CONTROL VARIATIONAL METHOD APPROACH TO BENDING AND CONTACT PROBLEMS FOR GAO BEAM

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This paper is dedicated to the memory of our former colleague,
doc. RNDr. Ing. Jiří Horák, CSc.,
who unexpectedly passed away this summer.
He initiated our research devoted to nonlinear beam problems.

Abstract. This paper deals with a nonlinear beam model which was published by D. Y. Gao in 1996. It is considered either pure bending or a unilateral contact with elastic foundation, where the normal compliance condition is employed. Under additional assumptions on data, higher regularity of solution is proved. It enables us to transform the problem into a control variational problem. For basic types of boundary conditions, suitable transformations of the problem are derived. The control variational problem contains a simple linear state problem and it is solved by the conditioned gradient method. Illustrative numerical examples are introduced in order to compare the Gao beam with the classical Euler-Bernoulli beam.

Keywords: nonlinear beam; elastic foundation; contact problem; normal compliance condition; control variational method; finite element method

MSC 2010: 49J15, 49S05, 65K10, 74K10, 74M15

1. Introduction

Beams are essential components of many engineering constructions, thus knowledge how to solve various beam problems is quite important. The classical Euler-Bernoulli model is still the most popular but as a linear model has certain limits of applicability. This paper deals with a mathematical beam model governed by a fourth-order nonlinear differential equation which was introduced by Gao in [9]. The beam is subjected to an axial constant force and transverse loads, i.e., loads that act perpendicular to the longitudinal axis of the beam.

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If a tension is prescribed in the axial direction then the energy potential is convex and the problem has a unique solution. The same holds for a sufficiently small axial compression. However, the energy potential is nonconvex for larger compression and has three local extremes. This leads to a physical phenomenon which is known as buckling. In [6] the nonconvex case with the axial compression was studied by means of the so-called canonical dual finite element method and Gao's canonical dual transformation, see [10].

Further, interaction between the beam and the foundation is often studied. The foundation can be either rigid or deformable. In the former case, we arrive at Signorini's conditions leading to variational inequalities (see, e.g., [13] or [14]). We will consider the *deformable foundation* based on the *normal compliance contact condition* (see, e.g., [22], [24]). This condition can be included into the beam models by an additional nonlinear term. The Euler-Bernoulli beam model enriched with this nonlinear term was studied in [15], [17], [26], [27], [23]. The Gao beam model with the normal compliance contact condition was introduced in a recent paper [11].

For the sake of completeness, we note that the dynamic contact of the Gao beam with a reactive or rigid foundation was described recently in [3]. Vibration characteristic of contacting one-dimensional structures with a Gao beam were dealt with in [2], where the model, existence of weak solutions, and computer simulations can be found. It is interesting to note that dynamic problems with the normal compliance condition have always a unique solution. Nevertheless, the dynamic models are out of the scope of this paper.

In this paper, we focus on a transformation of the static Gao beam model into an optimal control problem. This idea is motivated by the monograph [20]. The optimal control problems are studied, e.g., in [16], [28]. The transformation of the problem is called the *control variational method* (CVM). CVM was introduced in [4] for the first time. Later, it was applied to solve contact problems with the Euler-Bernoulli beam [25], [5]. Recently, CVM was used for solution of contact problems with the Gao beam [18] and [19].

Unlike [18] and [19], we transform the original beam model into the optimal control problem, which is convex and smooth and the related set of admissible control variables coincides with a linear space. Numerical solution of such a problem is therefore much simpler. Moreover, we present an abstract framework how to construct various transformations of the original problem into the optimal control problem by using higher regularity of the solution. This enables us to find an efficient transformation depending on prescribed boundary conditions.

The rest of the paper is organized as follows. In Section 2, the nonlinear Gao beam problem is introduced and analyzed. In Section 3, the transformations of the problem into the optimal control one are presented for various boundary conditions.

In Section 4, the contact problem including the Gao beam and the deformable foundation is formulated and the main results are extended to this case. Section 5 is devoted to numerical examples where the Gao beam is compared with the classical Euler-Bernoulli beam. Some concluding remarks are introduced in Section 6.

2. Gao beam

In this section we introduce a nonlinear mathematical beam model which was invented by D. Y. Gao. Next considerations will adhere to the following assumptions:

- ▷ the Euler-Bernoulli hypothesis is valid (i.e. straight lines orthogonal to the mid-surface remain straight and orthogonal to it even after a deformation),
- ▷ the material of the beam is isotropic,
- ▷ the beam has a uniform cross-section of a rectangular shape,
- ▷ in addition to a transverse load, an axial load will also be considered here.

Using the finite deformation theory for Hooke's material Gao proposed in [9] nonlinear beam model with moderately large elastic deformations. Contrary to the Euler-Bernoulli beam model, neither the stresses nor the deformations of the cross sections in the lateral direction are neglected in the Gao model.

Let us denote by E the Young modulus, I the area moment of inertia, q the distributed transverse load, ν the Poisson ratio, L the length of the beam, $2h$ its height and b its width. Further, we set

$$I = \tfrac{2}{3}h^3 b, \quad \alpha = 3hb(1-\nu^2), \quad \mu = (1+\nu)(1-\nu^2), \quad f = (1-\nu^2)q. \tag{2.1}$$

Finally, we introduce the constant axial force P and assume that $P > 0$ and $P < 0$ cause a compression and a tension, respectively, see Fig. 1.

