SOME STOCHASTIC COMPARISON RESULTS FOR SERIES AND PARALLEL SYSTEMS WITH HETEROGENEOUS PARETO TYPE COMPONENTS

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Abstract. We focus on stochastic comparisons of lifetimes of series and parallel systems consisting of independent and heterogeneous new Pareto type components. Sufficient conditions involving majorization type partial orders are provided to obtain stochastic comparisons in terms of various magnitude and dispersive orderings which include usual stochastic order, hazard rate order, dispersive order and right spread order. The usual stochastic order of lifetimes of series systems with possibly different scale and shape parameters is studied when its matrix of parameters changes to another matrix in certain sense.

Keywords: stochastic order; parallel system; series system; majorization; multivariate chain majorization; Pareto type distribution; T-transform matrix

MSC 2010: 60E15, 60K10

1. Introduction

The concept of majorization was introduced to study Schur-convexity of a function. Various key ideas on majorization were discussed by Hardy at al. [16]. Prior to the volume by Marshall and Olkin [19], scholars were unconscious of the literature related to majorization, in spite of the availability of several works in this direction. Since last two-three decades, the notion of majorization has played a prominent role to study various stochastic orders in different fields such as reliability theory, economics, quantum information theory, mathematics, probability and statistics. Let $\underline{x} = (x_1, x_2, \ldots, x_n)$ and $\underline{y} = (y_1, y_2, \ldots, y_n)$ be two vectors in $\mathbb{R}^n$, the set of real n-vectors. Further, let $x_{1:n} \leqslant x_{2:n} \leqslant \ldots \leqslant x_{n:n}$ and $y_{1:n} \leqslant y_{2:n} \leqslant \ldots \leqslant y_{n:n}$ be the ranked values of $\underline{x}$ and $\underline{y}$, respectively. Then $\underline{y}$ is said to be majorized

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by $\underline{x}$, denoted by $\underline{x} \succeq_m \underline{y}$, if $\sum_{i=1}^{j} x_{i:n} \leqslant \sum_{i=1}^{j} y_{i:n}$, $j = 1, 2, \ldots, n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$. Thus, $\underline{x} \succeq_m \underline{y}$ means that though sums of the components of two vectors are the same, the components in $\underline{x}$ are more dispersed compared to those of the vector $\underline{y}$. More specifically, the concept of majorization deals with the diversity of the components of the vectors. It is used as a measure of income inequality and species diversity. Further, various useful inequalities can be obtained by applying some order preserving function to a suitable majorization ordering. For a comprehensive survey on this topic, we refer to [19] and [1].

Order statistics are of great interest in operations research, reliability theory, data analysis, statistical inference and other areas of applied probability. They have received a lot of attention from many researchers. For details on order statistics, see [3] and [7]. Denote by $X_{1:n} \leqslant X_{2:n} \leqslant \ldots \leqslant X_{n:n}$ the order statistics from independent random variables $X_1, X_2, \ldots, X_n$. In reliability theory, we often encounter with a k-out-of-n system. It works if and only if at least k components out of n work. In particular, an n-out-of-n system and a 1-out-of-n system represent, respectively, series and parallel systems. Note that series and parallel systems are the simplest examples of coherent systems which have been widely considered in the literature. In general, the kth order statistic $X_{k:n}$ represents the lifetime of $(n-k+1)$-out-of-n system.

We often face various situations in reliability and survival studies, where stochastic comparisons of system lifetimes are useful to choose the most reliable system. These comparisons are also helpful to obtain various bounds of aging characteristics of a complex system. Because of their important applications, there has been a considerable interest in studying stochastic comparisons of lifetimes of systems having independent heterogeneous components. Many authors have considered several statistical models in this direction. Dykstra et al. in [8] studied the problem of stochastically comparing the order statistics of two sets of n exponential random variables, where the random variables in one set are independent and heterogeneous and the random variables in the other set are independent and identically distributed. Khaledi and Kochar in [18] stochastically compared order statistics corresponding to two sets of independent Weibull and gamma random variables with a common shape parameter under the condition that their scale parameters majorize each other. Zhao and Balakrishnan in [23] obtained dispersive ordering results of fail-safe systems comprising of heterogeneous exponential components. For two sets of Weibull random variables, where one set contains n independent but heterogeneous random variables and the other set contains independent but identical random variables, Fang and Zhang in [12] obtained sufficient conditions for dispersive ordering of the largest order statistics. Balakrishnan and Zhao in [4] proved some hazard rate comparison results of parallel systems associated with heterogeneous gamma distributed components. Fang and Zhang in [13] stochastically compared parallel systems with exponentiated-Weibull components in terms of the usual stochastic order, dispersive order and the likelihood ratio order. They provided sufficient conditions for stochastic comparisons between lifetimes of parallel systems. Gupta et al. in [15] obtained ordering results for parallel as well as series systems having Fréchet distributed components. Fang and Balakrishnan in [9] compared the largest order statistics arising from independent heterogeneous Weibull random variables based on the likelihood ratio order. Fang and Balakrishnan in [10] discussed stochastic comparisons of the smallest and largest order statistics from independent heterogeneous exponential-Weibull random variables. Further, they obtained sufficient conditions for the hazard rate ordering of the smallest order statistics. For some recent references on the problems related to stochastic comparisons, we refer to [14], [5], [11], [20] and the references therein.

In this paper, we consider stochastic comparisons of series and parallel systems with a new Pareto type components. This distribution was proposed by [6] as a generalization of the usual Pareto distribution. It has upside-down bathtub or a decreasing hazard rate function depending on the values of the parameters. For some applications of this distribution in reliability engineering and finance, one may refer to [6]. They pointed out that the newly proposed distribution fits better in various real life situations. Let a random variable X follow a new Pareto type distribution. Then the cumulative distribution function (cdf) and the probability density function (pdf) of X are given by

$$F(x;\alpha,\beta)=\frac{x^{\alpha}-\beta^{\alpha}}{x^{\alpha}+\beta^{\alpha}},\quad x>\beta,\ \alpha>0 \tag{1.1}$$

and

$$f(x;\alpha,\beta)=\frac{2\alpha\beta^{\alpha}x^{\alpha-1}}{(x^{\alpha}+\beta^{\alpha})^{2}},\quad x>\beta,\ \alpha>0, \tag{1.2}$$

respectively. Here, α is a shape parameter and β (>0) is a scale parameter. For convenience, we use the notation $X\sim \mathrm{NP}(\alpha,\beta)$.

The arrangement of the paper is as follows. In the next section, we present some definitions and results which are useful to derive our main results. In Section 3, based on vector majorization, we obtain some ordering results such as usual stochastic, hazard rate and dispersive orderings for comparisons of two series and parallel systems having heterogeneous NP type components. Further, for a series system, stochastic comparison has been studied with respect to multivariate chain majorization. In Section 4, we provide some applications of the established results. Finally, some concluding remarks are added in Section 5.

Throughout the paper, the terms increasing and decreasing are used in nonstrict sense. Prime "$'$" denotes ordinary derivative. The random variables used in this paper are taken to be nonnegative.

2. Preliminaries

In this section, we provide some preliminary definitions and lemmas which will be useful in the sequel. To compare lifetimes of series and parallel systems, stochastic orders have been extensively used in the literature. Below, we present a few of them. For more on various stochastic orders, see [22].

Definition 2.1. Let X_i, $i = 1, 2$ be two random variables with pdfs $f_{X_i}(\cdot)$, cdfs $F_{X_i}(\cdot)$, survival functions $\bar{F}_{X_i}(\cdot) = 1 - F_{X_i}(\cdot)$, failure rate functions $r_{X_i}(\cdot) = f_{X_i}(\cdot)/\bar{F}_{X_i}(\cdot)$, and reversed failure rate functions $\widetilde{r}_{X_i}(\cdot) = f_{X_i}(\cdot)/F_{X_i}(\cdot)$. Then X_1 is said to be smaller than X_2 in the

(a) failure rate ordering (written as $X_1 \leqslant_{\rm fr} X_2$) if $r_{X_1}(x) \geqslant r_{X_2}(x)$ for all x in $(0, \infty)$;
(b) usual stochastic ordering (written as $X_1 \leqslant_{\rm st} X_2$) if $F_{X_2}(x) \leqslant F_{X_1}(x)$ for all x in $(-\infty, \infty)$;
(c) dispersive ordering (written as $X_1 \leqslant_{\rm disp} X_2$) if $F_1^{-1}(u_2) - F_1^{-1}(u_1) \leqslant F_2^{-1}(u_2) - F_2^{-1}(u_1)$ for $0 \leqslant u_1 < u_2 \leqslant 1$;
(d) right spread ordering (written as $X_1 \leqslant_{\rm rs} X_2$) if $\int_{F_1^{-1}(p)}^{\infty} \bar{F}_1(t)\,{\rm d}t \leqslant \int_{F_2^{-1}(p)}^{\infty} \bar{F}_2(t)\,{\rm d}t$ for $0 \leqslant p \leqslant 1$.

It is well known that

$$X_1 \leqslant_{\rm disp} X_2 \Rightarrow X_1 \leqslant_{\rm rs} X_2 \Rightarrow {\rm Var}(X_1) \leqslant {\rm Var}(X_2),$$

where ${\rm Var}(X)$ denotes the variance of the random variable X. To establish various stochastic inequalities and bounds, the notion of majorization (see [19]) plays a vital role which are presented below.

