CONVERGENCE THEORY FOR THE EXACT INTERPOLATION SCHEME WITH APPROXIMATION VECTOR AS THE FIRST COLUMN OF THE PROLONGATOR AND RAYLEIGH QUOTIENT ITERATION NONLINEAR SMOOTHER

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Abstract. We extend the analysis of the recently proposed nonlinear EIS scheme applied to the partial eigenvalue problem. We address the case where the Rayleigh quotient iteration is used as the smoother on the fine-level. Unlike in our previous theoretical results, where the smoother given by the linear inverse power method is assumed, we prove nonlinear speed-up when the approximation becomes close to the exact solution. The speed-up is cubic. Unlike existent convergence estimates for the Rayleigh quotient iteration, our estimates take advantage of the powerful effect of the coarse-space.

Keywords: nonlinear multigrid; exact interpolation scheme

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1. Introduction

This paper is concerned with a convergence analysis of the nonlinear two-level method of [7] with a nonlinear Rayleigh quotient iteration smoother, applied to the partial eigenvalue problem. The matrix is assumed to be symmetric and positive definite with a simple minimal eigenvalue. We seek the minimum eigenvalue and the corresponding eigenvector. Our method belongs to the class of preconditioned eigensolvers using projected methods or subspace correction methods like Davidson, Jacobi-Davidson, and Generalized Davidson methods, see, for example, [3], [5], [9], [10], [11]. At the same time, we strongly exploit the multigrid structure of the solution spaces. In [4], we provided a convergence proof assuming the linear inverse power method is used on the fine-level. We showed that under reasonable assumptions, the

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method accelerates with the growing condition number of the matrix. On the other hand, there is no nonlinear speed-up when the iterate is close to the exact solution.

In this paper, we analyze the method of [7] with the Rayleigh quotient iteration used as the smoother and prove a cubic nonlinear speed-up. Unlike the known estimates for nonlinear inverse iteration (see e.g. [12]), our convergence estimates take advantage of the powerful effect of the coarse-space and improve with growing condition number of the matrix.

The convergence proof is limited to the two-level case. However, this limitation may not be too restrictive since the convergence proof, assuming a reasonable (and moderate) p-approximation property of the coarse-space, allows for radically aggressive coarsening while preserving good asymptotic convergence bounds.

The method of [7] is a special type of Exact Interpolation Scheme (EIS) proposed by Brandt with collaborators in [6], [1] and long before that, by Mandel and Sekerka in [8]. The coarse-level correction process is the Rayleigh-Ritz procedure with non-orthonormal approximation of the eigenvectors given by the columns of the prolongator. EIS is a nonlinear multigrid scheme with the prolongator constructed so that the current approximation $\mathbf{x}$ belongs to its range. While the authors of [6], [1] use a more complicated way of guaranteeing $\mathbf{x} \in \operatorname{Range} P$, we use a general purpose prolongator and simply add the current approximation $\mathbf{x}$ as its first column. The method was tested with extremely good results on problems of nuclear reactor criticality computations ([7]).

Our convergence analysis consists in proving the speed-up of the method with the Rayleigh quotient iteration smoother compared to the method that uses the linear inverse power method as the smoother. In other words, we estimate how many times faster is the method with the Rayleigh quotient iteration smoother compared to the method with the linear inverse power method used on the fine-level. As such, our new proof represents a natural extension of the convergence result of [4] that it invokes.

The analysis has been done for a partial eigenvalue problem assuming the matrix is symmetric, positive definite and the minimum eigenvalue is unique. The result can be easily extended to the case of a generalized eigenvalue problem where both matrices are symmetric and positive definite. The methodology of the generalization has been developed in [4].

Our numerical tests are performed on a model example. For numerical results of real-life problems, namely on the nuclear reactor criticallity computatioins on highly unstructured meshes, we refer to [7]. Here, we test the method on the partial eigenvalue problem obtained by Q1 discretization of the second order elliptic operator, in particular the discretization of the Laplace operator and its inisotropic singular perturbation. Both problems are discretized on the regular square grid on the square computational domain. The first observation is that adding the coarse-space of even a very moderate size (few degrees of freedom) significantly reduces the number of iterations compared to the Rayleigh quotient iteration applied to the same problem. This reflects the fact that the eigenvector corresponding to the minimal eigenvalue is, in general, smooth and as such well represented on a coarse level. The two-level method with the Raileigh quotient iteration used as the smoother performs excellently for highly anisotropic problems. The computational results for larger problems (bigger condition number of A) are generally better than for a smaller problems, mainly for multigrid method that uses linear inverse iteration as the smoother. This confirms our theoretical findings.

The paper is organized as follows. In Section 2 we present the algorithm and the convergence result of [4]. In the key Section 3 we analyze the nonlinear speed up of the Rayleigh quotient iteration smoother compared to the linear inverse power method. Section 4 contains the final convergence theorem. The numerical tests presented in final Section 5 confirm our theoretical findings. For the results of experiments with large problems that use inexact Rayleigh quotient iteration given by multigrid, see [7].

2. Algorithm

In this section, we explain our nonlinear multigrid scheme of [7], applied to the partial eigenvalue problem, and give the convergence result of [4].

The multigrid method consists of coarse-level correction along with smoothing, an iterative process performed on the fine level. The coarse-level correction consists in finding an approximation of the solution using the Ritz-Galerkin approximation to the solution on the coarse-space. The coarse-space is introduced as the range of a linear, injective prolongator $P\colon\ \mathbb{R}^m \to \mathbb{R}^n$, $m < n$, with $\mathbb{R}^n$ being the space where our fine-level problem is formulated.

Our method is a special type of Exact Interpolation Scheme (EIS) proposed in [1] and [6]. To treat a nonlinear problem, EIS updates the prolongator in each iteration so that the current approximation is contained in its range. While the authors of [6] and [1] use a quite complicated way of guaranteeing $\mathbf{x} \in \operatorname{Range} P$, we use a general purpose prolongator and simply add the current approximation $\mathbf{x}$ as its first column.

To explain the reasons for enriching the coarse-space by a current fine-level approximation, we first notice the following difficulty when solving a nonlinear problem by multigrid: in the linear case $A\mathbf{x} = \mathbf{f}$, we can take the current approximation $\mathbf{x}$ and formulate the residual equation $A\mathbf{u} = \mathbf{d}$, $\mathbf{d} = A\mathbf{x} - \mathbf{f}$. Due to the linearity of A, the exact solution of the residual equation then gives a correction to the approximation $\mathbf{x}$. For $\hat{\mathbf{u}} = A^{-1}\mathbf{d}$, $\mathbf{x} - \hat{\mathbf{u}}$ is the exact solution of the original system $A\mathbf{x} = \mathbf{f}$. In the standard multigrid, we repeatedly solve the residual problem approximately and use the correction. In the nonlinear case $A(\mathbf{x}) = \mathbf{f}(\mathbf{x})$, this is not possible and we are forced to use the coarse-level correction for the original problem rather than for the residual equation.

In our method, let P be the prolongator for now. The current fine-level approximation $\mathbf{x}$ must be available on the coarse-level, as we want to solve the coarse-level equation:

$$\text{find } \mathbf{x}_2\colon\ P^{\mathrm{T}} A(\mathbf{x} - P\mathbf{x}_2) = P^{\mathrm{T}}\mathbf{f}(\mathbf{x} - P\mathbf{x}_2), \tag{2.1}$$

where the final, corrected solution is $\mathbf{x} - P\mathbf{x}_2$. We resolve the difficulty discussed above by enriching the coarse-space $V = \mathrm{Range}(P)$ with the current fine eigenvector approximation $\mathbf{x}$ as its basis function in the first column of the prolongator. Thus, we use the prolongator $[\mathbf{x}|P]$ and instead of (2.1), we solve the problem

$$\text{find } \mathbf{x}_2\colon\ [\mathbf{x}|P]^{\mathrm{T}} A([\mathbf{x}|P]\mathbf{x}_2) = [\mathbf{x}|P]^{\mathrm{T}}\mathbf{f}([\mathbf{x}|P]\mathbf{x}_2)$$

that has the same final solution $[\mathbf{x}|P]\mathbf{x}_2$ as equation (2.1), with $[\mathbf{x}|P]$ in place of P. That is,

$$\text{find } \mathbf{x}_2'\colon\ [\mathbf{x}|P]^{\mathrm{T}} A(\mathbf{x} - [\mathbf{x}|P]\mathbf{x}_2') = [\mathbf{x}|P]^{\mathrm{T}}\mathbf{f}(\mathbf{x} - [\mathbf{x}|P]\mathbf{x}_2').$$

Here, the final corrected solution is $\mathbf{x} - [\mathbf{x}|P]\mathbf{x}_2'$. In the sequence of the coarse-spaces, each coarse-space contains the current approximation as a column of the prolongator. Thus, in the limit, the coarse-space contains the exact solution as its basis function and the solution of the coarse-level problem is the solution of the fine-level problem.

The description of our algorithm follows. Let A be a symmetric and positive definite $n \times n$ matrix with eigenvalues $\lambda_{\min} = \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \ldots \leqslant \lambda_n = \lambda_{\max}$. The particular case of interest is for A to be a finite element stiffness matrix.

We solve the partial eigenvalue problem:

$$\text{Find } \lambda_1,\ \mathbf{v}_1 \in \mathbb{R}^n \setminus \{\mathbf{0}\}\colon\ A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1. \tag{2.2}$$

We consider a linear injective prolongator $P\colon\ \mathbb{R}^m \to \mathbb{R}^n$, $m < n$. We are interested in aggressive coarsening, i.e. $m \ll n$.

Our two-level algorithm with evolving coarse-space for solving (2.2) proceeds as follows:

Algorithm 1.

▷ Perform Rayleigh-Ritz procedure with non-orthonormal approximation of eigenvectors given by the columns of the prolongator $[\mathbf{x}|P]$ as follows:

1. For given input iterate $\mathbf{x} \in \mathbb{R}^n$, construct/update the coarse-level matrices

$$A_2(\mathbf{x}) = [\mathbf{x}|P]^{\mathrm{T}} A[\mathbf{x}|P], \quad B_2(\mathbf{x}) = [\mathbf{x}|P]^{\mathrm{T}}[\mathbf{x}|P]. \tag{2.3}$$

See Remark 2.2.

