LINEAR COMPLEMENTARITY PROBLEMS AND BI-LINEAR GAMES

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Abstract. In this paper, we define bi-linear games as a generalization of the bimatrix games. In particular, we generalize concepts like the value and equilibrium of a bimatrix game to the general linear transformations defined on a finite dimensional space. For a special type of **Z**-transformation we observe relationship between the values of the linear and bi-linear games. Using this relationship, we prove some known classical results in the theory of linear complementarity problems for this type of **Z**-transformations.

Keywords: bimatrix game; nash equilibrium; **Z**-transformation; semi positive map

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1. Introduction

For a given matrix $A \in M_n(\mathbb{R})$ and a vector $q \in \mathbb{R}^n$, the linear complementarity problem ($\mathrm{LCP}(A,q)$) is to find an $x \in \mathbb{R}^n$ such that

$$x \geqslant 0, \quad y := Ax + q \geqslant 0 \quad \text{and} \quad x^\top y = 0.$$

Here $x \geqslant 0$ means entriwise non-negative vector in $\mathbb{R}^n$ ($x \in \mathbb{R}^n_+$). If such an x exists, we call it a *solution of the problem* $\mathrm{LCP}(A,q)$. LCPs can be considered as a unified model of linear and quadratic programs and bimatrix games. In particular, the Nash equilibrium of the bimatrix game could be identified by solving a corresponding LCP. The complementarity pivot algorithm was initially developed for solving LCP has been generalized to derive efficient algorithms for computing Kakutani and Brouwer fixed points, which in turn helps the computation of economic equilibria. Also, very

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large scale linear programs could be tackled by the iterative methods developed for solving LCP. For detailed literature on LCP theory and its applications we refer to [12], [2], [8], [3], and references therein.

In this paper, we consider the cone linear complementarity problem, which is a generalization of the above standard linear complementarity problem. That is, for a given closed convex cone or proper cone K and a vector q in the finite dimensional real inner product space $(V, \langle\cdot,\cdot\rangle)$ and a linear transformation $L\colon V \to V$, the cone linear complementarity problem ($\mathrm{LCP}(L, K, q)$) is to find an $x \in K$ such that $y := L(x) + q \in K^*$ and $\langle x, y\rangle = 0$. Here K and its dual cone $K^* := \{y \in V;\ \langle y, x\rangle \geqslant 0$ for all $x \in K\}$ play the role of $\mathbb{R}^n_+$ in the standard LCP setting. A linear transformation L is said to have the **Q**-*property* if for all $q \in V$, $\mathrm{LCP}(L, K, q)$ has a solution. A sufficient condition for a linear transformation to have the **Q**-property is given by Karamardian [11]. Similarly, L has the **GUS**-*property* if for all $q \in V$, $\mathrm{LCP}(L, K, q)$ has a unique solution. In the standard LCP setting, these transformations are known as the **Q**-*matrix* and **P**-*matrix*, and they play a prominent role in matrix theory and economic theory. For further details see [4], [9].

We say a matrix A is a **Z**-*matrix* if its off-diagonal entries are non-positive. **Z**-matrices are studied extensively by Fiedler and Pták [4]. Matrices of this type are studied not just in matrix theory but also in numerical analysis and economic theory. In particular, in numerical analysis, the study of matrix splitting and asymptotic rate of convergence of various iterative methods involves such matrices [17]. In economic theory, these kinds of matrices are termed "matrices of the Leontief type" and they play an important role in the study of Leontief's input-output system and factor-price equalization; for details see [5], [13] and references therein. Properties of a **Z**-matrix related to the LCP theory are found in [10], [1] and references therein.

It is easy to observe that for non-negative vectors x, y with $x^\top y = 0$; A being a **Z**-matrix, we get $y^\top Ax \leqslant 0$. This observation is used to extend the **Z**-matrix property to the general linear transformations in the following way. Given a closed convex cone K in V and a linear transformation L defined on V, we say that L is a **Z**-*transformation* on K(or has a **Z**-*property*) if

$$[x \in K,\ y \in K^* \text{ and } \langle x, y\rangle = 0] \Rightarrow \langle L(x), y\rangle \leqslant 0.$$

Linear transformations of this type are introduced in the form of cross positive matrices in [16]. In [7], Gowda and Tao introduced the above formal definition of **Z**-transformation on proper cone (closed, pointed, convex cone with a non-empty interior) and extended the **Z**-matrix results to the **Z**-transformations defined on a proper cone. In particular, they proved the following theorem.

