# CHARACTERIZATIONS OF CONTINUOUS DISTRIBUTIONS THROUGH INEQUALITIES INVOLVING THE EXPECTED VALUES OF SELECTED FUNCTIONS

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*Abstract.* Nanda (2010) and Bhattacharjee et al. (2013) characterized a few distributions with help of the failure rate, mean residual, log-odds rate and aging intensity functions. In this paper, we generalize their results and characterize some distributions through functions used by them and Glaser's function. Kundu and Ghosh (2016) obtained similar results using reversed hazard rate, expected inactivity time and reversed aging intensity functions. We also, via $w(\cdot)$-function defined by Cacoullos and Papathanasiou (1989), characterize exponential and logistic distributions, as well as Type 3 extreme value distribution and obtain bounds for the expected values of selected functions in reliability theory. Moreover, a bound for the varentropy of random variable $X$ is provided.

*Keywords*: characterization; hazard rate; mean residual life function; reversed hazard rate; expected inactivity time; log-odds rate; Glaser's function

*MSC 2010*: 60E15, 62E10

## 1. Introduction

Let $X$ be a random variable having absolutely continuous distribution function $F(t)$, survival function $\overline{F}(t) = 1 - F(t)$ and probability density function $f(t)$. Let $X$ take values in an interval $(a, b)$ with $-\infty \leqslant a < b \leqslant \infty$, where $a = \inf\{t\colon\ F(t) > 0\}$ and $b = \sup\{t\colon\ F(t) < 1\}$. Then the hazard rate (HR) function of $X$ is defined for $t < b$ as $r(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\ln \overline{F}(t) = f(t)/\overline{F}(t)$; besides, let the random variable $X$ have finite moments of all orders with variance $\operatorname{Var}(X) = \sigma^2$ and mean $E(X) = \mu$.

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A useful reliability measure of $X$ is the mean residual life (MRL) which is defined as the expectation of the residual life random variable $X_t = (X - t \mid X > t)$, given by

$$m(t) = \frac{1}{\overline{F}(t)} \int_t^{\infty} \overline{F}(x)\,\mathrm{d}x,$$

for $t < b$. The MRL function is usually of interest for a non-negative random variable. For instance, if $X$ is thought of as the lifetime of a device, then for every $t \geqslant 0$, $m(t)$ expresses the conditional expected residual life of the device at time $t$ given that the device is still alive at time $t$. Hence, we assume $F(t) = 0$ for $t < 0$ (i.e. $a = 0$, $b = \infty$).

Hall and Wellner [14] and Bhattacharjee [3] have characterized the class of mean residual life functions. It has been shown by Gupta [10] that the MRL function determines the distribution uniquely. In particular, it is well known that a constant MRL characterizes the exponential distribution, as was shown by Nanda [20]. For comprehensive review and applications of the mean residual function, we refer to Guess and Proschan [9].

The hazard rate and MRL function are related by

$$r(t) = \frac{1 + m'(t)}{m(t)}. \tag{1.1}$$

It is well known that $r(t)$ determines the distribution function uniquely and hence $m(t)$ also characterizes the distribution. In addition, $\overline{F}(t)$ and $r(t)$ are connected by

$$\overline{F}(t) = \exp\Bigl\{-\int_0^t r(x)\,\mathrm{d}x\Bigr\}. \tag{1.2}$$

As a dual notion to the residual life, the inactivity time at time $t$ is $X_{(t)} = (t - X \mid X < t)$. The expected inactivity time (EIT) function of $X$ is defined by

$$m^*(t) = E(X_{(t)}) = \frac{1}{F(t)} \int_a^t F(x)\,\mathrm{d}x.$$

The EIT function is a well-known reliability measure which has applications in many disciplines such as reliability theory, survival analysis and actuarial studies.

