method. Here, we use Newton's method to solve the eigenproblem (2.1): Find $(\lambda, u) \in \mathbb{R} \times V$ such that

$$\begin{cases} a(u, v) - \lambda b(u, v) = 0 \quad \forall\, v \in V, \\ b(u, u) - 1 = 0. \end{cases} \tag{3.1}$$

If we have an eigenpair approximation (μ_0, u_0) with $b(u_0, u_0) = 1$, Newton's method for (3.1) is to find $(\widetilde{\lambda}, \widetilde{u}) \in \mathbb{R} \times V$ such that

$$\begin{cases} a(\widetilde{u} - u_0, v) - \mu_0 \cdot b(\widetilde{u} - u_0, v) - (\widetilde{\lambda} - \mu_0) b(u_0, v) \\ \qquad\qquad = -(a(u_0, v) - \mu_0 \cdot b(u_0, v)) \quad \forall\, v \in V, \\ -b(\widetilde{u} - u_0, u_0) = 0. \end{cases} \tag{3.2}$$

After simplifying (3.2), we have the following equation for the new eigenpair approximation $(\widetilde{\lambda}, \widetilde{u}) \in \mathbb{R} \times V$:

$$\begin{cases} a(\widetilde{u}, v) - \mu_0 \cdot b(\widetilde{u}, v) - \widetilde{\lambda} b(u_0, v) = -\mu_0 b(u_0, v) \quad \forall\, v \in V, \\ -b(\widetilde{u} - u_0, u_0) = 0. \end{cases} \tag{3.3}$$

Now, we come to proving that the mixed problem (3.3) has only one solution. To this aim, we define the bilinear forms

$$A_{\mu_0}(u,v) = a(u,v) - \mu_0 b(u,v), \quad B(v,\nu) = -\nu b(u_0,v), \tag{3.4}$$

where $u \in V$, $v \in V$, $\nu \in W = \mathbb{R}$ and $\mu_0 = a(u_0,u_0)/b(u_0,u_0)$.

Assume that $f \in V'$ and $g \in W'$. We consider the following mixed problem: Find $(u,\lambda) \in V \times W$ such that

$$\begin{cases} A_{\mu_0}(u,v) + B(v,\lambda) = f(v) & \forall\, v \in V, \\ B(u,\nu) = g(\nu) & \forall\, \nu \in W. \end{cases} \tag{3.5}$$

Concerning the existence and uniqueness of problem (3.5), the following theorem holds.

Theorem 3.1. *Assume u_0 is an eigenfunction approximation to $M(\lambda_1)$ with sufficiently small error and $\|u_0\|_b = 1$. Then the bilinear forms defined in* (3.4) *satisfy the following conditions:*

(1) *There exists $\alpha > 0$ (depending on $\lambda_2 - \lambda_1$) such that*

$$A_{\mu_0}(v,v) \geqslant \alpha \|v\|_a^2 \quad \forall\, v \in V_0, \tag{3.6}$$

where $V_0 = \{v\colon\ B(v,\nu) = 0 \quad \forall\, \nu \in W\} = \{v\colon\ b(u_0,v) = 0\}$.

(2) *There exists $\sigma > 0$ (depending on $1/\mu_0$) such that*

$$\sup_{v \in V} \frac{B(v,\nu)}{\|v\|_a} \geqslant \sigma |\nu| \quad \forall\, \nu \in W. \tag{3.7}$$

Based on these two conditions, the mixed equation (3.5) *has only one solution.*

P r o o f. We decompose u_0 as $u_0 = w_1 + w_1^{\perp}$ such that $w_1 \in M(\lambda_1)$ and $w_1^{\perp} \perp M(\lambda_1)$.

Since u_0 ($\|u_0\|_b = 1$) is an eigenfunction approximation to $M(\lambda_1)$ with sufficiently small error, there exists a small enough positive number δ such that

$$\|u_0 - w_1\|_a \leqslant \delta. \tag{3.8}$$

From Lemma 3.1, we also have

$$|\mu_0 - \lambda_1| \leqslant C\delta^2. \tag{3.9}$$

Since (3.8) and $\|u_0\|_b^2 = \|w_1\|_b^2 + \|w_1^\perp\|_b^2$ hold, $w_1^\perp$ and w_1 have estimates

$$\|w_1^\perp\|_b \leqslant C\|w_1^\perp\|_a \leqslant C\delta, \quad \|w_1\|_b \geqslant 1 - C\delta.$$

We also do the decomposition $v = v_1 + v_1^\perp$ with $v_1 \in M(\lambda_1)$ and $v_1^\perp \perp M(\lambda_1)$ for $v \in V_0$. Since $b(w_1 + w_1^\perp, v_1 + v_1^\perp) = 0$, the following inequality holds:

$$\|v_1\|_b\|w_1\|_b = |b(v_1, w_1)| = |-b(v_1^\perp, w_1^\perp)| = |-b(v, w_1^\perp)| \leqslant C\delta\|v\|_b.$$

Then $\|v_1\|_b$ has the estimate

$$\|v_1\|_b \leqslant \frac{C\delta}{1 - C\delta}\|v\|_b \leqslant C\delta\|v\|_b. \tag{3.10}$$

From (3.10) and the property $\|v\|_b^2 = \|v_1\|_b^2 + \|v_1^\perp\|_b^2$, we obtain the estimates

$$\begin{aligned} b(v, v) = b(v_1, v_1) + b(v_1^\perp, v_1^\perp) &\leqslant C\delta^2 b(v, v) + \frac{1}{\lambda_2}a(v_1^\perp, v_1^\perp) \\ &\leqslant C\delta^2 b(v, v) + \frac{1}{\lambda_2}a(v, v). \end{aligned}$$

Thus we have the inequality

$$b(v, v) \leqslant \frac{1}{\lambda_2(1 - C\delta^2)}a(v, v). \tag{3.11}$$

By virtue of (3.9), (3.11) and the definition of $A_\mu(\cdot, \cdot)$, the following inequalities hold:

$$\begin{aligned} a(v, v) - \mu_0 b(v, v) &\geqslant \Big(1 - \frac{\mu_0}{\lambda_2(1 - C\delta^2)}\Big)a(v, v) \geqslant \frac{\lambda_2(1 - C\delta^2) - \mu_0}{\lambda_2(1 - C\delta^2)}a(v, v) \\ &\geqslant \frac{\lambda_2 - \lambda_1 - C\delta^2}{\lambda_2(1 - C\delta^2)}a(v, v). \end{aligned}$$

This means that (3.6) holds for $\alpha = (\lambda_2 - \lambda_1 - C\delta^2)/(\lambda_2(1 - C\delta^2))$ when δ is small enough.

