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ON PRESSURE BOUNDARY CONDITIONS FOR STEADY FLOWS OF INCOMPRESSIBLE FLUIDS WITH PRESSURE AND SHEAR RATE DEPENDENT VISCOSITIES*

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Abstract. We consider a class of incompressible fluids whose viscosities depend on the pressure and the shear rate. Suitable boundary conditions on the traction at the inflow/outflow part of boundary are given. As an advantage of this, the mean value of the pressure over the domain is no more a free parameter which would have to be prescribed otherwise. We prove the existence and uniqueness of weak solutions (the latter for small data) and discuss particular applications of the results.

Keywords: existence, weak solutions, incompressible fluids, non-Newtonian fluids, pressure dependent viscosity, shear dependent viscosity, inflow/outflow boundary conditions, pressure boundary conditions, filtration boundary conditions

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1. Introduction

A well-known property of the Navier-Stokes equations describing the motion of an incompressible Newtonian fluid is that the fluid pressure is determined to within a constant. This degree of freedom does not play important role as far as only the pressure gradient is present in the equations of motion. Some generalizations of the Navier-Stokes equations, such as the equations for fluids with shear rate dependent viscosity share this property as well.

It has been observed that under some circumstances the fluid viscosity may depend significantly both on the shear rate and on the pressure. In such case the value of the

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pressure affects the whole solution of the equations. In previous theoretical studies, such as [10], [16], [26], the mean value of the pressure either over the whole domain or over its nontrivial subdomain was prescribed as one of the input parameters. A difficulty of this approach lies in the fact that the pressure mean value is not a proper quantity from the practical point of view, i.e. there is no hint on the value which should be prescribed for a particular application. The objective of this paper is to propose an alternative way of fixing the pressure, namely to use a suitable inflow/outflow boundary condition.

Let us demonstrate the idea on a simple example: Consider the Navier-Stokes equations and the Poiseuille flow in a 2D channel $(0, L) \times (0, 1)$ of length L and height 1, for which the velocity and the pressure are given by

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{x}) &= (v_0 x_2(1 - x_2), 0), \quad v_0 \in \mathbb{R}, \\ p(\boldsymbol{x}) &= p_0 - 2\mu v_0 x_1, \quad p_0 \in \mathbb{R}. \end{aligned}$$

Here μ is the (constant) viscosity and $\frac{1}{4}v_0$ is the peak velocity in the channel centre. The parameter p_0 can be chosen arbitrarily and has no influence on the velocity. If we additionally prescribe a constant normal force h on the channel outlet $\{L\} \times (0, 1)$ by

$$-p + 2\mu \mathbf{D}(\boldsymbol{v})\boldsymbol{n} \cdot \boldsymbol{n} = h, \tag{1.1}$$

where $\mathbf{D}(\boldsymbol{v})$ is the symmetric velocity gradient and $\boldsymbol{n}$ the unit outer normal to the boundary, then we automatically obtain $p_0 = 2\mu v_0 L - h$ and the pressure is fixed. We will show (see Section 4) that boundary conditions similar to (1.1) have the same effect on weak solutions to fluids with shear rate and pressure dependent viscosity.

In many applications, induced force is prescribed on a part of the boundary:

$$\mathbf{T}\boldsymbol{n} = \boldsymbol{h}(\boldsymbol{x}), \tag{1.2}$$

where $\mathbf{T} = -p\mathbf{I} + \mathbf{S}$ denotes the Cauchy stress, $\boldsymbol{n}$ the outer normal to the boundary and $\boldsymbol{h}$ a given force. As a particular example, often a kind of natural outflow can be achieved in flow simulations by simply prescribing

$$\mathbf{T}\boldsymbol{n} = \mathbf{0};$$

this type of condition (usually referred to as the *do nothing* condition) is easy to use in numerical simulations and yields quite reliable results (see e.g. [20]).