2. Find the eigenvector $\mathbf{v}^2$ corresponding to the smallest eigenvalue of the coarse-level problem

$$A_2(\mathbf{x})\mathbf{v}^2 = \lambda B_2(\mathbf{x})\mathbf{v}^2. \tag{2.4}$$

   (If the coarse-level problem (2.4) is to be solved iteratively, a natural initial guess for $\mathbf{v}^2$ is the first canonical basis vector $\mathbf{e}_1 = (1, 0, \dots, 0)^{\mathrm{T}} \in \mathbb{R}^{m+1}$. See Remark 2.1.)

3. Prolongate $\mathbf{v} \leftarrow [\mathbf{x}|P]\mathbf{v}^2$.

▷ Post-smooth: either perform the inverse power method iteration (old variant of [4])

$$\mathbf{x}^{\mathrm{new}} \leftarrow A^{-\nu}\mathbf{v}, \tag{2.5}$$

or perform the Rayleigh quotient iteration

set $\mathbf{x}^0 = \mathbf{v}$;

(2.6) for $i = 1, \dots, \nu$ perform $\mathbf{x}^i \leftarrow (A - R(\mathbf{x}^{i-1})I)^{-1}\mathbf{x}^{i-1}$ with

(2.7) $R(\mathbf{x}) = \dfrac{\langle A\mathbf{x}, \mathbf{x}\rangle}{\|\mathbf{x}\|^2}$ (the Rayleigh quotient);

set $\mathbf{x}^{\mathrm{new}} = \mathbf{x}^{\nu}$.

▷ Normalize $\mathbf{x}^{\mathrm{new}} \leftarrow \|\mathbf{x}^{\mathrm{new}}\|^{-1}\mathbf{x}^{\mathrm{new}}$.

Remark 2.1. Clearly, for the first canonical basis vector $\mathbf{e}_1 = (1, 0, \dots, 0)^{\mathrm{T}} \in \mathbb{R}^{m+1}$ it holds that

$$\mathbf{x} = [\mathbf{x}|P]\mathbf{e}_1.$$

The vector $\mathbf{e}_1$ is therefore (assuming $\mathbf{x} \notin \mathrm{Range}(P)$) a coarse level isomorphic counterpart of the current approximation $\mathbf{x}$; the coarse-level iteration started from $\mathbf{e}_1$ is therefore essentially (via the isomorphism $[\mathbf{x}|P]\colon \mathbb{R}^{m+1} \to \mathrm{Range}([\mathbf{x}|P])$) started from $\mathbf{x}$.

Remark 2.2. Note that only the first column of the prolongator $[\mathbf{x}|P]$ changes from one iteration to the next. Therefore, only the first row and the first column of matrices $A_2(\mathbf{x})$ and $B_2(\mathbf{x})$ have to be recalculated in each iteration. If the coarse-level problem (2.4) is to be solved by the inverse power method

$$\mathbf{v}^2 \leftarrow A_2(\mathbf{x})^{-1}B_2(\mathbf{x})\mathbf{v}^2,$$

the action of the inverse $A_2(\mathbf{x})^{-1}$ can be performed using pre-calculated Choleski decomposition of the matrix $A_2(\mathbf{x})$ with the first column and the first row excluded.

This matrix is the same in every iteration. Note also that the columns of P can be (before the first iteration) A-orthonormalized, resulting in a very cheap action of $A_2(\mathbf{x})^{-1}$. An efficient way how to evaluate the action of $A_2(\mathbf{x})^{-1}$ using the action of $(PAP)^{-1}$ by a Schur complement technique is given in [7], see (8).

Remark 2.3. The coarse-level correction part 1–3 of Algorithm 1 is the Rayleigh-Ritz procedure with non-orthonormal approximation of eigenvectors given by the columns of the prolongator $[\mathbf{x}|P]$, resulting (due to the non-orthonormality of the columns) in generalized eigenvalue problem on the coarse level ($B_2(\mathbf{x}) \neq I$). For the vector $\mathbf{v}$ returned by Step 3 of Algorithm 1, the pair $(\lambda_{\min}(B_2(\mathbf{x})) = R(\mathbf{v}), \mathbf{v})$ is then a Ritz pair.

In more detail, the coarse-level problem (2.4) is the result (equivalent) of the Galerkin formulation

$$\text{(2.8)} \quad \text{find } \mathbf{v} \in \operatorname{Range}([\mathbf{x}|P]) \setminus \{\mathbf{0}\}\colon \ \langle (A - R(\mathbf{v})I)\mathbf{v}, \mathbf{w}\rangle = 0 \quad \forall\, \mathbf{w} \in \operatorname{Range}([\mathbf{x}|P]).$$

Indeed, (2.8) is equivalent to the problem

$$\begin{aligned}&\text{find } \mathbf{v}^2 \in \mathbb{R}^{m+1} \setminus \{\mathbf{0}\}\colon\\ &\left\langle \left(A - \frac{\langle A[\mathbf{x}|P]\mathbf{v}^2, [\mathbf{x}|P]\mathbf{v}^2\rangle}{\langle [\mathbf{x}|P]\mathbf{v}^2, [\mathbf{x}|P]\mathbf{v}^2\rangle} I \right) [\mathbf{x}|P]\mathbf{v}^2, [\mathbf{x}|P]\mathbf{w}^2 \right\rangle = 0 \quad \forall\, \mathbf{w^2} \in \mathbb{R}^{m+1}\end{aligned}$$

which, after transposing prolongators $[\mathbf{x}|P]$ in the right arguments of the inner products, becomes

$$\left\langle \left(A_2(\mathbf{x}) - \frac{\langle A_2(\mathbf{x})\mathbf{v}^2, \mathbf{v}^2\rangle}{\langle B_2(\mathbf{x})\mathbf{v}^2, \mathbf{v}^2\rangle} B_2(\mathbf{x}) \right) \mathbf{v}^2, \mathbf{w}^2 \right\rangle = 0 \quad \forall\, \mathbf{w}^2 \in \mathbb{R}^{m+1}$$

with matrices $A_2(\mathbf{x})$ and $B_2(\mathbf{x})$ given by (2.3). The above identity holds if and only if the left argument of the above inner product is zero, which happens if and only if $\mathbf{v}^2$ is an eigenvector of (2.4). Thus, (2.8) and (2.4) are equivalent.

Define the scaled residual norm $r(\mathbf{x})$ by

$$\text{(2.9)} \qquad r(\mathbf{x}) = \frac{\|A\mathbf{x} - R(\mathbf{x})\mathbf{x}\|}{\|\mathbf{x}\|}, \quad \mathbf{x} \neq \mathbf{0}.$$

The following convergence theorem gives the estimate for Algorithm 1 with old version of the smoother (2.5), in terms of $r(\cdot)$.

Theorem 2.4 ([4]). *Let A be a symmetric and positive definite $n \times n$ matrix with simple minimal eigenvalue $\lambda_{\min}$. We assume the eigenvalues are numbered so* *that* $\lambda_{\min} = \lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_n = \lambda_{\max}$. *Consider* $\alpha \in (0,1]$ *and* $\beta > 0$ *such that* $\alpha\beta - 1 > 0$. *We assume there is a linear mapping* $Q\colon \mathbb{R}^n \to \operatorname{Range}(P)$ *such that*

$$\forall\, \mathbf{u} \in \mathbb{R}^n\colon\ \lambda_{\min}^{\beta/2}\|\mathbf{u} - Q\mathbf{u}\| \leqslant \frac{C}{\operatorname{cond}(A)^{\alpha\beta/2}}\|\mathbf{u}\|_{A^\beta}. \tag{2.10}$$

In addition, assume that the input iterate $\mathbf{x}$ *is reasonably close to the first eigenvector* $\mathbf{v}_1$ *so that* $R(\mathbf{x}) \in [\lambda_1, \lambda_2)$. *Then the result* $\mathbf{x}^{\text{new}}$ *upon exit of Algorithm* 1 *with* $\nu \geqslant \beta/2$ *and the post-smoothing performed by* (2.5) *satisfies the estimate*

$$r(\mathbf{x}^{\text{new}}) = r(A^{-\nu}\mathbf{v}) \leqslant q_{\text{MGII}}(\mathbf{x}) r(\mathbf{x}),$$
$$q_{\text{MGII}}(\mathbf{x}) = \frac{C}{\operatorname{cond}(A)^{(\alpha\beta-1)/2}} \frac{\sqrt{\lambda_2/\lambda_{\min} - 1}}{(\lambda_2 - R(\mathbf{x}))/\lambda_{\min}}.$$

The rate of convergence $q_{\text{MGII}}(\mathbf{x})$ *satisfies*

$$\lim_{\operatorname{cond}(A)\to\infty} q_{\text{MGII}}(\mathbf{x}) = 0. \tag{2.11}$$

Remark 2.5. In specific applications, the parameter α is related to the mesh resolution H of the coarse level space ($H = h^\alpha$, h being the fine-level resolution). The parameter β corresponds to p-approximation quality of the coarse space ($\beta = p+1$). In specific applications, the constant C in (2.10) is indeed bounded. Consider a regular $2D$ finite difference discretization of the Poisson equation on a cube with the resolution h and a P1 finite element coarse-space with the resolution $H = h^\alpha$. Then A^2 is a regular finite difference discretization of the biharmonic equation (the problem with the H^2-equivalent form). There is a mapping Q ([2]) such that

$$h\|\mathbf{u} - Q\mathbf{u}\| \leqslant C' H^2 \|\mathbf{u}\|_{A^2}, \quad H = h^\alpha, \quad \operatorname{cond}(A) = h^{-2}, \quad \lambda_{\min} \approx h^2$$

with C' uniformly bounded. Then,

$$\lambda_{\min}\|\mathbf{u} - Q\mathbf{u}\| \leqslant \frac{C'h}{\operatorname{cond}(A)^\alpha}\|\mathbf{u}\|_{A^2}.$$

The condition (2.10) therefore holds with $\beta = 2$ and $C = C'h$.