The reversed hazard rate (RHR) function of $X$ is given by $r^*(t) = f(t)/F(t)$ for $t > a$, which is related to $F(t)$ by

$$F(t) = \exp\Bigl\{-\int_t^b r^*(x)\,\mathrm{d}x\Bigr\}. \tag{1.3}$$

Furthermore, Glaser's function (also known as eta-function) $\eta(t)$ for a random variable $X$ is defined as

$$\eta(t) = -\frac{f'(t)}{f(t)}.$$

This function contains useful information about $r(t)$, but it is simpler, because it does not involve $\overline{F}(t)$. The relation between $r(t)$ and $\eta(t)$ is given by $\frac{\mathrm{d}}{\mathrm{d}t}\ln r(t) = r(t) - \eta(t)$. Here we obviously assume that $f(t)$ is a differentiable density function on $(0, \infty)$. For further studies on the relationship between Glaser's function and the failure rate, see Gupta and Warren [12].

The aging intensity (AI) function, which analyzes the aging property of a system quantitatively (cf. Jiang et al., [16]), is defined as

$$L(t) = \frac{tr(t)}{\int_0^t r(u)\,\mathrm{d}u} = \frac{-tf(t)}{\overline{F}(t)\ln\overline{F}(t)}.$$

The log-odds rate (LOR) of $X$ is defined as

$$\mathrm{LOR}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{LO}(t) = \frac{f(t)}{F(t)\overline{F}(t)}, \tag{1.4}$$

where $\mathrm{LO}(t) = \ln(F(t)/\overline{F}(t))$ is the log-odds function.

A probability distribution can be characterized by various methods. Nanda [20], Bhattacharjee et al. [4] and Iwińska and Szymkowiak [15] studied the characterizations of distributions through the expected values of the failure rate, MRL, log-odds rate and aging intensity functions. It is pointed out that the functions considered by them are defined for left truncated random variables. For characterizations of distributions using EIT function, we refer to Chandra and Roy [7], Kundu et al. [19], and Asadi and Berred [1]. For the characterization of right truncated distributions, Kundu and Ghosh [18] have used the reversed hazard rate, expected inactivity time and reversed aging intensity functions.

Cacoullos and Papathanasiou [6] obtained a lower bound for the variance of a function of a random variable. They established that, if $X$ is a continuous random variable with density function $f(x)$, mean $\mu$ and variance $\sigma^2$ and $g\colon \mathbb{R} \to \mathbb{R}$ is any absolutely continuous function with derivative $g'$, then

$$\mathrm{Var}[g(X)] \geqslant \sigma^2(E[w(X)g'(X)])^2, \tag{1.5}$$

where the $w(\cdot)$-function is defined by

$$\sigma^2 w(x)f(x) = \int_{-\infty}^{x} (\mu - t)f(t)\,\mathrm{d}t. \tag{1.6}$$

The equality holds if and only if $g$ is linear.

In this article, we consider the particular cases of the function $g$, which is used in reliability analysis for aging.

## 2. Characterization through functions of HR, MRL and AI

In this section, we characterize probability distributions such as exponential, Pareto, Weibull, Makeham, and Gompertz distributions. First we generalize Theorem 2.3 of Nanda [20].

As stated in it, the exponential distribution can be characterized by $E[Xr(X)]$. Below we characterize this distribution in terms of $E[X^k r(X)]$, where $k > 0$ is a real constant.

**Proposition 2.1.** *For any non-negative random variable $X$,*

$$E[X^k r(X)] \geqslant \frac{k+1}{E(X^{k+1})}(E(X^k))^2; \tag{2.1}$$

*the equality holds if and only if $X$ is exponentially distributed.*

Proof. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}(E(X^k))^2 &= \left[\int_0^\infty x^k f(x)\,\mathrm{d}x\right]^2 = \left[\int_0^\infty \sqrt{x^k\overline{F}(x)}\sqrt{\frac{x^k}{\overline{F}(x)}}f(x)\,\mathrm{d}x\right]^2 \\ &\leqslant \int_0^\infty x^k\overline{F}(x)\,\mathrm{d}x \int_0^\infty \frac{x^k f^2(x)}{\overline{F}(x)}\,\mathrm{d}x.\end{aligned} \tag{2.2}$$

Since (see Gut [13]),

$$\int_0^\infty x^k\overline{F}(x)\,\mathrm{d}x = \frac{E(X^{k+1})}{k+1}, \tag{2.3}$$

and

$$\int_0^\infty \frac{x^k f^2(x)}{\overline{F}(x)}\,\mathrm{d}x = E[X^k r(X)],$$

(2.2) reduces to (2.1). The equality holds if and only if there exists a constant $A > 0$ such that, for all $x \geqslant 0$,

$$\sqrt{\frac{x^k f^2(x)}{\overline{F}(x)}} = A\sqrt{x^k\overline{F}(x)}.$$

This gives $r(x) = \text{constant}$, which holds if and only if $X$ is exponentially distributed.

