ON SYNGE-TYPE ANGLE CONDITION FOR d-SIMPLICES

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Abstract. The maximum angle condition of J. L. Synge was originally introduced in interpolation theory and further used in finite element analysis and applications for triangular and later also for tetrahedral finite element meshes. In this paper we present some of its generalizations to higher-dimensional simplicial elements. In particular, we prove optimal interpolation properties of linear simplicial elements in $\mathbb{R}^d$ that degenerate in some way.

Keywords: simplicial element; maximum angle condition; interpolation error; higher-dimensional problem; d-dimensional sine; semiregular family of simplicial partitions

MSC 2010: 65N30, 65N50, 65N12, 65N15

1. Introduction

The regularity of finite element meshes is important both in analysis and practical applications of the finite element method, as it greatly influences interpolation properties of finite element spaces and through Cea's lemma also the convergence rate of the finite element method. One of the most famous uses of the mesh regularity concept is the so-called Zlámal minimum angle condition (see Remark 4 and also [29], [8], [10], [28]), which can be roughly described as a general requirement on triangular elements to guarantee that their shape does not degenerate in the course of refinement.

However, in practical calculations we sometimes produce simplicial elements degenerating in some way [9], see Figure 1. Flat and narrow elements are also commonly used in covering thin slots, gaps or strips of different materials or to approximate functions that change more rapidly in one direction than in another direction [1].

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Therefore, much effort has been devoted to finding suitable (and practical) concepts, which are weaker than the Zlámal-type conditions. The first attempt was done by Synge [27] without any application to the convergence of the finite element method. Already in 1957, he proved that linear triangular elements yield the optimal interpolation order in the maximum norm under the so-called maximum angle condition (see Remark 5). This condition is weaker than the minimum angle condition. Later, Jamet [14] proved optimal intrepolation properties of triangular finite elements of k-th degree in the $W^{m,p}$-Sobolev norm provided

$$1 < p \leqslant \infty \quad \text{and} \quad k + 1 - m > \frac{2}{p}$$

and edges do not degenerate to one line. However, we immediately observe that the important case $p = 2$ and $k = m$, used e.g. in convergence analysis of the finite element method, is not covered by the above inequalities (see also [15], p. 493). This gap was resolved in [15]. The maximum angle condition for triangular elements was also investigated in terms of various norms in [2], [3], [4], [16], [15], [17], [18], [20], [22], [23], [24]. To obtain optimal interpolation properties of linear tetrahedral elements one has to impose (see [19]) the maximum angle condition for all triangular faces as well as a similar condition for all dihedral angles between faces. In [19] it is shown that these two conditions are independent. For the cap (and also sliver) tetrahedron from Figure 1, the Synge condition holds for any triangular face, but some of the dihedral angles between faces may converge to π. On the other hand, for the spike tetrahedron the Synge condition is violated for some faces, but all dihedral angles are less than some positive constant $C < \pi$ for a certain manner of degeneracy.

Finally note that the maximum angle condition is, in fact, not necessary for convergence of finite element approximations as shown in [13], [20], [21], and [24].

2. Notation and definitions

A *d-simplex* (or just simplex) S in $\mathbb{R}^d$, $d \in \{1, 2, 3, \ldots\}$, is the convex hull of $d + 1$ vertices $A_0, A_1, \ldots, A_d$ that do not belong to the same $(d-1)$-dimensional hyperplane, i.e.,

$$S = \operatorname{conv}\{A_0, A_1, \ldots, A_d\}.$$

We denote by h_S the length of the longest edge of S.