3. Acceleration effect of the Rayleigh quotient iteration method

In this section we prove that for $\mathbf{x}$ close to the first eigenvector, our method with the Rayleigh quotient iteration (2.6) used as the smoother converges much faster than the two-level method with the smoother given by inverse power method. The method with linear smoother (2.5) was analyzed in [4]. As in [4], the new result takes advantage of the powerful effect of the coarse-space and improves with growing condition number of the matrix A. The acceleration effect in the smoothing iteration is cubic.

We start our analysis with a straightforward observation.

Lemma 3.1. *Let A be $n \times n$ symmetric and positive definite matrix with simple minimal eigenvalue. For convenience, we assume that the matrix A is scaled so that this minimal eigenvalue satisfies $\lambda_{\min}(A) = \lambda_1(A) = 1$. Assume $\mathbf{x} \in \mathbb{R}^n$ is a vector such that $R(\mathbf{x}) \in [\lambda_1(A), \lambda_2(A))$. Define the operator Q by*

$$Q(\mathbf{x})\colon\ \mathbf{u} \equiv \sum_i c_i \mathbf{v}_i \mapsto \mathbf{v} = c_1(R(\mathbf{x}) - \lambda_1(A))\mathbf{v}_1 + \sum_{i>1} c_i \mathbf{v}_i. \tag{3.1}$$

Here, $\{\mathbf{v}_i\}$ are orthonormal eigenvectors with the eigenvalue $\lambda_i(A)$ corresponding to the eigenvector $\mathbf{v}_i$. The eigenvalues are assumed to be ordered so that

$$\lambda_1(A) < \lambda_2(A) \leqslant \ldots \leqslant \lambda_n(A).$$

Then, for all $\mathbf{u} \in \mathbb{R}^n$ it holds that

$$\|(AQ(\mathbf{x}))^{-1}\mathbf{u}\| \leqslant \|(A - R(\mathbf{x})I)^{-1}\mathbf{u}\| \leqslant \frac{\lambda_2(A)}{\lambda_2(A) - R(\mathbf{x})}\|(AQ(\mathbf{x}))^{-1}\mathbf{u}\|. \tag{3.2}$$

P r o o f. Clearly, the operator $Q(\cdot)$ defined by (3.1) is symmetric in the Euclidean inner product, commutes with A, and $AQ(\mathbf{x}) = Q(\mathbf{x})A$ is symmetric as well. The operators $AQ(\mathbf{x})$ and $A - R(\mathbf{x})I$ have the same eigenvectors $\{\mathbf{v}_i\}$ as A; by $\lambda_i(AQ(\mathbf{x}))$ or $\lambda_i(A - R(\mathbf{x})I)$ we simply mean the eigenvalue corresponding to $\mathbf{v}_i$. First we establish the inequality

$$\lambda_k^{-1}(AQ(\mathbf{x})) \leqslant |\lambda_k^{-1}(A - R(\mathbf{x})I)| \leqslant \frac{\lambda_2(A)}{\lambda_2(A) - R(\mathbf{x})}\lambda_k^{-1}(AQ(\mathbf{x})). \tag{3.3}$$

Obviously, $\lambda_1(AQ(\mathbf{x})) = R(\mathbf{x}) - \lambda_1(A) = |\lambda_1(A - R(\mathbf{x})I)|$ and

$$\frac{\lambda_2(A)}{\lambda_2(A) - R(\mathbf{x})} > 1.$$

Thus, for the first eigenvalue, (3.3) holds trivially. For $k > 1$, $\lambda_k(AQ(\mathbf{x})) = \lambda_k(A)$. Further,

$$\begin{aligned}\lambda_k(AQ(\mathbf{x})) &= \lambda_k(A) > \lambda_k(A - R(\mathbf{x})I) = \lambda_k(A) - R(\mathbf{x}) \\ &= \frac{\lambda_k(A) - R(\mathbf{x})}{\lambda_k(A)}\lambda_k(A) \geqslant \frac{\lambda_2(A) - R(\mathbf{x})}{\lambda_2(A)}\lambda_k(A) \\ &= \frac{\lambda_2(A) - R(\mathbf{x})}{\lambda_2(A)}\lambda_k(AQ(\mathbf{x})),\end{aligned}$$

proving (3.3) for $k > 1$.

Recall that the eigenvectors are orthonormalized, so for $\mathbf{u} = \sum\limits_i c_i\mathbf{v}_i$ we have

$$\|(AQ(\mathbf{x}))^{-1}\mathbf{u}\|^2 = \sum_i c_i^2\lambda_i^{-2}(AQ(\mathbf{x}))$$

and

$$\|(A - R(\mathbf{x})I)^{-1}\mathbf{u}\|^2 = \sum_i c_i^2\lambda_i^{-2}(A - R(\mathbf{x})I),$$

which together with (3.3) proves (3.2), completing the proof. □

The following lemma shows that for $\mathbf{v}$ such that $R(\mathbf{v}) \approx \lambda_1$, the smoother given by the Rayleigh quotient iteration yields the same residual $r(\cdot)$ as the inverse power method for an approximation vector with a much smaller error.

Lemma 3.2. *Let A be a symmetric and positive definite matrix with eigenvalues $1 = \lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_n$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ the corresponding orthonormalized eigenvectors. Consider a vector $\mathbf{v} \in \mathbb{R}^n$ in the form $\mathbf{v} = c_1\mathbf{v}_1 + \mathbf{e}$, $\mathbf{e} \in \operatorname{span}\{\mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ and a vector $\mathbf{x} \in \mathbb{R}^n$ such that $R(\mathbf{x}) < \lambda_2$. Then*

$$\frac{\lambda_2 - R(\mathbf{x})}{\lambda_2} r(A^{-1}\overline{\mathbf{v}}) \leqslant r((A - R(\mathbf{x})I)^{-1}\mathbf{v}) \leqslant \frac{\lambda_2}{\lambda_2 - R(\mathbf{x})} r(A^{-1}\overline{\mathbf{v}}), \tag{3.4}$$
$$\overline{\mathbf{v}} = c_1\mathbf{v}_1 + (R(\mathbf{x}) - \lambda_1)\mathbf{e}.$$

P r o o f. Let us prove first the second inequality of (3.4). Since $R(\mathbf{x})\mathbf{x}$ is the orthogonal projection of $A\mathbf{x}$ onto $\operatorname{span}\mathbf{x}$, we have

$$\|(A - R(\mathbf{x})I)\mathbf{x}\| \leqslant \|(A - R(\mathbf{x}')I)\mathbf{x}\|$$

for all vectors $\mathbf{x}' \in \mathbb{R}^n$. Using this property, Lemma 3.1, and the fact that the operators $A - R(\mathbf{v})I$ and $AQ(\mathbf{v})$ (see (3.1)) commute with A, we get

$$
\begin{aligned}
(3.5)\qquad r((A-R(\mathbf{x})I)^{-1}\mathbf{v}) &= \frac{\|[A-R((A-R(\mathbf{x})I)^{-1}\mathbf{v})I](A-R(\mathbf{x})I)^{-1}\mathbf{v}\|}{\|(A-R(\mathbf{x})I)^{-1}\mathbf{v}\|} \\
&\leqslant \frac{\|[A-R((AQ(\mathbf{x}))^{-1}\mathbf{v})I](A-R(\mathbf{x})I)^{-1}\mathbf{v}\|}{\|(A-R(\mathbf{x})I)^{-1}\mathbf{v}\|} \\
&= \frac{\|(A-R(\mathbf{x})I)^{-1}[A-R((AQ(\mathbf{x}))^{-1}\mathbf{v})I]\mathbf{v}\|}{\|(A-R(\mathbf{x})I)^{-1}\mathbf{v}\|} \\
&\leqslant \frac{\lambda_2}{\lambda_2-R(\mathbf{x})}\,\frac{\|(AQ(\mathbf{x}))^{-1}[A-R((AQ(\mathbf{x}))^{-1}\mathbf{v})I]\mathbf{v}\|}{\|(AQ(\mathbf{x}))^{-1}\mathbf{v}\|} \\
&= \frac{\lambda_2}{\lambda_2-R(\mathbf{x})}\,\frac{\|[A-R((AQ(\mathbf{x}))^{-1}\mathbf{v})I](AQ(\mathbf{x}))^{-1}\mathbf{v}\|}{\|(AQ(\mathbf{x}))^{-1}\mathbf{v}\|} \\
&= \frac{\lambda_2}{\lambda_2-R(\mathbf{x})}\, r((AQ(\mathbf{x}))^{-1}\mathbf{v}).
\end{aligned}
$$

By definition (3.1) of $Q(\cdot)$ and using the fact that the operators A and $Q(\mathbf{x})$ commute with A, we get

$$
\begin{aligned}
(AQ(\mathbf{x}))^{-1}\mathbf{v} &= (AQ(\mathbf{x}))^{-1}(c_1\mathbf{v}_1+\mathbf{e}) = A^{-1}Q^{-1}(\mathbf{x})(c_1\mathbf{v}_1+\mathbf{e}) \\
&= A^{-1}\Big(\frac{c_1}{R(\mathbf{x})-\lambda_1}\mathbf{v}_1+\mathbf{e}\Big) = \frac{1}{R(\mathbf{x})-\lambda_1}A^{-1}(c_1\mathbf{v}_1+(R(\mathbf{x})-\lambda_1)\mathbf{e}).
\end{aligned}
$$

Since $r(\cdot)$ is independent of the scaling of the argument, the previous identity yields

$$
(3.6)\qquad r((AQ(\mathbf{x}))^{-1}\mathbf{v}) = r(A^{-1}[c_1\mathbf{v}_1+(R(\mathbf{x})-\lambda_1)\mathbf{e}]).
$$

The proof of the second inequality of (3.4) now follows by (3.6).