□

The next theorem gives a useful lower bound for $E[c^{-X}r(X)]$; $c>1$, and characterizes the exponential distribution.

**Theorem 2.2.** *Let $X$ be an absolutely continuous non-negative random variable. Then*

$$E[c^{-X}r(X)] \geqslant \frac{(\ln c)(E[c^{-X}])^2}{1-E[c^{-X}]} \tag{2.4}$$

*for $c>1$, where the equality holds if and only if $X$ has exponential distribution.*

P r o o f. Using the Cauchy-Schwarz inequality, we get

$$\left(E[c^{-X}]\right)^2 \leqslant E[c^{-X}r(X)]E\Big[\frac{c^{-X}}{r(X)}\Big]. \tag{2.5}$$

On the other hand, since

$$E\Big[\frac{c^{-X}}{r(X)}\Big] = \int_0^\infty c^{-x}\overline{F}(x)\,\mathrm{d}x = \frac{1}{\ln c}(1-E(c^{-X})), \tag{2.6}$$

substituting (2.6) in (2.5) yields (2.4).

The equality is obtained if and only if there exists a constant $A>0$ such that, for all $x \geqslant 0$,

$$\sqrt{\frac{c^{-x}f^2(x)}{\overline{F}(x)}} = A\sqrt{c^{-x}\overline{F}(x)}.$$

This gives $r(x) = \text{constant}$, which again holds if and only if $X$ is exponentially distributed. □

**Corollary 2.3.** *Under the assumptions of Theorem 2.2, if $c=\mathrm{e}^t$ for $t>0$ then*

$$E[\mathrm{e}^{-tX}r(X)] \geqslant \frac{t\left(f^*(t)\right)^2}{1-f^*(t)}, \tag{2.7}$$

*where $f^*(t)=E[\mathrm{e}^{-tX}]=\int_0^\infty \mathrm{e}^{-tx}f(x)\,\mathrm{d}x$ is the Laplace transform of $X$.*

*The equality holds if and only if $F$ is an exponential distribution function.*

In the following theorem a characterization of Gompertz distribution is provided.

**Theorem 2.4.** *Let $X$ be an absolutely continuous non-negative random variable with $E[c^{-X}r(X)]<\infty$ and $E[(c^{-X}r(X))^{-1}]<\infty$, where $c>1$. Then*

$$E\Big[\frac{1}{c^{-X}r(X)}\Big] \geqslant \frac{1}{E[c^{-X}r(X)]}, \tag{2.8}$$

*and the equality holds if and only if $X$ follows the Gompertz distribution with probability density function*

$$f(x) = \theta c^x \mathrm{e}^{-\theta(c^x-1)/\ln c}, \quad x \geqslant 0,\ \theta > 0. \tag{2.9}$$

Proof. By the Cauchy-Schwarz inequality, we obtain (2.8). The equality in (2.8) holds if and only if there exists a constant $A > 0$ such that, for all $x \geqslant 0$,

$$\frac{f(x)}{c^{-x}r(x)} = Ac^{-x}r(x)f(x),$$

which is equivalent to the fact that $r(x) = \theta c^x$. Now, using (1.2), we obtain $\overline{F}(x) = \mathrm{e}^{-\theta(c^x-1)/\ln c}$. Since $\overline{F}(\infty) = 0$, we should have $c > 1$ and therefore the result is obtained. $\square$

Remark 2.5. In the previous theorem, it is clear that if any of the moments or both of them are infinite then inequality (2.8) will be trivial.

The next corollary characterizes Makeham distribution through $E[\mathrm{e}^{-tX}r(X)]$.