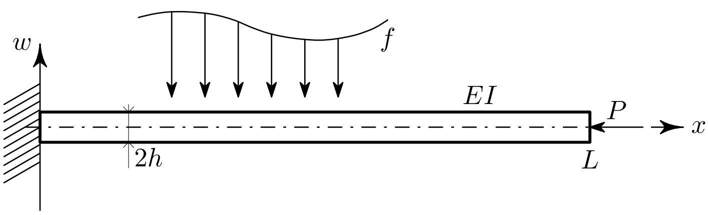

Figure 1. Nonlinear Gao beam.

The model can be described by the following fourth-order differential equation with respect to the unknown deflection w of the beam [9]:

$$EIw'''' - E\alpha(w')^2 w'' + P\mu w'' = f \quad \text{in } (0, L). \tag{2.2}$$

The model can be completed by one of the following sets of stable and unstable boundary conditions (the eventual unstable conditions are separated by comma within each set):

(B1) clamped (or fixed) beam
$w(0) = w'(0) = w(L) = w'(L) = 0$,

(B2) propped cantilever beam
$w(0) = w'(0) = w(L) = 0$, $w''(L) = 0$,

(B3) cantilever beam
$w(0) = w'(0) = 0$, $w''(L) = EIw'''(L) - \frac{1}{3}E\alpha(w'(L))^3 + P\mu w'(L) = 0$,

(B4) simply supported beam
$w(0) = w(L) = 0$, $w''(0) = w''(L) = 0$.

Remark 2.1. The classical Euler-Bernoulli beam model is known primarily in the form $EIw'''' = q$ in $(0, L)$. Including the axial force, we arrive at the equation $EIw'''' + Pw'' = q$ in $(0, L)$ (see, e.g., [7]).

From now on, we will assume:

(A1) f belongs to $L^2((0, L))$,

(A2) E, I, α, μ are positive constants.

In order to introduce the variational setting of the problem, we define the space V of admissible displacements, which satisfies $H_0^2((0, L)) \subset V \subset H^2((0, L))$ and contains the constraints on the stable boundary conditions specified above. For example, we have $V = H_0^2((0, L))$ for the boundary conditions (B1). Further, let $(\cdot, \cdot)_k$ and $\|\cdot\|_k$ denote the standard scalar product and the corresponding norm in $H^k((0, L))$, $k = 0, 1, 2, \ldots$, respectively, where $H^0((0, L)) = L^2((0, L))$. We will use the fact that the space $H^k((0, L))$, $k = 1, 2, \ldots$, can be continuously embedded into $C^{k-1}([0, L])$ (see [1]). Especially, we have:

$$\exists\, c_E > 0\colon\ \max_{x\in[0,L]} |v'(x)| \leqslant c_E \|v\|_2 \quad \forall\, v \in H^2((0, L)). \tag{2.3}$$

We will also use the Friedrichs-type inequality

$$\exists\, c_I > 0\colon\ \|v''\|_0^2 \geqslant c_I \|v\|_2^2 \quad \forall\, v \in V, \tag{2.4}$$

which holds for any V defined by the boundary conditions (B1), (B2), (B3), or (B4).

From the differential equation (2.2) and the related boundary conditions, we arrive at the nonlinear variational equation

$$\text{find } w \in V\colon\ a(w, v) - d(w, v) + \pi(w, v) = (f, v)_0 \quad \forall\, v \in V, \tag{2.5}$$

where the forms $a, d, \pi\colon V \times V \to \mathbb{R}$ are defined as follows:

$$a(w,v) = EI\int_0^L w''v''\,\mathrm{d}x,\ d(w,v) = P\mu\int_0^L w'v'\,\mathrm{d}x,\ \pi(w,v) = \frac{1}{3}E\alpha\int_0^L (w')^3v'\,\mathrm{d}x.$$

We see that the forms a, d are bilinear while π is nonlinear in the first component. The inequality (2.3) implies that π is well-defined in $V \times V$.

The variational problem related to (2.5) consists of the minimization of the energy potential (see [9]):

$$\text{find } w \in V\colon\ \Pi_G(w) = \min_{v\in V}\Pi_G(v). \tag{2.6}$$

Here

$$\Pi_G(v) = \Pi_0(v) + \Pi_N(v) + \Pi_P(v), \quad v \in V, \tag{2.7}$$

where

$$\Pi_0(v) = \frac{1}{2}a(v,v) - (f,v)_0 = \frac{1}{2}EI\int_0^L (v'')^2\,\mathrm{d}x - \int_0^L fv\,\mathrm{d}x, \tag{2.8}$$

$$\Pi_N(v) = \frac{1}{4}\pi(v,v) = \frac{1}{12}E\alpha\int_0^L (v')^4\,\mathrm{d}x, \tag{2.9}$$

$$\Pi_P(v) = -\frac{1}{2}d(v,v) = -\frac{1}{2}P\mu\int_0^L (v')^2\,\mathrm{d}x. \tag{2.10}$$

Notice that the functional Π_G is continuous in V. In particular, the continuity of the part Π_N follows from (2.3). In order to show the existence of a solution to (2.6), we shall investigate the convexity and the coercivity of Π_G in V.

Lemma 2.1. *Let the assumptions* (A1), (A2) *hold and define*

$$\overline{P} = \min_{\substack{v\in V\\ v\neq 0}} \frac{EI\|v''\|_0^2}{\mu\|v'\|_0^2}. \tag{2.11}$$

Then the functional Π_G *is*

(1) *coercive in* V,
(2) *convex in* V *if* $P \leqslant \overline{P}$,
(3) *strictly convex in* V *if* $P < \overline{P}$.