Theorem 1.1. *Let K be a proper cone in V and suppose $L\colon V \to V$ has the* **Z***-property with respect to K. Then the following are equivalent.*

(1) *L^{-1} exists and $L^{-1}(K) \subseteq K$.*
(2) *There exists an $x \in K^{\circ}$ such that $L(x) \in K^{\circ}$.*
(3) *L and $L^{\top}$ has* **Q***-property with respect to K and K^* respectively.*
(4) *There exists an $y \in (K^*)^{\circ}$ such that $L^{\top}(y) \in (K^*)^{\circ}$.*

In addition to the above theorem, we have the following equivalent conditions for a **Z**-transformation L defined on any proper cone K, see [16], [7].

(a) $\mathrm{e}^{-tL}(K) \subseteq K$ for all $t \geqslant 0$.
(b) $L = \lim\limits_{n\to\infty}(\alpha_n I - S_n)$ where $\alpha_n \in \mathbb{R}$ and S_n is a linear transformation on V with $S_n(K) \subseteq K$ for all n.

The last item helps to generate **Z**-transformations. In this paper, we consider transformations of the form $\alpha I - S$ with $S(K) \subseteq K$, which is the natural extension of the **Z**-matrix.

In [6], [14], linear games are introduced as a generalization of the two-person zero-sum games. In particular, the concept of the game theoretic value has been introduced for the general linear transformation and extended to some classical theorems in game theory. In addition, it is shown that various properties of the **Z**-transformation like the positive stability property and the semi positivity property are related to the game theoretic value of the linear transformation as in Theorem 2.2.

We can observe that for certain matrix games it is constructive to apply the process of iterative elimination of dominated strategies (IEDS) if we consider the game as a non-zero-sum game (bimatrix game), whereas it is not beneficial if we consider them as a zero-sum game. For instance, consider the game corresponding to the **Z**-matrix $A := sI - B$ where B is entrywise non-negative. If we see this game as a zero-sum game $\Gamma(A)$, the diagonal and off-diagonal entries of the pay-off matrix might have different signs and applying the IEDS process will not help in any way. On the other hand, if we consider the bimatrix game $\Gamma(A, B)$, player II's pay-off matrix has all non-negative entries and applying the IEDS process possibly reduces the dimension of the game as illustrated in Example 4.3, which might eventually help in finding the equilibrium strategies.

Motivated by the possible benefits of bimatrix game consideration, we ask, can a linear game corresponding to the **Z**-transformation be analyzed in the form of a non-zero-sum game? In this regard we introduce the bi-linear game as a generalization of the two-person non-zero-sum game (bimatrix game). We study some of the well known properties of **Z**-transformations by using the concept of bi-linear game and relating them to the equilibrium value of the bi-linear game. In particular, we provide a sufficient condition for L to have the semi positive property and **Q**-property. That is, we show that $\mathrm{LCP}(L, q)$ has a global solvable property if the bi-linear game corresponding to L has a positive equilibrium value and a completely mixed strategy in an equilibrium pair.

The organization of this paper is as follows. In Section 2, we recall some basic definitions and preliminary results which will be used in later sections. In Section 3, we define bi-linear game and state some basic game theory results in this general setting. In Section 4, we discuss the relationship between the game theoretic value of linear and bi-linear games corresponding to a given **Z**-transformation. As an application of this relationship, we deduce a sufficient condition for **Z**-transformations to hold some known results from matrix theory and the theory of complementarity problems.

2. Preliminaries

Let $(V, \langle\cdot,\cdot\rangle)$ and $(W, \langle\cdot,\cdot\rangle)$ be finite dimensional real inner product spaces, where $\langle\cdot,\cdot\rangle$ denotes the inner product on the respective spaces. For a linear map $L\colon V \to W$, we denote $L^\top$ as its transpose. For $x \in V$, let $x^\perp := \{y \in V;\ \langle y, x\rangle = 0\}$.