**Corollary 2.6.** *In Theorem 2.4 if $c = \mathrm{e}^t$ for $t > 0$ then*

$$E[\mathrm{e}^{-tX}r(X)] \geqslant \frac{1}{E[(\mathrm{e}^{-tX}r(X))^{-1}]}, \tag{2.10}$$

*and the equality holds if and only if $X$ follows the Makeham distribution with probability density function*

$$f(x) = \theta \exp\Big\{tx - \frac{\theta}{t}(\mathrm{e}^{tx}-1)\Big\}, \quad x \geqslant 0,\ \theta > 0,\ t > 0. \tag{2.11}$$

Now, in the particular case where $t = 1$, we would like to compare the lower bounds obtained for $E[\mathrm{e}^{-tX}r(X)]$ in inequalities (2.7) and (2.10).

Let $X$ have a Rayleigh distribution with density function $f(x) = 2x\mathrm{e}^{-x^2}$, $x > 0$. Then the lower bound (2.10) is

$$\frac{1}{E[(\mathrm{e}^{-X}r(X))^{-1}]} = \frac{2\mathrm{e}^{-1/4}}{\sqrt{\pi}(1+\operatorname{erf}(\frac{1}{2}))} \approx 0.578.$$

On the other hand, since $f^*(1) = 1 - \frac{1}{2}\sqrt{\pi}\mathrm{e}^{1/4}\operatorname{erfc}(\frac{1}{2})$, the lower bound (2.7) is

$$\frac{2\big(1 - \frac{1}{2}\sqrt{\pi}\mathrm{e}^{1/4}\operatorname{erfc}(\frac{1}{2})\big)^2}{\sqrt{\pi}\mathrm{e}^{1/4}\operatorname{erfc}(\frac{1}{2})} \approx 0.378.$$

Accordingly, comparing these values with the exact value $E[e^{-X}r(X)]$, for this distribution we conclude that the lower bound (2.10) is better than the lower bound (2.7).

In the next theorem, we characterize a distribution in which mean residual life is proportional to $1/x$.

**Theorem 2.7.** *Let $X$ be an absolutely continuous positive random variable with $E[Xm(X)] < \infty$ and $E[1/Xm(X)] < \infty$. Then*

$$E\Big[\frac{1}{Xm(X)}\Big] \geqslant \frac{1}{E[Xm(X)]}, \tag{2.12}$$

*and the equality holds if and only if $X$ follows the distribution with survival function*

$$\overline{F}(x) = \frac{x}{a}e^{-(x^2-a^2)/2\theta}, \quad x > a,\ a > 0,\ \theta > 0. \tag{2.13}$$

P r o o f. By applying the Cauchy-Schwarz inequality we obtain (2.12). The equality in (2.12) holds if and only if there exists a constant $A > 0$ such that

$$\frac{f(x)}{xm(x)} = Axm(x)f(x),$$

which is equivalent to the fact that $m(x) = \theta/x$. Lastly, applying (1.1) and (1.2), we get (2.13). $\square$

**Theorem 2.8.** *Let $X$ be an absolutely continuous positive random variable with $E[m(X)/X] < \infty$ and $E[X/m(X)] < \infty$. Then*

$$E\Big[\frac{m(X)}{X}\Big] \geqslant \frac{1}{E[X/m(X)]}, \tag{2.14}$$

*with equality if and only if $X$ has the Pareto distribution with probability density function*

$$f(x) = \frac{ca^c}{x^{c+1}}, \quad x > a,\ a > 0,\ c > 1. \tag{2.15}$$

P r o o f. The inequality (2.14) follows from the Cauchy-Schwarz inequality and equality holds if and only if there exists a constant $A > 0$ such that

$$\frac{xf(x)}{m(x)} = A\frac{m(x)f(x)}{x},$$

which is equivalent to the fact that $m(x) = \theta x$. Applying (1.1) and (1.2), we have $\overline{F}(x) = (x/a)^{-(1+\theta)/\theta}$ for $x > a$. Now, since $\theta > 0$ setting $c = (\theta+1)/\theta$, yields the desired result. $\square$

R e m a r k 2.9. Kundu and Ghosh [18] showed that inequalities (2.12) and (2.14) remain true when MRL is replaced by EIT, and equality characterizes the finite range distribution and three distributions as given in Theorem 2.1 of Kundu et al. [19] for suitable values of $p = r^*(t)m^*(t)$ respectively.