P r o o f. From the Hölder inequality, we have:

$$\|v'\|_0^2 \leqslant \sqrt{L}\|(v')^2\|_0 \quad \forall\, v \in H^2((0,L)).$$

Hence and from the Friedrichs inequality (2.4) we derive the coercivity of Π_G in V:

$$\begin{aligned}\Pi_G(v) &= \frac{1}{2}EI\|v''\|_0^2 + \frac{1}{12}E\alpha\|(v')^2\|_0^2 - \frac{1}{2}P\mu\|v'\|_0^2 - (f,v)_0 \\ &\geqslant \frac{1}{2}EIc_I\|v\|_2^2 + \frac{1}{12}E\alpha L^{-1/2}\|v'\|_0^4 - \frac{1}{2}P\mu\|v'\|_0^2 - \|f\|_0\|v\|_2 \to \infty \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{as } \|v\|_2 \to \infty.\end{aligned}$$

The convexity of Π_G in V can be shown by using the second Gâteaux differential

$$\begin{aligned}\Pi''_G(w;v,v) &= EI\|v''\|_0^2 - P\mu\|v'\|_0^2 + E\alpha\int_0^L (w')^2(v')^2\,\mathrm{d}x \\ &\geqslant 0 \quad \forall\, v,w \in V,\ \forall\, P \leqslant \overline{P}.\end{aligned} \tag{2.12}$$

Analogously, we derive the sufficient condition for the strict convexity of Π_G in V. □

R e m a r k 2.2. From (2.12), we see that the condition $P \leqslant \overline{P}$ need not be necessary for the convexity of Π_G in V. So the critical value, P_{cr}^G, of the compressive axial force for the Gao beam can be higher, i.e., $P_{\mathrm{cr}}^G \geqslant \overline{P}$. On the other hand, $\overline{P}$ defines the critical value of the Euler-Bernoulli beam (see Remark 2.1), i.e. $P_{\mathrm{cr}}^E = \overline{P}$. Beyond P_{cr}^E, the Euler-Bernoulli energy potential is nonconvex with two local minima and one local maximum. This phenomenon is called *buckling*, see e.g. [7]. Extension of this result for the Gao beam was done, e.g. in [12].

From now on, we will investigate only the strict convex case and assume

(A3) P is a constant such that $P < \overline{P}$.

We arrive at the following existence result.

Theorem 2.1. *Let the assumptions* (A1)–(A3) *be satisfied. Then the problems* (2.6) *and* (2.5) *are equivalent and have a unique solution* $w \in V$. *Moreover,* $w \in H^4((0,L))$.

P r o o f. Notice that the convex and continuous functional in a Hilbert space is also weakly lower semicontinuous. Therefore, the first part of the theorem follows from the well-known result of variational calculus, see, e.g. [8].

Further, the differential equation

$$EIw'''' - E\alpha(w')^2w'' + P\mu w'' = f \tag{2.13}$$

is meaningful in the sense of distributional derivatives. Clearly, f and w'' belong to $L^2((0,L))$. Moreover, from (2.3), it follows that $(w')^2 w'' \in L^2((0,L))$. Consequently, we obtain $w'''' \in L^2((0,L))$. Hence, $w \in H^4((0,L))$. $\square$

R e m a r k 2.3. We emphasize that the higher regularity of the solution follows from the additional assumptions (A1)–(A3) on the given data. Notice that the differential equation (2.13) holds almost everywhere in $(0,L)$. The boundary conditions (B1)–(B4) are also well-defined due to the continuous embedding of $H^4((0,L))$ into $C^3([0,L])$. So the higher regularity of the solution enables us to work with the classical setting of the problem. We use this fact for easier explanation of the transformations presented in the next section.

3. Optimal control problems and Gao beam bending

We have shown that the Gao beam problem (2.6) has a unique solution $w \in V \cap H^4((0,L))$ satisfying

$$\Pi_G(w) \leqslant \Pi_G(v) \quad \forall\, v \in V, \tag{3.1}$$

under the assumptions (A1)–(A3). Transformation of (3.1) into an optimal control problem is based on the following auxiliary result.

Theorem 3.1. *Let* (A1)–(A3) *hold and* U, W *be two Hilbert spaces such that:*

(i) $W \subset V \cap H^4((0,L))$,

(ii) *the solution* w *to* (3.1) *belongs to* W,

(iii) *there is a bijective mapping* $T\colon\ U \to W$ *with the inverse* $T^{-1}\colon\ W \to U$.

Then $u^* = T^{-1}w \in U$ *is a unique solution to the minimization problem*

$$\Pi_G(Tu^*) \leqslant \Pi_G(Tu) \quad \forall\, u \in U. \tag{3.2}$$

P r o o f. From assumptions (i) and (ii) it follows that $w \in W$ is also a unique solution to the problem

$$\Pi_G(w) \leqslant \Pi_G(v) \quad \forall\, v \in W. \tag{3.3}$$

Since $T\colon\ U \to W$ is bijective, the inequality (3.3) can be transformed to (3.2), where $u^* = T^{-1}w \in U$. Clearly, u^* is a solution to (3.2) if and only if w solves (3.3). Therefore, the uniqueness of w implies that there is a unique solution to (3.2). $\square$

As we shall see, the construction of T will require to solve a state problem for given $u \in U$. Therefore, the problem (3.2) can be interpreted as the optimal control problem. For the transformation T to be meaningful, the related state problem should be simpler than the original one. Especially, if the state problem is linear then the mapping T will be affine, i.e., $Tu = T_0 u + \widehat{w}$, where $\widehat{w} \in W$ and $T_0 \colon U \to W$ is linear. In such a case, one can easily show the following useful result.