Definition 2.2. Let $\underline{x} = (x_1, x_2, \ldots, x_n)$ and $\underline{y} = (y_1, y_2, \ldots, y_n)$ be two real vectors in $\mathbb{A} \subset \mathbb{R}^n$. Also let $x_{1:n} \leqslant x_{2:n} \leqslant \ldots \leqslant x_{n:n}$ and $y_{1:n} \leqslant y_{2:n} \leqslant \ldots \leqslant y_{n:n}$ be the ranked values of $\underline{x}$ and $\underline{y}$, respectively. Then

(a) $\underline{y}$ is said to be majorized by $\underline{x}$ (written as $\underline{x} \succeq_m \underline{y}$) if

$$\sum_{i=1}^{j} x_{i:n} \leqslant \sum_{i=1}^{j} y_{i:n}, \quad j = 1, 2, \ldots, n-1, \quad \text{and} \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i;$$

(b) $\underline{y}$ is said to be weak lower majorized by $\underline{x}$ (written as $\underline{x} \succeq_w \underline{y}$) if

$$\sum_{i=1}^{j} x_{i:n} \leqslant \sum_{i=1}^{j} y_{i:n}, \quad j = 1, 2, \ldots, n-1, \quad \text{and} \quad \sum_{i=1}^{n} x_i \geqslant \sum_{i=1}^{n} y_i.$$

Definition 2.3. A real valued function $\phi\colon \mathbb{A} \to \mathbb{R}$ is said to be Schur-concave (Schur-convex) on $\mathbb{A}$ if for $\underline{x}, \underline{y} \in \mathbb{A}$, $\underline{x} \succeq_m \underline{y} \Rightarrow \phi(\underline{x}) \leqslant \phi(\underline{y})$ $(\geqslant \phi(\underline{y}))$.

The following well-known lemma provides the relationship between two forms of majorization as described in Definition 2.2.

Lemma 2.1 ([21]). *Let $\underline{x}$ and $\underline{y}$ be two n-dimensional real vectors. Then $\underline{x} \succeq_w \underline{y}$ if and only if there exists an n-dimensional vector $\underline{z}$ such that $\underline{x} \succeq_m \underline{z}$ and $\underline{z} \geqslant \underline{y}$ (i.e., $z_i \geqslant y_i$, $i = 1, 2, \ldots, n$).*

Lemma 2.2 ([19]). *A permutation symmetric differentiable function $\phi(\underline{x})$ is Schur-concave (Schur-convex) if and only if*

$$(x_i - x_j)\Big(\frac{\partial \phi(\underline{x})}{\partial x_i} - \frac{\partial \phi(\underline{x})}{\partial x_j}\Big) \leqslant 0 \ (\geqslant 0),$$

for all $i \neq j$.

Lemma 2.3 ([17]). *Let $X_{\lambda_1}, X_{\lambda_2}, \ldots, X_{\lambda_n}$ be nonnegative random variables with $X_{\lambda_i} \sim F(\lambda_i x)$, where $\lambda_i > 0$, $i = 1, 2, \ldots, n$ and $F(\cdot)$ is an absolutely continuous distribution function. Let $r(\cdot)$ be the failure rate function corresponding to $F(\cdot)$. If $x^2 r'(x)$ is decreasing and $(\lambda_1, \lambda_2, \ldots, \lambda_n) \succeq_m (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)$, then*

(a) $X_{1:n}^{\lambda} \geqslant_{\rm fr} X_{1:n}^{\lambda^*}$;

(b) $X_{1:n}^{\lambda} \geqslant_{\rm disp} X_{1:n}^{\lambda^*}$, *provided $r(x)$ is decreasing.*

Lemma 2.4. *Let a function $\delta\colon (0, \infty) \times (0, 1) \to (-\infty, 0)$ be defined as $\delta(\alpha, t) = (1 + t^\alpha)^{-1} \ln t$. Then*

(a) *$\delta(\alpha, t)$ is increasing with respect to t;*

(b) *$\delta(\alpha, t)$ is decreasing with respect to α.*

P r o o f. Differentiating $\delta(\alpha, t)$ with respect to t, we get

$$\frac{\partial \delta(\alpha, t)}{\partial t} = \frac{1}{t(1 + t^\alpha)^2}\{1 + t^\alpha - \alpha t^\alpha \ln t\},$$

which is positive, since $0 < t < 1$. This shows that $\delta(\alpha, t)$ is increasing with respect to t. Next, differentiating $\delta(\alpha, t)$ with respect to α, we obtain

$$\frac{\partial \delta(\alpha, t)}{\partial \alpha} = -\frac{t^{\alpha}(\ln t)^2}{(1+t^{\alpha})^2} < 0.$$

Thus $\delta(\alpha, t)$ is decreasing with respect to α. □

Lemma 2.5. *Consider a function* $\varrho\colon (0,\infty) \times (0,1) \to (0,\infty)$ *defined as* $\varrho(\alpha, t) = \alpha/(t(1+t^{\alpha}))$. *Then*

(a) $\varrho(\alpha, t)$ *is decreasing with respect to* t*;*
(b) $\varrho(\alpha, t)$ *is increasing with respect to* α*.*

Proof. Differentiating $\varrho(\alpha, t)$ with respect to t, we get

$$\frac{\partial \varrho(\alpha, t)}{\partial t} = -\frac{\alpha}{t^2(1+t^{\alpha})^2}\{\alpha t^{\alpha} + (1+t^{\alpha})\} < 0.$$

This proves that $\varrho(\alpha, t)$ is decreasing with respect to t. Again differentiating $\varrho(\alpha, t)$ with respect to α, we have

$$\frac{\partial \varrho(\alpha, t)}{\partial \alpha} = \frac{1}{t(1+t^{\alpha})}\{1 + t^{\alpha} - \alpha t^{\alpha} \ln t\} > 0.$$

Hence, the result follows. □

It is well known that a square matrix Π is called a permutation matrix if each row and column has exactly one entry unity and zeros elsewhere. It is not hard to see that there exist $n!$ such matrices of size $n \times n$. A T-transform matrix is of the form $T_w = wI_n + (1-w)\Pi$, $0 \leqslant w \leqslant 1$, where Π is a permutation matrix that just interchanges two coordinates. Consider two T-transform matrices $T_{w_1} = w_1 I_n + (1-w_1)\Pi_1$ and $T_{w_2} = w_2 I_n + (1-w_2)\Pi_2$, where Π_1 and Π_2 are two permutation matrices that just interchange two coordinates. The matrices T_{w_1} and T_{w_2} have the same structure if $\Pi_1 = \Pi_2$, otherwise they have different structures. Below, we present the notion of multivariate majorization.

Definition 2.4. Consider two $m \times n$ matrices $M_1 = \{a_{ij}\}$ and $M_2 = \{b_{ij}\}$ with respective rows $a_1^R, \ldots, a_m^R$ and $b_1^R, \ldots, b_m^R$. Then M_1 is said to chain majorize M_2, abbreviated by $M_1 \gg M_2$, if there exists a finite set of $n \times n$ T-transform matrices $T_{w_1}, T_{w_2}, \ldots, T_{w_k}$ such that $M_2 = M_1 T_{w_1} T_{w_2} \ldots T_{w_k}$.

The following results whose proofs are analogous to Theorems 2 and 3 of [2] are useful to obtain stochastic comparisons based on multivariate chain majorization. For $i, j = 1, 2, \ldots, n$, let

$$(2.1)\quad S_n = \left\{ (\underline{x}, \underline{y}) = \begin{pmatrix} x_1 x_2 \ldots x_n \\ y_1 y_2 \ldots y_n \end{pmatrix} \colon\ x_i > 0, y_j > 0 \text{ and } (x_i - x_j)(y_i - y_j) \leqslant 0 \right\}.$$

Lemma 2.6. *A differentiable function* $\eta\colon \mathbb{R}^{+^4} \to \mathbb{R}^+$ *satisfies*

$$(2.2)\qquad \eta(A) \geqslant \eta(B) \text{ for all } A, B \text{ such that } A \in S_2, \text{ and } A \gg B$$

if and only if

(a) $\eta(A) = \eta(A\Pi)$ *for all permutation matrices* Π *and*

(b) $\sum\limits_{i=1}^{2} (a_{ik} - a_{ij})[\eta_{ik}(A) - \eta_{ij}(A)] \geqslant 0$ *for all* $j, k = 1, 2$, *where* $\eta_{ij}(A) = \partial\eta(A)/\partial a_{ij}$.

Lemma 2.7. *Let the function* $\varphi\colon \mathbb{R}^{+^2} \to \mathbb{R}^+$ *be differentiable and the function* $\eta_n\colon \mathbb{R}^{+^{2n}} \to \mathbb{R}^+$ *be defined as*

$$(2.3)\qquad \eta_n(A) = \prod_{i=1}^{n} \varphi(a_{1i}, a_{2i}).$$

Assume that η_2 *satisfies* (2.2). *Then for* $A \in S_n$ *and* $B = AT_w$, *we have* $\eta_n(A) \geqslant \eta_n(B)$.

3. Main results

In this section, we carry out stochastic comparisons of lifetimes of series and parallel systems having independent but not identically distributed NP type components. Note that any comparison of the random variables should be over a common domain. Below in a few cases, we will notice that the distributions of $X_{1:n}$ and $Y_{1:n}$ as well as $X_{n:n}$ and $Y_{n:n}$ are defined over domains with different lower end points. In our results, we always take a common domain with the lower end point being the maximum of the two lower limits. First we consider this problem based on vector majorization.

3.1. Stochastic comparisons based on vector majorization and other results. Our first result establishes that there exist usual stochastic orders between $X_{1:n}(X_{n:n})$ and $Y_{1:n}(Y_{n:n})$ under some conditions on parameters.

Theorem 3.1. *Let $X_1, X_2, \ldots, X_n$ be independent random variables with $X_i \sim \mathrm{NP}(\alpha, \beta_i)$, $i = 1, 2, \ldots, n$, and $Y_1, Y_2, \ldots, Y_n$ independent random variables with $Y_i \sim \mathrm{NP}(\alpha, \beta_i^*)$, $i = 1, 2, \ldots, n$. Then*

(a) $(\beta_1, \beta_2, \ldots, \beta_n) \succeq_m (\beta_1^*, \beta_2^*, \ldots, \beta_n^*)$ *implies* $X_{n:n} \geqslant_{\mathrm{st}} Y_{n:n}$ *for* $\alpha \geqslant 1$,

(b) $(\beta_1, \beta_2, \ldots, \beta_n) \succeq_m (\beta_1^*, \beta_2^*, \ldots, \beta_n^*)$ *implies* $X_{1:n} \leqslant_{\mathrm{st}} Y_{1:n}$ *for* $\alpha > 0$.