The following theorem is a generalization of Theorem 2.3 of Bhattacharjee et al. [4].

**Theorem 2.10.** *For any non-negative random variable $X$, if for some real $k \geqslant 0$, $E[X^k L(X)/r(X)] < \infty$ and $E[r(X)/(X^k L(X))] < \infty$, then*

$$E\Big[\frac{X^k L(X)}{r(X)}\Big] \geqslant \frac{1}{E[r(X)/(X^k L(X))]}. \tag{2.16}$$

*The equality holds in (2.16) if and only if $X$ has a Weibull distribution with probability density function*

$$f(x) = (k+1)\theta x^k \mathrm{e}^{-\theta x^{k+1}}, \quad x \geqslant 0,\ \theta > 0. \tag{2.17}$$

P r o o f. Using the Cauchy-Schwarz inequality, it follows that

$$E\Big[\frac{X^k L(X)}{r(X)}\Big] E\Big[\frac{r(X)}{X^k L(X)}\Big] \geqslant 1. \tag{2.18}$$

The equality in (2.16) holds if and only if there exists a constant $\theta > 0$ such that, for all $x \geqslant 0$,

$$\int_0^x r(t)\,\mathrm{d}t = \theta x^{k+1}. \tag{2.19}$$

Differentiating (2.19) with respect to $x$, we get $r(x) = (k+1)\theta x^k$ for all $x > 0$. Now, by applying (1.2), we obtain $\overline{F}(x) = \mathrm{e}^{-\theta x^{k+1}}$. Hence, the result follows. $\square$

In the next theorem we obtain a characterization of the two-parameter Weibull distribution through an inequality between $E[XL(X)]$ and $E(m(X))$.

**Theorem 2.11.** *For any non-negative random variable $X$,*

$$E[XL(X)] \geqslant \frac{\mu^2}{E(m(X))}. \tag{2.20}$$

*The equality holds if and only if the random variable $X$ follows two-parameter Weibull distribution.*

P r o o f. By the Cauchy-Schwarz inequality we have

$$\left[\int_0^\infty x f(x)\,\mathrm{d}x\right]^2 \leqslant \int_0^\infty \frac{-x^2 f^2(x)}{\overline{F}(x)\ln\overline{F}(x)}\,\mathrm{d}x \int_0^\infty -\overline{F}(x)\ln\overline{F}(x)\,\mathrm{d}x. \tag{2.21}$$

Now, since

$$\int_0^\infty \frac{-x^2 f^2(x)}{\overline{F}(x)\ln\overline{F}(x)}\,\mathrm{d}x = E[XL(X)], \tag{2.22}$$

and, as Asadi and Zohrevand [2] showed,

$$\int_0^\infty -\overline{F}(x)\ln\overline{F}(x)\,\mathrm{d}x = E(m(X)); \tag{2.23}$$

thus, substituting (2.22) and (2.23) into (2.21) gives (2.20).

The equality in (2.20) holds if and only if there exists a constant $A > 0$ such that

$$\sqrt{-\overline{F}(x)\ln\overline{F}(x)} = A\sqrt{\frac{-x^2 f^2(x)}{\overline{F}(x)\ln\overline{F}(x)}},$$

which is equivalent to $L(x) = \text{constant}$. Thus, the result follows from Lemma 2.1 in Bhattacharjee et al. [4]. □

R e m a r k 2.12. According to Proposition 2.2 in Navarro et al. [21], we get a further lower bound for $E[XL(X)]$:

$$E[XL(X)] \geqslant \frac{(\beta+1)(E(X))^{\beta+2}}{E(X^{\beta+1})(\Gamma(1+\beta^{-1}))^{\beta}},$$

for all $\beta > 0$.