Some existence analysis of the Navier-Stokes equations with the condition (1.2) is available: Local results (i.e. for small data or short time) were obtained e.g. in [24] and in [25] for stationary and for time dependent case, respectively. Global existence analysis is, however, an open problem because (1.2) does not prevent backward flow through the boundary and thus an uncontrolled amount of kinetic energy can be brought into the domain. In [23] the authors showed the existence of weak solutions to the variational inequality involving an explicit constraint imposed on the backward flows.

In this paper we will study boundary conditions involving a surface force depending on the velocity:

$$-\mathbf{T}\boldsymbol{n} = \boldsymbol{b}(\boldsymbol{x}, \boldsymbol{v}), \tag{1.3}$$

where the assumptions on $\boldsymbol{b}$ are specified in Subsection 2.2. Important examples and their motivation are given in Section 5. We follow the approach used e.g. in [13], where

$$\boldsymbol{b} = \boldsymbol{h}(\boldsymbol{x}) + \frac{1}{2}(\boldsymbol{v} \cdot \boldsymbol{n})^{-}\boldsymbol{v}$$

with $z^{-} := \max\{0, -z\}$ being the negative part of z. Namely, we restrict ourselves to such forms of $\boldsymbol{b}$ in (1.3) that expend all the kinetic energy brought in by the inflow, allowing us to establish standard energy estimates.

The paper is organized as follows. In Section 2 we specify the problem to be analyzed and state the main theorem. The existence and uniqueness of weak solutions is then proved in Section 3 and Section 4, respectively. Finally, Section 5 contains particular applications covered by the theory.

2. Definition of the problem and the main result

We investigate the system of PDEs

$$\left.\begin{aligned} \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \mathbf{S} + \nabla p &= \boldsymbol{f} \\ \operatorname{div} \boldsymbol{v} &= 0 \end{aligned}\right\} \quad \text{in } \Omega,$$

where

$$\mathbf{S} \equiv \mathbf{S}(p, \mathbf{D}(\boldsymbol{v})) = \nu(p, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}). \tag{2.1}$$

Here $\boldsymbol{v}$, p, $\boldsymbol{f}$, $\nu(p, |\mathbf{D}(\boldsymbol{v})|^2)$ is the velocity, the kinematic pressure, the body force and the kinematic viscosity, respectively. The equations describe the motion of an incompressible homogeneous fluid in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3. The domain boundary consists of three measurable and disjoint parts: $\partial\Omega := \Gamma_D \cup \Gamma_1 \cup \Gamma_2$, on which we prescribe the boundary conditions

$$\boldsymbol{v} = \mathbf{0} \qquad \text{on } \Gamma_D, \tag{2.2}$$

$$p\boldsymbol{n} - \mathbf{S}\boldsymbol{n} = \boldsymbol{b}_1(\boldsymbol{v}) \qquad \text{on } \Gamma_1, \tag{2.3}$$

$$\left.\begin{aligned} \boldsymbol{v} &= (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n} \\ p - \mathbf{S}\boldsymbol{n}\cdot\boldsymbol{n} &= b_2(\boldsymbol{v}) \end{aligned}\right\} \qquad \text{on } \Gamma_2. \tag{2.4}$$

Throughout the paper we will assume that $\partial\Omega$, Γ_D, Γ_1, and Γ_2 are Lipschitz continuous. Further we will denote $\Gamma := \Gamma_1 \cup \Gamma_2$ and suppose that $|\Gamma_D| > 0$ and $|\Gamma| > 0$, i.e., the Dirichlet condition (2.2) and at least one of the conditions (2.3), (2.4) are present. Note that $|\Gamma_D| > 0$ is needed in order to guarantee the validity of Korn's inequality.

The equations governing the flow of an incompressible fluid with the viscosity depending on the pressure and the shear rate were subject to a number of recent studies. For more details on models of the type (2.1), we refer the reader to [16], [27], [29], [30]. Simple flows and numerical simulations are discussed in [21], [22]. In [9], [10], [26], issues concerning various boundary conditions were studied. In [8], [11], some further generalizations are provided. The proof of existence presented here derives from the one developed in [16], where the existence theory was established for steady flows subject to homogeneous Dirichlet boundary condition only.