To prove the first inequality of (3.4) we estimate again using the minimizing property of the orthogonal projection, the fact that the operators $AQ(\mathbf{x})$ and $A - R(\mathbf{x})I$ commute, and Lemma 3.1:

$$
\begin{aligned}
r((A-R(\mathbf{x})I)^{-1}\mathbf{v}) &= \frac{\|[A-R((A-R(\mathbf{x})I)^{-1}\mathbf{v})I](A-R(\mathbf{x})I)^{-1}\mathbf{v}\|}{\|(A-R(\mathbf{x})I)^{-1}\mathbf{v}\|} \\
&= \frac{\|(A-R(\mathbf{x})I)^{-1}[A-R((A-R(\mathbf{x})I)^{-1}\mathbf{v})I]\mathbf{v}\|}{\|(A-R(\mathbf{x})I)^{-1}\mathbf{v}\|} \\
&\geqslant \frac{\lambda_2-R(\mathbf{x})}{\lambda_2}\,\frac{\|(AQ(\mathbf{x}))^{-1}[A-R((A-R(\mathbf{x})I)^{-1}\mathbf{v})I]\mathbf{v}\|}{\|(AQ(\mathbf{x}))^{-1}\mathbf{v}\|}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\lambda_2 - R(\mathbf{x})}{\lambda_2} \frac{\|[A - R((A - R(\mathbf{x})I)^{-1}\mathbf{v})I](AQ(\mathbf{x}))^{-1}\mathbf{v}\|}{\|(AQ(\mathbf{x}))^{-1}\mathbf{v}\|} \\
&\geqslant \frac{\lambda_2 - R(\mathbf{x})}{\lambda_2} \frac{\|[A - R((AQ(\mathbf{x})^{-1}\mathbf{v})I](AQ(\mathbf{x}))^{-1}\mathbf{v}\|}{\|(AQ(\mathbf{x}))^{-1}\mathbf{v}\|} \\
&= \frac{\lambda_2 - R(\mathbf{x})}{\lambda_2} r((AQ(\mathbf{x}))^{-1}\mathbf{v}).
\end{aligned}
$$

The proof of the first inequality of (3.4) now follows by (3.6). □

The following lemma and its corollary contain the key result of this section. Namely, we prove that for an approximation close to the first eigenvector, the method using the smoother given by the Rayleigh quotient iteration converges much faster than the method with the inverse power method given by A^{-1}. The acceleration effect is cubic. As in [4], where the method with the smoother A^{-1} is analyzed, the result takes advantage of the effect of the coarse-space. The lemma is very convenient: we simply take the recent result for the smoother $\mathbf{v} \leftarrow A^{-1}\mathbf{v}$ and immediately get the asymptotically accelerated result for the fine-level smoother given by the Rayleigh quotient iteration $\mathbf{v} \leftarrow (A - R(\mathbf{v})I)^{-1}\mathbf{v}$. In other words, we get the accelerated result in the terms of the former result of [4] for A^{-1}.

Lemma 3.3. *Let A be a symmetric and positive definite matrix with eigenvalues $1 = \lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_n$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ the corresponding orthonormal eigenvectors. Consider a vector $\mathbf{v}$ in the form $\mathbf{v} = c_1\mathbf{v}_1 + \mathbf{e}$, $\mathbf{e} \perp \mathbf{v}_1$ and set $\mathbf{e}' = \|\mathbf{v}\|^{-1}\mathbf{e}$. We assume that $|c_1| \geqslant c'_{1,\min}\|\mathbf{v}\|$, $c'_{1,\min} \in (0, 1]$. Let $\mathbf{x} \in \mathbb{R}^n$ be a vector such that $R(\mathbf{x}) < \lambda_2$ and $R(\mathbf{x}) - \lambda_1 \leqslant 1$. Then it holds that*

$$
\begin{aligned}
(3.7) \qquad r((A - R(\mathbf{x})I)^{-1}\mathbf{v})) \leqslant{} & \frac{\lambda_2}{(\lambda_2 - R(\mathbf{x}))|c'_{1,\min}|}(R(\mathbf{x}) - \lambda_1)r(A^{-1}\mathbf{v}) \\
& + C(R(\mathbf{x}) - \lambda_1)\|\mathbf{e}'\|^2.
\end{aligned}
$$

Here, $C > 0$ is a constant independent of $\mathbf{e}$, dependent exclusively on $c'_{1,\min}$.

Proof. In view of the previous lemma, we first investigate the dependence of $r(A^{-1}\mathbf{v})$ on the magnitude of $\mathbf{e}$. Let us set $\mathbf{v}' = \|\mathbf{v}\|^{-1}\mathbf{v}$, $c'_1 = c_1/\|\mathbf{v}\|$ and recall that $\mathbf{e}' = \|\mathbf{v}\|^{-1}\mathbf{e}$. Then

$$
\mathbf{v}' = c'_1\mathbf{v}_1 + \mathbf{e}', \ r(A^{-1}\mathbf{v}) = r(A^{-1}\mathbf{v}').
$$

Thus, we will investigate $r(A^{-1}\mathbf{v}')$ in terms of the scaled error $\mathbf{e}' \perp \mathbf{v}_1$. As $1 = \|\mathbf{v}'\|^2 = {c'_1}^2 + \|\mathbf{e}'\|^2$, we have $\|\mathbf{e}'\| \leqslant 1$. Further, we set $\mathbf{e}_u = \|\mathbf{e}\|^{-1}\mathbf{e}$. Then the parameter t in the expression $r(A^{-1}\mathbf{v}) = r(A^{-1}(c'_1\mathbf{v}_1 + t\mathbf{e}_u))$ stands for the Euclidean norm of the actual scaled error $\mathbf{e}'$. Since $R(\mathbf{w})\mathbf{w}$ is the orthogonal projection of $A\mathbf{w}$ onto span$\{\mathbf{w}\}$ w.r.t. the Euclidean inner product, we find that

$$r(\mathbf{w}) = \frac{\|A\mathbf{w} - R(\mathbf{w})\mathbf{w}\|}{\|\mathbf{w}\|} = \Big(\frac{\|A\mathbf{w}\|^2 - R^2(\mathbf{w})\|\mathbf{w}\|^2}{\|\mathbf{w}\|^2}\Big)^{1/2} = \Big(\frac{\|A\mathbf{w}\|^2}{\|\mathbf{w}\|^2} - R^2(\mathbf{w})\Big)^{1/2}. \tag{3.8}$$

Define the function

$$\varphi(t) = r(A^{-1}(c_1'\mathbf{v}_1 + t\mathbf{e}_u)). \tag{3.9}$$

By (3.8), $\mathbf{e}_u \perp \mathbf{v}_1$, and the fact that the eigenvectors are orthonormal, we have

$$\begin{aligned}\varphi(t) &= \Big(\frac{\|c_1'\mathbf{v}_1 + t\mathbf{e}_u\|^2}{\|A^{-1}(c_1'\mathbf{v}_1 + t\mathbf{e}_u)\|^2} - \Big(\frac{\|A^{-1/2}(c_1'\mathbf{v}_1 + t\mathbf{e}_u)\|^2}{\|A^{-1}(c_1'\mathbf{v}_1 + t\mathbf{e}_u)\|^2}\Big)^2\Big)^{1/2} \\ &= \Big(\frac{(c_1')^2 + t^2\|\mathbf{e}_u\|^2}{(c_1')^2 + t^2\|A^{-1}\mathbf{e}_u\|^2} - \Big(\frac{(c_1')^2 + t^2\|A^{-1/2}\mathbf{e}_u\|^2}{(c_1')^2 + t^2\|A^{-1}\mathbf{e}_u\|^2}\Big)^2\Big)^{1/2}.\end{aligned} \tag{3.10}$$

Let us set $a = 1 - \|A^{-1}\mathbf{e}_u\|^2$ and $b = 1 - \|A^{-1/2}\mathbf{e}_u\|^2$. Clearly, $a, b \leqslant 1$. As $\|\mathbf{e}_u\| = 1$ and $\lambda_1 = \lambda_{\min}(A) = 1$, it also holds that $\|A^{-1}\mathbf{e}_u\| \leqslant \lambda_1^{-1}\|\mathbf{e}_u\| \leqslant 1$ and $\|A^{-1/2}\mathbf{e}_u\| \leqslant \lambda_1^{-1/2}\|\mathbf{e}_u\| \leqslant 1$, hence $a, b \geqslant 0$. Since $t = \|\mathbf{e}'\|$ and $1 = \|\mathbf{v}'\|^2 = (c_1')^2 + \|\mathbf{e}'\|^2$, we have $(c_1')^2 + t^2 = 1$ and therefore,

$$\varphi(t) = \Big[\frac{1}{1 - at^2} - \Big(\frac{1 - bt^2}{1 - at^2}\Big)^2\Big]^{1/2}, \quad a, b \in [0, 1], \tag{3.11}$$

and $t^2 = 1 - (c_1')^2 \in [0, 1 - (c_{1,\min}')^2]$. We have

$$(\varphi^2)'(0) = 0, \tag{3.12}$$

$$(\varphi^2)''(0) = 4b - 2a, \tag{3.13}$$

$$(\varphi^2)^{(3)}(0)) = 0, \tag{3.14}$$

and

$$\begin{aligned}(\varphi^2)^{(4)}(t) = {} & 384\frac{a^4t^4}{(1 - at^2)^5} + 288\frac{a^3t^2}{(1 - at^2)^4} \\ & - 6\Big[24\frac{a^2t^2}{(1 - at^2)^4} + 4\frac{a}{(1 - at^2)^3}\Big][8b^2t^2 - 4b(1 - bt^2)] \\ & + 24\frac{a^2}{(1 - at^2)^3} + 16bt\Big[192\frac{a^3t^3}{(1 - at^2)^5} - 72\frac{a^2t}{(1 - at^2)^4}\Big](1 - bt^2) \\ & - \Big[1920\frac{a^4t^4}{(1 - at^2)^6} + 1152\frac{a^3t^2}{(1 - at^2)^5} + 72\frac{a^2}{(1 - at^2)^4}\Big](1 - bt^2)^2 \\ & - 384\frac{ab^2t^2}{(1 - at^2)^3} - 24\frac{b^2}{(1 - at^2)^2}.\end{aligned} \tag{3.15}$$

Set

$$C = \max\{|(\varphi^2)^{(4)}(t)|:\ a, b \in [0,1],\ t^2 \leqslant 1 - (c'_{1,\min})^2\}.$$

It follows from (3.15) that C is uniformly bounded, dependent exclusively on $c'_{1,\min}$. Since $t = \|\mathbf{e}'\|$, by (3.12)–(3.14) we have a Taylor series

$$r^2(A^{-1}\mathbf{v}) = \varphi^2(t) = (\varphi^2)''(0)t^2 + (\varphi^2)^{(4)}(\hat{t})t^4 \tag{3.16}$$

for some $\hat{t} \in [0,1]$ and $|\varphi^2(\hat{t}))^{(4)}| \leqslant C$.