Corollary 3.1. *Let all assumptions of Theorem* 3.1 *be satisfied and* $T \colon U \to W$ *be affine, in addition. Then the functional* $J \colon U \to \mathbb{R}$, $J(u) = \Pi_G(Tu)$, *is strictly convex, continuous, coercive and Gâteaux differentiable in* U. *In particular, the Gâteaux differential of* J *at* u *in the direction* z *reads*

$$J'(u; z) = a(Tu, T_0 z) + \pi(Tu, T_0 z) - d(Tu, T_0 z) - (f, T_0 z)_0. \tag{3.4}$$

In Sections 3.1–3.4, we derive a suitable transformation $T \colon U \to W$ for the boundary conditions (B1)–(B4), respectively. In order to simplify the derivation, we will work with the classical setting of the problem which is meaningful due to the higher regularity of the solution to (3.1), see Remark 2.3.

3.1. Transformation of the Gao problem for boundary conditions (B1). The Gao beam problem for the boundary conditions (B1) can be written under the assumptions (A1)–(A3) as follows:

$$\begin{cases} \text{find } w \in H^4((0,L)) \text{ such that} \\ EIw'''' - E\alpha(w')^2 w'' + P\mu w'' = f \quad \text{a.e. in } (0,L), \\ w(0) = w'(0) = w(L) = w'(L) = 0. \end{cases} \tag{3.5}$$

Here, we set $V := H_0^2((0,L))$, $W := V \cap H^4((0,L))$ and $U := L^2((0,L))$. The idea of the transformation is based on a simplification of the fourth-order nonlinear differential equation by the substitution:

$$EIw'''' = f + u^*, \quad u^* = E\alpha(w')^2 w'' - P\mu w''. \tag{3.6}$$

The related state problem reads:

$$\begin{cases} \text{given } u \in U, \text{ find } w_u \in W \text{ such that} \\ EI{w_u}'''' = f + u \quad \text{a.e. in } (0,L), \\ w_u(0) = w_u'(0) = w_u(L) = w_u'(L) = 0. \end{cases} \tag{3.7}$$

Clearly, the problem (3.7) has a unique solution w_u under the assumptions (A1)–(A3) and defines the bijective mapping $T \colon U \to W$, $T \colon u \mapsto w_u$, with the inverse $T^{-1}v = EIv'''' - f$ for any $v \in W$. Moreover, T is affine, i.e., $Tu = T_0 u + \widehat{w}$, where T_0 is linear and $\widehat{w} = w_u \in W$ is a solution to (3.7) for $u = 0$.

R e m a r k 3.1. From (3.6) it follows that the solution $u^* = T^{-1}w$ of the optimal control problem belongs to $C^1([0,L])$, in addition.

3.2. Transformation of the Gao problem for boundary conditions (B2)**.** The Gao beam problem for the boundary conditions (B2) can be written under the assumptions (A1)–(A3) as follows:

$$
\begin{cases} \text{find } w \in H^4((0,L)) \text{ such that} \\ EIw'''' - E\alpha(w')^2 w'' + P\mu w'' = f \quad \text{a.e. in } (0,L), \\ w(0) = w'(0) = w(L) = 0, \quad w''(L) = 0. \end{cases} \tag{3.8}
$$

Analogously to the previous problem with conditions (B1), we set

$$
\begin{aligned} V &:= \{v \in H^2((0,L));\ v(0) = v'(0) = v(L) = 0\}, \\ W &:= \{v \in H^4((0,L));\ v(0) = v'(0) = v(L) = v''(L) = 0\}, \end{aligned}
$$

and define the following state problem based on the substitution (3.6):

$$
\begin{cases} \text{given } u \in U := L^2((0,L)), \text{ find } w_u \in W \text{ such that} \\ EI{w_u}'''' = f + u \quad \text{a.e. in } (0,L), \\ w_u(0) = w_u'(0) = w_u(L) = {w_u}''(L) = 0. \end{cases} \tag{3.9}
$$

Clearly, the problem (3.9) has a unique solution w_u under the assumptions (A1)–(A3) and defines the bijective affine mapping $T\colon U \to W$, $T\colon u \mapsto w_u$, with the inverse $T^{-1}v = EIv'''' - f$ for any $v \in W$.

R e m a r k 3.2. From (3.6) it follows that the solution $u^* = T^{-1}w$ of the optimal control problem belongs to $C^1([0,L])$, in addition.

3.3. Transformation of the Gao problem for boundary conditions (B3)**.** The Gao beam problem for the boundary conditions (B3) can be written under the assumptions (A1)–(A3) as follows:

$$
\begin{cases} \text{find } w \in H^4((0,L)) \text{ such that} \\ EIw'''' - E\alpha(w')^2 w'' + P\mu w'' = f \quad \text{a.e. in } (0,L), \\ w(0) = w'(0) = 0, \quad w''(L) = EIw'''(L) - \frac{1}{3}E\alpha(w'(L))^3 + P\mu w'(L) = 0. \end{cases} \tag{3.10}
$$

We set

$$
\begin{aligned} V &:= \{v \in H^2((0,L));\ v(0) = v'(0) = 0\}, \\ W &:= \{v \in H^4((0,L));\ v(0) = v'(0) = v''(L) = 0\}. \end{aligned}
$$

