REMARKS ON BALANCED NORM ERROR ESTIMATES FOR SYSTEMS OF REACTION-DIFFUSION EQUATIONS

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Abstract. Error estimates of finite element methods for reaction-diffusion problems are often realized in the related energy norm. In the singularly perturbed case, however, this norm is not adequate. A different scaling of the H^1 seminorm leads to a balanced norm which reflects the layer behavior correctly. We discuss the difficulties which arise for systems of reaction-diffusion problems.

Keywords: singular perturbation; finite element method; layer-adapted mesh; balanced norm

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1. Introduction

We will examine the finite element method for the numerical solution of systems of reaction-diffusion equations

$$-Eu'' + Au = f \quad \text{in } \Omega = (0,1), \tag{1.1a}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.1b}$$

where $E = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_l)$ with small real parameters $\varepsilon_1, \ldots, \varepsilon_l$. A is a symmetric, strictly diagonally dominant matrix with sufficiently smooth components a_{ij} and $a_{ii} > 0$; we assume also f to be sufficiently smooth.

The finite element discretization uses the bilinear form

$$B(u,v) := \sum_m \varepsilon_m (u'_m, v'_m) + \sum_m \sum_{i=1}^{m} (a_{mi} u_i, v_m).$$

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The related energy norm is

$$\|v\|_e^2 := \sum_m \varepsilon_m (v_m', v_m') + \|v\|_0^2.$$

Linß [7] proved error estimates for linear elements on Bakhvalov and Shishkin meshes, for instance,

$$\|u - u^N\|_e \leqslant CN^{-1} \ln N$$

for a Shishkin mesh.

However, the typical boundary layer function $\exp(-x/\varepsilon_l^{1/2})$ measured in the norm $\|\cdot\|_\varepsilon$ is of order $\mathcal{O}(\varepsilon_l^{1/4})$. Consequently, error estimates in the energy norm are less valuable. Therefore, we ask the fundamental question:

Is it possible to prove error estimates in the balanced norm

$$\|v\|_b^2 := \sum_m \varepsilon_m^{1/2} (v_m', v_m') + \|v\|_0^2\,? \tag{1.2}$$

The first balanced error estimate was presented by Lin and Stynes [5] using a first order system least squares (FOSLS) mixed method. But it is also possible to use a direct mixed method [10]. Several further results concerning balanced norm estimates for finite element methods and second order reaction-diffusion problems are presented in [11] (for instance, weakly nonlinear problems, different classes of layer-adapted meshes, the 3D case, supercloseness). For the *hp*-FEM on spectral boundary layer meshes we refer to [8] and [2].

Convection-diffusion problems with different layers in the x- and y-direction are examined in [3], fourth order problems discretized by mixed finite element methods in [4].

As discussed in [11], it is open how to handle problems with different layers in one coordinate direction or systems of reaction-diffusion equations.

In Section 2 we will repeat a basic idea to prove error estimates in a balanced norm from [12].

In Section 3.1 we start to discuss systems in the case $\varepsilon_1 = \varepsilon_2$ (for simplicity, we only discuss two equations, i.e., $l = 2$), and in Section 3.2 sketch the difficulties for different parameters.

2. The basic error estimate in a balanced norm in the scalar case

We consider Shishkin meshes. In the scalar case (replace the matrix A by a scalar c) the mesh distributes $N/4$ points (assuming N is divisible by 4) equidistantly within each of the subintervals $[0, \lambda_x]$, $[1-\lambda_x, 1]$ and the remaining points within the third subinterval. For simplicity, assume

$$\lambda = \lambda_x = \lambda_y = \min\left\{1/4, \lambda_0\sqrt{\varepsilon/c^*}\ln N\right\} \quad \text{with } \lambda_0 = 2 \text{ and } c^* < c.$$

We remark that the choice of λ_0 mainly depends on the polynomial degree of the finite element space. For systems, see [7] for the description of a related Shishkin mesh.

We use the step sizes

$$h := \frac{4\lambda}{N} \quad \text{and} \quad H := \frac{2(1-2\lambda)}{N}.$$

Let $V^N \subset H_0^1(\Omega)$ be the space of linear finite elements on Ω^N. A standard weak formulation of the scalar version of problem (1.1) reads: Find $u \in V$ such that

$$\varepsilon(u', v') + (cu, v) = (f, v) \quad \forall\, v \in V. \tag{2.1}$$

Replacing V in (2.1) by V^N one obtains a standard discretization that yields the FEM-solution u^N.