In view of Lemma 3.2, we also have to investigate $r(A^{-1}(c'_1\mathbf{v}_1 + (R(\mathbf{x}) - \lambda_1)\mathbf{e}_u)$. We stress that we arrived at the form of the function φ using the assumption that $(c'_1)^2 + t^2 = 1$. To be able to use (3.16) for $r^2((A - R(\mathbf{x})I)^{-1}\mathbf{v})$, we need a simple substitution. Let us set

$$c''_1 = \frac{c'_1}{\sqrt{(c'_1)^2 + (R(\mathbf{x}) - \lambda_1)^2 t^2}}, \quad \xi = t\frac{R(\mathbf{x}) - 1}{\sqrt{(c'_1)^2 + (R(\mathbf{x}) - \lambda_1)^2 t^2}}.$$

Clearly $\xi \approx (R(\mathbf{x}) - \lambda_1)t$ for $\mathbf{x}$ close to the first eigenvector. Then, $(c''_1)^2 + \xi^2 = 1$ and we can use (3.16) to get

$$\begin{aligned} r^2(A^{-1}(c'_1\mathbf{v}_1 + (R(\mathbf{x}) - \lambda_1)\mathbf{e}_u)) &= r^2(A^{-1}(c''_1\mathbf{v}_1 + \xi\mathbf{e}_u)) = \varphi^2(\xi) \\ &= (\varphi^2)''(0)\xi^2 + (\varphi^2)^{(4)}(\hat{\xi})\xi^4 \end{aligned}$$

for some $\hat{\xi} \in [0,1]$ with $|\varphi^2(\hat{\xi}))^{(4)}| \leqslant C$. We have

$$\xi^2 = \frac{(R(\mathbf{x}) - \lambda_1)^2}{(c'_1)^2 + (R(\mathbf{x}) - \lambda_1)^2 t^2}t^2 \leqslant \frac{(R(\mathbf{x}) - \lambda_1)^2}{(c'_{1,\min})^2}t^2$$

and we conclude that

$$r^2(A^{-1}(c'_1\mathbf{v}_1 + (R(\mathbf{x}) - \lambda_1)\mathbf{e}_u)) \leqslant \frac{(\varphi^2)''(0)}{(c_{1,\min})^2}(R(\mathbf{x}) - \lambda_1)^2 t^2 + O((R(\mathbf{x}) - \lambda_1)^4 t^4), \tag{3.17}$$

where the constant hidden in the symbol $O((R(\mathbf{x}) - \lambda_1)^4 t^4)$ is dependent exclusively on $c'_{1,\min} > 0$. Multiplying (3.16) by $-(R(\mathbf{x}) - 1)^2/(c'_{1,\min})^2$ and adding this to (3.17) yields

$$r^2(A^{-1}(c'_1\mathbf{v}_1 + (R(\mathbf{x}) - \lambda_1)\mathbf{e}_u)) - \frac{(R(\mathbf{x}) - 1)^2}{(c'_{1,\min})^2}r^2(A^{-1}\mathbf{v}) \leqslant O(R(\mathbf{x}) - \lambda_1)^2 t^4.$$

Since $t = \|\mathbf{e}'\|$, we get

$$r(A^{-1}(c'_1\mathbf{v}_1 + (R(\mathbf{x}) - \lambda_1)\mathbf{e}_u)) \leqslant \frac{R(\mathbf{x}) - \lambda_1}{|c'_{1,\min}|}r(A^{-1}\mathbf{v}) + O(R(\mathbf{x}) - \lambda_1)\|\mathbf{e}'\|^2. \tag{3.18}$$

Statement (3.7) now follows by Lemma 3.2. □

It remains to avoid the assumption $\lambda_1 = 1$.

Corollary 3.4. *Let A be a symmetric and positive definite $n \times n$ matrix with eigenvalues $\lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_n$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ the corresponding orthonormal eigenvectors. Consider a vector $\mathbf{v}$ in the form $\mathbf{v} = c_1\mathbf{v}_1 + \mathbf{e}$, $\mathbf{e} \perp \mathbf{v}_1$ and set $\mathbf{e}' = \|\mathbf{v}\|^{-1}\mathbf{e}$. We assume that $|c_1| \geqslant c'_{1,\min}\|\mathbf{v}\|$, $c'_{1,\min} \in (0,1]$. Let $\mathbf{x} \in \mathbb{R}^n$ be a vector satisfying $R(\mathbf{x}) < \lambda_2$ and $R(\mathbf{x}) - \lambda_1 \leqslant \lambda_1$. Then we have*

$$r((A - R(\mathbf{x})I)^{-1}\mathbf{v})) \leqslant \frac{\lambda_2}{(\lambda_2 - R(\mathbf{x}))|c'_{1,\min}|}\Big(\frac{R(\mathbf{x})}{\lambda_1} - 1\Big) r(A^{-1}\mathbf{v}) + C\lambda_1\Big(\frac{R(\mathbf{x})}{\lambda_1} - 1\Big)\|\mathbf{e}'\|^2. \tag{3.19}$$

Here, $C > 0$ is a constant independent of $\mathbf{e}$, *dependent exclusively on $c'_{1,\min}$.*

P r o o f. Set $A' = \lambda_1{}^{-1}A$, and $r_{A'}(\cdot)$, $R_{A'}(\cdot)$ to be $r(\cdot)$, $R(\cdot)$, respectively, with A replaced by A'. Lemma 3.3 gives

$$r_{A'}((A' - R_{A'}(\mathbf{x})I)^{-1}\mathbf{v})) \leqslant \frac{\lambda_2/\lambda_1}{(\lambda_2/\lambda_1 - R_{A'}(\mathbf{x}))|c'_{1,\min}|}(R_{A'}(\mathbf{x}) - 1) r_{A'}((A')^{-1}\mathbf{v}) + C(R_{A'}(\mathbf{x}) - 1)\|\mathbf{e}'\|^2. \tag{3.20}$$

Clearly, $r_{A'}(\cdot) = \lambda_1^{-1}r(\cdot)$, $R_{A'}(\cdot) = 1/\lambda_1 R(\cdot)$ and $A' - R_{A'}(\mathbf{x})I = \lambda_1^{-1}(A - R(\mathbf{x})I)$. Hence,

$$r_{A'}((A' - R_{A'}(\mathbf{x})I)^{-1}\mathbf{v})) = \frac{1}{\lambda_1} r((A - R(\mathbf{x})I)^{-1}\mathbf{v}))$$

and

$$r_{A'}((A')^{-1}\mathbf{v}) = \frac{1}{\lambda_1} r(A^{-1}\mathbf{v}).$$

The proof of (3.19) follows from (3.20), the last two identities, and $R_{A'}(\cdot) = \lambda_1^{-1}R(\cdot)$. □

Corollary 3.4 proves the acceleration effect in the estimate for $r(\mathbf{v})$ in terms of $R(\mathbf{v})/\lambda_1 - 1$. To eliminate the inconsistency, we prove that $R(\mathbf{v})/\lambda_1 - 1$ can be controlled by $Cr^2(\mathbf{v})$.

Lemma 3.5. *Let A be a symmetric and positive definite $n \times n$ matrix and $\{\mathbf{v}_i\}$ its system of orthonormal eigenvectors with λ_i the eigenvalue corresponding to $\mathbf{v}_i$. We assume the usual numbering $\lambda_i \leqslant \lambda_{i+1}$, $i = 1, \ldots n-1$. We consider a vector $\mathbf{x} = \sum_i c_i\mathbf{v}_i$ such that $R(\mathbf{x}) \leqslant \lambda_2$. Then*

$$\frac{R(\mathbf{x})}{\lambda_1} - 1 \leqslant \frac{\|\mathbf{x}\|^2}{(\lambda_2/\lambda_1 - 1)c_1^2}\Big(\frac{r(\mathbf{x})}{\lambda_1}\Big)^2. \tag{3.21}$$

In addition, it holds that

$$c_1^2 \geqslant \Big(1 - \frac{R(\mathbf{x})/\lambda_1 - 1}{\lambda_2/\lambda_1 - 1}\Big)\|\mathbf{x}\|^2. \tag{3.22}$$

R e m a r k 3.6. In the above estimates, $(c_1')^2 \equiv (c_1/\|\mathbf{x}\|)^2 \in (0,1]$.

P r o o f. Without loss of generality, we assume that $\lambda_1 = 1$ and $\|\mathbf{x}\| = 1$. The generalization follows by a trivial argument.