Let

$$F_i = \operatorname{conv}\{A_0, \ldots, A_{i-1}, A_{i+1}, \ldots, A_d\}$$

be the facet of S opposite to vertex A_i for $i \in \{0, \ldots, d\}$. The dihedral angle α between two such facets is defined by means of the inner product of their outward

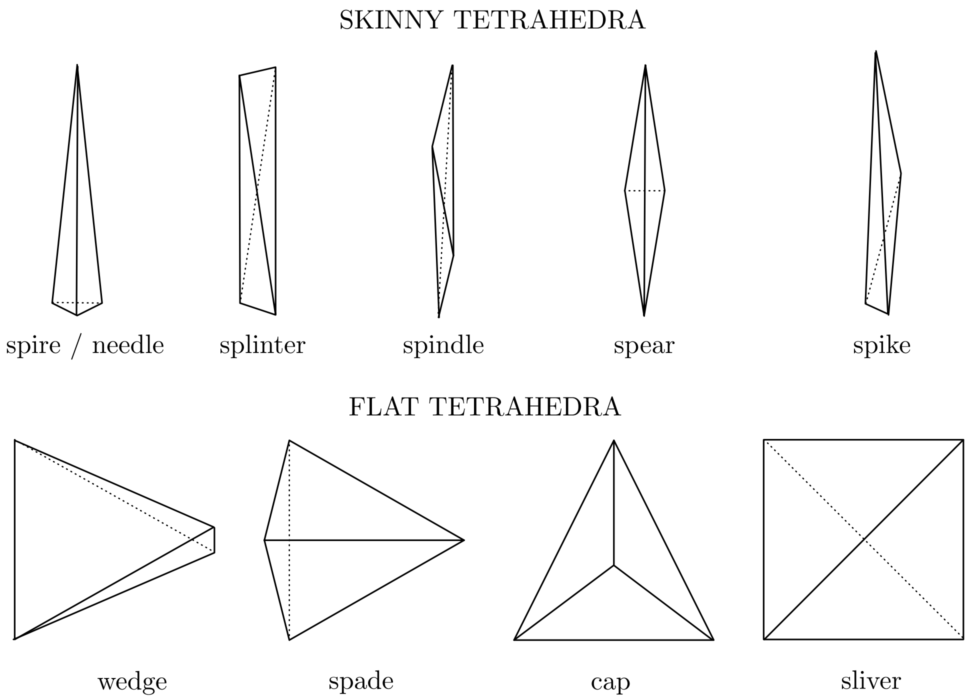

Figure 1. Classification of degenerated tetrahedra according to [9] and [11].

unit normals n_1 and n_2 (see also [12], p. 74),

$$\cos\alpha = -n_1 \cdot n_2.$$

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Assume that $\overline{\Omega}$ is *polytopic* (or just a *polytope*). By this we mean that $\overline{\Omega}$ is the closure of a domain whose boundary $\partial\overline{\Omega}$ is contained in a finite number of $(d-1)$-dimensional hyperplanes.

Next we define a simplicial partition of a bounded closed polytopic domain $\overline{\Omega} \subset \mathbb{R}^d$ as follows. We subdivide $\overline{\Omega}$ into a finite number of simplices (called *elements* and denoted by S), so that their union is $\overline{\Omega}$, any two distinct simplices have disjoint interiors, and any facet of any simplex is either a facet of another simplex from the partition or belongs to the boundary $\partial\overline{\Omega}$. The set of such simplices will be called a *simplicial face-to-face partition* (shortly only *partition*) and denoted by $\mathcal{T}_h$, where

$$h = \max_{S \in \mathcal{T}_h} h_S.$$

The sequence of partitions $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of $\overline{\Omega}$ is called a *family of partitions* if for every $\varepsilon > 0$ there exists

$$\mathcal{T}_h \in \mathcal{F} \quad \text{with} \quad h < \varepsilon.$$

In what follows, all constants C_i are independent of S and h, but can depend on the dimension d. By meas_p we denote the p-dimensional volume ($p \leqslant d$).