## 3. Characterization through functions of $w(\cdot)$-function

What we have discussed so far, gives us motivation to find an easier lower bound for inequality (1.5) for some particular functions $g$. In fact, by getting a lower bound for $E[w(X)g'(X)]$, we will obtain a new lower bound for $\mathrm{Var}[g(X)]$. In Theorems 3.1 and 3.3, $g(x)$ is considered to be $\int_0^x r(u)\,\mathrm{d}u$ and $\int_x^b r^*(u)\,\mathrm{d}u$ respectively. If $r(x)$ and $r^*(x)$ exist, then $-\ln\overline{F}(x) = \int_0^x r(u)\,\mathrm{d}u$ and $-\ln F(x) = \int_x^b r^*(u)\,\mathrm{d}u$ represent the cumulative hazard (failure) rate and the cumulative reversed hazard rate respectively.

**Theorem 3.1.** *Let $X$ be a non-negative random variable. Then*

$$E[w(X)r(X)] \geqslant \frac{1}{E\big[w(X)/r(X)\big]}, \tag{3.1}$$

*where $w(x)$ is the $w(\cdot)$-function corresponding to the random variable $X$. The equality is achieved if and only if $X$ has exponential distribution.*

P r o o f. Using the Cauchy-Schwarz inequality, we have

$$(E[w(X)])^2 \leqslant E[w(X)r(X)]E\Big[\frac{w(X)}{r(X)}\Big]. \tag{3.2}$$

On the other hand, as Cacoullos and Papathanasiou [6] showed, since $E[w(X)] = 1$, the inequality (3.2) yields (3.1). The equality holds in (3.1) if and only if $r(x) =$ constant, or equivalently if and only if $X$ has the exponential distribution. □

R e m a r k 3.2. Referring to Cacoullos and Papathanasiou [5] and [6], it can be seen that if $X$ has a gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$ then $w(x) = x/\alpha\beta$ and thus Theorem 3.1 will be a special case of Proposition 2.1 for $k = 1$.

**Theorem 3.3.** *For any absolutely continuous random variable $X$,*

$$E[w(X)r^*(X)] \geqslant \frac{1}{E\big[w(X)/r^*(X)\big]}, \tag{3.3}$$

*where $w(x)$ is the $w(\cdot)$-function corresponding to the random variable $X$. The equality holds if and only if $X$ follows the distribution given in equation* (2.3) *of Kundu and Ghosh* [18].

P r o o f. Like in Theorem 3.1, it is easy to check that inequality (3.3) holds. The equality holds in (3.3) if and only if $r^*(x) =$ constant, or equivalently, by using (1.3), if and only if $X$ is distributed as Type 3 extreme value distribution defined on $(-\infty, b]$; $b \geqslant 0$, specified as

$$F(x) = \exp\Big(\frac{x-b}{b-\mu}\Big), \quad x \in (-\infty, b], \tag{3.4}$$

as given in equation (2.3) of Kundu and Ghosh [18]. □

In the next theorem, we assume that $g(x)$ used in inequality (1.5) is $\mathrm{LO}(x)$.

**Theorem 3.4.** *Let $X$ be a random variable with $E[w(X)\,\mathrm{LOR}(X)] < \infty$ and $E[w(X)/\,\mathrm{LOR}(X)] < \infty$. Then*

$$E[w(X)\,\mathrm{LOR}(X)] \geqslant \frac{1}{E[w(X)/\,\mathrm{LOR}(X)]}. \tag{3.5}$$

*The equality holds if and only if $X$ follows logistic distribution.*

P r o o f. Using the Cauchy-Schwarz inequality and the fact that $E[w(X)] = 1$, the proof of inequality (3.5) is trivial. Moreover, the equality holds in (3.5) if and only if $\mathrm{LOR}(x) =$ constant, or equivalently, $X$ follows logistic distribution (see Bhattacharjee et al. [4]). $\square$

The next remark gives an upper bound for $E[w(X)\,\mathrm{LOR}(X)]$.

R e m a r k 3.5. In the inequality (1.5) if $g(x) = \mathrm{LO}(x)$ then

$$\mathrm{Var}[\mathrm{LO}(X)] \geqslant \sigma^2 (E[w(X)\,\mathrm{LOR}(X)])^2.$$

Since $\mathrm{Var}[\mathrm{LO}(X)] = E[\mathrm{LO}(X)]^2 = \int_a^b \bigl(\ln \frac{F(x)}{\overline{F}(x)}\bigr)^2 f(x)\,\mathrm{d}x = \int_0^1 (\ln \frac{u}{1-u})^2\,\mathrm{d}u = \frac{1}{3}\pi^2$, thus

$$E[w(X)\,\mathrm{LOR}(X)] \leqslant \frac{\pi}{\sigma\sqrt{3}}.$$

In the following remark, we obtain an upper bound for $E[w(X)r(X)]$.