Proof. (a) Denote $\underline{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$. The cdf of $X_{n:n}$ is

$$F_{X_{n:n}}(x; \alpha, \underline{\beta}) = \prod_{i=1}^{n} \frac{x^\alpha - \beta_i^\alpha}{x^\alpha + \beta_i^\alpha}, \quad x > \max_{1 \leqslant i \leqslant n} \beta_i$$

and the corresponding survival function is given by

$$\bar{F}_{X_{n:n}}(x; \alpha, \underline{\beta}) = 1 - \prod_{i=1}^{n} \frac{x^\alpha - \beta_i^\alpha}{x^\alpha + \beta_i^\alpha}, \quad x > \max_{1 \leqslant i \leqslant n} \beta_i. \tag{3.1}$$

Differentiating (3.1) with respect to β_i, we get

$$\frac{\partial \bar{F}_{X_{n:n}}(x; \alpha, \underline{\beta})}{\partial \beta_i} = \frac{2\alpha x^\alpha \beta_i^{\alpha-1}}{x^{2\alpha} - \beta_i^{2\alpha}} \bar{F}_{X_{n:n}}(x; \alpha, \underline{\beta}). \tag{3.2}$$

Define

$$g(z) = 2\alpha x^\alpha \frac{z^{\alpha-1}}{x^{2\alpha} - z^{2\alpha}}, \quad z > 0. \tag{3.3}$$

Then for $\alpha \geqslant 1$ it can be shown that $g'(z) \geqslant 0$, i.e., $g(z)$ in (3.3) is increasing in $z > 0$. Now using (3.2) and after some simplifications, it is not hard to see that

$$\begin{aligned}(\beta_i - \beta_j)&\left(\frac{\partial \bar{F}_{X_{n:n}}(x; \alpha, \underline{\beta})}{\partial \beta_i} - \frac{\partial \bar{F}_{X_{n:n}}(x; \alpha, \underline{\beta})}{\partial \beta_j}\right) \\ &= 2(\beta_i - \beta_j)\alpha x^\alpha \bar{F}_{X_{n:n}}(x; \alpha, \underline{\beta})\left(\frac{\beta_i^{\alpha-1}}{x^{2\alpha} - \beta_i^{2\alpha}} - \frac{\beta_j^{\alpha-1}}{x^{2\alpha} - \beta_j^{2\alpha}}\right) \geqslant 0.\end{aligned}$$

Hence, from Lemma 2.2, $\bar{F}_{X_{n:n}}(x; \alpha, \underline{\beta})$ is Schur-convex in $\underline{\beta}$. Thus utilizing Definition 2.3, the desired result follows.

(b) For $x > \max_{1 \leqslant i \leqslant n} \beta_i$, the survival function of $X_{1:n}$ is given by

$$\bar{F}_{X_{1:n}}(x; \alpha, \underline{\beta}) = 2^n \prod_{i=1}^{n} \frac{\beta_i^\alpha}{x^\alpha + \beta_i^\alpha}. \tag{3.4}$$

Differentiating (3.4) with respect to β_i we get

$$\frac{\partial \bar{F}_{X_{1:n}}(x;\alpha,\underline{\beta})}{\partial \beta_i} = \frac{1}{x}\bar{F}_{X_{1:n}}(x;\alpha,\underline{\beta})\varrho\Big(\alpha,\frac{\beta_i}{x}\Big), \tag{3.5}$$

where $\varrho(\alpha,t) = \alpha/(t(1+t^{\alpha}))$, $\alpha > 0$, $0 < t < 1$. Note that using Lemma 2.5, for $0 < t < 1$, it can be shown that $\varrho(\alpha,t)$ is decreasing in t. Now, from (3.5) we get

$$\begin{aligned}(\beta_i - \beta_j)&\Big(\frac{\partial \bar{F}_{X_{1:n}}(x;\alpha,\underline{\beta})}{\partial \beta_i} - \frac{\partial \bar{F}_{X_{1:n}}(x;\alpha,\underline{\beta})}{\partial \beta_j}\Big)\\ &= (\beta_i - \beta_j)\frac{1}{x}\bar{F}_{X_{1:n}}(x;\alpha,\underline{\beta})\Big(\varrho\Big(\alpha,\frac{\beta_i}{x}\Big) - \varrho\Big(\alpha,\frac{\beta_j}{x}\Big)\Big) \leqslant 0,\end{aligned}$$

which implies that $\bar{F}_{X_{1:n}}(x;\alpha,\underline{\beta})$ is Schur-concave in $\underline{\beta}$. Hence, the result follows. □

The following counterexample shows that the usual stochastic order as in the first part of Theorem 3.1 need not hold if either $\underline{\beta} \succeq_m \underline{\beta}^*$ or $\alpha \geqslant 1$ does not hold.

Counterexample 3.1. (i) Let (X_1, X_2, X_3) be a set of independent random variables such that $X_i \sim \mathrm{NP}(1.5, \beta_i)$, $i = 1,2,3$ with $\beta_1 = 1$, $\beta_2 = 8$, and $\beta_3 = 1.1$. Further, let (Y_1, Y_2, Y_3) be a set of independent random variables following $Y_i \sim \mathrm{NP}(1.5, \beta_i^*)$, $i = 1,2,3$ with $\beta_1^* = 8$, $\beta_2^* = 1.1$, and $\beta_3^* = 4$. Clearly, $(\beta_1,\beta_2,\beta_3) \not\succeq_m (\beta_1^*,\beta_2^*,\beta_3^*)$, though $\alpha > 1$. Now, we plot the difference $F_{X_{3:3}}(x) - F_{Y_{3:3}}(x)$ in Figure 1(a) which shows that $X_{3:3} \not\geq_{\mathrm{st}} Y_{3:3}$.

(ii) Let X_1, X_2, X_3 be three independent random variables with $X_i \sim \mathrm{NP}(0.8, \beta_i)$, $i = 1,2,3$, with $\beta_1 = 0.1$, $\beta_2 = 1$, and $\beta_3 = 9$. Also, let Y_1, Y_2, Y_3 be three independent random variables with $Y_i \sim \mathrm{NP}(0.8, \beta_i^*)$, $i = 1,2,3$, with $\beta_1^* = 0.1$, $\beta_2^* = 4$, and $\beta_3^* = 6$. It is easy to observe that $(\beta_1,\beta_2,\beta_3) \succeq_m (\beta_1^*,\beta_2^*,\beta_3^*)$. Note that $\alpha = 0.8 < 1$. In Figure 1(b), we plot the difference $F_{X_{3:3}}(x) - F_{Y_{3:3}}(x)$ which ensures that there is a cut between the plots of $F_{X_{3:3}}(x)$ and $F_{Y_{3:3}}(x)$, that is, $X_{3:3} \not\geq_{\mathrm{st}} Y_{3:3}$.

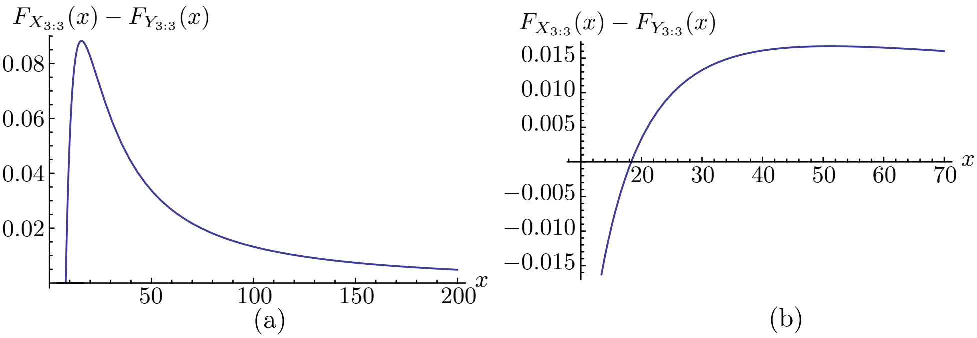

Figure 1. (a) presents plot of $F_{X_{3:3}}(x) - F_{Y_{3:3}}(x)$ as in Counterexample 3.1(i).
(b) presents plot of $F_{X_{3:3}}(x) - F_{Y_{3:3}}(x)$ as considered in Counterexample 3.1(ii).

We consider the following counterexample to show that in the second part of Theorem 3.1 if $\underline{\beta} \not\succeq_m \underline{\beta}^*$ then, $X_{1:n} \leqslant_{\rm st} Y_{1:n}$ need not hold for $\alpha > 0$.

Counterexample 3.2. Let X_1, X_2, X_3 be three independent random variables with $X_i \sim \mathrm{NP}(0.5, \beta_i)$, $i = 1, 2, 3$. Further, let Y_1, Y_2, Y_3 be three independent random variables with $Y_i \sim \mathrm{NP}(0.5, \beta_i^*)$, $i = 1, 2, 3$. Assume that $\beta_1 = 2.1$, $\beta_2 = 1$, $\beta_3 = 1.8$, $\beta_1^* = 3.5$, $\beta_2^* = 0.8$, and $\beta_3^* = 0.9$. Clearly, $(\beta_1, \beta_2, \beta_3) \not\succeq_m (\beta_1^*, \beta_2^*, \beta_3^*)$. Now, using graphical plot, it can be shown that $X_{1:3} \nleqslant_{\rm st} Y_{1:3}$. The graph has not been provided for brevity.