Now, we come to proving (3.7). From the definitions of $B(\cdot, \cdot)$ and μ, we have

$$\sup_{v \in V} \frac{B(v, \nu)}{\|v\|_a} \geqslant |\nu| \frac{b(u_0, u_0)}{\|u_0\|_a} = \frac{|\nu|}{\mu_0} \quad \forall\, \nu \in W.$$

It means that (3.7) holds for

$$\sigma = \frac{1}{\mu_0}.$$

According to the theory for the mixed finite element method [4], there exists only one solution for equation (3.5). □

Corollary 3.1. *Under the conditions of Theorem* 3.1*, the inequality*

$$\|w\|_a + |\gamma| \leqslant C_4 \sup_{0\neq(v,\nu)\in V\times W} \frac{A_{\mu_0}(w,v) + B(v,\gamma) + B(w,\nu)}{\|v\|_a + |\nu|}, \tag{3.12}$$

holds for any $(w,\gamma) \in V \times W$*. The constant* C_4 *depends on* $1/(\lambda_2 - \lambda_1)$*,* λ_1 *and* λ_2 *as*

$$C_4 = C\Big(\lambda_1 + \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\Big). \tag{3.13}$$

3.2. One Newton iteration step. Based on previous discussion, we propose a one correction step to improve the given eigenpair approximation. Assume we have obtained an eigenpair approximation $(\lambda_{1,h_k}, u_{1,h_k}) \in \mathbb{R}\times V_{h_k}$ with $\|u_{1,h_k}\|_b = 1$. Now we introduce a type of iteration step to improve the accuracy of the current eigenpair approximation $(\lambda_{1,h_k}, u_{1,h_k})$. Let $V_{h_{k+1}} \subset V$ be a finer finite element space such that $V_{h_k} \subset V_{h_{k+1}}$. Based on this finer finite element space, we define the following one Newton iteration step.

A l g o r i t h m 3.1. One Newton Iteration Step

(1) Solve the augmented mixed problem: Find $(\widehat{\lambda}_{1,h_{k+1}}, \widehat{u}_{1,h_{k+1}}) \in \mathbb{R} \times V_{h_{k+1}}$ such that

$$\begin{cases} a(\widehat{u}_{1,h_{k+1}}, v_{h_{k+1}}) - \lambda_{1,h_k} b(\widehat{u}_{1,h_{k+1}}, v_{h_{k+1}}) - \widehat{\lambda}_{1,h_{k+1}} b(u_{1,h_k}, v_{h_{k+1}}) \\ \qquad\qquad = -\lambda_{1,h_k} b(u_{1,h_k}, v_{h_{k+1}}) \quad \forall\, v_{h_{k+1}} \in V_{h_{k+1}}, \\ b(\widehat{u}_{1,h_{k+1}}, u_{1,h_k}) = b(u_{1,h_k}, u_{1,h_k}). \end{cases} \tag{3.14}$$

(2) Do the normalization for $\widehat{u}_{1,h_{k+1}}$ as

$$u_{1,h_{k+1}} = \frac{\widehat{u}_{1,h_{k+1}}}{\|\widehat{u}_{1,h_{k+1}}\|_b} \tag{3.15}$$

and compute the Rayleigh quotient for $u_{1,h_{k+1}}$

$$\lambda_{1,h_{k+1}} = \frac{a(u_{1,h_{k+1}}, u_{1,h_{k+1}})}{b(u_{1,h_{k+1}}, u_{1,h_{k+1}})}. \tag{3.16}$$

Then we obtain a new eigenpair approximation $(\lambda_{1,h_{k+1}}, u_{1,h_{k+1}}) \in \mathbb{R} \times V_{h_{k+1}}$. Summarize the above two steps into

$$(\lambda_{1,h_{k+1}}, u_{1,h_{k+1}}) = \texttt{Newton\_Iteration}(\lambda_{1,h_k}, u_{1,h_k}, V_{h_{k+1}}).$$

Theorem 3.2. *Assume $(\lambda_{1,h_k}, u_{1,h_k})$ is a good enough approximation to (λ_1, u_1) such that (3.6), (3.7) hold and $\lambda_{1,h_k} = a(u_{1,h_k}, u_{1,h_k})/b(u_{1,h_k}, u_{1,h_k})$. After one iteration step, the resulting approximation $(\lambda_{1,h_{k+1}}, u_{1,h_{k+1}}) \in \mathbb{R} \times V_{h_{k+1}}$ has the following error estimates*

$$\|\bar{u}_{1,h_{k+1}} - u_{1,h_{k+1}}\|_a \leqslant C_5 \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_a^2, \tag{3.17}$$

$$|\bar{\lambda}_{1,h_{k+1}} - \lambda_{1,h_{k+1}}| \leqslant C_6 \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_a^4, \tag{3.18}$$

where C_5 and C_6 are constants which depend on $1/(\lambda_2 - \lambda_1)$, λ_1 and λ_2 (similarly to (3.13)) but are independent of the mesh sizes h_k and h_{k+1}.

P r o o f. From the definition (2.3), we know that the eigenpair approximation $(\bar{\lambda}_{1,h_{k+1}}, \bar{u}_{1,h_{k+1}})$ satisfies the equations

$$\begin{cases} a(\bar{u}_{1,h_{k+1}}, v_{h_{k+1}}) - \lambda_{1,h_k} b(\bar{u}_{1,h_{k+1}}, v_{h_{k+1}}) - \bar{\lambda}_{1,h_{k+1}} b(u_{1,h_k}, v_{h_{k+1}}) \\ \qquad = (\bar{\lambda}_{1,h_{k+1}} - \lambda_{1,h_k}) b(\bar{u}_{1,h_{k+1}}, v_{h_{k+1}}) \\ \qquad\quad - \bar{\lambda}_{1,h_{k+1}} b(u_{1,h_k}, v_{h_{k+1}}) \quad \forall\, v_{h_{k+1}} \in V_{h_{k+1}}, \\ b(\bar{u}_{1,h_{k+1}}, u_{1,h_k}) = b(\bar{u}_{1,h_{k+1}}, u_{1,h_k}). \end{cases} \tag{3.19}$$