Let us recall the definition of the sine of a d-dimensional angle in $\mathbb{R}^d$ from [12], p. 72 (see also [5]). Denote by $\widehat{A}_i$ the angle at the vertex A_i of the simplex S. The d-sine of $\widehat{A}_i$ for $d > 1$ is defined as

$$(1) \qquad \sin_d(\widehat{A}_i | A_0 A_1 \dots A_d) = \frac{d^{d-1} (\operatorname{meas}_d S)^{d-1}}{(d-1)! \prod\limits_{j=0, j\neq i}^{d} \operatorname{meas}_{d-1} F_j}.$$

R e m a r k 1. For $d = 2$, $\sin_2(\widehat{A}_i | A_0 A_1 A_2)$ is the standard sine of the angle $\widehat{A}_i$ in the triangle $A_0 A_1 A_2$, due to the following well-known formula, e.g. for $i = 0$,

$$(2) \qquad \operatorname{meas}_2(A_0 A_1 A_2) = \frac{1}{2} |A_0 A_1| |A_0 A_2| \sin \widehat{A}_0.$$

R e m a r k 2. Consider the standard cube-corner tetrahedron with vertices

$$A_0 = (0, 0, 0), \ A_1 = (1, 0, 0), \ A_2 = (0, 1, 0), \ \text{and } A_3 = (0, 0, 1).$$

Its solid (spatial) angle at A_0, i.e., the solid angle of the first octant, is $\frac{4}{8}\pi = \frac{1}{2}\pi$. Its 3-sine is by (1) equal to (see [12], p. 76)

$$\sin_3(\widehat{A}_0 | A_0 A_1 A_2 A_3) = \frac{3^2 (1/6)^2}{2! (1/2)^3} = 1.$$

For any $d > 1$ and a general hypercube-corner d-simplex S with vertices

$$A_0 = (0, \dots, 0), \ A_1 = (1, 0, \dots, 0), \dots, A_d = (0, \dots, 0, 1),$$

we have $\operatorname{meas}_d S = 1/d!$ and $\operatorname{meas}_{d-1} F_j = 1/(d-1)!$ for $j = 1, \dots, d$. Substituting these values into (1), a simple calculation also leads to

$$\sin_d(\widehat{A}_0 | A_0 A_1 \dots A_d) = 1.$$

R e m a r k 3. Denoting by A_0, A_1, A_2, A_3 the vertices of the cap or sliver tetrahedron from Figure 1 in an arbitrary way, we find by (1) that

$$\sin_3(\widehat{A}_0 | A_0 A_1 A_2 A_3) \to 0.$$

Definition 1. A family $\mathcal{F}$ is called a regular family of partitions of a polytope into simplices if there exists $C > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $S = \operatorname{conv}\{A_0, \dots, A_d\} \in \mathcal{T}_h$ we have

$$(3) \qquad \forall\, i \in \{0, 1, \dots, d\} \qquad \sin_d(\widehat{A}_i | A_0 A_1 \dots A_d) \geqslant C > 0,$$

where $\sin_d$ is defined in (1).

R e m a r k 4. The above definition is a natural d-dimensional generalization (see [8]) of the Zlámal minimum angle condition, which is formulated as follows: There exists a constant α_0 such that for any triangulation $\mathcal{T}_h \in \mathcal{F}$ and any triangle $T \in \mathcal{T}_h$ we have

$$\alpha_T \geqslant \alpha_0 > 0, \qquad \text{(Zlámal condition)}$$

where α_T is the minimum angle of the triangle T. Further (equivalent) definitions of the regularity of a family of partitions of a polytope into simplices are presented in [6], [7].

Definition 2. A family $\mathcal{F}$ is called a semiregular family of partitions of a polytope into simplices if there exists $C > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $S = \operatorname{conv}\{A_0, \ldots, A_d\} \in \mathcal{T}_h$ we can always find d edges of S which, when considered as vectors, constitute a (higher-dimensional) angle whose d-sine is bounded from below by the constant C.

R e m a r k 5. We observe that for the case $d = 2$, Definition 2 is equivalent to the maximum angle condition of Synge (see e.g. [27] and [3]): There exists a constant γ_0 such that for any triangulation $\mathcal{T}_h \in \mathcal{F}$ and any triangle $T \in \mathcal{T}_h$ we have

$$\gamma_T \leqslant \gamma_0 < \pi, \qquad \text{(Synge condition)}$$

where γ_T is the maximum angle of the triangle T. To see this, we denote by $\widehat{A}_0$ the largest angle of T and then apply formula (2). We observe that in the family of triangulations which satisfies the maximum angle condition of Synge, we can have, in principle, an infinite sequence of triangles with decaying sizes whose minimum angles approach zero, which is not possible for the Zlámal condition, see e.g. [18] for some examples on this.