Unlike the two previous cases, we have one nonlinear boundary condition, here. Therefore, we suggest the substitution

$$(3.11)\qquad \begin{cases} EIw'''' = f + \tilde{u}^*, & \tilde{u}^* = E\alpha(w')^2w'' - P\mu w'',\\ EIw'''(L) = p^*, & p^* = \frac{1}{3}E\alpha(w'(L))^3 - P\mu w'(L). \end{cases}$$

Then we arrive at $U := L^2((0,L)) \times \mathbb{R}$ and the state problem

$$(3.12)\qquad \begin{cases} \text{given } u = (\tilde{u}, p) \in U, \text{ find } w_u \in W \text{ such that}\\ EI{w_u}'''' = f + \tilde{u} \quad \text{a.e. in } (0,L),\\ w_u(0) = w_u'(0) = {w_u}''(L), \quad EI{w_u}'''(L) = p. \end{cases}$$

Clearly, the problem (3.12) has a unique solution w_u under the assumptions (A1)–(A3) and defines the bijective mapping $T\colon U \to W$, $T\colon u \mapsto w_u$, with the inverse $T^{-1}v = (EIv'''' - f,\ EIv'''(L))$ for any $v \in W$. Moreover, T is affine, i.e., $Tu = T_0u + \widehat{w}$, where T_0 is linear and $\widehat{w} = w_u \in W$ is a solution to (3.12) for $u = (\tilde{u}, p)$, where $\tilde{u} = 0$ and $p = 0$.

R e m a r k 3.3. From (3.6) it follows that the solution $u^* = T^{-1}w$ of the optimal control problem belongs to $C^1([0,L])$, in addition.

3.4. Transformation of the Gao problem for boundary conditions (B4)**.** The Gao beam problem for the boundary conditions (B4) can be written under the assumptions (A1)–(A3) as follows:

$$(3.13)\qquad \begin{cases} \text{find } w \in H^4((0,L)) \text{ such that}\\ EIw'''' - E\alpha(w')^2w'' + P\mu w'' = f \quad \text{a.e. in } (0,L),\\ w(0) = w(L) = 0, \quad w''(0) = w''(L) = 0. \end{cases}$$

We set

$$\begin{aligned} V &:= \{v \in H^2((0,L));\ v(0) = v(L) = 0\},\\ W &:= \{v \in H^4((0,L));\ v(0) = v''(0) = v(L) = v''(L) = 0\}. \end{aligned}$$

As in the case (B1) or (B2), one can choose $U := L^2((0,L))$ and the substitution (3.6) to transform the problem (3.13). Nevertheless, we will present another possible transformation of the problem with a simpler state problem.

To this end, we introduce the auxiliary functions $g, u^* \in H^2((0,L)) \cap H_0^1((0,L))$, which are uniquely defined by the differential equations

$$(3.14)\qquad g'' = f, \quad (u^*)'' = E\alpha(w')^2w'' - P\mu w'' \quad \text{a.e. in } (0,L).$$

If we substitute (3.14) into (3.13) and use the conditions $w''(0) = w''(L) = 0$, we arrive at the equation

$$EIw'' = g + u^* \quad \text{a.e. in } (0, L).$$

This substitution enables us to set $U = H^2((0,L)) \cap H_0^1((0,L))$ and define the following state problem:

$$\text{(3.15)} \qquad \begin{cases} \text{given } u \in U, \text{ find } w_u \in W \text{ such that} \\ EI{w_u}'' = g + u. \end{cases}$$

It is easy to verify that the solution w_u is uniquely defined and in addition, it belongs to W. Therefore, one can introduce the bijective mapping $T\colon U \to W$, $T\colon u \mapsto w_u$, with the inverse $T^{-1}v = EIv'' - g$ for any $v \in W$. Moreover, T is affine, i.e., $Tu = T_0 u + \widehat{w}$, where T_0 is linear and $\widehat{w} = w_u \in W$ is a solution to (3.15) for $u = 0$.

Remark 3.4. From (3.14) it follows that the solution $u^* = T^{-1}w$ of the optimal control problem belongs to $C^1([0, L])$, in addition.

4. Contact problems for Gao beam and deformable foundation

The aim of this section is to extend the results from Sections 2 and 3 to the contact problem including the Gao beam and the deformable foundation. We assume that the beam is situated above the foundation, which is represented by a smooth function $g \leqslant 0$, see Figure 2.

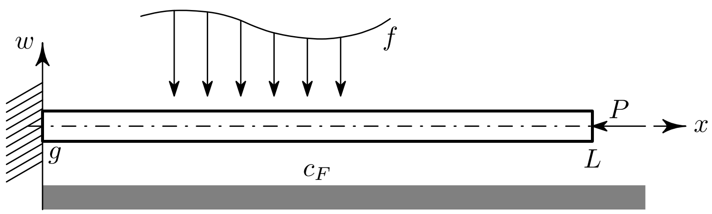

Figure 2. Gao beam and deformable foundation.

We consider the interaction between the beam and the foundation based on the normal compliance contact conditions (see e.g. [22], [24]). Then we arrive at the following nonlinear differential equation for the deflection of the beam:

$$\text{(4.1)} \qquad EIw'''' - E\alpha(w')^2 w'' + P\mu w'' = f + c_F(g - w)^+ \quad \text{in } (0, L).$$

Here, $v^+(x) = \max\{0, v(x)\}$ and $c_F = (1 - \nu^2)k_F$ with the foundation modulus $k_F > 0$. To be in accordance with (A1), we assume that $c_F > 0$ is constant. The additional last term represents the contact forces on the foundation. We see that the foundation is active only at $x \in (0, L)$, where $w(x) < g(x)$. Notice that such penetration is not possible for the rigid foundation.