2.1. Structural assumptions

The following assumptions on $\mathbf{S}$ are considered.

(A1) For a given $r \in (1,2)$, there exist positive constants C_1 and C_2 such that for all symmetric linear transformations $\mathbf{B}, \mathbf{D} \in \mathbb{R}^{d\times d}$ and all $p \in \mathbb{R}$:

$$\begin{aligned} C_1(1+|\mathbf{D}|^2)^{(r-2)/2}|\mathbf{B}|^2 &\leqslant \frac{\partial \mathbf{S}(p,\mathbf{D})}{\partial \mathbf{D}} \cdot (\mathbf{B}\otimes\mathbf{B}) \\ &\leqslant C_2(1+|\mathbf{D}|^2)^{(r-2)/2}|\mathbf{B}|^2, \end{aligned}$$

where $(\mathbf{B}\otimes\mathbf{B})_{ijkl} = \mathbf{B}_{ij}\mathbf{B}_{kl}$.

(A2) For all symmetric linear transformations $\mathbf{D} \in \mathbb{R}^{d\times d}$ and for all $p \in \mathbb{R}$:

$$\left|\frac{\partial \mathbf{S}(p,\mathbf{D})}{\partial p}\right| \leqslant \gamma_0(1+|\mathbf{D}|^2)^{(r-2)/4} \leqslant \gamma_0,$$

with $\gamma_0 > 0$ to be specified later.

For particular examples see the references given above.

We state some useful inequalities following from (A1) and (A2). First, it was proved in [28], Lemma 1.19 of Chapter 5, that for every $p \in \mathbb{R}$ and $\mathbf{D} \in \mathbb{R}^{d\times d}_{\rm sym}$

$$|\mathbf{S}(p,\mathbf{D}) : \mathbf{D}| \leqslant \frac{C_2}{r-1}(1+|\mathbf{D}|)^{r-1}, \tag{2.5}$$

$$\mathbf{S}(p,\mathbf{D}) : \mathbf{D} \geqslant C_3 \min\{|\mathbf{D}|^2, |\mathbf{D}|^r\}, \tag{2.6}$$

with $C_3 = C_3(r, C_1)$. Next, defining

$$I^{1,2} := |\mathbf{D}^1 - \mathbf{D}^2|^2 \int_0^1 (1 + |\mathbf{D}^1 + s(\mathbf{D}^2 - \mathbf{D}^1)|^2)^{(r-2)/2}\,\mathrm{d}s, \tag{2.7}$$

one can show that (see e.g. Lemma 1.4 in [10])

$$\frac{C_1}{2} I^{1,2} \leqslant (\mathbf{S}(p^1,\mathbf{D}^1) - \mathbf{S}(p^2,\mathbf{D}^2)) : (\mathbf{D}^1 - \mathbf{D}^2) + \frac{\gamma_0^2}{2C_1}|p^1 - p^2|^2, \tag{2.8}$$

$$|\mathbf{S}(p^1,\mathbf{D}^2) - \mathbf{S}(p^2,\mathbf{D}^2)| \leqslant C_2\sqrt{I^{1,2}} + \gamma_0|p^1 - p^2|, \tag{2.9}$$

$$\|1 + |\mathbf{D}^1| + |\mathbf{D}^2|\|_r^{r-2} \|\mathbf{D}^1 - \mathbf{D}^2\|_r^2 \leqslant \int_\Omega I^{1,2}\,\mathrm{d}\boldsymbol{x}. \tag{2.10}$$

We use the inequality (2.6) in the form

Lemma 2.1. *Assume that* (A1), (A2) *are fulfilled. Let*[1] $\boldsymbol{u} \in \mathbf{W}^{1,r}(\Omega)$ *and* $F \geqslant 0$. *Then*

$$\int_\Omega \mathbf{S}(p,\mathbf{D}(\boldsymbol{u})) : \mathbf{D}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} - F\|\mathbf{D}(\boldsymbol{u})\|_r \geqslant C_4 \min\{\|\mathbf{D}(\boldsymbol{u})\|_r^2, \|\mathbf{D}(\boldsymbol{u})\|_r^r\} - C_5(F^2 + F^{r'}), \tag{2.11}$$

where $r' := r/(r-1)$, *and the constants* $C_4, C_5 > 0$ *depend solely on* Ω, r *and* C_3.