A set $K \subseteq V$ is said to be a *convex cone* if $px + qy \in K$ for all $x, y \in K$ and $p, q \geqslant 0$. A convex cone K is said to be *self-dual* if its dual $K^* := \{x \in V;\ \langle x, y\rangle \geqslant 0$ for all $y \in K\}$ is equal to K. For a set K, we use K° to denote the interior of K.

Let K be a self-dual cone in V. For a fixed $e \in K^\circ$, consider the subset $\Delta := \{x \in K;\ \langle x, e\rangle = 1\}$. It is clear that Δ is a compact convex set and forms a base for K. That is, every element in K is a positive scalar multiple of an element in Δ.

Definition 2.1. A linear map $L\colon V \to V$ is said to be **Z**-transformation on K if

$$[x \in K,\ y \in K^* \text{ and } \langle x, y\rangle = 0] \Rightarrow \langle L(x), y\rangle \leqslant 0.$$

When the context is clear, we simply say that L is a **Z**-*transformation* or that L has the **Z**-*property*.

2.1. Linear games. For a given linear map $L\colon V \to V$ and a fixed vector e in the interior of the self-dual cone $K \subseteq V$, the linear game $\Gamma(L) := (L, K, e)$ is a game played by two players I and II as defined in [6]. That is, if player I and II chooses $x \in \Delta$ and $y \in \Delta$ respectively as their strategies, then the pay-off for player I is $\langle y, L(x)\rangle$ and the pay-off for player II is $-\langle y, L(x)\rangle$. A pair of strategies (x^*, y^*) is said to be *optimal* for $\Gamma(L)$ if the following inequality holds for all $x, y \in \Delta$.

$$\langle y^*, L(x)\rangle \leqslant \langle y^*, L(x^*)\rangle \leqslant \langle y, L(x^*)\rangle.$$

The payoff $\langle y^*, L(x^*)\rangle$ at the optimal is called the *value of the game* $\Gamma(L)$ and it is denoted by $v(L)$. The strategies in the interior of Δ are called *completely mixed strategies*. The following result due to Gowda and Ravindran provides the relationship between the value of the linear games and the corresponding LCPs.

Theorem 2.2 ([6], Theorem 6). *Let $L\colon V \to V$ be a $\mathbf{Z}$-transformation. Then the following are equivalent:*

(1) *L is positive stable (that is, the real parts of all the eigenvalues of L are positive).*
(2) *L is semi positive on K (that is, there exists $x \in K^\circ$ such that $L(x) \in K^\circ$).*
(3) *L is invertible with $L^{-1}(K) \subseteq K$.*
(4) *L has $\mathbf{Q}$-property (that is, for every $q \in V$, the linear complementarity problem $\mathrm{LCP}(L, K, q)$ has a solution).*
(5) *A dynamical system $\dot{x}+L(x) = 0$ is asymptotically stable (that is, any trajectory starting from an arbitrary point in $\mathbb{R}^n$ converges to the origin).*
(6) *$v(L) > 0$.*

3. Bi-linear games

The following defines the bi-linear game in its most general form.

Let V, W be real inner product spaces of dimension n and m, respectively. Let K_1 and K_2 be self dual cones in V and W, respectively. For a fixed $e_1 \in K_1^\circ$ and $e_2 \in K_2^\circ$, consider the sets Δ_1 and Δ_2 defined as follows:

$$\Delta_1 := \{x \in K_1;\ \langle x, e_1\rangle = 1\},$$
$$\Delta_2 := \{y \in K_2;\ \langle y, e_2\rangle = 1\}.$$

Let $L_1, L_2\colon V \to W$ be two linear transformations. The bi-linear game corresponding to L_1 and L_2 denoted as $\Gamma(L_1, L_2)$ is defined to be the game played by two players, say player I and II. If the players I and II choose $x \in \Delta_1$ and $y \in \Delta_2$ respectively as their strategies, then their payoffs $P_I(x, y)$ and $P_{II}(x, y)$ are defined as follows:

$$P_I(x, y) := \langle y, L_1(x)\rangle,$$
$$P_{II}(x, y) := \langle y, L_2(x)\rangle.$$

A strategy x of player I (II) is said to be completely mixed if $x \in \Delta_1^\circ$ ($x \in \Delta_2^\circ$).