Let us define $w_{h_{k+1}} := \bar{u}_{1,h_{k+1}} - \widehat{u}_{1,h_{k+1}}$ and $\gamma := \bar{\lambda}_{1,h_{k+1}} - \widehat{\lambda}_{1,h_{k+1}}$. Due to (3.14) and (3.19), the following equations hold:

$$\begin{cases} a(w_{h_{k+1}}, v_{h_{k+1}}) - \lambda_{1,h_k} b(w_{h_{k+1}}, v_{h_{k+1}}) - \gamma b(u_{1,h_k}, v_{h_{k+1}}) \\ \qquad = (\bar{\lambda}_{1,h_{k+1}} - \lambda_{1,h_k}) b(\bar{u}_{1,h_{k+1}} - u_{1,h_k}, v_{h_{k+1}}) \quad \forall\, v_{h_{k+1}} \in V_{h_{k+1}}, \\ \nu b(w_{h_{k+1}}, u_{1,h_k}) = \nu b(\bar{u}_{1,h_{k+1}} - u_{1,h_k}, u_{1,h_k}) \\ \qquad = -\frac{1}{2} \nu b(\bar{u}_{1,h_{k+1}} - u_{1,h_k}, \bar{u}_{1,h_{k+1}} - u_{1,h_k}) \quad \forall\, \nu \in W. \end{cases} \tag{3.20}$$

Then combining Lemma 3.1, Corollary 3.1, (3.20) and

$$\|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_b \lesssim \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_a,$$

we obtain the inequalities

$$\begin{aligned} \|w_{h_{k+1}}\|_a + |\gamma| &\lesssim |\bar{\lambda}_{1,h_{k+1}} - \lambda_{1,h_k}| \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_b + \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_b^2 \\ &\lesssim \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_a^2, \end{aligned} \tag{3.21}$$

where the hidden constant depends on $1/(\lambda_2 - \lambda_1)$, λ_1 and λ_2 as in (3.13).

The inequality (3.21) means the following estimate holds:

$$\|\bar{u}_{1,h_{k+1}} - \widehat{u}_{1,h_{k+1}}\|_a \lesssim \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_a^2. \tag{3.22}$$

Combining the above inequality (3.22), the definition (3.15), $\|\bar{u}_{1,h_{k+1}}\|_b = 1$, and $\|\widehat{u}_{1,h_{k+1}}\|_b \geqslant \|\bar{u}_{1,h_{k+1}}\|_b - \|\bar{u}_{1,h_{k+1}} - \widehat{u}_{1,h_k}\|_b$ having a lower bound greater than zero, we have the inequalities

$$\begin{aligned}
&\|\bar{u}_{1,h_{k+1}} - u_{1,h_{k+1}}\|_a \\
&\quad\leqslant \left\|\bar{u}_{1,h_{k+1}} - \frac{\bar{u}_{1,h_{k+1}}}{\|\widehat{u}_{1,h_{k+1}}\|_b}\right\|_a + \frac{\|\bar{u}_{1,h_{k+1}} - \widehat{u}_{1,h_{k+1}}\|_a}{\|\widehat{u}_{1,h_{k+1}}\|_b} \\
&\quad\leqslant \frac{\|\bar{u}_{1,h_{k+1}}\|_a}{\|\widehat{u}_{1,h_{k+1}}\|_b}|\|\widehat{u}_{1,h_{k+1}}\|_b - \|\bar{u}_{1,h_{k+1}}\|_b| + \frac{\|\bar{u}_{1,h_{k+1}} - \widehat{u}_{1,h_{k+1}}\|_a}{\|\widehat{u}_{1,h_{k+1}}\|_b} \\
&\quad\leqslant \frac{\|\bar{u}_{1,h_{k+1}}\|_a}{\|\widehat{u}_{1,h_{k+1}}\|_b}\|\bar{u}_{1,h_{k+1}} - \widehat{u}_{1,h_{k+1}}\|_b + \frac{\|\bar{u}_{1,h_{k+1}} - \widehat{u}_{1,h_{k+1}}\|_a}{\|\widehat{u}_{1,h_{k+1}}\|_b} \\
&\quad\lesssim \|\bar{u}_{1,h_{k+1}} - \widehat{u}_{1,h_{k+1}}\|_a \lesssim \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_a^2.
\end{aligned}$$

This is the desired result (3.17). Furthermore, from (3.17) and Lemma 3.1, the other desired result (3.18) can be obtained easily and the proof is complete. □

R e m a r k 3.1. Theorem 3.2 shows that Newton's method has second order convergence rate when the initial approximation has enough accuracy. We would like to point out that Theorem 3.2 and its proof also give the analysis for the algebraic eigenvalue problems by Newton's method.

4. Multilevel iteration method

In this section, we introduce a type of multilevel scheme based on the *One Newton Iteration Step* defined by Algorithm 3.1. The proposed multilevel method can obtain eigenpair approximation with the optimal accuracy and with much smaller computational work compared with solving the eigenvalue problem directly in the finest finite element space.

Before introducing the multigrid scheme, we define a sequence of triangulations $\mathcal{T}_{h_k}$ of Ω. Suppose $\mathcal{T}_{h_1}$ is given and let $\mathcal{T}_{h_k}$ be obtained from $\mathcal{T}_{h_{k-1}}$ via regular refinement (produce β^d subelements) such that

$$h_k = \frac{1}{\beta}h_{k-1}.$$

Based on this sequence of meshes, we construct the corresponding nested linear finite element spaces such that

$$V_{h_1} \subset V_{h_2} \subset \ldots \subset V_{h_n}, \tag{4.1}$$

and the following relation of approximation errors holds:

$$\frac{1}{\beta}\eta_a(h_{k-1}) \leqslant C_7\eta_a(h_k), \quad \frac{1}{\beta}\delta_{h_{k-1}}(\lambda) \leqslant C_7\delta_{h_k}(\lambda), \quad k = 2, \ldots, n. \tag{4.2}$$

From the error estimate results in Proposition 2.1, we have

$$\|\bar{u}_{1,h_k} - \bar{u}_{1,h_{k+1}}\|_a \leqslant C_8\delta_{h_k}(\lambda_1), \quad k = 1, \ldots, n-1, \tag{4.3}$$

where the constant C_8 is a constant independent of the mesh size h_k.

Algorithm 4.1. Multilevel Eigenvalue Iteration Scheme

(1) Construct a sequence of nested finite element spaces $V_{h_1}, V_{h_2}, \ldots, V_{h_n}$ such that (4.1) and (4.2) hold.