R e m a r k 6. Let us point out that the Synge condition guarantees the optimal interpolation order for triangles, which is not true if the maximum angle tends to π, see [13], [26].

R e m a r k 7. Each d-simplex has $\binom{d+1}{2} = \frac{1}{2}(d+1)d$ edges. The d edges mentioned in the above definition do not necessarily emanate from the same vertex. An example is a path simplex with its d orthogonal edges forming a path (in the sense of graph theory). The path element can degenerate (e.g. the d-dimesional sine of some of its angles can be close to zero) but the d-sine made by these orthogonal d edges stays the same.

3. Main results

Below we will use the standard definitions of norms and seminorms of various function spaces, see e.g. [10] for details.

Lemma 1. *Any regular family of partitions of a polytope into simplices is semiregular.*

Proof. For any simplex from the regular family of partitions the required d edges in the definition of semiregularity are just d edges emanating from some of its vertices. $\square$

Lemma 2. *Consider a simplex S with vertices $A_0, A_1, \ldots, A_d$. Let A_{i_l} denote any of its vertices. We have the following relations between sines of angles of various dimensions based on A_{i_l}:*

$$\sin_2(\widehat{A}_{i_l}|A_{i_0}A_{i_1}A_{i_2}) \geqslant \sin_3(\widehat{A}_{i_l}|A_{i_0}A_{i_1}A_{i_2}A_{i_3}) \geqslant \ldots \geqslant \sin_d(\widehat{A}_{i_l}|A_0 \ldots A_d), \tag{4}$$

where $i_0, i_1, i_2, i_3, \ldots$ are distinct indices from the set $\{0, 1, \ldots, d\}$ and $i_l \in \{i_0, i_1, i_2\}$.

For the proof see [12], p. 76. By Remark 2 we see that the inequalities in (4) may become equalities for the hypercube-corner simplex.

Lemma 3. *Let $\mathcal{F}$ be a semiregular family of partitions of a polytope into simplices. Then there exists a constant $C > 0$ (depending on d only) such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $S \in \mathcal{T}_h$ we have*

$$|\det \mathcal{M}_S| > C, \tag{5}$$

where the entries of the matrix $\mathcal{M}_S$ are the scalar products $t_i^{\mathrm{T}} t_j$, $i, j = 1, \ldots, d$, of the unit vectors $t_1, t_2, \ldots, t_d$ parallel to those d edges of S which are required in Definition 2.

Proof. It is well-known that the volume of the parallelotope $\mathcal{P}$ generated by the vectors $t_1, t_2, \ldots, t_d$ is equal to $|\det \mathcal{M}|$, where

$$\mathcal{M} = (t_i^{\mathrm{T}} t_j)_{i,j=1}^d.$$

For convenience, we can suppose that these d vectors originate from the same vertex of the parallelotope. Let us denote it as A_i, where $i \in \{0, 1, \ldots, d\}$, and let the d-simplex formed by A_i and the endpoints of these d vectors be denoted by $\widetilde{S}$. Then

$$|\det \mathcal{M}_{\widetilde{S}}| = \operatorname{meas}_d \mathcal{P} = d! \operatorname{meas}_d \widetilde{S}. \tag{6}$$

Let us use the notations $A_0, A_1, \dots, A_d$ for the vertices of the simplex $\widetilde{S}$. From (1), we have

$$(\text{meas}_d\, \widetilde{S})^{d-1} = \frac{(d-1)!}{d^{d-1}} \prod_{j=0, j\neq i}^{d} \text{meas}_{d-1}\, \widetilde{F}_j \cdot \sin_d(\widehat{A}_i | A_0 A_1 \dots A_d), \tag{7}$$

where $\widetilde{F}_j$ are the facets of $\widetilde{S}$ forming the d-dimensional angle at the vertex A_i. Further, due to the property of semiregularity, $\sin_d(\widehat{A}_i | A_0 A_1 \dots A_d)$ is bounded from below by a positive constant, i.e.