Further, the presence of the foundation does not influence boundary conditions. Therefore, one can consider, e.g., the conditions (B1), (B2), (B3), or (B4) introduced in Section 2.

The weak formulation of the problem consisting of (4.1) and the boundary conditions reads

$$\text{find } w \in V\colon\ a(w, v) - d(w, v) + \pi(w, v) - \kappa(w, v) = (f, v)_0 \quad \forall\, v \in V, \tag{4.2}$$

where

$$\kappa(w, v) = \int_0^L c_F (g - w)^+ v \,\mathrm{d}x, \quad w, v \in V. \tag{4.3}$$

This additional term is associated with the potential

$$\Pi_F(v) = \frac{1}{2} \int_0^L c_F \big((g - v)^+\big)^2 \,\mathrm{d}x. \tag{4.4}$$

Hence, the total potential energy is given by the sum of (2.7) and (4.4):

$$\Pi(v) = \Pi_G(v) + \Pi_F(v) = \Pi_0(v) + \Pi_N(v) + \Pi_P(v) + \Pi_F(v), \quad v \in V. \tag{4.5}$$

The corresponding minimization problem reads

$$\text{find } w \in V\colon\ \Pi(w) = \min_{v \in V} \Pi(v). \tag{4.6}$$

Since the additional functional Π_F is nonnegative, convex, continuous and Gâteaux differentiable in V, one can straightforwardly extend Theorem 2.1.

Theorem 4.1. *Let the assumptions* (A1)–(A3) *be satisfied. Then the problems* (4.6) *and* (4.2) *are equivalent and have a unique solution* $w \in V$. *Moreover,* $w \in H^4((0, L))$.

Theorem 3.1 can be also easily modified so that the optimal control problem is in the form

$$\Pi(Tu^*) \leqslant \Pi(Tu) \quad \forall\, u \in U, \tag{4.7}$$

instead of (3.2). The state problems (3.7), (3.9), (3.12), (3.15), which define the operator T depending on the boundary conditions, remain the same even in the case of the contact problem. The substitutions (3.6), (3.11), (3.14) are modified as follows:

$$EIw'''' = f + u^*, \quad u^* = E\alpha(w')^2 w'' - P\mu w'' + c_F(g-w)^+, \tag{4.8}$$

$$\begin{cases} EIw'''' = f + \tilde{u}^*, \tilde{u}^* = E\alpha(w')^2 w'' - P\mu w'' + c_F(g-w)^+, \\ EIw'''(L) = p^*, p^* = \frac{1}{3}E\alpha(w'(L))^3 - P\mu w'(L), \end{cases} \tag{4.9}$$

$$\begin{cases} u^* \in H^2((0,L)) \cap H_0^1((0,L)), \\ (u^*)'' = E\alpha(w')^2 w'' - P\mu w'' + c_F(g-w)^+, \end{cases} \tag{4.10}$$

respectively.

5. Numerical realization and examples

Details on numerical solution of the Gao beam problem based on the control variational method including sensitivity analysis can be found in [18]. We recapitulate briefly its main principles. The optimal control problem is discretized by the standard finite element method (see, e.g., [21]) and solved by the conditioned gradient method (see [28]). The sensitivity analysis is based on formula (3.4). In particular, the cubic Hermite elements (see [21]) are used for solution of the state problem while conforming linear elements are considered for the control variable.

Below we present Examples 1–4 with the boundary conditions (B1), (B2), (B3), and (B4), respectively. We always set: $E = 21 \cdot 10^4\,\mathrm{MPa}$, $\nu = 0.3$, $h = 0.1\,\mathrm{m}$, $I = \frac{2}{3}h^3 = 0.666{,}667 \cdot 10^{-3}\,\mathrm{m}^4$, $L = 1\,\mathrm{m}$. The lateral load q is assumed to be uniformly distributed, i.e., the function f is constant. In each example, we compare the results for the pure bending problem (2.6) and the contact problem (4.6) with the deformable foundation given by the constant gap $g = 0.001\,\mathrm{m}$ and $k_F = 5 \cdot 10^8\,\mathrm{N\,m^{-2}}$. We choose the equidistant partition $[0, L]$ with 32 elements.

In Examples 1–4, we set the following transverse and axial forces:

$$\begin{aligned} q &= -1 \cdot 10^8\,\mathrm{N\,m^{-1}}, \quad P = -10^8\,\mathrm{N}, \\ q &= -5 \cdot 10^7\,\mathrm{N\,m^{-1}}, \quad P = +10^8\,\mathrm{N}, \\ q &= -2 \cdot 10^6\,\mathrm{N\,m^{-1}}, \quad P = +10^8\,\mathrm{N}, \\ q &= -5 \cdot 10^7\,\mathrm{N\,m^{-1}}, \quad P = -10^8\,\mathrm{N}, \end{aligned}$$

respectively. The results of Examples 1–4 are depicted in Figures 3–6, respectively. The beam deflection is visualized there. For comparison, we also depict the corresponding deflection of the classical Euler-Bernoulli beam (dashed line). The dotted

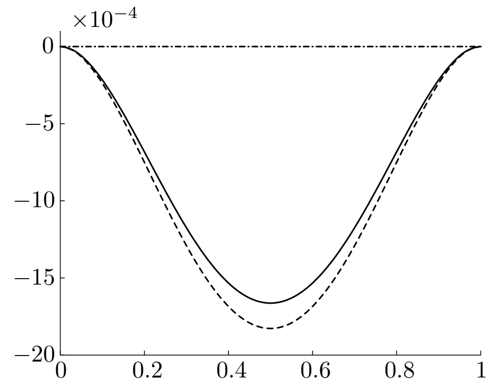

Figure 3. Example 1 for (B1): left—pure bending; right—contact with foundation.