(2) Solve the following eigenvalue problem: Find $(\lambda_{1,h_1}, u_{1,h_1}) \in \mathbb{R} \times V_{h_1}$ such that $b(u_{1,h_1}, u_{1,h_1}) = 1$ and

$$a(u_{1,h_1}, v_{h_1}) = \lambda_{1,h_1} b(u_{1,h_1}, v_{h_1}) \quad \forall\, v_{h_1} \in V_{h_1}. \tag{4.4}$$

(3) Do $k = 1, \ldots, n-1$

Obtain a new eigenpair approximation $(\lambda_{1,h_{k+1}}, u_{1,h_{k+1}}) \in \mathbb{R} \times V_{h_{k+1}}$ by a Newton iteration step

$$(\lambda_{1,h_{k+1}}, u_{1,h_{k+1}}) = \texttt{Newton\_Iteration}(\lambda_{1,k}, u_{1,h_k}, V_{h_{k+1}}). \tag{4.5}$$

End do

Finally, we obtain an eigenpair approximation $(\lambda_{1,h_n}, u_{1,h_n}) \in \mathbb{R} \times V_{h_n}$.

Theorem 4.1. *Assume h_1 is small enough such that $(\lambda_{1,h_1}, u_{1,h_1})$ satisfies conditions* (3.6) *and* (3.7). *After implementing Algorithm* 4.1, *the resulting eigenpair approximation $(\lambda_{1,h_n}, u_{1,h_n})$ has the error estimates*

$$\|u_{1,h_n} - \bar{u}_{1,h_n}\|_a \leqslant \delta_{h_n}(\lambda_1), \tag{4.6}$$

$$|\lambda_{1,h_n} - \bar{\lambda}_{1,h_n}| \leqslant C_9\delta^2_{h_n}(\lambda_1), \tag{4.7}$$

when the mesh size h_1 is small enough.

Besides, there exists an eigenfunction u_1 of (2.1) *corresponding to λ_1 such that the following final convergence results hold:*

$$\|u_1 - u_{1,h_n}\|_a \leqslant 2\delta_{h_n}(\lambda_1), \tag{4.8}$$

$$|\lambda_1 - \lambda_{1,h_n}| \leqslant 2C_{10}\delta^2_{h_n}(\lambda_1). \tag{4.9}$$

P r o o f. Let us prove (4.6) by the method of induction. First, it is obvious that (4.6) holds for $n = 1$ according to (4.4). Then we assume that (4.6) holds for $n = k$. It means we have the estimate

$$\|\bar{u}_{1,h_k} - u_{1,h_k}\|_a \leqslant \delta_{h_k}(\lambda_1). \tag{4.10}$$

Now let us consider the case of $n = k + 1$. Combining (4.3), (4.10) and the triangle inequality leads to the estimates

$$\begin{aligned} \|\bar{u}_{1,h_{k+1}} - u_{1,h_{k+1}}\|_a &\leqslant C_5\|u_{1,h_k} - \bar{u}_{1,h_{k+1}}\|_a^2 \\ &\leqslant 2C_5\|u_{1,h_k} - \bar{u}_{1,h_k}\|_a^2 + 2C_5\|\bar{u}_{1,h_k} - \bar{u}_{1,h_{k+1}}\|_a^2 \\ &\leqslant 2C_5\delta_{h_k}^2(\lambda_1) + 2C_5C_8^2\delta_{h_k}^2(\lambda_1) \\ &\leqslant 2C_5(1 + C_8^2)\delta_{h_k}^2(\lambda_1) \\ &= 2\beta C_5(1 + C_8^2)\delta_{h_k}(\lambda_1)\delta_{h_k}(\lambda_1)/\beta \\ &\leqslant 2\beta C_5C_7\big(1 + C_8^2\big)\delta_{h_k}(\lambda_1)\delta_{h_{k+1}}(\lambda_1). \end{aligned} \tag{4.11}$$

This means that the result (4.6) also holds for $n = k+1$ if $2\beta C_5C_7(1+C_8^2)\delta_{h_k}(\lambda_1) < 1$. Thus we prove the desired result (4.6). From Lemma 3.1 and (4.6), we can obtain the desired result (4.7). Finally, (4.8) and (4.9) can be proved from (2.7), (2.9), (4.6), (4.7) and the triangle inequality. □

Now, we turn to extending Newton's iteration (3.2) for solving one eigenvalue to the corresponding version for more eigenvalues (including simple and semisimple eigenvalues). Assume that $\lambda_m < \lambda_{m+1}$ and we have obtained the first m eigenpairs approximations $\{(\mu_j, u_{0,j})\}_{j=1}^m$ to the problem (3.1), which satisfy

$$b(u_{0,i}, u_{0,j}) = \delta_{ij}, \quad i, j = 1, \ldots, m,$$

where μ_j is the Rayleigh quotient of $u_{0,j}$.

Similarly to the one eigenvalue case, Newton's method for more eigenvalues of (3.1) is to find $(x_j, \widetilde{u}_j) \in \mathbb{R}^m \times V$ $(j = 1, \ldots, m)$ such that

$$\begin{cases} a(\widetilde{u}_j, v) - \mu_j \cdot b(\widetilde{u}_j, v) - \sum\limits_{i=1}^m x_{ij} b(u_{0,i}, v) = -\mu_j b(u_{0,j}, v) \quad \forall\, v \in V, \\ b(\widetilde{u}_j, u_{0,i}) = b(u_{0,j}, u_{0,i}) \quad \forall\, i = 1, \ldots, m, \end{cases} \tag{4.12}$$

where x_{ij} is the ith component of x_j.

Now, we come to proving (4.12) has only one solution for any $j = 1, \ldots, m$. To this aim, we define the bilinear forms

$$A_{\mu_j}(u, v) = a(u, v) - \mu_j b(u, v), \quad B(v, y) = -\sum_{i=1}^m y_i b(u_{0,i}, v). \tag{4.13}$$

Here and hereafter in this section $u \in V$, $v \in V$, $y \in W = \mathbb{R}^m$.

Assume that $f_{\mu_j} \in V'$, $g_j \in W'$ are defined as

$$f_{\mu_j}(v) = -\mu_j b(u_{0,j}, v), \quad g_j(y) = -\sum_{i=1}^{m} y_i b(u_{0,i}, u_{0,j}).$$

We consider the following mixed problems: Find $(x_j, \widetilde{u}_j) \in \mathbb{R}^m \times V$, $(j = 1, \ldots, m)$, such that

$$\begin{cases} A_{\mu_j}(\widetilde{u}_j, v) + B(v, x) = f_{\mu_j}(v) \quad \forall\, v \in V, \\ B(\widetilde{u}_j, y) = g_j(y) \quad \forall\, y \in W. \end{cases} \tag{4.14}$$

Define $\mathcal{K} = M(\lambda_1) \cup \ldots \cup M(\lambda_m)$. About the existence and uniqueness of problem (4.14), the following theorem holds.