Certain assumptions on f allow a decomposition of u into smooth components S and layer terms E such that the following estimates for the interpolation error of the Lagrange interpolant hold true (see [13]):

$$\|u - u^I\|_0 \preceq N^{-2}, \quad \varepsilon^{1/4}|u - u^I|_1 \preceq N^{-1}\ln N \tag{2.2}$$

and

$$\|u - u^I\|_{\infty,\Omega_0} \preceq N^{-2}, \quad \|u - u^I\|_{\infty,\Omega\setminus\Omega_0} \preceq (N^{-1}\ln N)^2, \tag{2.3}$$

where $\Omega_0 = (\lambda_x, 1-\lambda_x)$. Let us also introduce $\Omega_f := \Omega \setminus \Omega_0$. We have used the notation that if $a \preceq b$ there exists a constant C independent of ε such that $a \leqslant Cb$.

Instead of the Lagrange interpolant we introduce into the error analysis the L_2 projection $\pi u \in V^N$ from u. Based on

$$u - u^N = u - \pi u + \pi u - u^N$$

we estimate $\xi := \pi u - u^N$:

$$\|\xi\|_e^2 \preceq \varepsilon|\nabla\xi|_1^2 + c\,\|\xi\|_0^2 = \varepsilon(\nabla(\pi u - u), \nabla\xi) + (c\,(\pi u - u), \xi).$$

Because our projection is defined by $(c\,(\pi u - u), \xi) = 0$, it follows that

$$|\pi u - u^N|_1 \preceq |u - \pi u|_1. \tag{2.4}$$

If we now could prove an estimate similar to (2.2) for the error of the L_2 projection, we would obtain an estimate in the balanced norm if also a fitting estimate of $\|u - u_N\|_0$ is available. The standard error estimation in the energy norm yields for the L_2 part $\|u - u_N\|_0 \preceq \varepsilon^{1/4}(N^{-1}\ln N + N^{-2})$, which is sufficient for our aims. Alternatively, one can also prove $\|u - u_N\|_0 \preceq N^{-2}$, very easily in 1D, while in 2D one uses the supercloseness techniques assuming additionally $\lambda_0 \geqslant 2.5$.

If πu has some representation $\pi u = \sum_i V_i \varphi_i$, the V_j satisfy the tridiagonal system (with $\bar{h}_i := (h_i + h_{i+1})/2$)

$$\frac{1}{6}\frac{h_i}{\bar{h}_i}\tilde{c}_{i-1}V_{i-1} + \frac{2}{3}\tilde{c}_i V_i + \frac{1}{6}\frac{h_{i+1}}{\bar{h}_i}\tilde{c}_{i+1}V_{i+1} = \frac{1}{\bar{h}_i}\int_{x_{i-1}}^{x_{i+1}} c u \varphi_i. \tag{2.5}$$

The coefficient matrix M of this system is strictly diagonal dominant. It follows that $|V_i| \preceq \|u\|_\infty$, hence we have the stability property

$$\|\pi u\|_\infty \preceq \|u\|_\infty. \tag{2.6}$$

As a consequence we obtain

Lemma 1. *Assuming the validity of* (2.2) *and* (2.3), *the error of the L_2 projection on the Shishkin mesh satisfies*

$$\|u - \pi u\|_\infty \preceq \|u - u^I\|_\infty, \quad \varepsilon^{1/4}|u - \pi u|_1 \preceq N^{-1}(\ln N)^{3/2}. \tag{2.7}$$

From (2.4), Lemma 1 and the estimates for $\|u - u_N\|_0$ we get

Theorem 1. *Assuming* (2.2) *and* (2.3), *the error of the Galerkin finite element method with linear elements on a Shishkin mesh satisfies*

$$\|u - u^N\|_b \preceq N^{-1}(\ln N)^{3/2} + N^{-2}. \tag{2.8}$$

In 2D, the L_∞ stability of L_2 projection is an interesting topic [1], we used in [12] a result of Oswald [9] for meshes with a special structure. Inverse inequalities are used to move from estimates in W^1_∞ to L_∞, for details see [12]. Finally, in 2D one obtains the estimate (2.8) for linear as well as bilinear finite elements.

3.1. The case $\varepsilon_1 = \varepsilon_2$. First let us remark that for systems

$$-\varepsilon u'' + Au = f \quad \text{in } \Omega = (0,1), \tag{3.1a}$$

$$u(0) = u(1) = 0 \quad \text{on } \partial\Omega, \tag{3.1b}$$

so far there exists only a result of Lin and Stynes [6] in a balanced norm. Following the basic idea from [5], but using C^1 elements instead of mixed finite elements, they introduce the bilinear form

$$\varepsilon(w', v') + (Aw, v) + \varepsilon^{3/2}(w'', v'') + \varepsilon^{1/2}((Aw)', v')$$

and analyze the finite element method for quadratic C^1 elements. The analysis for the Galerkin method with C^0 elements is open.