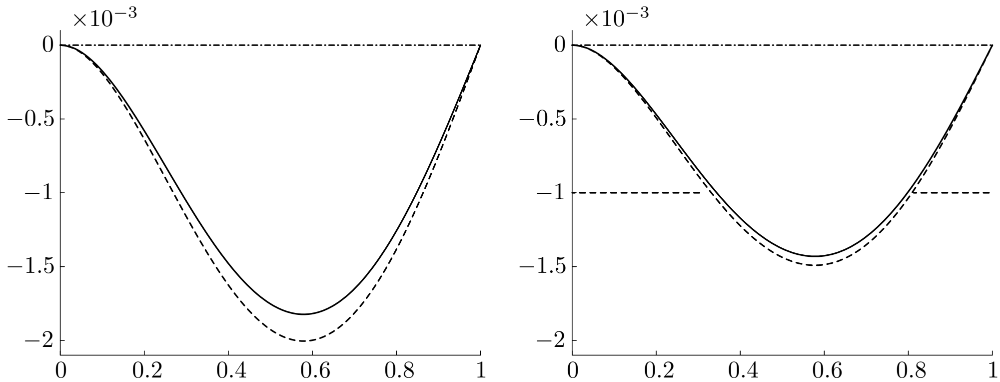

Figure 4. Example 2 for (B2): left—pure bending; right—contact with foundation.

lines represent the foundation. We see that the Gao beam is tougher than the classical Euler-Bernoulli beam. Further, we observed that the results remain visually the same even for much finer partitions of $[0, L]$.

6. Conclusion

We have presented a suitable transformation of the Gao beam problem to the optimal control problem depending on prescribed boundary conditions. The transformation has been derived under higher regularity of a solution to the beam problem. The optimal control problem remains convex and smooth, and the corresponding state problem is linear. Therefore, this transformation is convenient even for numerical solution.

In this work, we have assumed that the axial force is such that the energy potential is convex. This need not be true for larger compressive axial forces. Nevertheless, it

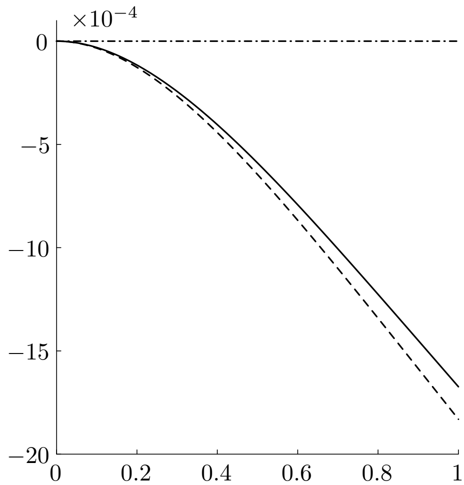

Figure 5. Example 3 for (B3): left—pure bending; right—contact with foundation.

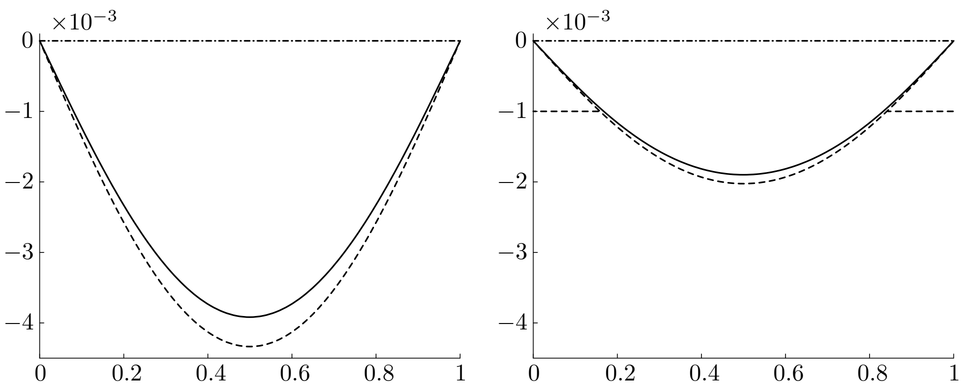

Figure 6. Example 4 for (B4): left—pure bending; right—contact with foundation.

seems that the presented transformation of the problem remains meaningful even for the nonconvex case and thus this technique could be promising for numerical solution. The extension of the results to the nonconvex case is the aim of our ongoing research.