Let us set $c = R(\mathbf{x}) - 1$. Thus, to prove (3.21) and (3.22) means to prove

$$r^2(\mathbf{x}) \geqslant (\lambda_2 - 1)cc_1^2 \tag{3.23}$$

and

$$c_1^2 \geqslant 1 - \frac{c}{\lambda_2 - 1}. \tag{3.24}$$

We first prove the auxiliary estimate

$$\frac{\|A\mathbf{x}\|^2}{\|\mathbf{x}\|_A^2} - \frac{\|\mathbf{x}\|_A^2}{\|\mathbf{x}\|^2} \geqslant (\lambda_2 - 1)\frac{c}{1+c}c_1^2. \tag{3.25}$$

By assumption, $\mathbf{x} = \sum\limits_i c_i \mathbf{v}_i$, $\sum\limits_i c_i^2 = 1$ and $c_1^2 + \sum\limits_{i>1} c_i^2\lambda_i = 1 + c$. Set

$$q = \frac{\sum\limits_{i>1} c_i^2\lambda_i}{\sum\limits_{i>1} c_i^2} \geqslant \lambda_2.$$

Then $c_1^2 + q\sum\limits_{i>1} c_i^2 = 1 + c$, or, $c_1^2 + q(1 - c_1^2) = 1 + c$. Thus, we have

$$q = 1 + \frac{c}{1 - c_1^2}, \quad c_1^2 \geqslant 1 - \frac{c}{\lambda_2 - 1}. \tag{3.26}$$

This proves (3.24). Further, for any nonzero $\mathbf{w} \in \mathbb{R}^n$,

$$\frac{\|A\mathbf{w}\|}{\|\mathbf{w}\|_A} \geqslant \frac{\|\mathbf{w}\|_A}{\|\mathbf{w}\|}. \tag{3.27}$$

Indeed, $\|\mathbf{w}\|_A^2 = \langle A\mathbf{w}, \mathbf{w}\rangle \leqslant \|A\mathbf{w}\|\|\mathbf{w}\|$. Dividing the above estimate by $\|\mathbf{w}\|\|\mathbf{w}\|_A$ yields (3.27). Setting $\mathbf{w} = \sum\limits_{i>1} c_i\mathbf{v}_i$, we get

$$\frac{\sum\limits_{i>1} c_i^2\lambda_i^2}{\sum\limits_{i>1} c_i^2\lambda_i} \geqslant \frac{\sum\limits_{i>1} c_i^2\lambda_i}{\sum\limits_{i>1} c_i^2} = q. \tag{3.28}$$

Using (3.26) and (3.28), we have

$$q-1=\frac{c}{1-c_1^2} \quad \text{and} \quad \frac{1}{1-c_1^2} \geqslant \frac{\lambda_2-1}{c}.$$

To conclude, from the fact that $\sum\limits_{i>1} c_i^2 = 1-c_1^2$ and from (3.28) it follows that

$$\begin{aligned}
\frac{\|A\mathbf{x}\|^2}{\|\mathbf{x}\|_A^2}-\frac{\|\mathbf{x}\|_A^2}{\|\mathbf{x}\|^2} &\geqslant \frac{c_1^2+q\sum\limits_{i>1}c_i^2\lambda_i}{c_1^2+\sum\limits_{i>1}c_i^2\lambda_i}-\frac{c_1^2+q\sum\limits_{i>1}c_i^2}{c_1^2+\sum\limits_{i>1}c_i^2}\\
&=\frac{c_1^2+q^2\sum\limits_{i>1}c_i^2}{c_1^2+q\sum\limits_{i>1}c_i^2}-\frac{c_1^2+q\sum\limits_{i>1}c_i^2}{c_1^2+\sum\limits_{i>1}c_i^2}\\
&=\frac{\Big(c_1^2+q^2\sum\limits_{i>1}c_i^2\Big)\Big(c_1^2+\sum\limits_{i>1}c_i^2\Big)-\Big(c_1^2+q\sum\limits_{i>1}c_i^2\Big)\Big(c_1^2+q\sum\limits_{i>1}c_i^2\Big)}{\Big(c_1^2+q\sum\limits_{i>1}c_i^2\Big)\Big(c_1^2+\sum\limits_{i>1}c_i^2\Big)}\\
&=\frac{(q-1)^2c_1^2\sum\limits_{i>1}c_i^2}{c_1^2+q\sum\limits_{i>1}c_i^2}\geqslant\frac{c^2c_1^2}{(1-c_1^2)(1+c)}\geqslant\frac{cc_1^2(\lambda_2-1)}{1+c}.
\end{aligned}$$

This constitutes the proof of (3.25).

The estimate (3.23) is a more or less straightforward consequence of (3.25). Indeed, we first notice that in the definition of r in (2.9), the term $R(\mathbf{x})\mathbf{x}$ is the projection of $A\mathbf{x}$ onto span$\{\mathbf{x}\}$ orthogonal in the Euclidean inner product. Therefore, we have by the Pythagorean Theorem

$$r^2(\mathbf{x})=\frac{\|A\mathbf{x}\|^2}{\|\mathbf{x}\|^2}-R^2(\mathbf{x})=\frac{\|\mathbf{x}\|_A^2}{\|\mathbf{x}\|^2}\left(\frac{\|A\mathbf{x}\|^2}{\|\mathbf{x}\|_A^2}-\frac{\|\mathbf{x}\|_A^2}{\|\mathbf{x}\|^2}\right), \tag{3.29}$$

and (3.23) follows by (3.25) and $R(\mathbf{x})=1+c$. Statements (3.21) and (3.22) follow by substitution $A \leftarrow 1/\lambda_1 A$ and $\mathbf{x} \leftarrow \|\mathbf{x}\|^{-1}\mathbf{x}$. □

Remark 3.7. Let $\mathbf{v}=c_1\mathbf{v}_1+\mathbf{e}$, $\mathbf{e}\perp\mathbf{v}_1$. Let us set $\mathbf{v}'=\|\mathbf{v}\|^{-1}\mathbf{v}$, $c_1'=c_1/\|\mathbf{v}\|$, and $\mathbf{e}'=\|\mathbf{v}\|^{-1}\mathbf{e}$. Viewing $\mathbf{v}$ as an input iterate of the Rayleigh quotient iteration (2.6), let us make a realistic assumption that

$$R(\mathbf{v})-\lambda_1 \leqslant \lambda_2-R(\mathbf{v}). \tag{3.30}$$

This assumption represents a minimal requirement on the input iterate of the Rayleigh quotient iteration (2.6), if we expect it to work reasonably. By (3.30),

$R(\mathbf{x}) \leqslant (\lambda_1 + \lambda_2)/2$ and therefore (3.22) gives

$$(c_1')^2 \geqslant \frac{1}{2}.$$

In the assumption of Lemma 3.3, we can therefore take $c'_{1,\min} = \sqrt{1/2}$.

Also, note that since $\|\mathbf{e}'\|^2 = 1 - (c_1')^2$, (3.22) gives

$$\|\mathbf{e}'\|^2 = \frac{\|\mathbf{e}\|^2}{\|\mathbf{v}\|^2} \leqslant \frac{R(\mathbf{v}) - \lambda_1}{\lambda_2 - \lambda_1}. \tag{3.31}$$

Hence under assumption (3.30), $\|\mathbf{e}'\| \leqslant \sqrt{1/2}$.

4. Final result

In this section we give the proof establishing our final convergence estimate. To this end, we need one technical result proved in [4] and one more straightforward technical result.

Lemma 4.1 ([4], it follows also from the Courant-Fischer principles)**.** *Let* $\mathbf{x}$ *be the iterate upon entry to Algorithm* 1 *and* $\mathbf{v} = [\mathbf{x}|P]\mathbf{v}^2$ *the prolongated solution of the coarse-level problem* (2.4). *Then*

$$R(\mathbf{v}) = \inf_{\mathbf{u}\in\mathrm{Range}([\mathbf{x}|P])} R(\mathbf{u}) \leqslant R(\mathbf{x}). \tag{4.1}$$

Proof. We will derive the coarse-level problem (2.4) by minimizing $R(\mathbf{u})$ on the subspace $\mathrm{Range}([\mathbf{x}|P])$.

The minimizer $\mathbf{v}$ of $R(\mathbf{u})$ on $\mathrm{Range}([\mathbf{x}|P])$ satisfies the condition

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} R(\mathbf{v} + t\mathbf{w}) = 0 \quad \forall\, \mathbf{w} \in \mathrm{Range}([\mathbf{x}|P]).$$

We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} R(\mathbf{v} + t\mathbf{w}) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \frac{\langle A(\mathbf{v}+t\mathbf{w}), \mathbf{v}+t\mathbf{w}\rangle}{\|\mathbf{v}+t\mathbf{w}\|^2} \\ &= \frac{2\langle A\mathbf{v}, \mathbf{w}\rangle\|\mathbf{v}\|^2 - 2\langle \mathbf{v}, \mathbf{w}\rangle\|\mathbf{v}\|_A^2}{\|\mathbf{v}\|^4} \\ &= \frac{2}{\|\mathbf{v}\|^2}\langle A\mathbf{v} - R(\mathbf{v})\mathbf{v}, \mathbf{w}\rangle.\end{aligned}$$

Thus the minimizer $\mathbf{v} \in \text{Range}([\mathbf{x}|P])$ is the solution of the Galerkin problem (2.8) that leads to the coarse-level problem (2.4); see Remark 2.3. Let $\mathbf{v}^{2,i}$ be the generalized eigenvectors of (2.4) and $\lambda^{2,i}$ the corresponding eigenvalues. We assume the natural numbering $\lambda^{2,i} \leqslant \lambda^{2,i+1}$. Clearly, $\lambda^{2,i} = R([\mathbf{x}|P]\mathbf{v}^{2,i})$. The eigenvalues $\lambda^{2,i} = R([\mathbf{x}|P]\mathbf{v}^{2,i})$ are (all) extremes and the values in the saddle points of $R(\mathbf{v})$ on $\text{Range}[\mathbf{x}|P]$. The value $\lambda^{2,1} = R([\mathbf{x}|P]\mathbf{v}^{2,1}) = R([\mathbf{x}|P]\mathbf{v}^{2})$ is therefore the global minimum. □

Before formulating the final convergence theorem, we need one more technical result.