Proof. Define $\hat\Omega := \{\boldsymbol{x} \in \Omega\colon\ |\mathbf{D}(\boldsymbol{u})| > 1\}$ and $\underline{\Omega} := \Omega \setminus \hat\Omega$. Then (2.6) gives

$$\int_\Omega \mathbf{S}(p,\mathbf{D}(\boldsymbol{u})) : \mathbf{D}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} - F\|\mathbf{D}(\boldsymbol{u})\|_r \geqslant C_3\|\mathbf{D}(\boldsymbol{u})|_{\underline{\Omega}}\|_2^2 + C_3\|\mathbf{D}(\boldsymbol{u})|_{\hat\Omega}\|_r^r - F(\|\mathbf{D}(\boldsymbol{u})|_{\underline{\Omega}}\|_r + \|\mathbf{D}(\boldsymbol{u})|_{\hat\Omega}\|_r).$$

Hölder's inequality $\|\mathbf{D}(\boldsymbol{u})|_{\underline{\Omega}}\|_r^2 \leqslant \|\mathbf{D}(\boldsymbol{u})|_{\underline{\Omega}}\|_2^2 |\underline{\Omega}|^{\frac12(2-r)\cdot 2/r} \leqslant \|\mathbf{D}(\boldsymbol{u})|_{\underline{\Omega}}\|_2^2 |\Omega|^{\frac12(2-r)\cdot 2/r}$, Young's inequality and the fact that $\frac12 \min\{\|\mathbf{D}(\boldsymbol{u})\|_r^2, \|\mathbf{D}(\boldsymbol{u})\|_r^r\} \leqslant \|\mathbf{D}(\boldsymbol{u})|_{\underline{\Omega}}\|_r^2 + \|\mathbf{D}(\boldsymbol{u})|_{\hat\Omega}\|_r^2$ then lead to (2.11). □

[1] In this paper, $\mathrm{W}^{1,r}(\Omega)$, $\mathrm{W}^{1,r}_0(\Omega)$, $\mathrm{L}^q(\Omega)$, $\mathrm{L}^q_0(\Omega)$ stand for the Sobolev space, its subspace of functions with zero trace, the Lebesgue space, and its subspace of functions with zero mean value, respectively. Bold symbols denote the vector counterparts of these spaces. The norms of $\mathrm{W}^{1,r}(\Omega)$, $\mathrm{L}^q(\Omega)$ will be denoted by $\|\cdot\|_{1,r}$, $\|\cdot\|_q$ respectively.

2.2. Boundary assumptions

Concerning the boundary conditions (2.3)–(2.4), we define

$$\langle \boldsymbol{b}(\boldsymbol{v}), \boldsymbol{\varphi}\rangle := \langle \boldsymbol{b}_1(\boldsymbol{v}), \boldsymbol{\varphi}\rangle_{\Gamma_1} + \langle b_2(\boldsymbol{v}\cdot\boldsymbol{n}), \boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_{\Gamma_2}$$

and assume the following conditions:

(B1) With some $\gamma_1 \in \langle 3, r^*)$, the mapping

$$\boldsymbol{b}_1(\cdot)\colon\ \mathbf{L}^{\gamma_1}(\Gamma_1) \to \mathbf{L}^{\gamma_1}(\Gamma_1)^* \tag{2.12}$$

is continuous and bounded. Here $r^* := (d-1)r/(d-r)$ denotes the exponent for which $\mathbf{W}^{1,r}(\Omega) \hookrightarrow \mathbf{L}^{r^*}(\partial\Omega)$.