Definition 3.1 (Equilibrium pair). A pair of strategies $(x^*, y^*) \in \Delta_1 \times \Delta_2$ is said to be an *equilibrium pair* for $\Gamma(L_1, L_2)$ if it satisfies the following inequalities:

$$\begin{aligned} \langle y^*, L_1(x^*)\rangle &\geqslant \langle y^*, L_1(x)\rangle \quad \forall\, x \in \Delta_1, \\ \langle y^*, L_2(x^*)\rangle &\geqslant \langle y, L_2(x^*)\rangle \quad \forall\, y \in \Delta_2. \end{aligned} \tag{3.1}$$

The payoffs $v_1 := \langle y^*, L_1(x^*)\rangle$ and $v_2 := \langle y^*, L_2(x^*)\rangle$ at the equilibrium pair (x^*, y^*) is called the equilibrium value of player I and II respectively at (x^*, y^*).

Let $\mathcal{E}$ denote the set of all equilibrium pairs of the game $\Gamma(L_1, L_2)$ and we say that $\mathcal{E}$ is completely mixed if the strategies of both players in all the equilibrium pairs are completely mixed. In this case we say that the bi-linear game $\Gamma(L_1, L_2)$ is *completely mixed*. We define the following subset of Δ_1:

$$S(y^*) := \{x \in \Delta_1;\ (x, y^*) \in \mathcal{E}\}.$$

Remark 3.2. In this definition of bi-linear game, we get the classical bimatrix game concepts when $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, $K_1 = \mathbb{R}^n_+$, $K_2 = \mathbb{R}^m_+$, e_1 and e_2 are the vector of ones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Here $\langle x, y\rangle := x^\top y$.

Remark 3.3. In a bi-linear game $\Gamma(L_1, L_2)$ if $V = W$, $K_1 = K_2$, $L_2 = -L_1$ and $e_1 = e_2$ then the bi-linear game is reduced to the linear game $\Gamma(L_1)$ defined as in [6].

Theorem 3.4 ([15], Theorem 7.2.2)**.** *Let Δ_1 and Δ_2 be two compact convex sets in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Let $f_1(x, y)$ and $f_2(x, y)$ be two continuous functions defined on $\Delta_1 \times \Delta_2$. Consider $f_1(x, y)$ is concave in x for fixed y and $f_2(x, y)$ is concave in y for fixed x. Then there exists an equilibrium pair $(x^*, y^*) \in \Delta_1 \times \Delta_2$.*

The existence of an equilibrium for this general setting is proved in the following theorem.

Theorem 3.5. *For any bi-linear game $\Gamma(L_1, L_2)$ there exists a equilibrium pair (x^*, y^*).*

Proof. It is immediate from Theorem 3.4, since Δ_1, Δ_2 are compact convex sets and P_I, P_{II} are concave functions in each coordinate. □

4. Z-transformation and bi-linear games

Let K be a self-dual cone in a finite dimensional real inner product space V and $\widehat{L}$ be a linear map defined on V such that $\widehat{L}(K) \subseteq K$. Consider a linear map L defined as $L := sI - \widehat{L}$ where s is a fixed scalar and I is the identity map on V. Clearly, this is a **Z**-transformation. Any such linear map is called an **M**-transformation if $\varrho(\widehat{L}) \leqslant s$ and a nonsingular **M**-transformation if $\varrho(\widehat{L}) < s$.

In this section, for a linear transformation of the form $L = sI - \widehat{L}$, we fix $e \in K^\circ$ and consider the set $\Delta := \{x \in K;\ \langle x, e\rangle = 1\}$ as the strategy set for both the players in $\Gamma(L)$ as well as $\Gamma(L, \widehat{L})$.

Since $\widehat{L}(K) \subseteq K$, we observe that the equilibrium value of the player II(v_2) in the bi-linear game $\Gamma(L, \widehat{L})$ is always non-negative.

Definition 4.1. Let L be a $\mathbf{Z}$-transformation such that $L = sI - \widehat{L}$ with $\widehat{L}(K) \subseteq K$. Then we say that *value of the bi-linear game is positive* if there exists an equilibrium pair such that the value of the player I corresponding to that equilibrium pair is positive.