Lemma 4.2. *We have*

$$R(A^{-1}\mathbf{v}) \leqslant R(\mathbf{v}). \tag{4.2}$$

At the same time, assuming $R(\mathbf{x}) - \lambda_1 \leqslant \lambda_2 - R(\mathbf{x})$,

$$R((A - R(\mathbf{x})I)^{-1}\mathbf{v}) \leqslant R(\mathbf{v}). \tag{4.3}$$

P r o o f. We prove a more difficult statement (4.3). The proof of (4.2) is analogous. Without loss of generality, we assume that the matrix A is scaled so that $\lambda_{\min}(A) = 1$. The generalization follows by a trivial argument, namely, by substitution $A \leftarrow \lambda_{\min}^{-1} A$. Assume the eigenvalues $\{\lambda_i\}$ of A are numbered so that $\lambda_i \leqslant \lambda_{i+1}$ and the corresponding eigenvectors $\{\mathbf{v}_i\}$ are orthonormalized. Set

$$S(\mathbf{x}) = \left[\left|\frac{1}{\lambda_{\min}(A - R(\mathbf{x})I)}\right|(A - R(\mathbf{x})I)\right]^{-1}.$$

Denote by λ_i^S the eigenvalue of $S(\mathbf{x})$ corresponding to the eigenvector $\mathbf{v}_i$. Clearly, $(\lambda_{i+1}^S)^2 \leqslant (\lambda_i^S)^2$. The vector $\mathbf{v}$ can be expressed as a linear combination of the eigenvectors $\mathbf{v}_i$ as $\mathbf{v} = \sum_i c_i \mathbf{v}_i$. Thus, to prove (4.3) means to prove $R(S(\mathbf{x})\mathbf{v}) \leqslant R(\mathbf{v})$, that is,

$$\frac{\sum_i c_i^2 (\lambda_i^S)^2 \lambda_i}{\sum_i c_i^2 (\lambda_i^S)^2} \leqslant \frac{\sum_i c_i^2 \lambda_i}{\sum_i c_i^2}. \tag{4.4}$$

We proceed by induction; assume that

$$\frac{\sum_{i=1}^k c_i^2 (\lambda_i^S)^2 \lambda_i}{\sum_{i=1}^k c_i^2 (\lambda_i^S)^2} \leqslant \frac{\sum_{i=1}^k c_i^2 \lambda_i}{\sum_{i=1}^k c_i^2} \tag{4.5}$$

for some $k < n$. Then, since $(\lambda_k^S)^2$ are non-increasing, (4.5) holds also for $k+1$ in the place of k. Indeed, set $q = \sum_{i=1}^{k} c_i^2 (\lambda_i^S)^2 \lambda_i \Big/ \sum_{i=1}^{k} c_i^2 (\lambda_i^S)^2 \leqslant \lambda_{k+1}$. By induction assumption (4.5) we get

$$\sum_{i=1}^{k} c_i^2 \lambda_i \geqslant q \sum_{i=1}^{k} c_i^2. \tag{4.6}$$

We have

$$\frac{\sum_{i=1}^{k+1} c_i^2 (\lambda_i^S)^2 \lambda_i}{\sum_{i=1}^{k+1} c_i^2 (\lambda_i^S)^2} = \frac{q \sum_{i=1}^{k} c_i^2 (\lambda_i^S)^2 + c_{k+1}^2 (\lambda_{k+1}^S)^2 \lambda_{k+1}}{\sum_{i=1}^{k} c_i^2 (\lambda_i^S)^2 + c_{k+1}^2 (\lambda_{k+1}^S)^2}. \tag{4.7}$$

The function on the right-hand side of the above inequality is a convex combination of q and λ_{k+1} and $q \leqslant \lambda_{k+1}$, that is, it is a non-increasing function of $\sum_{i=1}^{k} c_i^2 (\lambda_i^S)^2 / \Big[\sum_{i=1}^{k} c_i^2 (\lambda_i^S)^2 + c_{k+1}^2 (\lambda_{k+1}^S)^2\Big]$, hence a non-increasing function of $\sum_{i=1}^{k} c_i^2 (\lambda_i^S)^2$ (enlarging $\sum_{i=1}^{k} c_i^2 (\lambda_i^S)^2$ makes larger the weight $\sum_{i=1}^{k} c_i^2 (\lambda_i^S)^2 / \Big[\sum_{i=1}^{k} c_i^2 (\lambda_i^S)^2 + c_{k+1}^2 (\lambda_{k+1}^S)^2\Big]$ in front of the smaller or equal q at the expense of the weight of the larger or equal λ_{k+1}). Therefore, since $\sum_{i=1}^{k} c_i^2 (\lambda_i^S)^2 \geqslant (\lambda_{k+1}^S)^2 \sum_{i=1}^{k} c_i^2$, we have

$$\frac{\sum_{i=1}^{k+1} c_i^2 (\lambda_i^S)^2 \lambda_i}{\sum_{i=1}^{k+1} c_i^2 (\lambda_i^S)^2} \leqslant \frac{q (\lambda_{k+1}^S)^2 \sum_{i=1}^{k} c_i^2 + c_{k+1}^2 (\lambda_{k+1}^S)^2 \lambda_{k+1}}{(\lambda_{k+1}^S)^2 \sum_{i=1}^{k} c_i^2 + c_{k+1}^2 (\lambda_{k+1}^S)^2} = \frac{q \sum_{i=1}^{k} c_i^2 + c_{k+1}^2 \lambda_{k+1}}{\sum_{i=1}^{k} c_i^2 + c_{k+1}^2}.$$

The proof of (4.5) with $k+1$ in the place of k now follows by (4.6). The proof of (4.4) follows by induction with the fact that for $k = 1$, both sides of (4.5) are equal to λ_1.

This proves (4.3). The proof of (4.2) follows by setting $S = A^{-1}$, that is, $\lambda_i^S = 1/\lambda_i$. □

Theorem 4.3. *Let A be a symmetric and positive definite $n \times n$ matrix with simple minimal eigenvalue $\lambda_{\min}$. We assume the eigenvalues are numbered so that $\lambda_{\min} = \lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_n = \lambda_{\max}$. Consider $\alpha \in (0,1]$ and $\beta > 0$ such that $\alpha\beta - 1 > 0$. We assume there is a linear mapping $Q\colon \mathbb{R}^n \to \mathrm{Range}(P)$ and a constant $C_A > 0$ such that*

$$\forall\, \mathbf{u} \in \mathbb{R}^n\colon\ \lambda_{\min}^{\beta/2} \|\mathbf{u} - Q\mathbf{u}\| \leqslant \frac{C_A}{\mathrm{cond}(A)^{\alpha\beta/2}} \|\mathbf{u}\|_{A^\beta}.$$

In addition, assume that the input iterate $\mathbf{x}$ *is reasonably close to the first eigenvector* $\mathbf{v}_1$ *so that* $R(\mathbf{x}) - \lambda_1 \leqslant \lambda_2 - R(\mathbf{x})$ *and* $R(\mathbf{x})/\lambda_1 - 1 \leqslant 1$. *Then the result* $\mathbf{x}^{\mathrm{new}}$ *on exit of Algorithm* 1 *with* $\nu \geqslant \beta/2$ *and post-smoothing performed by the Rayleigh quotient iteration* (2.6) *satisfies the estimate*

$$\frac{r(\mathbf{x}^{\mathrm{new}})}{\lambda_{\min}} \leqslant q_{\mathrm{MGRQI}}(\mathbf{x})\Big(\frac{r(\mathbf{x})}{\lambda_{\min}}\Big)^3 + C\nu\Big(\frac{r(\mathbf{x})}{\lambda_{\min}}\Big)^4, \tag{4.8}$$

where

$$q_{\mathrm{MGRQI}}(\mathbf{x}) = 2\frac{C_A}{\mathrm{cond}(A)^{(\alpha\beta-1)/2}}\Big(\frac{\lambda_{\min}}{\lambda_2 - \lambda_{\min}}\Big)^{1/2}\Big(\frac{2\sqrt{2}\lambda_{\min}\lambda_2}{(\lambda_2 - \lambda_{\min})^2}\Big)^{\nu}.$$

The quantity $q_{\mathrm{MGRQI}}(\mathbf{x})$ *satisfies*

$$\lim_{\mathrm{cond}(A)\to\infty} q_{\mathrm{MGRQI}}(\mathbf{x}) = 0.$$

The constant C *depends only on* $\lambda_2/\lambda_{\min}$.

P r o o f. Let $\mathbf{x}$ be the iterate upon entry of Algorithm 1, $\mathbf{v} = [\mathbf{x}|P]\mathbf{v}^2$ the prolongated solution of the coarse-level problem (2.4), and $\mathbf{x}^i$ the i-th iterate in smoothing procedure (2.6). Clearly,

$$\mathbf{x}^i = (A - R(\mathbf{x}^{i-1})I)^{-1}\mathbf{x}^{i-1}.$$

By Lemma 4.1 and Lemma 4.2 we have that

$$R(\mathbf{x}^{\mathrm{new}}) = R(\mathbf{x}^{\nu}) \leqslant R(\mathbf{x}^{\nu-1}) \leqslant \ldots \leqslant R(\mathbf{x}^0) = R(\mathbf{v}) \leqslant R(\mathbf{x}). \tag{4.9}$$

For a vector $\mathbf{w} \in \mathbb{R}^n$, define $c'(\mathbf{w})$ and $\mathbf{e}'(\mathbf{w})$ by

$$\frac{1}{\|\mathbf{w}\|}\mathbf{w} = c'(\mathbf{w})\mathbf{v}_1 + \mathbf{e}'(\mathbf{w}), \quad \mathbf{e}'(\mathbf{w}) \perp \mathbf{v}_1.$$

Set

$$\mathbf{y}^k = \Big(\prod_{i=k,k-1,\ldots}^{0}(A - R(\mathbf{x}^{i-1})I)^{-1}\Big)A^{-(\nu-k)}\mathbf{x}^0, \quad k = 0,\ldots,\nu,$$

and

$$\mathbf{z}^k = \Big(\prod_{i=k-1,k-2,\ldots}^{0}(A - R(\mathbf{x}^{i-1})I)^{-1}\Big)A^{-(\nu-k)}\mathbf{x}^0, \quad k = 1,\ldots,\nu$$

(with $\prod_{i=0}^{0} \equiv I$). By Lemma 4.2 we have

$$R(\mathbf{y}^k) \leqslant R(\mathbf{x}^0), \quad R(\mathbf{z}^k) \leqslant R(\mathbf{x}^0), \quad k = 0, \dots, \nu. \tag{4.10}$$