$$(\text{meas}_d\, \widetilde{S})^{d-1} \geqslant C \frac{(d-1)!}{d^{d-1}} \prod_{j=0, j\neq i}^{d} \text{meas}_{d-1}\, \widetilde{F}_j, \tag{8}$$

where C is a constant from the definition of semiregularity. Now, we notice that in (8) each term $\text{meas}_{d-1}\, \widetilde{F}_j$ can be computed (and further estimated from below via a product of areas of the relevant $(d-2)$-dimensional facets) in a similar way to (7) and (8) using the semiregularity property and relations between sines of various dimensions (4). Finally, in order to get (5) proved, we apply the above arguments recursively to all involved areas of facets of dimensions $d-2, d-3, \dots, 2$ and notice at the end that after finitely many steps the areas of one-dimensional facets are just $\|t_j\| = 1$. Lemma 3 thus follows from (6) and (8) by induction. □

The next theorem gives the optimal interpolation order in the following maximum norm over S

$$\|w\|_{1,\infty} = \max\Big\{ \max_{x\in S} |w(x)|,\ \max_{x\in S} \Big|\frac{\partial w(x)}{\partial x_1}\Big|, \dots, \max_{x\in S} \Big|\frac{\partial w(x)}{\partial x_d}\Big| \Big\} \tag{9}$$

for any Lipschitz function w. In what follows, we will also use the matrix norm

$$\|A\|_1 = \max_j \sum_i |a_{ij}|.$$

Theorem 1. *Let $\mathcal{F}$ be a semiregular family of partitions of a polytope into simplices. Then there exists a constant $C > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $S \in \mathcal{T}_h$ we have*

$$\|v - \pi_S v\|_{1,\infty} \leqslant C h_S |v|_{2,\infty} \quad \forall\, v \in \mathcal{C}^2(S), \tag{10}$$

where π_S is the standard Lagrange linear interpolant and $h_S = \text{diam}\, S$.

P r o o f. Consider a simplex $S \in \mathcal{T}_h \in \mathcal{F}$. Let $v \in \mathcal{C}^2(S)$ be arbitrary and let e be an arbitrary edge of S. Set

$$w = v - \pi_S v \quad \text{on } S. \tag{11}$$

Since $w = 0$ at all vertices of S, there exists by the Rolle's theorem [25], p. 387 a point $Q \in e$ such that

$$t^{\mathrm{T}} \operatorname{grad} w(Q) = 0, \tag{12}$$

where t is the unit vector parallel to e. Set

$$z = t^{\mathrm{T}} \operatorname{grad} w \tag{13}$$

in S. Let P be an arbitrary fixed point in the interior of S and let u be the unit vector parallel to QP. Then by (12) and (13) we have $z(Q) = 0$ and

$$z(P) = \int_Q^P u^{\mathrm{T}} \operatorname{grad} z \, \mathrm{d}s = \int_Q^P u^{\mathrm{T}} (\operatorname{hes} w) t \, \mathrm{d}s, \tag{14}$$

where $\operatorname{hes} w$ is the matrix of the second derivatives (Hessian) of w. Thus, by (13), (14), and (11),

$$|t^{\mathrm{T}} \operatorname{grad} w(P)| \leqslant d h_S |w|_{2,\infty} = d h_S |v|_{2,\infty}, \tag{15}$$

where the upper estimate does not depend on P.