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References

[1] *R. A. Adams*: Sobolev Spaces. Pure and Applied Mathematics 65, Academic Press, New York, 1975. zbl MR

[2] *K. T. Andrews, Y. Dumont, M. F. M'Bengue, J. Purcell, M. Shillor*: Analysis and simulations of a nonlinear dynamic beam. Z. Angew. Math. Phys. *63* (2012), 1005–1019. zbl MR doi

[3] *K. T. Andrews, K. L. Kuttler, M. Shillor*: Dynamic Gao beam in contact with a reactive or rigid foundation. Advances in Variational and Hemivariational Inequalities (W. Han et al., eds.). Advances in Mechanics and Mathematics 33, Springer, Cham, 2015, pp. 225–248. zbl MR doi

[4] *V. Arnautu, H. Langmach, J. Sprekels, D. Tiba*: On the approximation and the optimization of plates. Numer. Funct. Anal. Optimization *21* (2000), 337–354. zbl MR doi

[5] *M. Barboteu, M. Sofonea, D. Tiba*: The control variational method for beams in contact with deformable obstacles. ZAMM, Z. Angew. Math. Mech. *92* (2012), 25–40. zbl MR doi

[6] *K. Cai, D. Y. Gao, Q. H. Qin*: Post-buckling solutions of hyper-elastic beam by canonical dual finite element method. Math. Mech. Solids *19* (2014), 659–671. zbl MR doi

[7] *J. G. Eisley, A. M. Waas*: Analysis of Structures. An Introduction Including Numerical Methods. John Wiley & Sons, Hoboken, 2011. zbl doi

[8] *S. Fučík, A. Kufner*: Nonlinear Differential Equations. Studies in Applied Mechanics 2. Elsevier Scientific Publishing Company, Amsterdam, 1980. zbl MR

[9] *D. Y. Gao*: Nonlinear elastic beam theory with application in contact problems and variational approaches. Mech. Res. Commun. *23* (1996), 11–17. zbl MR doi

[10] *D. Y. Gao*: Duality Principles in Nonconvex Systems. Theory, Methods and Applications. Nonconvex Optimization and Its Applications 39, Kluwer Academic Publihers, Dordrecht, 2000. zbl MR doi

[11] *D. Y. Gao, J. Machalová, H. Netuka*: Mixed finite element solutions to contact problems of nonlinear Gao beam on elastic foundation. Nonlinear Anal., Real World Appl. *22* (2015), 537–550. zbl MR doi

[12] *D. Y. Gao, R. W. Ogden*: Closed-form solutions, extremality and nonsmoothness criteria in a large deformation elasticity problem. Z. Angew. Math. Phys. *59* (2008), 498–517. zbl MR doi

[13] *R. Glowinski, J. L. Lions, R. Trémolières*: Numerical Analysis of Variational Inequalities. Studies in Mathematics and Its Applications 8, North-Holland Publishing, Amsterdam, 1981. zbl MR doi

[14] *I. Hlaváček, J. Haslinger, J. Nečas, J. Lovíšek*: Solution of Variational Inequalities in Mechanics. Applied Mathematical Sciences 66, Springer, New York, 1988. zbl MR doi

[15] *J. V. Horák, H. Netuka*: Mathematical model of pseudointeractive set: 1D body on non-linear subsoil. I. Theoretical aspects. Engineering Mechanics *14* (2007), 311–325.

[16] *J. L. Lions*: Optimal Control of Systems Governed by Partial Differential Equations. Die Grundlehren der mathematischen Wissenschaften 170, Springer, Berlin, 1971. zbl MR

[17] *J. Machalová, H. Netuka*: Solving the beam bending problem with an unilateral Winkler foundation. Numerical Analysis and Applied Mathematics ICNAAM 2011 (T. E. Simos et al., eds.). AIP Conference Proceedings 1389, AIP-Press, Springer, 2011, pp. 1820–1824. doi

[18] *J. Machalová, H. Netuka*: Solution of contact problems for nonlinear Gao beam and obstacle. J. Appl. Math. *2015* (2015), Article ID 420649, 12 pages. MR doi

[19] *J. Machalová, H. Netuka*: Solution of contact problems for Gao beam and elastic foundation. To appear in Math. Mech. Solids. Special issue on Inequality Problems in Contact Mechanics (2017). doi

[20] *P. Neittaanmäki, J. Sprekels, D. Tiba*: Optimization of Elliptic Systems. Theory and Applications. Springer Monographs in Mathematics, Springer, New York, 2006. zbl MR doi

[21] *J. N. Reddy*: An Introduction to the Finite Element Method. McGraw-Hill Book Company, New York, 2006. zbl

[22] *M. Shillor, M. Sofonea, J. J. Telega*: Models and Analysis of Quasistatic Contact: Variational Methods. Lecture Notes in Physics 655, Springer, Berlin, 2004. zbl doi

[23] *R. Šimeček*: Optimal design of an elastic beam with a unilateral elastic foundation: semicoercive state problem. Appl. Math., Praha *58* (2013), 329–346. zbl MR doi

[24] *M. Sofonea, A. Matei*: Mathematical Models in Contact Mechanics. London Mathematical Society Lecture Note Series 398, Cambridge University Press, Cambridge, 2012. zbl doi
[25] *M. Sofonea, D. Tiba*: The control variational method for contact of Euler-Bernoulli beams. Bull. Trans. Univ. Braşov, Ser. III, Math. Inform. Phys. *2* (2009), 127–136. zbl MR
[26] *S. Sysala*: Unilateral elastic subsoil of Winkler's type: Semi-coercive beam problem. Appl. Math., Praha *53* (2008), 347–379. zbl MR doi
[27] *S. Sysala*: Numerical modelling of semi-coercive beam problem with unilateral elastic subsoil of Winkler's type. Appl. Math., Praha *55* (2010), 151–187. zbl MR doi
[28] *F. Tröltzsch*: Optimal Control of Partial Differential Equations. Theory, Methods and Applications. Graduate Studies in Mathematics 112, American Mathematical Society, Providence, 2010. zbl MR doi

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