Theorem 3.2. *Let $X_1, X_2, \ldots, X_n$ be independent random variables with $X_i \sim \mathrm{NP}(\alpha, \beta_i)$, $i = 1, 2, \ldots, n$, and $Y_1, Y_2, \ldots, Y_n$ independent random variables with $Y_i \sim \mathrm{NP}(\alpha, \beta_i^*)$, $i = 1, 2, \ldots, n$. Then for $\alpha \leqslant 1$, $(1/\beta_1, 1/\beta_2, \ldots, 1/\beta_n) \succeq_m (1/\beta_1^*, 1/\beta_2^*, \ldots, 1/\beta_n^*)$ implies $X_{1:n} \geqslant_{\rm st} Y_{1:n}$.*

Proof. Denote $\xi_i = 1/\beta_i$ and $\xi_i^* = 1/\beta_i^*$, $i = 1, 2, \ldots, n$. Then the given condition is equivalent to $(\xi_1, \xi_2, \ldots, \xi_n) \succeq_m (\xi_1^*, \xi_2^*, \ldots, \xi_n^*)$. For $x > \max\limits_{1\leqslant i\leqslant n} 1/\xi_i$, the survival function of $X_{1:n}$ is given by

$$\bar{F}_{X_{1:n}}(x; \alpha, \underline{\xi}) = \prod_{i=1}^{n} \frac{2}{1 + (x\xi_i)^\alpha}, \tag{3.6}$$

where $\underline{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$. Differentiating (3.6) with respect to ξ_i, $i = 1, 2, \ldots, n$, we get

$$\frac{\partial \bar{F}_{X_{1:n}}(x; \alpha, \underline{\xi})}{\partial \xi_i} = -\frac{\alpha x^\alpha \xi_i^{\alpha-1}}{(x\xi_i)^\alpha + 1} \bar{F}_{X_{1:n}}(x; \alpha, \underline{\xi}). \tag{3.7}$$

Let us define $h(z) = z^{\alpha-1}((xz)^\alpha + 1)^{-1}$ for $z > 0$. Note that this function $h(z)$ is decreasing in $z > 0$ for $\alpha \leqslant 1$. Hence, from (3.7) it is not hard to check that

$$\begin{aligned}(\xi_i - \xi_j)&\left(\frac{\partial \bar{F}_{X_{1:n}}(x; \alpha, \underline{\xi})}{\partial \xi_i} - \frac{\partial \bar{F}_{X_{1:n}}(x; \alpha, \underline{\xi})}{\partial \xi_j}\right) \\ &= -(\xi_i - \xi_j)\alpha x^\alpha \bar{F}_{X_{1:n}}(x; \alpha, \underline{\xi})\left(\frac{\xi_i^{\alpha-1}}{(x\xi_i)^\alpha + 1} - \frac{\xi_j^{\alpha-1}}{(x\xi_j)^\alpha + 1}\right) \geqslant 0.\end{aligned}$$

Thus, from Lemma 2.2, we have that $\bar{F}_{X_{1:n}}(x; \alpha, \underline{\xi})$ is Schur-convex in $\underline{\xi}$, and then the desired result readily follows. □

Theorem 3.3. *Let $X_1, X_2, \ldots, X_n$ be independent random variables with $X_i \sim \mathrm{NP}(\alpha, \beta_i)$, $i = 1, 2, \ldots, n$, and $Y_1, Y_2, \ldots, Y_n$ independent random variables with $Y_i \sim \mathrm{NP}(\alpha, \beta_i^*)$, $i = 1, 2, \ldots, n$. If*

(a) $(\beta_1, \beta_2, \ldots, \beta_n) \geqslant (\beta_1^*, \beta_2^*, \ldots, \beta_n^*)$, *i.e.*, $\beta_i \geqslant \beta_i^*$, *then* $X_{n:n} \geqslant_{\rm st} Y_{n:n}$;

(b) $(1/\beta_1, 1/\beta_2, \ldots, 1/\beta_n) \geqslant (1/\beta_1^*, 1/\beta_2^*, \ldots, 1/\beta_n^*)$, *i.e.*, $1/\beta_i \geqslant 1/\beta_i^*$, *then* $Y_{1:n} \geqslant_{\rm st} X_{1:n}$.

Proof. The survival functions of $X_{n:n}$ and $X_{1:n}$ are given by (3.1) and (3.4), respectively. Moreover, it can be shown that the function $(x^\alpha - \beta^\alpha)(x^\alpha + \beta^\alpha)^{-1}$ is decreasing in β. Hence, using Definition 2.1, the results follow. □

Remark 3.1. Theorem 3.3(a) is a generalization of Theorem 3.1(a) to a wider range of scale parameters.

The following consecutive counterexamples show that if the conditions made in Theorem 3.3 are not satisfied then the mentioned stochastic orders may not hold.

Counterexample 3.3. Consider three independent random variables X_1, X_2, X_3 such that $X_i \sim {\rm NP}(0.5, \beta_i)$, $i = 1,2,3$. Assume $\beta_1 = 0.2$, $\beta_2 = 0.5$, and $\beta_3 = 0.7$. Take another set of three random variables Y_1, Y_2, Y_3 such that $Y_i \sim {\rm NP}(0.5, \beta_i^*)$, $i = 1,2,3$. Here, let $\beta_1^* = 0.4$, $\beta_2^* = 0.3$, and $\beta_3^* = 1.2$. Clearly, $(\beta_1, \beta_2, \beta_3) \not\geqslant (\beta_1^*, \beta_2^*, \beta_3^*)$. The difference between $F_{X_{3:3}}(x)$ and $F_{Y_{3:3}}(x)$ is depicted in Figure 2(a) which shows that $X_{3:3} \not\geqslant_{\rm st} Y_{3:3}$.

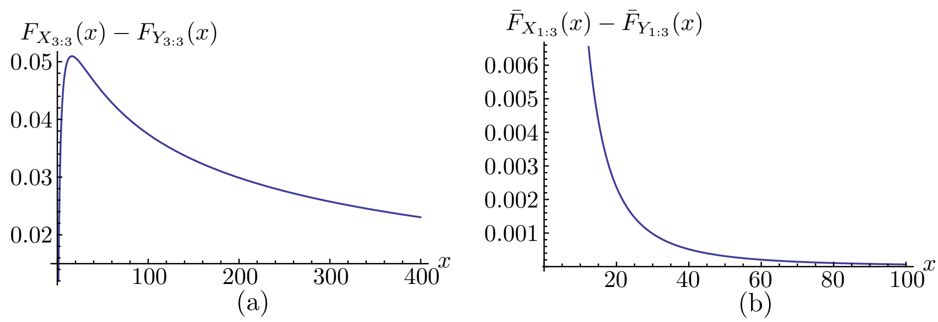

Figure 2. (a) presents plot of $F_{X_{3:3}}(x) - F_{Y_{3:3}}(x)$ as considered in Counterexample 3.3. (b) presents plot of $\bar{F}_{X_{1:3}}(x) - \bar{F}_{Y_{1:3}}(x)$ as in Counterexample 3.4.

Counterexample 3.4. Let X_1, X_2, X_3 be a set of independent random variables such that $X_i \sim {\rm NP}(0.8, \beta_i)$, $i = 1,2,3$. Here, we take $\beta_1 = 1.2$, $\beta_2 = 0.5$, and $\beta_3 = 1.7$. Take another set of three random variables Y_1, Y_2, Y_3 such that $Y_i \sim {\rm NP}(0.8, \beta_i^*)$, $i = 1,2,3$. Let $\beta_1^* = 0.4, \beta_2^* = 0.8$, and $\beta_3^* = 1.2$. Clearly, $(1/\beta_1, 1/\beta_2, 1/\beta_3) \not\geqslant (1/\beta_1^*, 1/\beta_2^*, 1/\beta_3^*)$. Based on this data, we have plotted $\bar{F}_{X_{1:3}}(x) - \bar{F}_{Y_{1:3}}(x)$ in Figure 2(b) which guarantees that $Y_{1:3} \not\geqslant_{\rm st} X_{1:3}$.

The following result provides a stochastic comparison for the lifetimes of two parallel systems having independently distributed new Pareto type components with varying scale parameters but fixed shape parameters. This comparison is studied based on the vector weak lower majorization of the scale parameters.

Theorem 3.4. *Let* $X_1, X_2, \ldots, X_n$ *be independent random variables with* $X_i \sim \mathrm{NP}(\alpha, \beta_i)$, $i = 1, 2, \ldots, n$, *and* $Y_1, Y_2, \ldots, Y_n$ *independent random variables with* $Y_i \sim \mathrm{NP}(\alpha, \beta_i^*)$, $i = 1, 2, \ldots, n$. *Then for* $\alpha \geqslant 1$, $(\beta_1, \beta_2, \ldots, \beta_n) \succeq_w (\beta_1^*, \beta_2^*, \ldots, \beta_n^*)$ *implies* $X_{n:n} \geqslant_{\mathrm{st}} Y_{n:n}$.

Proof. If $(\beta_1, \beta_2, \ldots, \beta_n) \succeq_w (\beta_1^*, \beta_2^*, \ldots, \beta_n^*)$ holds then by Lemma 2.1 there exists a vector $(\mu_1, \mu_2, \ldots, \mu_n)$ such that $(\beta_1, \beta_2, \ldots, \beta_n) \succeq_m (\mu_1, \mu_2, \ldots, \mu_n)$ and $(\mu_1, \mu_2, \ldots, \mu_n) \geqslant (\beta_1^*, \beta_2^*, \ldots, \beta_n^*)$. Now let $Z_1, Z_2, \ldots, Z_n$ be independent random variables with $Z_i \sim \mathrm{NP}(\alpha, \mu_i)$. Then from Theorem 3.1(a) we obtain $X_{n:n} \geqslant_{\mathrm{st}} Z_{n:n}$. Moreover, we have $(\mu_1, \mu_2, \ldots, \mu_n) \geqslant (\beta_1^*, \beta_2^*, \ldots, \beta_n^*)$, i.e., $\mu_i \geqslant \beta_i^*$, $i = 1, 2, \ldots, n$. Hence, according to Theorem 3.3(a), we get $Z_{n:n} \geqslant_{\mathrm{st}} Y_{n:n}$. Thus $X_{n:n} \geqslant_{\mathrm{st}} Y_{n:n}$. This completes the proof of the theorem. □

Theorem 3.5. *Let* $X_1, X_2, \ldots, X_n$ *be independent random variables with* $X_i \sim \mathrm{NP}(\alpha, \beta_i)$, $i = 1, 2, \ldots, n$, *and* $Y_1, Y_2, \ldots, Y_n$ *independent random variables with* $Y_i \sim \mathrm{NP}(\alpha, \beta_i^*)$, $i = 1, 2, \ldots, n$. *Then, for* $0 < \alpha \leqslant 1$,

(a) $(\beta_1, \beta_2, \ldots, \beta_n) \succeq_m (\beta_1^*, \beta_2^*, \ldots, \beta_n^*)$ *implies* $X_{1:n} \geqslant_{\mathrm{fr}} Y_{1:n}$;
(b) $(\beta_1, \beta_2, \ldots, \beta_n) \succeq_m (\beta_1^*, \beta_2^*, \ldots, \beta_n^*)$ *implies* $X_{1:n} \geqslant_{\mathrm{disp}} Y_{1:n}$.