Theorem 4.2. *If L is semi positive, then $\Gamma(L, \widehat{L})$ has positive value.*

Proof. Let L be semi positive. That is, there exists an $x' \in \Delta$ such that $L(x') \in \Delta^\circ$. We claim $\Gamma(L, \widehat{L})$ has positive value. Let us assume that player I chooses x' as his strategy. Then for all $y \in \Delta$, $\langle y, L(x')\rangle > 0$. Suppose (x^*, y^*) is an equilibrium pair. Then from the definition of equilibrium pair we have that $\langle y^*, L(x^*)\rangle \geqslant \langle y^*, L(x)\rangle$ for all $x \in \Delta$. In particular, $\langle y^*, L(x^*)\rangle \geqslant \langle y^*, L(x')\rangle > 0$. Thus player I has positive value in $\Gamma(L, \widehat{L})$. $\square$

From the above theorem we can state the relation between linear and bi-linear game as "For a $\mathbf{Z}$-transformation $L = sI - \widehat{L}$, if $\Gamma(L)$ has positive value then $\Gamma(L, \widehat{L})$ has positive value". It is evident from the following example that the converse of the Theorem 4.2 need not be true.

Example 4.3. Let $\widehat{L}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by the matrix $[\widehat{L}] = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$ and $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $[L] = 6I_2 - [\widehat{L}] = \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}$ with respect to the standard basis, where I_2 denotes the identity matrix of order 2. Here we note that player II's pay-off matrix has all entries of non-negative sign and applying that IEDS process eliminates the dominated strategy (2nd row) of player II; by that the dimension of the game reduced to 1×2 from 2×2. In the next iteration, player I's dominated 2nd column eliminated. By this the dimension of the game gets reduced to 1×1 from 1×2. This does not happen if we see it as a linear game.

In linear game settings, Gowda and Ravindran showed that for a $\mathbf{Z}$-transformation, semi positivity is equivalent to the value being positive, but this is not true in the bi-linear game setting. However, the converse of Theorem 4.2 holds with the assumption of some weaker conditions that we will show in the following theorem.

Theorem 4.4. *For some $(x^*, y^*) \in \mathcal{E}$, if $\Gamma(L, \widehat{L})$ has positive value and $S(y^*)$ has at least one completely mixed strategy then L is semi positive.*

Proof. Let $(x^*, y^*) \in \mathcal{E}$ such that x^* is completely mixed strategy and the bi-linear game $\Gamma(L, \widehat{L})$ has positive value. That is, $\langle y^*, L(x^*)\rangle > 0$ where $x^* \in \Delta^\circ$ and by the definition of $\widehat{L}$ we have $\langle y^*, \widehat{L}(x^*)\rangle \geqslant 0$. We claim that there exists $x \in K^\circ$

such that $L(x) \in K^\circ$. Since L is continuous, it is enough to show that there exists an $x \in K$ such that $L(x) \in K^\circ$. Suppose there exists no $x \in K$ such that $L(x) \in K^\circ$. This says for all $x \in K$, $L(x) \notin K^\circ$. By Theorem 1.1, this implies that for all $y \in K$, $L^\top(y) \notin K^\circ$, since L is a $\mathbf{Z}$-transformation. In particular, $L^\top(y^*) \notin K^\circ$. This implies that there exists an $x \in K$ such that $\langle y^*, L(x)\rangle \leqslant 0$. Since $x^* \in K^\circ$, we can choose α sufficiently small such that $0 < \alpha < 1/\langle x, e\rangle$ and $z := x^* - \alpha x \in K$, and for some $\beta \geqslant 0$, $\beta z \in \Delta$. That is, $\beta(\langle x^*, e\rangle - \alpha\langle x, e\rangle) = 1$. Since $\langle x^*, e\rangle = 1$, we have $\beta > 1$. This implies $\langle y^*, L(\beta z)\rangle = \beta\langle y^*, L(x^*)\rangle - \beta\alpha\langle y^*, L(x)\rangle > \langle y^*, L(x^*)\rangle$ which is a contradiction to the assumption that (x^*, y^*) is an equilibrium pair. Thus there exists a $x \in K$ such that $L(x) \in K^\circ$. □

The following example illustrates that the converse of the Theorem 4.4 need not be true.