R e m a r k 3.6. If in the inequality (1.5) we put $g(x) = -\ln\bigl(\overline{F}(X)\bigr)$ then

$$\mathrm{Var}[-\ln(\overline{F}(X))] \geqslant \sigma^2 (E[w(X)r(X)])^2.$$

Note that $\mathrm{Var}[-\ln \overline{F}(X)] = \int_0^1 (\ln u)^2\,\mathrm{d}u - \bigl(\int_0^1 \ln u\,\mathrm{d}u\bigr)^2 = 1$, thus

$$E[w(X)r(X)] \leqslant \frac{1}{\sigma}. \tag{3.6}$$

E x a m p l e 3.7. If $X$ has the inverse-gamma distribution with probability density function $f(x) = \beta^\alpha \Gamma(\alpha)^{-1} x^{-\alpha-1} \exp(-\beta/x)$, $x > 0$ and parameters $\alpha > 2$ and $\beta > 0$, then by using (1.6), we can easily obtain $w(x) = (\alpha-1)(\alpha-2)\beta^{-2}x^2$. Now if $\alpha = 4$ and $\beta = 1$ then $w(x) = 6x^2$ and by using (3.1) and (3.6), we can get $3 \leqslant E[w(X)r(X)] \leqslant 3\sqrt{2}$.

**Theorem 3.8.** *Let $X$ be an absolutely continuous non-negative random variable with $E[w(X)/\eta(X)] < \infty$ and $w$ be a smooth function. Then if for all $x > 0$,*

(i) $\eta(x) > 0 \Rightarrow E\bigl[w(X)/\eta(X)\bigr] \geqslant 1/E[w'(X)]$,

(ii) $\eta(x) < 0 \Rightarrow E\bigl[w(X)/\eta(X)\bigr] \leqslant 1/E[w'(X)]$,

*where $w(x)$ is the $w(\cdot)$-function corresponding to the random variable $X$. The equality holds if and only if $X$ has an exponential distribution.*

P r o o f. Using the Cauchy-Schwarz inequality and the fact that $E[w(X)] = 1$, we get

$$1 \leqslant E[w(X)\eta(X)]E\Bigl[\frac{w(X)}{\eta(X)}\Bigr]. \tag{3.7}$$

Now since $w$ is a smooth function, using equation (1.94) of Johnson [17], we have

$$E(w(X)\eta(X)) = E[w'(X)], \tag{3.8}$$

and since it can be verified from (1.6) that $w(x)$ is positive, this completes the proof. Equality in (3.7) holds if and only if $\eta(x) = \text{constant}$, or equivalently if and only if $X$ is exponentially distributed.

It should be noted that although the eta function is constant for uniform distribution ($\eta(x) = 0$), however, since $\eta$ appears in the denominator the condition $\eta(x) \neq 0$ does not hold. □

Remark 3.9. The varentropy of a random variable $X$ is defined as

$$\operatorname{Var}[-\ln f(X)] = \int_{\mathbb{R}} f(x)(\ln f(x))^2 \,\mathrm{d}x - \left(\int_{\mathbb{R}} f(x)\ln f(x)\,\mathrm{d}x\right)^2.$$

In the inequality (1.5) if $g(x) = -\ln f(x)$ then a lower bound for the varentropy $X$ is obtained by

$$\operatorname{Var}[-\ln f(X)] \geqslant \sigma^2 (E[w'(X)])^2. \tag{3.9}$$

Example 3.10. Let $X$ have beta distribution with parameters $a = 2$ and $b = 1$. It is easy to show that $f$ is log-concave and $\eta(x) = -1/x$ and thus $\eta(x) < 0$ for all $x$. As mentioned in Example 1, since $w(x) = 6x(1-x)$, so $E[w(X)/\eta(X)] \leqslant -\frac{1}{2}$. Also by using Theorem 2.3 of Fradelizi et al. [8] and (3.9),

$$\frac{2}{9} \leqslant \operatorname{Var}[-\ln f(X)] \leqslant 1.$$

Motivated by Theorem 2.5 of Nanda [20] we have the following result.