Consider now those d edges of S required in Definition 2 and d corresponding unit vectors $t_1, t_2, \ldots, t_d$ parallel with them. Writing

$$\operatorname{grad} w(P) = \sum_{j=1}^{d} c_j(P) t_j, \quad c_j \in \mathbb{R}^1, \tag{16}$$

we see that the coefficients $c_j = c_j(P)$ fulfil the linear system of algebraic equations

$$\sum_{j=1}^{d} t_i^{\mathrm{T}} t_j c_j(P) = t_i^{\mathrm{T}} \operatorname{grad} w(P), \quad i = 1, 2, \ldots, d, \tag{17}$$

with the Gram matrix

$$\mathcal{M}_S = (t_i^{\mathrm{T}} t_j)_{i,j=1}^{d}.$$

We can concisely write the above system as $\mathcal{M}_S \mathbf{c}(P) = \mathbf{b}(P)$.

Using standard definitions of vector and matrix norms and applying the constants from the relevant equivalence relations, we observe from the equalities (16) and (17) that

$$\|\mathrm{grad}\, w(P)\|_\infty \leqslant \|\mathbf{c}(P)\|_1 \leqslant \|\mathcal{M}_S^{-1}\|_1 \|\mathbf{b}(P)\|_1 \leqslant d\|\mathcal{M}_S^{-1}\|_1 \|\mathbf{b}(P)\|_\infty \tag{18}$$

is valid for any point P. (In the above we also used the fact that $\|t_j\|_2 = 1$ for $j = 1, 2, \dots, d$.) Let $\mathcal{M}_S^\star$ be the algebraic adjoint of the entries of $\mathcal{M}_S$. Then

$$\mathcal{M}_S^{-1} = \frac{1}{\det \mathcal{M}_S} \mathcal{M}_S^\star,$$

and a straightforward calculation leads to the following estimation of the $\|\cdot\|_1$-norm of $M_S^\star$:

$$\|\mathcal{M}_S^\star\|_1 \leqslant d \cdot (d-1)! = d!. \tag{19}$$

Hence, from (18), (19), and (15) we get

$$\|\mathrm{grad}\, w(P)\|_\infty \leqslant d\|\mathcal{M}_S^{-1}\|_1 \|\mathbf{b}(P)\|_\infty \leqslant \frac{d^2 d!}{|\det \mathcal{M}_S|} h_S |v|_{2,\infty}, \tag{20}$$

valid for any point $P \in S$. Now, using (20) and Lemma 3, we get that for any $i = 1, \dots, d$

$$\max_{x \in S} \left| \frac{\partial w(x)}{\partial x_i} \right| \leqslant C h_S |v|_{2,\infty}. \tag{21}$$

By [10], pp. 118–120, we have the estimate

$$\max_{x \in S} |w(x)| \leqslant C h_S^2 |v|_{2,\infty} \quad \forall\, v \in W_\infty^2(S) \tag{22}$$

without any regularity assumption on the family $\mathcal{F}$. The relations (9), (21), and (22) now imply (10). □

4. Examples

E x a m p l e 1. The needle, splinter, and wedge elements from Figure 1 satisfy Definition 2. They yield the optimal interpolation order of linear elements provided the lengths of their edges are as indicated in Figure 2. For the splinter tetrahedron from Figure 1, we have to take two short opposite edges and one long edge to satisfy Definition 2.

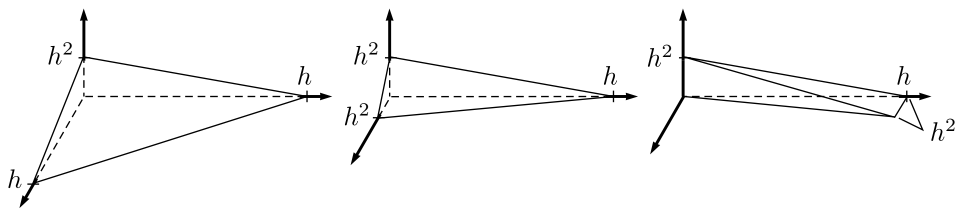

Figure 2. Three types of degenerating tetrahedra which do not deteriorate the optimal interpolation order [19]. The length h^2 can be replaced by $h^{1+\varepsilon}$ for any $\varepsilon > 0$.