Theorem 4.2. *Assume that there exists a decomposition of eigenspace $\mathcal{K}$ satisfying $\mathcal{K} = M(\lambda_1) \oplus \ldots \oplus M(\lambda_m)$ such that $u_{0,j}$ is an eigenfunction approximation to $M(\lambda_j)$ $(j = 1, \ldots, m)$. Then the bilinear forms defined in* (4.13) *satisfy the following conditions:*

(1) *There exists $\alpha > 0$ such that*

$$A_{\mu_j}(v, v) \geqslant \alpha \|v\|_a^2 \quad \forall\, v \in V_0, \tag{4.15}$$

where $V_0 = \{v\colon\ B(v, y) = 0$ for all $y \in W\} = \{v\colon\ b(u_{0,i}, v) = 0$ for all $i = 1, \ldots, m\}$.

(2) *There exists $\sigma > 0$ such that*

$$\sup_{v \in V} \frac{B(v, y)}{\|v\|_a} \geqslant \sigma \|y\| \quad \forall\, y \in W, \tag{4.16}$$

where $\|y\| := \max\limits_{i \in \{1, \ldots, m\}} |y_i|$.

Based on these two conditions, for any j $(j = 1, \ldots, m)$, the mixed equation (4.14) *has only one solution.*

P r o o f. We decompose $u_{0,j}$ as $u_{0,j} = w_{0,j} + w_{0,j}^{\perp}$ such that $w_{0,j} \in M(\lambda_j)$ and $w_{0,j}^{\perp} \perp_b w_{0,j}$. Then $\operatorname{span}\{w_{0,1}, \ldots, w_{0,m}\}$ is an orthogonal basis of the eigenspace $\mathcal{K}$.

Since $u_{0,j}$ $(\|u_{0,j}\|_b = 1)$ is an eigenfunction approximation to $M(\lambda_j)$ with sufficiently small error, there is a small enough number δ such that

$$\|u_{0,j} - w_{0,j}\|_a = \|w_{0,j}^{\perp}\|_a \leqslant \delta, \quad u_{0,j} - w_{0,j} \perp_b \operatorname{span}\{w_{0,j}\}, \quad j = 1, \ldots, m. \tag{4.17}$$

From Lemma 3.1, we also have

$$|\mu_j - \lambda_j| \leqslant C\delta^2, \quad j = 1, \ldots, m. \tag{4.18}$$

Since (4.17) and $\|u_{0,j}\|_b^2 = \|w_{0,j}\|_b^2 + \|w_{0,j}^\perp\|_b^2$, $w_{0,j}^\perp$ and $w_{0,j}$ have estimates

$$\|w_{0,j}^\perp\|_b \leqslant C\|w_{0,j}^\perp\|_a \leqslant C\delta, \quad \|w_{0,j}\|_b \geqslant 1 - C\delta, \quad j = 1, \ldots, m.$$

Similarly, we also decomposite $v \in V_0$ as

$$v = v_1 + \ldots + v_m + v^* = v_j + v_j^\perp, \quad j = 1, \ldots, m,$$

satisfying

$$v^* \perp_b \mathcal{K}, \quad v_j \in \operatorname{span}\{w_{0,j}\}, \quad v_j^\perp = \sum_{i=1, i\neq j}^{m} v_i + v^*, \quad v_j^\perp \perp_b \operatorname{span}\{w_{0,j}\}.$$

According to the definition of $v \in V_0$, i.e., $b(w_{0,j} + w_{0,j}^\perp, v_j + v_j^\perp) = 0$, we have

$$\begin{aligned}\|v_j\|_b\|w_{0,j}\|_b = |b(v_j, w_{0,j})| &= |-b(v_j^\perp, w_{0,j}^\perp)| = |b(v, w_{0,j}^\perp)| \\ &\leqslant C\delta\|v\|_b, \quad j = 1, \ldots, m.\end{aligned}$$

Therefore,

$$\|v_j\|_b \leqslant \frac{C\delta}{1 - C\delta}\|v\|_b \leqslant C\delta\|v\|_b, \quad j = 1, \ldots, m. \tag{4.19}$$

From (4.19) and the property $\|v\|_b^2 = \|v_1\|_b^2 + \ldots + \|v_m\|_b^2 + \|v^*\|_b^2$, the following estimates hold:

$$\begin{aligned}b(v, v) &= b(v_1, v_1) + \ldots + b(v_m, v_m) + b(v^*, v^*) \\ &\leqslant mC\delta^2 b(v, v) + \frac{1}{\lambda_{m+1}}a(v^*, v^*) \leqslant mC\delta^2 b(v, v) + \frac{1}{\lambda_{m+1}}a(v, v).\end{aligned}$$

Thus we have the inequality

$$b(v, v) \leqslant \frac{1}{\lambda_{m+1}(1 - mC\delta^2)}a(v, v). \tag{4.20}$$

From (4.18), (4.20) and the definition of $A_{\mu_j}(\cdot, \cdot)$, the following inequalities hold:

$$\begin{aligned}a(v, v) - \mu_j b(v, v) &\geqslant \Big(1 - \frac{\mu_j}{\lambda_{m+1}(1 - mC\delta^2)}\Big)a(v, v) \\ &\geqslant \frac{\lambda_{m+1}(1 - mC\delta^2) - \mu_j}{\lambda_{m+1}(1 - mC\delta^2)}a(v, v) \\ &\geqslant \frac{\lambda_{m+1} - \lambda_j - C\delta^2}{\lambda_{m+1}(1 - mC\delta^2)}a(v, v).\end{aligned}$$

It means (4.15) holds for $\alpha = (\lambda_{m+1} - \lambda_j - C\delta^2)/\big(\lambda_{m+1}(1 - mC\delta^2)\big) > 0$ $(j = 1, \ldots, m)$ when δ is small enough.

Now, we come to proving (4.16). Assume that the index s satisfies $\|y\| = |y_s|$. From $b(u_{0,i}, u_{0,j}) = \delta_{ij}$ $(i, j = 1, \ldots, m)$ and the definition of $B(\cdot, \cdot)$ and μ_j, taking $v = -\mathrm{sign}(y_s)u_{0,s}$, we have

$$\sup_{v \in V} \frac{B(v, y)}{\|v\|_a} \geqslant \frac{|y_s| b(u_{0,s}, u_{0,s})}{\|u_{0,s}\|_a} = \frac{\|y\|}{\mu_s} \geqslant \frac{\|y\|}{\mu} > 0 \quad \forall\, y \in W,$$

where $\mu = \max\limits_{t \in \{1,2,\ldots,m\}} \{\mu_t\}$. This means that (4.16) holds for

$$\sigma = \frac{1}{\mu}.$$

According to the theory for the mixed finite element method [4], there exists only one solution for the equations (4.14) for any $j = 1, \ldots, m$. □

Based on the previous discussion, we extend the one iteration step to improve the given approximations to the first m eigenpairs. Assume we have obtained the first m eigenpairs approximations $(\lambda_{i,h_k}, u_{i,h_k}) \in \mathbb{R} \times V_{h_k}$ with $\|u_{i,h_k}\|_b = 1$ $(i = 1, \ldots, m)$. Now we introduce a type of iteration step to improve the accuracy of the current eigenpair approximation $(\lambda_{i,h_k}, u_{i,h_k})$. Let $V_{h_{k+1}} \subset V$ be a finer finite element space such that $V_{h_k} \subset V_{h_{k+1}}$. Based on this finer finite element space, we define the following one Newton iteration step for m eigenvalues and then state the following version of *Multilevel Eigenvalue Iteration Scheme* for m eigenvalues.