(B2) With some $\beta_1 \geqslant 0$,

$$\langle \boldsymbol{b}_1(\boldsymbol{u}), \boldsymbol{u}\rangle_{\Gamma_1} \geqslant -\frac{1}{2}\int_{\Gamma_1} (\boldsymbol{u}\cdot\boldsymbol{n})|\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x} - \beta_1\|\boldsymbol{u}\|_{\gamma_1,\Gamma_1} \tag{2.13}$$

for all $\boldsymbol{u} \in \mathbf{L}^{\gamma_1}(\Gamma_1)$.

(B3) With some $\gamma_2 \geqslant 3$, the mapping

$$b_2(\cdot)\colon\ \mathrm{L}^{\gamma_2}(\Gamma_2) \to \mathrm{L}^{\gamma_2}(\Gamma_2)^* \tag{2.14}$$

is continuous and bounded.

(B4) With some $\beta_2 \geqslant 0$ and $\underline{\beta_2} > 0$,

$$\langle b_2(\boldsymbol{u}\cdot\boldsymbol{n}), \boldsymbol{u}\cdot\boldsymbol{n}\rangle_{\Gamma_2} \geqslant -\frac{1}{2}\int_{\Gamma_2} (\boldsymbol{u}\cdot\boldsymbol{n})|\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x} + \underline{\beta_2}\|\boldsymbol{u}\|_{\gamma_2,\Gamma_2}^{\gamma_2} - \beta_2 \tag{2.15}$$

for all $\boldsymbol{u} \in \mathbf{L}^{\gamma_2}(\Gamma_2)$.

(B5) With some continuous function $m\colon \mathbb{R}^+ \to \mathbb{R}^+$, where $\lim_{x\searrow 0} m(x) = 0$, b_2 is uniformly[2] monotone:

$$\langle b_2(w) - b_2(z), w - z\rangle_{\Gamma_2} \geqslant m(\|w - z\|_{\gamma_2,\Gamma_2}) \tag{2.16}$$

for all $w \neq z \in \mathrm{L}^{\gamma_2}(\Gamma_2)$.

Additionally, in order to prove the uniqueness of solutions we will require that the following stronger conditions hold:

[2] For the sake of simplicity, the uniform monotonicity is assumed here. The readers can verify themselves that the monotonicity of b_2 would also allow to show the existence of a weak solution, with help of the Minty trick.

(B6) With some $\lambda_1 > 0$ and $K_1 > 0$ (to be specified later),

$$\|\boldsymbol{b}_1(\boldsymbol{u}^1) - \boldsymbol{b}_1(\boldsymbol{u}^2)\|_{\gamma_1',\Gamma_1} \leqslant \lambda_1 \|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_{\gamma_1,\Gamma_1} \tag{2.17}$$

for all $\boldsymbol{u}^1, \boldsymbol{u}^2 \in \mathbf{L}^{\gamma_1}(\Gamma_1)$, $\|\boldsymbol{u}^i\|_{\gamma_1,\Gamma_1} \leqslant K_1$, $i = 1, 2$.

(B7) With some $\lambda_2 > 0$ and $K_2 > 0$ (to be specified later),

$$\|b_2(\boldsymbol{u}^1 \cdot \boldsymbol{n}) - b_2(\boldsymbol{u}^2 \cdot \boldsymbol{n})\|_{1,\Gamma_2} \leqslant \lambda_2 \|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_{r^*,\Gamma_2} \tag{2.18}$$

for all $\boldsymbol{u}^1, \boldsymbol{u}^2 \in \mathrm{L}^{\gamma_2}(\Gamma_2)$, $\|\boldsymbol{u}^i\|_{\gamma_2,\Gamma_2} \leqslant K_2$, $i = 1, 2$.