E x a m p l e 2. A higher-dimensional example can be constructed as follows. Consider positive numbers $r_1, r_2, \ldots, r_d$ and a simplex with vertices $A_0, A_1, \ldots, A_d$. We fix some number k such that $0 \leqslant k \leqslant d$. The first $k+1$ vertices of the simplex are defined as follows. Let $A_0 = (0, 0, \ldots, 0, \ldots, 0)$. Further, let

$$\begin{aligned} A_1 &= (r_1, 0, \ldots, 0, \ldots, 0), \\ A_2 &= (0, r_2, \ldots, 0, \ldots, 0), \\ &\vdots \\ A_k &= (0, \ldots, 0, r_k, 0, \ldots, 0). \end{aligned}$$

The remaining vertices are:

$$\begin{aligned} A_{k+1} &= (0, \ldots, 0, r_{k+1}, 0, 0, \ldots, 0), \\ A_{k+2} &= (0, \ldots, 0, r_{k+1}, r_{k+2}, 0, \ldots, 0), \\ &\vdots \\ A_{k+3} &= (0, \ldots, 0, r_{k+1}, r_{k+2}, r_{k+3}, 0, \ldots, 0), \\ A_d &= (0, \ldots, 0, r_{k+1}, r_{k+2}, r_{k+3}, \ldots, r_d). \end{aligned}$$

Therefore, for $k = 0$, we get the path-simplex, and for $k = d$ the hypercube-corner simplex. Allowing some of the r_k's to approach zero with different rates, in general, we arrive at various degenerated simplices still satisfying the semiregularity property.

E x a m p l e 3. Let $A_0 = (h, 0, 0)$, $A_1 = (0, h^3, h)$, $A_2(-h, 0, 0)$, and $A_3 = (0, h^3, -h)$ be the vertices of the tetrahedron S for some $h \in (0, 1)$. From Figure 1 we observe that S is a sliver tetrahedron when $h \to 0$, i.e., the condition (5) is violated. We show that the optimal interpolation order of the linear interpolant π_S is not guaranteed (cf. Theorem 1). Setting

$$v(x_1, x_2, x_3) = x_1^2,$$

we immediately find that

$$v(A_0) - v(A_2) = h^2, \quad v(A_1) = v(A_3) = 0.$$

Using the linearity of $\pi_S v$, we obtain

$$\begin{aligned} \frac{\partial}{\partial x_2}(v - \pi_S v) = -\frac{\partial(\pi_S v)}{\partial x_2} &= \frac{(\pi_S v)((A_0 + A_2)/2) - (\pi_S v)((A_1 + A_3)/2)}{h^3} \\ &= \frac{v(A_0) + v(A_2) - v(A_1) - v(A_3)}{2h^3} = h^{-1} \to \infty \quad \text{as} \quad h \to 0. \end{aligned}$$

Hence, the estimate (10) does not hold.

E x a m p l e 4. Let $A_0 = (0, 0, 0)$, $A_1 = (h, 0, 0)$, $A_2(h, 0, h^3)$, and $A_3 = (-h, h^3, 0)$ be the vertices of the tetrahedron S for some $h \in (0, 1)$. From Figure 1 we see that S is a spike tetrahedron when $h \to 0$. We again show that the optimal interpolation order of π_S is violated, since (5) is not true. Putting

$$v_1(x_1, x_2, x_3) = x_1^2,$$

we observe that

$$v(A_0) = 0, \quad v(A_1) = v(A_2) = v(A_3) = h^2.$$

Since $\pi_S v$ is linear, we deduce that

$$\begin{aligned} \frac{\partial}{\partial x_2}(v - \pi_S v) = -\frac{\partial(\pi_S v)}{\partial x_2} &= \frac{(\pi_S v)(A_0) - (\pi_S v)((A_1 + A_3)/2)}{\frac{1}{2}h^3} \\ &= -\frac{v(A_1) + v(A_3)}{h^3} = \frac{2h^{-2}}{h^3} \to -\infty \end{aligned}$$

whenever $h \to 0$. Consequently, the estimate (10) is not valid.

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