Example 4.5. Let $\widehat{L}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by the matrix $[\widehat{L}] = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$ and $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $[L] = 7.1 I_2 - [\widehat{L}] = \begin{bmatrix} 3.1 & -6 \\ -2 & 4.1 \end{bmatrix}$ with respect to the standard basis.

From the definition of the equilibrium value we observe the following lemma.

Lemma 4.6. *For a* $\mathbf{Z}$*-transformation* $L := sI - \widehat{L}$, *if* $\Gamma(L, \widehat{L})$ *has value positive then* $s > 0$.

From Example 4.3, we can observe that it is not necessary to hold $s \geqslant \varrho(\widehat{L})$. However, it does hold with some condition on $\Gamma(L, \widehat{L})$ that we will state and prove in the following theorem.

Theorem 4.7. *Consider a* $\mathbf{Z}$*-transformation* $L := sI - \widehat{L}$. *Suppose there exists* $(x^*, y^*) \in \mathcal{E}$ *such that the value of* $\Gamma(L, \widehat{L})$ *is positive and* $x^* \in K^\circ$. *Then the following statements hold.*

(1) *The linear complementarity problem* (L, K, q) *has global solvable property.*
(2) *The dynamical system* $\dot{x} + L(x) = 0$ *is asymptotically stable.*
(3) L *is invertible with* $L^{-1}(K) \subseteq K$.
(4) L *is a non-singular* $\mathbf{M}$*-transformation (i.e.* $s > \varrho(\widehat{L})$*).*

Proof. Let L be a $\mathbf{Z}$-transformation defined as $L := sI - \widehat{L}$. Assume there exist $(x^*, y^*) \in \mathcal{E}$ such that $x^* \in K^\circ$ and the value of $\Gamma(L, \widehat{L})$ is positive. Statements (1), (2), and (3) are clear from Theorems 4.4 and 2.2. Now we claim L is a non-singular $\mathbf{M}$-transformation, that is $s > \varrho(\widehat{L})$. Suppose $s \leqslant \varrho(\widehat{L})$. From Theorems 4.4 and 2.2 we see that L is positive stable. Since $\widehat{L}(K) \subseteq K$, by the Perron-Frobenius theorem for a proper cone there exists an eigenvector x such that $\widehat{L}x = \varrho(\widehat{L})x$. This implies

$L(x) = (s - \varrho(\widehat{L}))x$. Thus L has a non-positive eigenvalue $s - \varrho(\widehat{L})$ which is a contradiction to the fact that L is positive stable. Hence, $s > \varrho(\widehat{L})$. $\square$

Example 4.8. Consider the following **Z**-transformations defined on $\mathbb{R}^2$ and $\mathbb{R}^3$.

$$\text{(i)}\quad [L_1] := \begin{bmatrix} 5 & -4 \\ -3 & 4 \end{bmatrix}; \quad \text{(ii)}\quad [L_2] := \begin{bmatrix} 6 & -4 & -2 \\ -5 & 5 & -1 \\ -1 & 0 & 4 \end{bmatrix}.$$

(i) By considering

$$[\widehat{L}_1] = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix},$$

we can see that the bi-linear game $\Gamma(L_1, \widehat{L}_1)$ has $(\frac{1}{2}, \frac{1}{2})$ as a equilibrium strategy and $\frac{1}{2}$ as a equilibrium value for player I.

(ii) By considering

$$[\widehat{L}_2] = \begin{bmatrix} 1 & 4 & 2 \\ 5 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix},$$

we can see that for the bi-linear game $\Gamma(L_2, \widehat{L}_2)$, $(\frac{3}{13}, \frac{2}{13}, \frac{8}{13})$ and $\frac{13}{64}$ are equilibrium strategy and equilibrium value respectively for player I.

Thus we can conclude from the above theorem that both L_1 and L_2 have **Q**-property.

For a given **Z**-transformation $L := sI - \widehat{L}$, the following theorems are analogous to Theorem 6 in [6].

Theorem 4.9. *Suppose both L and $\widehat{L}$ are invertible with $L^{-1}(K) \subseteq K$. Then,*

$$v_1 = \frac{1}{\langle (L^\top)^{-1}(e), e\rangle} \quad \text{and} \quad v_2 = \frac{1}{\langle (\widehat{L})^{-1}(e), e\rangle},$$

respectively are equilibrium values of player I and II corresponding to the equilibrium pair (x^*, y^*), where $x^* := v_2(\widehat{L})^{-1}(e)$ and $y^* := v_1(L^\top)^{-1}(e)$.