2.3. Weak formulation

We define the following function spaces:

$$\begin{aligned} \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega) &:= \{\boldsymbol{v} \in \mathbf{W}^{1,r}(\Omega);\ \mathrm{tr}\,\boldsymbol{v}|_{\Gamma_D} = \mathbf{0},\ \mathrm{tr}\,\boldsymbol{v}|_{\Gamma_2} = (\mathrm{tr}\,\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n} \in \mathrm{L}^{\gamma_2}(\Gamma_2)\}, \\ \mathbf{W}^{1,r}_{\mathrm{b.c.,div}}(\Omega) &:= \{\boldsymbol{v} \in \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega);\ \mathrm{div}\,\boldsymbol{v} = 0 \text{ a.e. in } \Omega\}. \end{aligned}$$

Note that, due to embedding, $\boldsymbol{v} \in \mathbf{W}^{1,r}(\Omega)$ implies $\boldsymbol{v} \in \mathbf{L}^{\gamma_1}(\Gamma_1)$. Given $\boldsymbol{f} \in \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega)^*$, we consider the following weak formulation:

Definition 2.2 (Problem (P)). A pair $(\boldsymbol{v}, p) \in \mathbf{W}^{1,r}_{\mathrm{b.c.,div}}(\Omega) \times \mathrm{L}^{r'}(\Omega)$ is called a weak solution of Problem (P) if and only if

$$\begin{aligned} \int_\Omega \mathrm{div}(\boldsymbol{v} \otimes \boldsymbol{v}) \cdot \boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} &+ \int_\Omega \mathbf{S}(p, \mathbf{D}(\boldsymbol{v})) : \mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} \\ &- \int_\Omega p\,\mathrm{div}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{b}(\boldsymbol{v}), \boldsymbol{\varphi} \rangle = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle \end{aligned} \tag{2.19}$$

for all $\boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega)$.

We close this subsection by recalling the properties of the Bogovskiĭ operator (see [32] or [1], [3] for the reference) and by stating its corollary.

Lemma 2.3 (Bogovskiĭ's operator; [32], Lemma 3.17)**.** *Let $1 < q < \infty$. Then there exists a continuous linear operator $\mathcal{B}\colon \mathrm{L}^q_0(\Omega) \to \mathbf{W}^{1,q}_0(\Omega)$ such that for all $f \in \mathrm{L}^q_0(\Omega)$*

$$\begin{cases} \mathrm{div}(\mathcal{B}f) = f & \text{a.e. in } \Omega, \\ \|\mathcal{B}f\|_{1,q} \leqslant C_{\mathrm{div}}(\Omega, q)\|f\|_q. \end{cases} \tag{2.20}$$

Lemma 2.4. *Let $q \in (1,\infty)$, $s \in \langle 1,\infty\rangle$. Then there exists a continuous bounded linear operator $\tilde{\mathcal{B}}\colon \mathrm{L}^q(\Omega) \to \mathbf{W}^{1,q}_{\mathrm{b.c.}}(\Omega)$ such that for all $f \in \mathrm{L}^q(\Omega)$*

$$
\begin{cases} \operatorname{div}(\tilde{\mathcal{B}} f) = f \ \text{ a.e. in } \Omega, \\ \|\tilde{\mathcal{B}} f\|_{1,q} \leqslant \tilde{C}_{\mathrm{div}}(\Omega,\Gamma_1,\Gamma_2,q)\|f\|_q, \\ \|\tilde{\mathcal{B}} f\|_{s,\Gamma} \leqslant C'_{\mathrm{div}}(\Omega,\Gamma_1,\Gamma_2,s)|\int_\Omega f|. \end{cases} \tag{2.21}
$$

P r o o f. Let us take an arbitrary function $\boldsymbol{\xi} \in \mathcal{C}^\infty(\overline{\Omega})^d$ such that $\boldsymbol{\xi}|_{\Gamma_D} = \mathbf{0}$, $\boldsymbol{\xi}|_{\Gamma_2} = (\boldsymbol{\xi}\cdot\boldsymbol{n})\boldsymbol{n}$ and $\int_\Gamma \boldsymbol{\xi}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{x} = 1$. Then for any $f \in \mathrm{L}^q(\Omega)$ we define $\tilde{\mathcal{B}}(f) := \mathcal{B}\big(f - \big(\int_\Omega f\,\mathrm{d}\boldsymbol{x}\big)\operatorname{div}\boldsymbol{\xi}\big) + \big(\int_\Omega f\,\mathrm{d}\boldsymbol{x}\big)\boldsymbol{\xi}$. Since $\tilde{\mathcal{B}}(f)|_{\partial\Omega} = \big(\int_\Omega f\,\mathrm{d}\boldsymbol{x}\big)\boldsymbol{\xi}$, we have that $\tilde{\mathcal{B}}(f) \in \mathbf{W}^{1,q}_{\mathrm{b.c.}}(\Omega)$. It is then easy to verify with help of Lemma 2.3 that such choice meets the statement (2.21). □