Proof. (a) Note that for known α, the NP variable belongs to the scale model. The failure rate function of a random variable X having $\mathrm{NP}(\alpha, \beta)$ distribution is

$$r_X(x) = \frac{\alpha x^{\alpha-1}}{x^\alpha + \beta^\alpha}. \tag{3.8}$$

Denote $g(x) = x^2 r'_X(x)$. Then we have

$$g'(x) = \frac{-x^{2\alpha-1}\beta^\alpha \alpha(\alpha+1) + x^{\alpha-1}\beta^{2\alpha}\alpha(\alpha-1)}{(x^\alpha + \beta^\alpha)^3}.$$

For $\alpha \leqslant 1$, it is clear that $g'(x) < 0$, and hence $g(x)$ is decreasing. Thus, by Lemma 2.3 (a), the result follows.

(b) It is easy to see that for $\alpha \leqslant 1$, $r_X(x)$ given by (3.8) is decreasing. So by Lemma 2.3 (b), Part (b) follows immediately. This completes the proof. □

The next corollary immediately follows from Theorem 3.5(b).

Corollary 3.1. *Let* $(X_1, X_2, \ldots, X_n)$ *and* $(Y_1, Y_2, \ldots, Y_n)$ *be two sets of independent random variables as described in Theorem* 3.5. *Then, for* $0 < \alpha \leqslant 1$, $(\beta_1, \beta_2, \ldots, \beta_n) \succeq_m (\beta_1^*, \beta_2^*, \ldots, \beta_n^*) \Rightarrow X_{1:n} \geqslant_{\mathrm{rs}} Y_{1:n} \Rightarrow \mathrm{Var}(X_{1:n}) \geqslant \mathrm{Var}(Y_{1:n})$.

In our next result, we obtain a stochastic comparison of lifetimes of parallel and series systems based on vector majorization on shape parameters.

Theorem 3.6. *Let $X_1, X_2, \ldots, X_n$ be independent random variables with $X_i \sim \mathrm{NP}(\alpha_i, \beta)$, $i = 1, 2, \ldots, n$ and $Y_1, Y_2, \ldots, Y_n$ independent random variables with $Y_i \sim \mathrm{NP}(\alpha_i^*, \beta)$, $i = 1, 2, \ldots, n$. Then*

(a) $(\alpha_1, \alpha_2, \ldots, \alpha_n) \succeq_m (\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*)$ *implies* $X_{n:n} \geqslant_{\mathrm{st}} Y_{n:n}$;
(b) $(\alpha_1, \alpha_2, \ldots, \alpha_n) \succeq_m (\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*)$ *implies* $X_{1:n} \leqslant_{\mathrm{st}} Y_{1:n}$.

P r o o f. (a) Note that the cdf of $X_{n:n}$ is

$$F_{X_{n:n}}(x; \underline{\alpha}, \beta) = \prod_{i=1}^{n} \frac{x^{\alpha_i} - \beta^{\alpha_i}}{x^{\alpha_i} + \beta^{\alpha_i}}, \quad x > \beta. \tag{3.9}$$

Differentiating (3.9) with respect to α_i, we get

$$\frac{\partial F_{X_{n:n}}(x; \underline{\alpha}, \beta)}{\partial \alpha_i} = 2(\ln x - \ln \beta) \frac{\beta^{\alpha_i} x^{\alpha_i}}{x^{2\alpha_i} - \beta^{2\alpha_i}} F_{X_{n:n}}(x; \underline{\alpha}, \beta).$$

Thus,

$$\begin{aligned}(\alpha_i - \alpha_j)&\Big(\frac{\partial F_{X_{n:n}}(x; \underline{\alpha}, \beta)}{\partial \alpha_i} - \frac{\partial F_{X_{n:n}}(x; \underline{\alpha}, \beta)}{\partial \alpha_j}\Big) \\ &= 2(\alpha_i - \alpha_j)(\ln x - \ln \beta)\Big\{\frac{\beta^{\alpha_i} x^{\alpha_i}}{x^{2\alpha_i} - \beta^{\alpha_i}} - \frac{\beta^{\alpha_j} x^{\alpha_j}}{x^{2\alpha_j} - \beta^{2\alpha_j}}\Big\} F_{X_{n:n}}(x; \underline{\alpha}, \beta).\end{aligned}$$

Let us define

$$\phi(\alpha) = \frac{(\beta x)^{\alpha}}{x^{2\alpha} - \beta^{2\alpha}}. \tag{3.10}$$

Differentiating (3.10) with respect to α, we get

$$\phi'(\alpha) = \frac{(\beta x)^{\alpha}(x^{2\alpha} + \beta^{2\alpha})}{(x^{2\alpha} - \beta^{2\alpha})^2}(\ln \beta - \ln x). \tag{3.11}$$

Thus, for $x > \beta$, $\phi'(\alpha) < 0$. This implies that $\phi(\alpha)$ is decreasing in $\alpha > 0$ for $x > \beta$. Hence for $\alpha_i \geqslant \alpha_j$ ($\alpha_i \leqslant \alpha_j$) we have $\phi(\alpha_i) \leqslant \phi(\alpha_j)$ ($\phi(\alpha_i) \geqslant \phi(\alpha_j)$), which implies $(\alpha_i - \alpha_j)(\phi(\alpha_i) - \phi(\alpha_j)) \leqslant 0$. Thus

$$(\alpha_i - \alpha_j)\Big(\frac{\partial F_{X_{n:n}}(x; \underline{\alpha}, \beta)}{\partial \alpha_i} - \frac{\partial F_{X_{n:n}}(x; \underline{\alpha}, \beta)}{\partial \alpha_j}\Big) \leqslant 0.$$

From Lemma 2.2, we can easily conclude that $F_{X_{n:n}}(x;\underline{\alpha},\beta)$ is Schur-concave in $\underline{\alpha}$. Thus under the given hypothesis, we get $F_{X_{n:n}}(x;\underline{\alpha},\beta) \leqslant F_{Y_{n:n}}(x;\underline{\alpha}^*,\beta)$ which in-turn implies that $X_{n:n} \geqslant_{\mathrm{st}} Y_{n:n}$. This completes the proof of the first part.

(b) The survival function of $X_{1:n}$ can be written as

$$\bar{F}_{X_{1:n}}(x;\underline{\alpha},\beta) = \prod_{i=1}^{n} \frac{2\beta^{\alpha_i}}{x^{\alpha_i}+\beta^{\alpha_i}}, \quad x > \beta. \tag{3.12}$$

Differentiating (3.12) with respect to α_i, we get

$$\frac{\partial \bar{F}_{X_{1:n}}(x;\underline{\alpha},\beta)}{\partial \alpha_i} = \bar{F}_{X_{1:n}}(x;\underline{\alpha},\beta)(\ln\beta - \ln x)\frac{x^{\alpha_i}}{x^{\alpha_i}+\beta^{\alpha_i}}.$$

Thus,

$$\begin{aligned}(\alpha_i-\alpha_j)&\left(\frac{\partial \bar{F}_{X_{1:n}}(x;\underline{\alpha},\beta)}{\partial \alpha_i} - \frac{\partial \bar{F}_{X_{1:n}}(x;\underline{\alpha},\beta)}{\partial \alpha_j}\right)\\ &= (\alpha_i-\alpha_j)(\ln\beta-\ln x)\left\{\frac{x^{\alpha_i}}{x^{\alpha_i}+\beta^{\alpha_i}} - \frac{x^{\alpha_j}}{x^{\alpha_j}+\beta^{\alpha_j}}\right\}\bar{F}_{X_{1:n}}(x;\underline{\alpha},\beta).\end{aligned}$$

Now, for fixed x and β, let us define the function $\psi(\alpha)$ as $\psi(\alpha) = x^{\alpha}/(x^{\alpha}+\beta^{\alpha})$. To study its monotonicity, we differentiate $\psi(\alpha)$ with respect to α, which is given by

$$\psi'(\alpha) = \frac{(\beta x)^{\alpha}(\ln x - \ln\beta)}{(x^{\alpha}+\beta^{\alpha})^2}. \tag{3.13}$$

Note that for $x > \beta$, $\psi'(\alpha) > 0$. So, $\psi(\alpha)$ is increasing in $\alpha > 0$. Thus, for any $\alpha_i \geqslant \alpha_j$ ($\alpha_i \leqslant \alpha_j$), we have $\psi(\alpha_i) \geqslant \psi(\alpha_j)$ ($\psi(\alpha_i) \leqslant \psi(\alpha_j)$). Hence, $(\alpha_i-\alpha_j)\times(\ln\beta-\ln x)(\psi(\alpha_i)-\psi(\alpha_j)) \leqslant 0$, which implies that

$$(\alpha_i-\alpha_j)\left(\frac{\partial \bar{F}_{X_{1:n}}(x)}{\partial \alpha_i} - \frac{\partial \bar{F}_{X_{1:n}}(x)}{\partial \alpha_j}\right) \leqslant 0.$$

So, according to Lemma 2.2, $\bar{F}_{X_{1:n}}(x;\underline{\alpha},\beta)$ is Schur-concave in $\underline{\alpha}$ and hence $X_{1:n} \leqslant_{\mathrm{st}} Y_{1:n}$. This completes the proof. □

The following result generalizes Theorem 3.6(b) to a wide range of shape parameters.