**Theorem 3.11.** *Let $X$ be an absolutely continuous non-negative random variable with $E\big[r(X)/\eta(X)\big] < \infty$ and $E[\eta(X)/r(X)] < \infty$. Then*

$$E\Big[\frac{r(X)}{\eta(X)}\Big] \geqslant \frac{1}{E[\eta(X)/r(X)]}, \tag{3.10}$$

*and the equality holds if and only if $X$ follows the distribution having the survival function*

$$\overline{F}(x) = (1+\beta x)^{-1/\alpha}. \tag{3.11}$$

*For $\alpha, \beta > 0$, it becomes a Pareto distribution of the second kind (the Lomax). When $\alpha = 0$ then the above distribution gives an exponential distribution. When $\alpha < 0$ ($\neq -1$) and $\beta < 0$ then (3.11) is a finite range distribution.*

P r o o f. Using the Cauchy-Schwarz inequality, it follows that

$$E\Big[\frac{r(X)}{\eta(X)}\Big]E\Big[\frac{\eta(X)}{r(X)}\Big] \geqslant 1. \tag{3.12}$$

The equality in (3.12) holds if and only if there exists a constant $A > 0$ such that, for all $x \geqslant 0$,

$$\frac{r(x)}{\eta(x)}f(x) = A\frac{\eta(x)}{r(x)}f(x), \tag{3.13}$$

which is equivalent to $\eta(x)/r(x) = \theta$.

Thus, the equality holds if and only if there exists a constant $\theta(\neq 0)$ such that, for all $x \geqslant 0$,

$$\overline{F}(x)f'(x) + \theta f^2(x) = 0.$$

Writing $\overline{F}(x) = y$, the above equation can be written as

$$y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \theta\Big(\frac{\mathrm{d}y}{\mathrm{d}x}\Big)^2 = 0. \tag{3.14}$$

For $\theta \neq 1$, the solution of this second-order differential equation is

$$\overline{F}(x) = [(1-\theta)(k_1x + k_2)]^{1/(1-\theta)}. \tag{3.15}$$

Since $\overline{F}(0) = 1$, the above equation reduces to

$$\overline{F}(x) = [k_1(1-\theta)x + 1]^{1/(1-\theta)}, \tag{3.16}$$

where when considering $\theta - 1 = \alpha$ and $k_1(1-\theta) = \beta$, we have $k_1 = -\beta/\alpha$. Hence, when $\alpha, \beta > 0$, the given distribution is a Pareto distribution. Also according to Gupta and Kirmani [11], if $\alpha, \beta < 0$ then $X$ has a finite range distribution. Note that, since $\theta \neq 0$, hence $\alpha \neq -1$.

Finally, if $\theta = 1$, by simple calculation it can be shown that the solution of equation (3.14) is $\overline{F}(x) = k_2\mathrm{e}^{k_1x}$. Since $\overline{F}(0) = 1$, by writing $E(X) = \mu$ we have

$$\overline{F}(x) = \mathrm{e}^{-x/\mu}, \quad x \geqslant 0.$$

Hence, the result follows. □

## 4. Conclusion

In this article, we characterized probability distributions such as exponential, Pareto, Weibull, Makeham, Gompertz, and finite range distributions through inequalities involving expectation of functions of failure rate, mean residual, log-odds rate, aging intensity functions, and also Glaser's function. It should be noted that we first generalized the results of Nanda [20] and Bhattacharjee et al. [4]. Moreover, we characterized exponential and logistic distributions, as well as Type 3 extreme value distribution by using the $w(\cdot)$-function defined by Cacoullos and Papathanasiou [6]. We also obtained bounds for the expected values of selected functions in reliability theory by this function. Wang [22] proved that for every random variable $X$ with log-concave density $f$, varentropy is bounded from above by the number 1 and in fact the probability bound does not depend on $f$, whereas we have provided a lower bound for it that depends on the probability density function $X$.

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