Similarly, we first give a type of *One Iteration Step for m Eigenvalues* for the given eigenpair approximations $\{\lambda_{i,h_k}, u_{i,h_k}\}_{i=1}^m$.

A l g o r i t h m 4.2. One Newton Iteration Step for m Eigenvalues

(1) Do $i = 1, \ldots, m$

Find $(x_{i,h_{k+1}}, \widetilde{u}_{i,h_{k+1}}) \in \mathbb{R} \times V_{h_{k+1}}$ such that

$$\text{(4.21)} \quad \begin{cases} a(\widetilde{u}_{i,h_{k+1}}, v_{h_{k+1}}) - \lambda_{i,h_k} b(\widetilde{u}_{i,h_{k+1}}, v_{h_{k+1}}) - \sum\limits_{s=1}^{m} x_{si,h_{k+1}} b(u_{s,h_k}, v_{h_{k+1}}) \\ \qquad = -\lambda_{i,h_k} b(u_{i,h_k}, v_{h_{k+1}}) \quad \forall\, v_{h_{k+1}} \in V_{h_{k+1}}, \\ b(\widetilde{u}_{i,h_{k+1}}, u_{j,h_k}) = \delta_{ij} \quad \forall\, j = 1, \ldots, m, \end{cases}$$

where $x_{si,h_{k+1}}$ is the sth component of $x_{i,h_{k+1}}$.

End Do

(2) Build the finite dimensional space $\widetilde{V}_{h_{k+1}} = \mathrm{span}\{\widetilde{u}_{1,h_{k+1}}, \ldots, \widetilde{u}_{m,h_{k+1}}\}$ and solve the following eigenvalue problem: Find $(\lambda_{i,h_{k+1}}, u_{i,h_{k+1}}) \in \mathbb{R} \times \widetilde{V}_{h_{k+1}}$, $i = 1, 2, \ldots, m$, such that $b(u_{i,h_{k+1}}, u_{i,h_{k+1}}) = 1$ and

$$a(u_{i,h_{k+1}}, v_{h_{k+1}}) = \lambda_{i,h_{k+1}} b(u_{i,h_{k+1}}, v_{h_{k+1}}) \quad \forall\, v_{h_{k+1}} \in \widetilde{V}_{h_{k+1}}.$$

We summarize the above two steps into

$$\{\lambda_{i,h_{k+1}}, u_{i,h_{k+1}}\}_{i=1}^{m} = \texttt{Newton\_Iteration}(\{\lambda_{i,h_k}, u_{i,h_k}\}_{i=1}^{m}, V_{h_{k+1}}).$$

Based on Algorithm 4.2, we come to state the corresponding multilevel correction method.

Algorithm 4.3. Multilevel Eigenvalue Iteration Scheme for m Eigenvalues

(1) Construct a series of nested finite element spaces $V_{h_1}, V_{h_2}, \ldots, V_{h_n}$ such that (4.1) and (4.2) hold.

(2) Solve the eigenvalue problem in the initial finite element space V_{h_1}: Find $(\lambda_{h_1}, u_{h_1}) \in \mathbb{R} \times V_{h_1}$ such that $b(u_{h_1}, u_{h_1}) = 1$ and

$$a(u_{i,h_1}, v_{h_1}) = \lambda_{i,h_1} b(u_{i,h_1}, v_{h_1}) \quad \forall\, v_{h_1} \in V_{h_1}.$$

Choose the first m eigenpairs $\{\lambda_{i,h_1}, u_{i,h_1}\}_{i=1}^{m}$ which approximate the desired eigenpairs.

(3) Do $k = 1, \ldots, n-1$

Obtain new eigenpair approximations $\{\lambda_{i,h_{k+1}}, u_{i,h_{k+1}}\}_{i=1}^{m} \in \mathbb{R} \times V_{h_{k+1}}$ by the one Newton iteration step defined in Algorithm 4.2

$$\{\lambda_{i,h_{k+1}}, u_{i,h_{k+1}}\}_{i=1}^{m} = \texttt{Newton\_Iteration}(\{\lambda_{i,h_k}, u_{i,h_k}\}_{i=1}^{m}, V_{h_{k+1}}).$$

End do

Finally, we obtain m eigenpair approximations $\{\lambda_{i,h_n}, u_{i,h_n}\}_{i=1}^{m} \in \mathbb{R} \times V_{h_n}$.

In Algorithm 4.2, the parallel computation can be used to solve (4.21) for different i. Similarly to Theorems 3.2 and 4.1, the analysis of the scheme for m eigenvalues will be given in our future work.

5. Numerical results

In this section, three numerical examples are presented to illustrate the efficiency of the multilevel iteration scheme proposed in this paper.

5.1. Model eigenvalue problem. Here we give the numerical results of the multilevel iteration scheme for the Laplace eigenvalue problem on the two dimensional domain $\Omega = (0,1) \times (0,1)$. The sequence of finite element spaces is constructed by using linear element on the series of meshes which are produced by the regular refinement with $\beta = 2$ (producing β^2 subelements). In this example, we use two meshes

which are generated by the Delaunay method as the initial mesh $\mathcal{T}_{h_1}$ ($H = h_1$) to produce two sequences of finite element spaces for investigating the convergence behavior. Figure 1 shows the corresponding initial meshes: one is coarse and the other is fine.

Algorithm 4.1 is applied to solve the eigenvalue problem. For comparison, we also solve the eigenvalue problem by the direct method.