Let us consider the case of two equations and let us write the system (2.5) as

$$M(\pi u) = g.$$

Now we also define in the matrix case the generalized vector-valued L_2 projection by

$$(A(\Pi u), \xi) = (A\,u, \xi).$$

If A is a constant matrix, we get the desired L_∞ stability immediately. But if not, we get for the vector of the values of Πu in the mesh points a linear system, where the coefficient matrix $\widehat{M}$ has the structure

$$\widehat{M} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{3.2}$$

Here every matrix A_{ij} has the structure of M corresponding to (2.5), one has just to replace c by the components of A, i.e., $a_{11}, a_{12}, \ldots$

The question is: *Which assumptions on A guarantee that $\|\widehat{M}^{-1}\|_\infty$ is bounded?*

Remark that in the case of constant coefficients we have

$$\widehat{M} = \begin{bmatrix} a_{11}M & a_{12}M \\ a_{21}M & a_{22}M \end{bmatrix}.$$

Now $\widehat{M}$ is the product of the matrices

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{11}E & a_{12}E \\ a_{21}E & a_{22}E \end{bmatrix}.$$

Because M is strictly diagonally dominant, we have $\|\widehat{M}^{-1}\|_\infty \leqslant C$, and the same properties of A imply $\|\widehat{M}^{-1}\|_\infty \leqslant C$. We conjecture that perturbation arguments should yield results in the case of nonconstant coefficients.

3.2. Different small parameters. If the two small parameters are different there appear new difficulties. Consider the simplest case of two equations with *constant* coefficients:

$$-\varepsilon_1 u_1'' + u_1 + a_{12}u_2 = f_1, \tag{3.3a}$$

$$-\varepsilon_2 u_2'' + a_{21}u_1 + a_{22}u_2 = f_2 \tag{3.3b}$$

and discretize by the Galerkin method on the corresponding Shishkin mesh. We assume $\varepsilon_1 < \varepsilon_2$.

With some projections $\hat{u}_1, \hat{u}_2$ into the finite element space we introduce $\xi_1 = u_1^N - \hat{u}_1$ and $\xi_2 = u_2^N - \hat{u}_2$. Then

$$\begin{aligned}\varepsilon_1|\xi_1|_1^2 + \varepsilon_2|\xi_2|_1^2 + \|\xi\|_0^2 \preceq{}& \varepsilon_1((\hat{u}_1 - u_1)', \xi_1') + \varepsilon_2((\hat{u}_2 - u_2)', \xi_2') \\ &+ (\hat{u}_1 - u_1, \xi_1) + a_{12}(\hat{u}_2 - u_2, \xi_1) + a_{21}(\hat{u}_1 - u_1, \xi_2) + a_{22}(\hat{u}_2 - u_2, \xi_2).\end{aligned}$$

First we choose for the projections the L_2 projections of u_1, u_2.

Then the four terms on the second line disappear and we get

$$|\xi_2|_1 \preceq \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^{1/2} |u_1 - \hat{u}_1|_1 + |u_2 - \hat{u}_2|_1,$$

consequently we obtained the desired estimate for $\varepsilon_2^{1/4}|\xi_2|_1$ because $\varepsilon_1 < \varepsilon_2$.

But, unfortunately, this approach does not yield the desired estimate for $\varepsilon_1^{1/4}|\xi_1|_1$.

In the second step we define $\hat{u}_1$ and $\hat{u}_2$ by

$$(\hat{u}_1 + a_{12}\hat{u}_2, \xi_1) = (u_1 + a_{12}u_2, \xi_1) \tag{3.4}$$

and

$$\varepsilon_2(\hat{u}_2', \xi_2') + (a_{21}\hat{u}_1 + a_{22}\hat{u}_2, \xi_2) = \varepsilon_2(u_2', \xi_2') + (a_{21}u_1 + a_{22}u_2, \xi_2). \tag{3.5}$$

Using (3.4), we can eliminate the first component and get with $\beta = a_{22} - a_{12}a_{21}$

$$\varepsilon_2(\hat{u}_2', \xi_2') + \beta(\hat{u}_2, \xi_2) = \varepsilon_2(u_2', \xi_2') + \beta(u_2, \xi_2). \tag{3.6}$$

This means: $\hat{u}_2$ is just the Ritz projection of a standard scalar reaction-diffusion operator, and we have the desired estimate for $\varepsilon_2^{1/4}|u_2 - \hat{u}_2|_1$. From (3.4) we get (introducing L_2 projections)

$$u_1 - \hat{u}_1 = (u_1 - \pi u_1) + a_{12}(u_2 - \pi u_2) + a_{12}(\hat{u}_2 - u_2).$$

This yields the desired estimate for $\varepsilon_1^{1/4}|u_1 - \hat{u}_1|_1$.

Hence, the following natural question arises:

How that basic idea can be generalized to problems with nonconstant coefficients?

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