Proof. Since K is self-dual and $L^\top$ is invertible, $(L^\top)^{-1}(K^\circ) \subseteq K^\circ$. By the definition of $\widehat{L}$ we have $(\widehat{L})^{-1}(K^\circ) \subseteq K^\circ$. Hence, we see that $(\widehat{L})^{-1}(e)$ and $(L^\top)^{-1}(e)$ belongs to K° for $e \in K^\circ$. This implies that both $\langle (\widehat{L})^{-1}(e), e\rangle$ and $\langle (L^\top)^{-1}(e), e\rangle$ are non-zero. Define x^* and y^* as follows.

$$x^* := v_{x^*}(\widehat{L})^{-1}(e) \quad \text{where} \quad v_{x^*} = \frac{1}{\langle (\widehat{L})^{-1}(e), e\rangle},$$
$$y^* := v_{y^*}(L^\top)^{-1}(e) \quad \text{where} \quad v_{y^*} = \frac{1}{\langle (L^\top)^{-1}(e), e\rangle}.$$

It is clear that x^* and y^* belongs to Δ. Now, we claim that (x^*, y^*) satisfies the inequalities in (3.1). Now, $\langle y^*, L(x^*)\rangle = \langle v_{y^*}(L^\top)^{-1}(e), L(x^*)\rangle = v_{y^*}\langle e, x^*\rangle$. Since $x^* \in \Delta$, $\langle e, x^*\rangle = 1$ we have $\langle y^*, L(x^*)\rangle = v_{y^*}$. This implies, for $x \in \Delta$, $\langle y^*, L(x)\rangle = \langle v_{y^*}(L^\top)^{-1}(e), L(x)\rangle = v_{y^*}\langle e, x\rangle = \langle y^*, L(x^*)\rangle$. That is, for all $x \in \Delta$, $\langle y^*, L(x^*)\rangle \geqslant \langle y^*, L(x)\rangle$. Similarly, for $y \in \Delta$, $\langle y^*, \widehat{L}(x^*)\rangle = \langle y^*, \widehat{L}[v_{x^*}(\widehat{L})^{-1}(e)]\rangle = v_{x^*} = v_{x^*}\langle y, e\rangle = \langle y, \widehat{L}(x^*)\rangle$. Thus, for all $y \in \Delta$, $\langle y^*, \widehat{L}(x^*)\rangle \geqslant \langle y, \widehat{L}(x^*)\rangle$. Hence, (x^*, y^*) is an equilibrium pair and we can observe that corresponding to (x^*, y^*) the equilibrium value of the player $\mathrm{I}(v_1)$ is $1/\langle (L^\top)^{-1}(e), e\rangle$ and the equilibrium value of the player $\mathrm{II}(v_2)$ is $1/\langle (\widehat{L})^{-1}(e), e\rangle$. $\square$

In the above theorem, it is to be noted that strategies in the equilibrium pair are completely mixed. In the following proposition, we discuss the converse of Theorem 4.4 and the equivalence of the statements in Theorem 4.7.

Theorem 4.10. *Consider a* $\mathbf{Z}$*-transformation* $L := sI - \widehat{L}$. *Suppose* $\widehat{L}$ *is invertible. Then the following are equivalent:*

(1) *L is invertible with $L^{-1}(K) \subseteq K$.*

(2) *There exists an equilibrium pair (x^*, y^*) such that the value of $\Gamma(L, \widehat{L})$ is positive and x^* is completely mixed.*

(3) *L is semipositive.*

Proof. (1) $\Rightarrow$ (2): Assume L^{-1} exists and $L^{-1}(K) \subseteq K$. Now, consider x^*, y^* defined as in Theorem 4.9. It is clear that (x^*, y^*) is an equilibrium pair with x^* completely mixed and the equilibrium value corresponding to (x^*, y^*) is positive. (2) $\Rightarrow$ (1) follows from Theorem 4.7. We know the equivalence of (1) and (3) from Theorem 2.2. $\square$

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