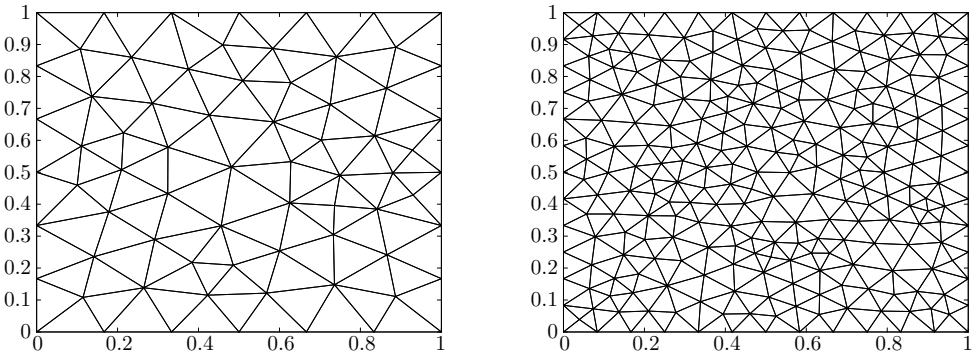

Figure 1. The initial coarse $H = \frac{1}{6}$ and fine $H = \frac{1}{12}$ meshes for Example 1.

Figure 2 gives the corresponding numerical results for the first eigenvalue $\lambda_1 = 2\pi^2$ and the corresponding eigenfunction on the two initial meshes illustrated in Figure 1. From Figure 2, we find the multilevel iteration scheme can obtain the same optimal error estimates as the direct eigenvalue solving method for the eigenvalue and the corresponding eigenfunction approximations.

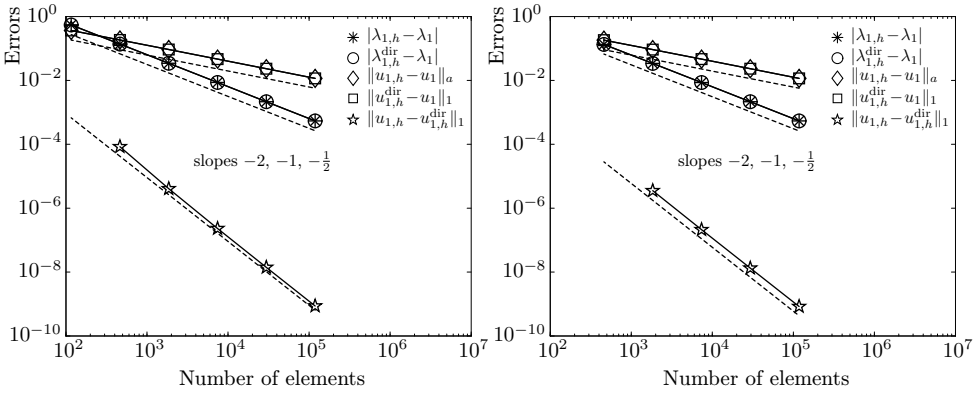

Figure 2. The errors of the multilevel iteration algorithm for the first eigenvalue $2\pi^2$ and the corresponding eigenfunction, where $u_h^{\rm dir}$ and $\lambda_h^{\rm dir}$ denote respectively the eigenfunction and eigenvalue approximation by direct eigenvalue solving (the left subfigure is for the coarse initial mesh to the left of Figure 1 and the right one for the fine initial mesh to the right of Figure 1).

We also check the convergence behavior for more eigenvalue approximations by Algorithm 4.1. Here the first six eigenvalues $\lambda = 2\pi^2, 5\pi^2, 5\pi^2, 8\pi^2, 10\pi^2, 10\pi^2$ are investigated. We adopt the meshes in Figure 1 as the initial ones and the corresponding numerical results are shown in Figure 3. Figure 3 also exhibits the optimal convergence rate of the multilevel iteration scheme.

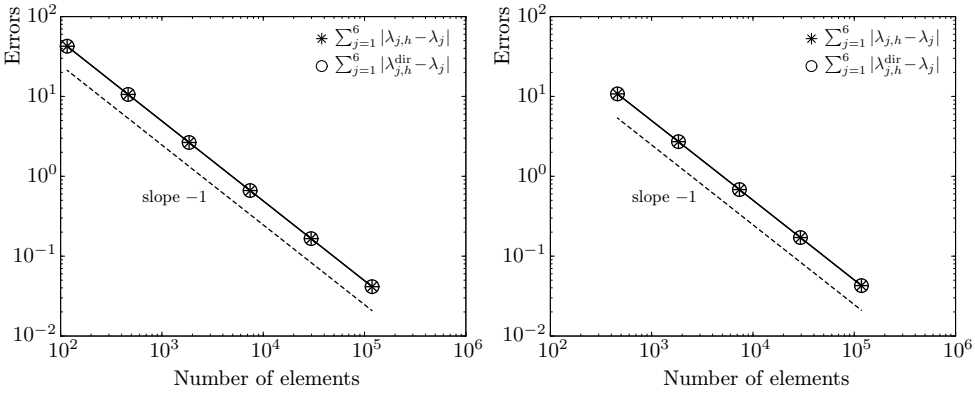

Figure 3. The errors of the multilevel iteration algorithm for the first six eigenvalues on the unit square (the left subfigure is for the coarse initial mesh to the left of Figure 1 and the right one for the fine initial mesh to the right of Figure 1).

5.2. More general eigenvalue problem. Here we give the numerical results of the multilevel iteration scheme for solving a more general eigenvalue problem on the unit square domain $\Omega = (0,1) \times (0,1)$: Find (λ, u) such that

$$\begin{cases} -\nabla \cdot \mathcal{A}\nabla u + \varphi u = \lambda \varrho u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \int_\Omega \varrho u^2 \, \mathrm{d}\Omega = 1, \end{cases} \tag{5.1}$$

where

$$\mathcal{A} = \begin{pmatrix} 1 + (x_1 - \frac{1}{2})^2 & (x_1 - \frac{1}{2})(x_2 - \frac{1}{2}) \\ (x_1 - \frac{1}{2})(x_2 - \frac{1}{2}) & 1 + (x_2 - \frac{1}{2})^2 \end{pmatrix},$$

$\varphi = \mathrm{e}^{(x_1 - 1/2)(x_2 - 1/2)}$ and $\varrho = 1 + (x_1 - \frac{1}{2})(x_2 - \frac{1}{2})$.

We first solve the eigenvalue problem (5.1) in the linear finite element space on the coarse mesh $\mathcal{T}_{h_1}$. Then refine the mesh in the regular way to produce a series of meshes $\mathcal{T}_{h_k}$ $(k = 2, \ldots, n)$ with $\beta = 2$ (connecting the midpoints of each edge) and solve the augmented mixed problem (3.14) in the finer linear finite element space V_{h_k} defined on $\mathcal{T}_{h_k}$.