Theorem 3.7. *Let $X_1, X_2, \ldots, X_n$ be independent random variables with $X_i \sim \mathrm{NP}(\alpha_i,\beta)$, $i = 1,2,\ldots,n$ and $Y_1, Y_2, \ldots, Y_n$ independent random variables with $Y_i \sim \mathrm{NP}(\alpha_i^*,\beta)$, $i = 1,2,\ldots,n$. If $(\alpha_1,\alpha_2,\ldots,\alpha_n) \geqslant (\alpha_1^*,\alpha_2^*,\ldots,\alpha_n^*)$, i.e. $\alpha_i \geqslant \alpha_i^*$, $i = 1,2,\ldots,n$, then $X_{1:n} \leqslant_{\mathrm{fr}} Y_{1:n}$.*

P r o o f. The failure rates of $X_{1:n}$ and $Y_{1:n}$ can be obtained as

$$r_{X_{1:n}}(x;\underline{\alpha},\beta)=\frac{1}{x}\sum_{i=1}^{n}\frac{\alpha_i}{1+(\beta/x)^{\alpha_i}},\qquad x>\beta$$

and

$$r_{Y_{1:n}}(x;\underline{\alpha}^*,\beta)=\frac{1}{x}\sum_{i=1}^{n}\frac{\alpha_i^*}{1+(\beta/x)^{\alpha_i^*}},\qquad x>\beta,$$

respectively. Now consider the difference

$$r_{X_{1:n}}(x;\underline{\alpha},\beta)-r_{Y_{1:n}}(x;\underline{\alpha}^*,\beta)=\frac{1}{x}\sum_{i=1}^{n}\Big\{\frac{\alpha_i}{1+(\beta/x)^{\alpha_i}}-\frac{\alpha_i^*}{1+(\beta/x)^{\alpha_i^*}}\Big\}. \tag{3.14}$$

Let $\eta(\alpha)=\alpha/(1+(\beta/x)^{\alpha})$ for $x>\beta$. Then it can be shown that $\eta(\alpha)$ is an increasing function in $\alpha>0$. Hence, under the hypothesis made, and from (3.14), we have that $r_{X_{1:n}}(x;\underline{\alpha},\beta)-r_{Y_{1:n}}(x;\underline{\alpha}^*,\beta)\geqslant 0$ implies $X_{1:n}\leqslant_{\rm fr} Y_{1:n}$. This completes the proof. □

C o u n t e r e x a m p l e 3.5. Let X_1,X_2,X_3,X_4 be independent random variables with $X_i\sim \mathrm{NP}(\alpha_i,1.6)$, $i=1,2,3,4$ with $\alpha_1=0.6$, $\alpha_2=1.9$, $\alpha_3=0.3$, and $\alpha_4=1.8$. Further, let Y_1,Y_2,Y_3,Y_4 be another set of independent random variables with $Y_i\sim \mathrm{NP}(\alpha_i^*,1.6)$, $i=1,2,3,4$ with $\alpha_1^*=0.4$, $\alpha_2^*=2.1$, $\alpha_3^*=0.8$, and $\alpha_4^*=1.6$. It is easy to see that $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\not\succeq(\alpha_1^*,\alpha_2^*,\alpha_3^*,\alpha_4^*)$. Now, we have plotted $r_{X_{1:n}}(x)-r_{Y_{1:n}}(x)$ in Figure 3(a). Thus, $X_{1:4}\not\leqslant_{\rm fr} Y_{1:4}$.

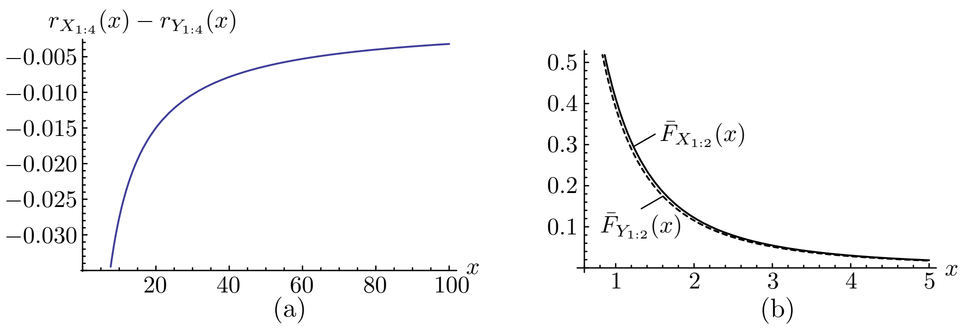

Figure 3. (a) Plot of $r_{X_{1:4}}(x)-r_{Y_{1:4}}(x)$ for the random variables as described in Counterexample 3.5. (b) Graphs of the survival functions of $X_{1:2}$ and $Y_{1:2}$ considered in Counterexample 3.6.

Theorem 3.8. *Let $X_1,X_2,\dots,X_n$ be n independent random variables with $X_i\sim \mathrm{NP}(\alpha_i,\beta)$, $i=1,2,\dots,n$, and let $Y_1,Y_2,\dots,Y_n$ be n independent random variables with $Y_i\sim \mathrm{NP}(\alpha_i^*,\beta)$, $i=1,2,\dots,n$. Then, for $(\alpha_1,\alpha_2,\dots,\alpha_n)\succeq_w(\alpha_1^*,\alpha_2^*,\dots,\alpha_n^*)$, we have $X_{1:n}\leqslant_{\rm st} Y_{1:n}$.*

P r o o f. It is given that $(\alpha_1, \alpha_2, \ldots, \alpha_n) \succeq_w (\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*)$. Thus by virtue of Lemma 2.1, there exists a vector $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $(\alpha_1, \alpha_2, \ldots, \alpha_n) \succeq_m (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $(\alpha_1, \alpha_2, \ldots, \alpha_n) > (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Let $W_1, W_2, \ldots, W_n$ be another set of independent random variables with $W_i \sim \mathrm{NP}(\lambda_i, \beta)$, $i = 1, 2, \ldots, n$. Thus from Theorem 3.6(b) we have

$$(\alpha_1, \alpha_2, \ldots, \alpha_n) \succeq_m (\lambda_1, \lambda_2, \ldots, \lambda_n) \Rightarrow X_{1:n} \leqslant_{\mathrm{st}} W_{1:n}. \tag{3.15}$$

Further, from Theorem 3.7 we get

$$(\alpha_1, \alpha_2, \ldots, \alpha_n) > (\lambda_1, \lambda_2, \ldots, \lambda_n) \Rightarrow W_{1:n} \leqslant_{\mathrm{fr}} Y_{1:n} \Rightarrow W_{1:n} \leqslant_{\mathrm{st}} Y_{1:n}. \tag{3.16}$$

By combining (3.15) and (3.16), the required result follows. □

R e m a r k 3.2. One can use the results obtained in this subsection to get lower (upper) bounds for the survival functions, failure rates, variance of parallel (series) system consisting of independent and heterogeneous NP type components in terms of the corresponding functions of the parallel (series) system consisting of independent and heterogeneous NP type components.

3.2. Stochastic comparisons based on multivariate chain majorization. In this part of the paper, we consider a system with independent NP type components. We assume heterogeneity in both the scale and shape parameters. In this case, the parameters can be represented in the form of a matrix. Here, we study stochastic comparisons of lifetimes of two series systems when the matrix of the parameters changes to another matrix of parameters in the sense of multivariate chain majorization. Below we consider series systems with two components.

Theorem 3.9. *Let X_1, X_2 be independent random variables with $X_i \sim \mathrm{NP}(\alpha_i, \beta_i)$, $i = 1, 2$ and Y_1, Y_2 other independent random variables with $Y_i \sim \mathrm{NP}(\alpha_i^*, \beta_i^*)$, $i = 1, 2$. Let $\left(\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{smallmatrix}\right) \in S_2$ be defined as in (2.1). Then, for*

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \gg \begin{pmatrix} \alpha_1^* & \alpha_2^* \\ \beta_1^* & \beta_2^* \end{pmatrix},$$

we have $X_{1:2} \leqslant_{\mathrm{st}} Y_{1:2}$.