Note that $\mathbf{x}^0 = \mathbf{v}$. Clearly, $\mathbf{y}^\nu = \mathbf{x}^\nu$, $\mathbf{y}^0 = A^{-\nu}\mathbf{x}^0 = A^{-\nu}\mathbf{v}$, $A^{-1}\mathbf{z}^k = \mathbf{y}^{k-1}$, and $\mathbf{y}^k = (A - R(\mathbf{x}^{k-1})I)^{-1}\mathbf{z}^k$. By Corollary 3.4, we get

$$\begin{aligned} r(\mathbf{y}^k) &\leqslant \frac{\lambda_2}{(\lambda_2 - R(\mathbf{x}^{k-1}))|c'_{1,\min}|}\Big(\frac{R(\mathbf{x}^{k-1})}{\lambda_1} - 1\Big) r(A^{-1}\mathbf{z}^k) \\ &\quad + C\Big(\frac{R(\mathbf{x}^{k-1})}{\lambda_1} - 1\Big)\|\mathbf{e}'(\mathbf{z}^k)\|^2 \\ &\leqslant \frac{\lambda_2}{(\lambda_2 - R(\mathbf{x}^{k-1}))|c'_{1,\min}|}\Big(\frac{R(\mathbf{x}^{k-1})}{\lambda_1} - 1\Big) r(\mathbf{y}^{k-1}) \\ &\quad + C\Big(\frac{R(\mathbf{x}^{k-1})}{\lambda_1} - 1\Big)\|\mathbf{e}'(\mathbf{z}^k)\|^2. \end{aligned} \tag{4.11}$$

Further, by virtue of (3.31) and (4.10),

$$\|\mathbf{e}'(\mathbf{z}^k)\|^2 \leqslant \frac{1}{\lambda_2/\lambda_1 - 1}\Big(\frac{R(\mathbf{z}^k)}{\lambda_1} - 1\Big) \leqslant \frac{1}{\lambda_2/\lambda_1 - 1}\Big(\frac{R(\mathbf{x}}{\lambda_1} - 1\Big).$$

Substituting this estimate and (4.9) into (4.11), we get

$$r(\mathbf{y}^k) \leqslant \frac{\lambda_2}{(\lambda_2 - R(\mathbf{x}))|c'_{1,\min}|}\Big(\frac{R(\mathbf{x})}{\lambda_1} - 1\Big) r(\mathbf{y}^{k-1}) + \frac{C}{\lambda_2/\lambda_1 - 1}\Big(\frac{R(\mathbf{x})}{\lambda_1} - 1\Big)^2$$

and therefore,

$$r(\mathbf{x}^\nu) \leqslant \Big(\frac{\lambda_2}{(\lambda_2 - R(\mathbf{x}))|c'_{1,\min}|}\Big)^\nu \Big(\frac{R(\mathbf{x})}{\lambda_1} - 1\Big)^\nu r(A^{-\nu}\mathbf{x}^0) + C\nu\Big(\frac{R(\mathbf{x})}{\lambda_1} - 1\Big)^2. \tag{4.12}$$

Statement (4.8) now follows by Theorem 2.4, Lemma 3.5, and Remark 3.7. □

R e m a r k 4.4. The convergence results can be easily extended to the case of the generalized eigenvalue problem assuming both matrices are symmetric, positive definite. The methodology has been developed in Section 4 of [4].

In the following numerical examples, we use the bilinear FE on the fine level and the bilinear FE on the coarse level. No orthogonalization process is applied to the coarse basis vectors. The computational domain is rectangular with the uniform square grid. The coarse-level mesh is also uniform. There is n (fine) DOFs and m coarse DOFs.

Tables show the numbers of iterations of several methods:

▷ inverse iteration method (II),
▷ Rayleigh quotient iteration (RQI),
▷ multigrid Rayleigh-Ritz method with one step of inverse iteration in every cycle (MGII),
▷ multigrid Rayleigh-Ritz method with one step of the Rayleigh quotient iteration in every cycle (MGRQI).

The last three columns in every table present reduction ratios of norms of the last two residuals $r(\mathbf{x}^j)$, i.e. $r(\mathbf{x}^{k+1})/r(\mathbf{x}^k)$, where k is the number of iterations to achieve the prescribed threshold of $r(\mathbf{x}^j)$. In particular, there are reduction ratios of residuals of MGII method (rMGII), reduction ratios of the cubic powers of residuals of MGII method (rMGII^3), and the reduction ratios of residuals of MGRQI method (rMGRQI), respectively.

To allow for the reproducibility of the presented numerical results, we start with the vector of ones. In our numerical experiments, the presented results are fully comparable to the ones that start with the random vectors.

We use Matlab routines for solving systems of linear equations. Of course, solving systems with almost singular matrices becomes costly. This is the usual drawback of the Rayleigh quotient iterations and we thus do not consider this issue in this paper. We only note that preconditioning methods for such cases may exploit the multigrid scheme presented here. For the numerical results with the non-exact Rayleigh quotient iteration given by multigrid, see [7].

Example 1. Equation $-\Delta u = \lambda u$ on a rectangle with a uniform mesh and homogeneous Dirichlet boundary conditions.

Example 2. Equation $-\nabla \cdot A\nabla u = \lambda u$ on a rectangle with a uniform mesh and homogeneous Dirichlet boundary conditions, where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}.$$

Adding the coarse-space of even a very moderate size (few degrees of freedom) significantly reduces the number of iterations compared to RQI. The MGRQI method

n	m	it. MGII	it. MGRQI	rMGII	rMGII^3	rMGRQI
9801	9	8	4	1.18e-01	1.66e-03	1.35e-05
	16	6	3	7.73e-02	4.62e-04	1.04e-07
	81	5	3	1.84e-02	6.28e-06	3.01e-10
	361	4	3	1.98e-03	1.01e-11	2.25e-10
39601	9	7	4	1.18e-01	1.63e-03	6.26e-05
	16	6	3	7.47e-02	4.16e-04	1.07e-07
	81	5	3	1.53e-02	3.57e-06	4.38e-10
	361	4	3	1.02e-03	1.06e-09	1.19e-09
	1521	4	3	7.42e-04	4.09e-10	4.90e-09

Table 5.1. Example 1. Tolerance 1e-11. For $n = 9801$ we have 14 steps of II method and 7 steps of RQI method. For $n = 39601$ we have 13 steps of II method and 5 steps of RQI method.

n	m	it. MGII	it. MGRQI	rMGII	rMGII^3	rMGRQI
9801	9	15	4	4.01e-01	6.43e-02	1.06e-06
	16	12	4	3.12e-01	3.03e-02	1.12e-05
	81	7	3	9.66e-02	9.02e-04	2.76e-09
	361	5	3	1.15e-02	1.53e-06	1.49e-10
39601	9	12	4	4.00e-01	6.39e-02	4.90e-06
	16	10	3	3.08e-01	2.93e-02	7.86e-07
	81	6	3	6.27e-02	2.47e-04	2.63e-09
	361	5	3	8.54e-03	6.23e-07	6.73e-10
	1521	4	3	1.85e-03	6.35e-09	2.78e-09

Table 5.2. Example 2 with $\alpha = 0.1$. Tolerance 1e-11. For $n = 9801$ we have 33 steps of II method and 8 steps of RQI method. For $n = 39601$ we have 30 steps of II method and 7 steps of RQI method.

n	m	it. MGII	it. MGRQI	rMGII	rMGII^3	rMGRQI
9801	9	61	4	8.47e-01	6.07e-01	6.48e-08
	16	46	4	8.11e-01	5.33e-01	8.35e-07
	81	15	3	5.02e-01	1.26e-01	3.08e-08
	361	7	3	1.69e-01	4.84e-03	2.03e-10
39601	9	48	4	8.45e-01	6.03e-01	3.02e-07
	16	35	4	8.09e-01	5.29e-01	3.66e-06
	81	12	3	4.94e-01	1.20e-01	2.83e-08
	361	6	3	8.19e-02	5.50e-04	5.50e-10
	1521	5	3	1.84e-02	6.25e-06	2.21e-09

Table 5.3. Example 2 with $\alpha = 0.01$. Tolerance 1e-11. For $n = 9801$ we have 198 steps of II method and 10 steps of RQI method. For $n = 39601$ we have 178 steps of II method and 9 steps of RQI method.

performs excellently for highly anisotropic problems. The computational results for larger problems (bigger condition number of A) are generally better than for a smaller problems, mainly for MGII. This confirms our theoretical findings.

n	m	it. MGII	it. MGRQI	rMGII	rMGII^3	rMGRQI
9801	9	488	5	9.84e-01	9.57e-01	6.00e-06
	16	346	4	9.80e-01	9.42e-01	9.30e-08
	81	81	4	9.21e-01	7.80e-01	8.88e-06
	361	23	3	7.47e-01	4.17e-01	2.01e-09
39601	9	315	5	9.82e-01	9.47e-01	9.93e-07
	16	215	4	9.77e-01	9.34e-01	1.30e-07
	81	50	3	9.08e-01	7.49e-01	4.54e-06
	361	15	3	6.94e-01	3.34e-01	1.32e-09
	1521	7	3	1.70e-01	4.95e-03	2.16e-09

Table 5.4. Example 2 with $\alpha = 0.001$. Tolerance 1e-11. For $n = 9801$ we have 1851 steps of II method and 6 steps of RQI method. For $n = 39601$ we have 1449 steps of II method and 11 steps of RQI method.

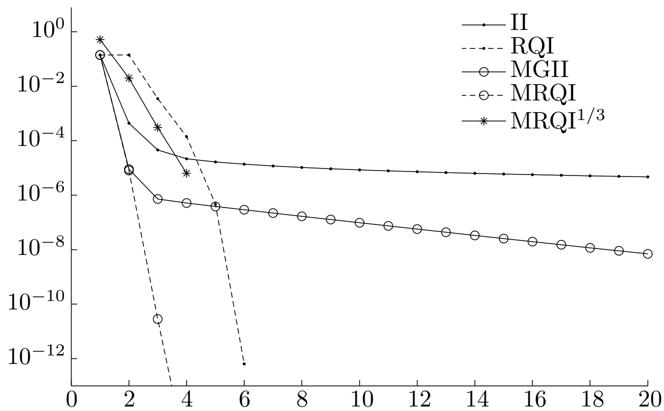

Figure 5.1. Graphical plot of errors corresponding to the third row in Table 5.4 (20 steps).

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