In this example, we also use two coarse meshes which are shown in Figure 1 as the initial meshes to investigate the convergence behavior. Since the exact solution is unknown, we choose an adequately accurate eigenvalue approximations by

the extrapolation method (see e.g. [13]) as the exact eigenvalue. Figure 4 gives the corresponding numerical results for the first six eigenvalue approximations and their corresponding eigenfunction approximations. Here we also compare the numerical results with those of the direct algorithm. Figure 4 also exhibits the optimal convergence rate of Algorithm 4.1.

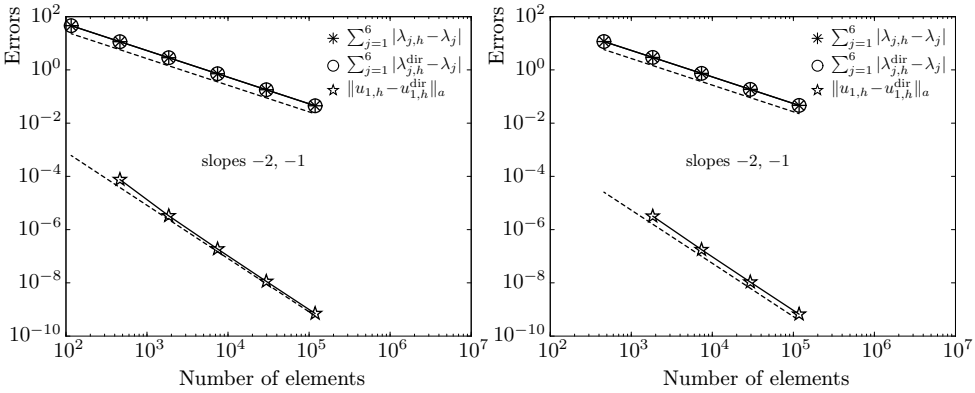

Figure 4. The errors of the multilevel iteration algorithm for the first six eigenvalues and the corresponding first eigenfunction, where $u_h^{\rm dir}$ and $\lambda_h^{\rm dir}$ denote respectively the eigenfunction and eigenvalue approximation by direct eigenvalue solving (the left subfigure is for the coarse initial mesh to the left of Figure 1 and the right one for the fine initial mesh to the right of Figure 1).

5.3. Eigenvalue problem on dumbbell shaped domain. To show our approach can also work well for a complex domain, we consider the Laplace eigenvalue problem on the two dimensional dumbbell shaped domain $\Omega = (0,\pi)^2 \cup [\pi, \frac{5}{4}\pi] \times (\frac{3}{8}\pi, \frac{5}{8}\pi) \cup (\frac{5}{4}\pi, \frac{9}{4}\pi) \times (0,\pi)$ (Figure 5). For this eigenvalue problem, the first eigenvalue is close to the second ($1.95532 \leqslant \lambda_1 \leqslant 1.95646$, $1.96025 \leqslant \lambda_2 \leqslant 1.96129$).

Since the dumbbell domain has reentrant corners, eigenfunctions with singularities are expected. The convergence order for eigenvalue approximations is less than 2 by the linear finite element method, which is the order predicted by the theory for regular eigenfunctions. Then the sequence of meshes is produced by the adaptive refinement based on the residual type of a posteriori error estimator (see e.g. [7], [12]). Figure 5 shows the initial mesh and the one after 10 adaptive refinements.

Algorithm 4.1 is applied to solve the eigenvalue problem.

Figure 6 presents the a posteriori error estimates for the first eigenfunction and the first six eigenfunctions. From Figure 6, we find the multilevel iteration scheme can also obtain the optimal accuracy.

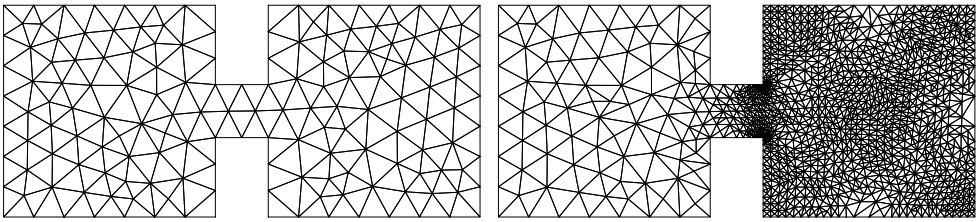

Figure 5. The initial mesh and the one after 10 adaptive refinements for the first eigenfunction.

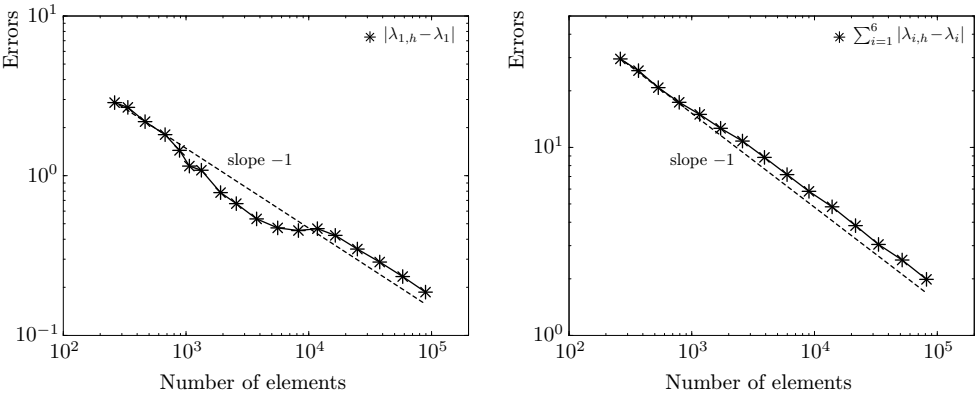

Figure 6. The errors of the multilevel iteration algorithm for the Laplace eigenvalue problem on the dumbbell domain (the left subfigure is for the first eigenvalue and the right one for the first six eigenvalues).

6. Concluding remarks

In this paper, we propose a type of multilevel method for eigenvalue problems based on Newton's method. In this type of iteration method, solving eigenvalue problem on the finest finite element space is decomposed into solving a small scale eigenvalue problem in the coarsest space and solving a sequence of augmented linear problems, derived by the Newton iteration step in the corresponding sequence of finite element spaces. The proposed scheme improves the overall efficiency of eigenvalue problem solving by the finite element method.

The quadratic convergence property of Newton's method improves the accuracy of the numerical solution. On the other hand, the multilevel technique overcomes the sensitivity of the initial guess of the Newton scheme in some sense. The method and idea here can be extended to nonlinear eigenvalue problems with some type of linearization for the nonlinear terms.

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