P r o o f. The survival function of $X_{1:2}$ is given by

$$\bar{F}_{X_{1:2}}(x; \underline{\alpha}, \underline{\beta}) = \prod_{i=1}^{2} \frac{2\beta_i^{\alpha_i}}{x^{\alpha_i} + \beta_i^{\alpha_i}}, \quad x > \max\{\beta_1, \beta_2\}. \tag{3.17}$$

Differentiating (3.17) with respect to α_i, we get

$$\frac{\partial \bar{F}_{X_{1:2}}(x;\underline{\alpha},\underline{\beta})}{\partial \alpha_i} = \bar{F}_{X_{1:2}}(x;\underline{\alpha},\underline{\beta})\frac{\ln(\beta_i/x)}{1+(\beta_i/x)^{\alpha_i}} = \bar{F}_{X_{1:2}}(x;\underline{\alpha},\underline{\beta})\delta(\alpha_i,\beta_i/x),$$

where $\delta(\cdot,\cdot)$ is defined in Lemma 2.4. Further, differentiating $\bar{F}_{X_{1:2}}(x;\underline{\alpha},\underline{\beta})$ given by (3.17) with respect to β_i, we obtain

$$\begin{aligned}\frac{\partial \bar{F}_{X_{1:2}}(x;\underline{\alpha},\underline{\beta})}{\partial \beta_i} &= \frac{1}{x}\bar{F}_{X_{1:2}}(x;\underline{\alpha},\underline{\beta})\frac{\alpha_i}{(\beta_i/x)(1+(\beta_i/x)^{\alpha_i})}\\ &= \frac{1}{x}\bar{F}_{X_{1:2}}(x;\underline{\alpha},\underline{\beta})\varrho(\alpha_i,\beta_i/x),\end{aligned}$$

where $\varrho(\cdot,\cdot)$ is defined in Lemma 2.5. Now, let us define $\varphi(\underline{\alpha},\underline{\beta})$ as

$$\begin{aligned}\varphi(\underline{\alpha},\underline{\beta}) &= (\alpha_1-\alpha_2)\Big(\frac{\partial \bar{F}_{X_{1:2}}(x;\underline{\alpha},\underline{\beta})}{\partial \alpha_1} - \frac{\partial \bar{F}_{X_{1:2}}(x;\underline{\alpha},\underline{\beta})}{\partial \alpha_2}\Big)\\ &\quad + (\beta_1-\beta_2)\Big(\frac{\partial \bar{F}_{X_{1:2}}(x;\underline{\alpha},\underline{\beta})}{\partial \beta_1} - \frac{\partial \bar{F}_{X_{1:2}}(x;\underline{\alpha},\underline{\beta})}{\partial \beta_2}\Big)\\ &= (\alpha_1-\alpha_2)\bar{F}_{X_{1:2}}(x)(\delta(\alpha_1,\beta_1/x)-\delta(\alpha_2,\beta_2/x))\\ &\quad + \frac{1}{x}(\beta_1-\beta_2)\bar{F}_{X_{1:2}}(x)(\varrho(\alpha_1,\beta_1/x)-\varrho(\alpha_2,\beta_2/x)).\end{aligned} \tag{3.18}$$

We have assumed that $\left(\begin{smallmatrix}\alpha_1 & \alpha_2\\ \beta_1 & \beta_2\end{smallmatrix}\right) \in S_2$, i.e., if $\alpha_1 \leqslant \alpha_2$ ($\alpha_1 \geqslant \alpha_2$), then $\beta_1 \geqslant \beta_2$ ($\beta_1 \leqslant \beta_2$). Further, according to Lemma 2.4, $\delta(\alpha,t)$ is increasing with respect to t and decreasing with respect to α. Thus, we have $\delta(\alpha_1,\beta_1/x) \geqslant \delta(\alpha_1,\beta_2/x) \geqslant \delta(\alpha_2,\beta_2/x)$ ($\delta(\alpha_1,\beta_1/x) \leqslant \delta(\alpha_1,\beta_2/x) \leqslant \delta(\alpha_2,\beta_2/x)$), which implies

$$(\alpha_1-\alpha_2)(\delta(\alpha_1,\beta_1/x)-\delta(\alpha_2,\beta_2/x)) \leqslant 0. \tag{3.19}$$

Similarly, according to Lemma 2.5, $\varrho(\alpha,t)$ is decreasing with respect to t and increasing with respect to α. Hence, $\varrho(\alpha_1,\beta_1/x) \leqslant \varrho(\alpha_1,\beta_2/x) \leqslant \varrho(\alpha_2,\beta_2/x)$ ($\varrho(\alpha_1,\beta_1/x) \geqslant \varrho(\alpha_1,\beta_2/x) \geqslant \varrho(\alpha_2,\beta_2/x)$) which implies

$$(\beta_1-\beta_2)(\varrho(\alpha_1,\beta_1/x)-\varrho(\alpha_2,\beta_2/x)) \leqslant 0. \tag{3.20}$$

By virtue of the inequalities obtained in (3.19) and (3.20), it is not hard to conclude from (3.18) that $\varphi(\underline{\alpha},\underline{\beta}) \leqslant 0$. Also, $\bar{F}_{X_{1:2}}(x,\underline{\alpha},\underline{\beta})$ is a permutation symmetric function. Thus using Lemma 2.6, we get

$$\begin{aligned}\begin{pmatrix}\alpha_1 & \alpha_2\\ \beta_1 & \beta_2\end{pmatrix} \gg \begin{pmatrix}\alpha_1^* & \alpha_2^*\\ \beta_1^* & \beta_2^*\end{pmatrix} &\Rightarrow \bar{F}_{X_{1:2}}(x;\underline{\alpha},\underline{\beta}) \leqslant \bar{F}_{Y_{1:2}}(x;\underline{\alpha}^*,\underline{\beta}^*)\\ &\Rightarrow X_{1:2} \leqslant_{\rm st} Y_{1:2}.\end{aligned}$$

This completes the proof. □

The following counterexample shows that the result in Theorem 3.9 need not hold if both the matrices of parameters do not belong to S_2.

C o u n t e r e x a m p l e 3.6. Let X_1, X_2 be two independent random variables associated with $X_i \sim \mathrm{NP}(\alpha_i, \beta_i)$, $i = 1, 2$. Also, let Y_1, Y_2 be two independent random variables associated with $Y_i \sim \mathrm{NP}(\alpha_i^*, \beta_i^*)$, $i = 1, 2$. Let

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} = \begin{pmatrix} 1.5 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}.$$

We take a T-transform matrix $T_{0.3} = \left(\begin{smallmatrix} 0.3 & 0.7 \\ 0.7 & 0.3 \end{smallmatrix}\right)$. Then

$$B = \begin{pmatrix} \alpha_1^* & \alpha_2^* \\ \beta_1^* & \beta_2^* \end{pmatrix} = AT_{0.3} = \begin{pmatrix} 1.01 & 1.29 \\ 0.46 & 0.54 \end{pmatrix}.$$

Thus, according to Definition 2.4, we have $A \gg B$. Here, the matrices A and B do not belong to S_2. Now from Figure 3(b), we see that $X_{1:2} \nleqslant_{\mathrm{st}} Y_{1:2}$.

The next theorem is an extension of Theorem 3.9 for $n > 2$.

Theorem 3.10. *Let $X_1, X_2, \ldots, X_n$ be independent random variables with $X_i \sim \mathrm{NP}(\alpha_i, \beta_i)$, $i = 1, 2, \ldots, n$. Further, let $Y_1, Y_2, \ldots, Y_n$ be another set of independent random variables with $Y_i \sim \mathrm{NP}(\alpha_i^*, \beta_i^*)$. Then, for*

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \beta_1 & \beta_2 & \ldots & \beta_n \end{pmatrix} \in S_n,$$

we have

$$\begin{pmatrix} \alpha_1^* & \alpha_2^* & \ldots & \alpha_n^* \\ \beta_1^* & \beta_2^* & \ldots & \beta_n^* \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \beta_1 & \beta_2 & \ldots & \beta_n \end{pmatrix} T_w \Rightarrow X_{1:n} \leqslant_{\mathrm{st}} Y_{1:n}.$$

P r o o f. Let us take $\eta_n(\underline{\alpha}, \underline{\beta}) = \bar{F}_{X_{1:n}}(x; \underline{\alpha}, \underline{\beta})$ and $\varphi(\alpha_i, \beta_i) = 2\beta_i^{\alpha_i}/(x^{\alpha_i} + \beta_i^{\alpha_i})$ for $i = 1, 2, \ldots, n$. Then

$$\bar{F}_{X_{1:n}}(x; \underline{\alpha}, \underline{\beta}) = 2^n \prod_{i=1}^{n} \frac{\beta_i^{\alpha_i}}{x^{\alpha_i} + \beta_i^{\alpha_i}} \Rightarrow \eta_n(\underline{\alpha}, \underline{\beta}) = \prod_{i=1}^{n} \varphi(\alpha_i, \beta_i). \tag{3.21}$$

We have proved that $\eta_2(\underline{\alpha}, \underline{\beta})$ satisfies (2.2). So, by Lemma 2.7 we have

$$\eta_n(\underline{\alpha}, \underline{\beta}) \leqslant \eta_n(\underline{\alpha}^*, \underline{\beta}^*) \Rightarrow \bar{F}_{X_{1:n}}(x; \underline{\alpha}, \underline{\beta}) \leqslant \bar{F}_{Y_{1:n}}(x; \underline{\alpha}^*, \underline{\beta}^*) \Rightarrow X_{1:n} \leqslant_{\mathrm{st}} Y_{1:n}.$$

This completes the proof. □

Further, it can be easily shown that the finite product of T-transform matrices having the same structure is also a T-transform matrix (see [2]). Hence, the following corollary immediately follows from Theorem 3.10.

Corollary 3.2. *Let $X_1, X_2, \ldots, X_n$ be independent random variables with $X_i \sim \mathrm{NP}(\alpha_i, \beta_i)$, $i = 1, 2, \ldots, n$. Further, let $Y_1, Y_2, \ldots, Y_n$ be another set of independent random variables with $Y_i \sim \mathrm{NP}(\alpha_i^*, \beta_i^*)$. Then, for*

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \beta_1 & \beta_2 & \ldots & \beta_n \end{pmatrix} \in S_n$$

we have

$$\begin{pmatrix} \alpha_1^* & \alpha_2^* & \ldots & \alpha_n^* \\ \beta_1^* & \beta_2^* & \ldots & \beta_n^* \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \beta_1 & \beta_2 & \ldots & \beta_n \end{pmatrix} T_{w_1} \ldots T_{w_k},$$

where T_{w_i}, $i = 1, 2, \ldots, k$, have the same structure, we get $X_{1:n} \leqslant_{\mathrm{st}} Y_{1:n}$.

Our next theorem shows that the result in Corollary 3.2 holds for T-transform matrices with different structures.