2.4. Main result

Theorem 2.5 (Well-posedness of (P))**.**

Let $\boldsymbol{f} \in \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega)^*$ and assume that (A1)–(A2) *hold for the viscosity,* (B1)–(B5) *hold for the boundary data, with*

$$
\frac{3d}{d+2} < r < 2 \quad \text{and} \quad \gamma_0 < \frac{1}{\tilde{C}_{\mathrm{div}}(\Omega,\Gamma_1,\Gamma_2,2)}\,\frac{C_1}{C_1+C_2}. \tag{2.22}
$$

Then

(i) *there exists a weak solution to* (P)*;*

(ii) *for any weak solution* $(\boldsymbol{v},p)$ *of* (P)*, the velocity* $\boldsymbol{v}$ *satisfies the estimate*

$$
\|\boldsymbol{v}\|_{1,r} + \|\boldsymbol{v}\|_{\gamma_2,\Gamma_2} \leqslant K, \tag{2.23}
$$

where $K \searrow 0$ whenever $(\|\boldsymbol{f}\|_{W^{1,r}_{\mathrm{b.c.}}(\Omega)^*}, \beta_1, \beta_2) \searrow \mathbf{0}$, the other problem data being fixed;

(iii) *if additionally* (B6), (B7) *are satisfied and if K and λ_1, λ_2 are small enough, then the weak solution to* (P) *is unique.*

The proof of (i) has the same structure as the proof given in [16] for the problem with the homogeneous Dirichlet boundary condition on $\partial\Omega$: In 3.1, we define an approximate problem (P^ε), derive energy estimates and show the existence of a weak solution to (P^ε) via Galerkin approximations. Also, (ii) follows from the estimates derived in here. In 3.2, we show estimates for the pressure p^ε uniform with respect to ε. This allows us to find sequences $\{(\boldsymbol{v}^{\varepsilon_n}, p^{\varepsilon_n})\}$, $\varepsilon_n \searrow 0$, weakly converging to a limit $(\boldsymbol{v}, p)$. In 3.3, the strong convergence of p^{ε_n} and $\mathbf{D}(\boldsymbol{v}^{\varepsilon_n})$ is shown and $(\boldsymbol{v}, p)$ is identified as the weak solution to problem (P).

3.1. Approximate problem (P^ε)

We relax the incompressibility constraint and look for a pair $(\boldsymbol{v}^\varepsilon, p^\varepsilon) \in \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega) \times W^{1,2}(\Omega)$ satisfying

$$\varepsilon \int_\Omega \nabla p^\varepsilon \cdot \nabla \xi \,\mathrm{d}\boldsymbol{x} + \varepsilon \int_\Omega p^\varepsilon \xi \,\mathrm{d}\boldsymbol{x} + \int_\Omega (\operatorname{div} \boldsymbol{v}^\varepsilon) \xi \,\mathrm{d}\boldsymbol{x} = 0 \quad \text{for all } \xi \in W^{1,2}(\Omega), \tag{3.1}$$

together with

$$\begin{aligned} \int_\Omega \operatorname{div}(\boldsymbol{v}^\varepsilon \otimes \boldsymbol{v}^\varepsilon) \cdot \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} - \frac{1}{2} \int_\Omega (\operatorname{div} \boldsymbol{v}^\varepsilon)(\boldsymbol{v}^\varepsilon \cdot \boldsymbol{\varphi}) \,\mathrm{d}\boldsymbol{x} - \int_\Omega p^\varepsilon \operatorname{div} \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} \\ + \int_\Omega \mathbf{S}(p^\varepsilon, \mathbf{D}(\boldsymbol{v}^\varepsilon)) : \mathbf{D}(\boldsymbol{\varphi}) \,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{b}(\boldsymbol{v}^\varepsilon), \boldsymbol{\varphi} \rangle = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle \quad \text{for all } \boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega). \end{aligned} \tag{3.2}$$