Theorem 3.11. *Let $X_1, X_2, \ldots, X_n$ be independent random variables with $X_i \sim \mathrm{NP}(\alpha_i, \beta_i)$, $i = 1, 2, \ldots, n$. Further, let $Y_1, Y_2, \ldots, Y_n$ be another set of independent random variables with $Y_i \sim \mathrm{NP}(\alpha_i^*, \beta_i^*)$, $i = 1, 2, \ldots, n$. For $i = 1, 2, \ldots, k$, where $k \geqslant 1$, let*

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \beta_1 & \beta_2 & \ldots & \beta_n \end{pmatrix} \in S_n \quad \text{and} \quad \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \beta_1 & \beta_2 & \ldots & \beta_n \end{pmatrix} T_{w_1} \ldots T_{w_i} \in S_n.$$

Then

$$\begin{pmatrix} \alpha_1^* & \alpha_2^* & \ldots & \alpha_n^* \\ \beta_1^* & \beta_2^* & \ldots & \beta_n^* \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \beta_1 & \beta_2 & \ldots & \beta_n \end{pmatrix} T_{w_1} \ldots T_{w_k} \Rightarrow X_{1:n} \leqslant_{\mathrm{st}} Y_{1:n}.$$

Proof. Let

$$\begin{pmatrix} \alpha_1^{(i)} & \alpha_2^{(i)} & \ldots & \alpha_n^{(i)} \\ \beta_1^{(i)} & \beta_2^{(i)} & \ldots & \beta_n^{(i)} \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \beta_1 & \beta_2 & \ldots & \beta_n \end{pmatrix} T_{w_1} \ldots T_{w_i},$$

for $i = 1, 2, \ldots, k$. Further, assume $Z_1^{(i)}, \ldots, Z_n^{(i)}$, $i = 1, 2, \ldots, k$, are independent sets of random variables with $Z_j^{(i)} \sim \mathrm{NP}(\alpha_j^{(i)}, \beta_j^{(i)})$, $j = 1, 2, \ldots, n$ and $i =$

$1,2,\dots,k$. If $\left(\begin{smallmatrix}\alpha_1 & \alpha_2 & \dots & \alpha_n\\ \beta_1 & \beta_2 & \dots & \beta_n\end{smallmatrix}\right) \in S_n$, then from the assumption, it can be easily seen that $\left(\begin{smallmatrix}\alpha_1^{(i)} & \alpha_2^{(i)} & \dots & \alpha_n^{(i)}\\ \beta_1^{(i)} & \beta_2^{(i)} & \dots & \beta_n^{(i)}\end{smallmatrix}\right) \in S_n$ for $i = 1,2,\dots,k$. Now

$$\begin{aligned}
&\begin{pmatrix}\alpha_1^* & \alpha_2^* & \dots & \alpha_n^*\\ \beta_1^* & \beta_2^* & \dots & \beta_n^*\end{pmatrix} = \left\{\begin{pmatrix}\alpha_1 & \alpha_2 & \dots & \alpha_n\\ \beta_1 & \beta_2 & \dots & \beta_n\end{pmatrix} T_{w_1}\dots T_{w_{k-1}}\right\} T_{w_k}\\
\Rightarrow &\begin{pmatrix}\alpha_1^* & \alpha_2^* & \dots & \alpha_n^*\\ \beta_1^* & \beta_2^* & \dots & \beta_n^*\end{pmatrix} = \begin{pmatrix}\alpha_1^{(k-1)} & \alpha_2^{(k-1)} & \dots & \alpha_n^{(k-1)}\\ \beta_1^{(k-1)} & \beta_2^{(k-1)} & \dots & \beta_n^{(k-1)}\end{pmatrix} T_{w_k}\\
\Rightarrow &Z_{1:n}^{(k-1)} \leqslant_{\mathrm{st}} Y_{1:n} \text{ (from Theorem 3.10)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\begin{pmatrix}\alpha_1 & \alpha_2 & \dots & \alpha_n\\ \beta_1 & \beta_2 & \dots & \beta_n\end{pmatrix} T_{w_1}\dots T_{w_{k-1}} = \left\{\begin{pmatrix}\alpha_1 & \alpha_2 & \dots & \alpha_n\\ \beta_1 & \beta_2 & \dots & \beta_n\end{pmatrix} T_{w_1}\dots T_{w_{k-2}}\right\} T_{w_{k-1}}\\
\Rightarrow &\begin{pmatrix}\alpha_1^{(k-1)} & \alpha_2^{(k-1)} & \dots & \alpha_n^{(k-1)}\\ \beta_1^{(k-1)} & \beta_2^{(k-1)} & \dots & \beta_n^{(k-1)}\end{pmatrix} = \begin{pmatrix}\alpha_1^{(k-2)} & \alpha_2^{(k-2)} & \dots & \alpha_n^{(k-2)}\\ \beta_1^{(k-2)} & \beta_2^{(k-2)} & \dots & \beta_n^{(k-2)}\end{pmatrix} T_{w_{k-1}}\\
\Rightarrow &Z_{1:n}^{(k-2)} \leqslant_{\mathrm{st}} Z_{1:n}^{(k-1)}.
\end{aligned}$$

Repeating this, we get

$$X_{1:n} \leqslant_{\mathrm{st}} Z_{1:n}^{(1)} \leqslant_{\mathrm{st}} \dots \leqslant_{\mathrm{st}} Z_{1:n}^{(k-2)} \leqslant_{\mathrm{st}} Z_{1:n}^{(k-1)} \leqslant_{\mathrm{st}} Y_{1:n},$$

which completes the proof. □

4. Application

It was mentioned earlier that the stochastic comparison results are useful in various areas of research. In this section, we discuss applications of a few of the established results. Note that the new Pareto type distribution has been shown to be a better model among various other lifetime models by [6] based on the Akaike information criterion, Bayesian information criterion and consistent Akaike information criterion. Consider a parallel system consisting of n independently working components. It is known that the system fails if all the components fail. Let us assume that one is interested in stochastically comparing the performance of parallel systems comprising of new Pareto type components.

(i) Consider two parallel systems, say A and B comprising n components each. Suppose that X_i, $i = 1,2,\dots,n$, is the failure time of the ith component of system A and Y_i, $i = 1,2,\dots,n$ is the failure time of the ith component of system B. For $i = 1,2,\dots,n$, let $X_i \sim \mathrm{NP}(\alpha,\beta_i)$ and $Y_i \sim \mathrm{NP}(\alpha,\beta_i^*)$. Here, assume $\alpha \geqslant 1$. The first part of Theorem 3.1 ensures that for two parallel systems of components having independent new Pareto type distributed lifetimes, if the scale parameter vector $\underline{\beta}$ of system A is more dispersed than the scale parameter vector $\underline{\beta}^*$ of system B, then the lifetime of system A will be larger than that of system B in the usual stochastic order. For the systems considered above, the first part of Theorem 3.3 tells that if the scale parameter of the ith component of system A is greater than or equal to that of the ith component of system B, $i = 1, 2, \ldots, n$, then the lifetime of system A will be larger than that of system B in the usual stochastic order when $\alpha > 0$. For $\alpha \geqslant 1$, Theorem 3.4 states that the lifetime of system A is larger than that of system B in the usual stochastic order when there is weak lower majorization between the vectors $\underline{\beta}$ and $\underline{\beta}^*$.

(ii) As in Part (i), we further consider two parallel systems A and B. Assume that they each have n independently distributed components. Let $X_i \sim \mathrm{NP}(\alpha_i, \beta)$ and $Y_i \sim \mathrm{NP}(\alpha_i^*, \beta)$, where $i = 1, 2, \ldots, n$. Also, note that X_i and Y_i represent the failure times of the ith component of the systems A and B, respectively. Thus, for a fixed β, using the first part of Theorem 3.6, we say that the system corresponding to the majorized shape parameter vector leads to a parallel system having smaller lifetime.

(iii) Let us consider two series systems, say C and D. Assume that they have n components each. We know that a series system will fail if at least one of the components fails. If X_i, $i = 1, 2, \ldots, n$, denotes the failure time of the ith component of system C and Y_i, $i = 1, 2, \ldots, n$, the failure time of the ith component of system D, then $X_{1:n}$ and $Y_{1:n}$ represent the lifetimes of the systems C and D, respectively. For $i = 1, 2, \ldots, n$, let $X_i \sim \mathrm{NP}(\alpha, \beta_i)$ and $Y_i \sim \mathrm{NP}(\alpha, \beta_i^*)$. For fixed α, if the scale parameter vector $\underline{\beta}$ of system C is more dispersed than the scale parameter vector $\underline{\beta}^*$ of system D, then the second part of Theorem 3.1 ensures that the lifetime of system C will be smaller than that of system D in the usual stochastic order. Analogously, for fixed $\alpha \leqslant 1$, Theorem 3.2 tells that if $\underline{\beta}^{-1}$ is more dispersed than $\underline{\beta}^{*-1}$, then the lifetime of system C is smaller than that of system D in the usual stochastic ordering.

Example 4.1. Suppose there are two systems, say system-I and system-II. Let each system have three components which are connected in series. Denote the components of system-I as I_1, I_2 and I_3 and those of system-II as II_1, II_2 and II_3. Let the failure times of I_1, I_2 and I_3 have $\mathrm{NP}(0.5, 0.1)$, $\mathrm{NP}(0.5, 1)$ and $\mathrm{NP}(0.5, 9)$ distributions, respectively. Further, let the failure times of II_1, II_2, and II_3 have $\mathrm{NP}(0.5, 0.1)$, $\mathrm{NP}(0.5, 4)$, and $\mathrm{NP}(0.5, 6)$ distributions, respectively. Clearly, $(0.1, 1, 9) \succeq_m (0.1, 4, 6)$. Thus, as an application of Corollary 3.1, system-II is preferable to use, since its lifetime has smaller variance.

5. Concluding remarks

The new Pareto type distribution has been proposed by [6] in the literature to study reliability and income data. In the present communication, we have studied stochastic comparisons of the lifetimes of series and parallel systems with independent heterogeneous NP type components. In this context, sufficient conditions associated with vector majorization and multivariate chain majorization have been provided. In many practical situations, extremes of more than two random lifetimes are often encountered. The contribution made in this paper is expected to be of interest to the reliability theorists and practitioners to obtain better systems.

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