Note that, contrary to the case studied in [16], equation (3.1) does not determine the mean value of the pressure $\frac{1}{|\Omega|} \int_\Omega p^\varepsilon \,\mathrm{d}\boldsymbol{x}$. This is a consequence of the fact that $\boldsymbol{v}^\varepsilon \cdot \boldsymbol{n}|_\Gamma$ is not prescribed.

We show that $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$ can be found as a limit of the Galerkin approximations $(\boldsymbol{v}^N, p^N)$ defined as

$$p^N := \sum_{k=1}^N c_k^N \alpha_k \quad \text{and} \quad \boldsymbol{v}^N := \sum_{k=1}^N d_k^N \boldsymbol{a}_k \qquad \text{for } N = 1, 2, \dots,$$

where $\{\alpha_k\}_{k=1}^\infty$ and $\{\boldsymbol{a}_k\}_{k=1}^\infty$ are bases of $W^{1,2}(\Omega)$ and $\mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega)$, respectively, and where $\boldsymbol{c}^N := (c_1^N, \dots, c_N^N)$ and $\boldsymbol{d}^N := (d_1^N, \dots, d_N^N)$ solve the algebraic system

$$\varepsilon \int_\Omega \nabla p^N \cdot \nabla \alpha_k \,\mathrm{d}\boldsymbol{x} + \varepsilon \int_\Omega p^N \alpha_k \,\mathrm{d}\boldsymbol{x} + \int_\Omega (\operatorname{div} \boldsymbol{v}^N) \alpha_k \,\mathrm{d}\boldsymbol{x} = 0, \quad k = 1, \dots, N, \tag{3.3a}$$

$$\begin{aligned} \int_\Omega \operatorname{div}(\boldsymbol{v}^N \otimes \boldsymbol{v}^N) \cdot \boldsymbol{a}_l \,\mathrm{d}\boldsymbol{x} - \frac{1}{2} \int_\Omega (\operatorname{div} \boldsymbol{v}^N)(\boldsymbol{v}^N \cdot \boldsymbol{a}_l) \,\mathrm{d}\boldsymbol{x} - \int_\Omega p^N \operatorname{div}(\boldsymbol{a}_l) \,\mathrm{d}\boldsymbol{x} \\ + \int_\Omega \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) : \mathbf{D}(\boldsymbol{a}_l) \,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{b}(\boldsymbol{v}^N), \boldsymbol{a}_l \rangle = \langle \boldsymbol{f}, \boldsymbol{a}_l \rangle, \quad l = 1, \dots, N. \end{aligned} \tag{3.3b}$$

Multiplying the kth equation in (3.3a) by c_k^N and the lth equation in (3.3b) by d_l^N and summing for $k, l = 1, \dots, N$, we obtain

$$
\begin{aligned}
(3.4) \qquad \varepsilon \|p^N\|_{1,2}^2 &+ \int_\Omega \operatorname{div}(\boldsymbol{v}^N \otimes \boldsymbol{v}^N) \cdot \boldsymbol{v}^N \,\mathrm{d}\boldsymbol{x} - \frac{1}{2} \int_\Omega (\operatorname{div} \boldsymbol{v}^N) |\boldsymbol{v}^N|^2 \,\mathrm{d}\boldsymbol{x} \\
&+ \int_\Omega \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) : \mathbf{D}(\boldsymbol{v}^N) \,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{b}(\boldsymbol{v}^N), \boldsymbol{v}^N \rangle = \langle \boldsymbol{f}, \boldsymbol{v}^N \rangle .
\end{aligned}
$$

Using Green's theorem, we observe that

$$
(3.5) \qquad \int_\Omega \operatorname{div}(\boldsymbol{v}^N \otimes \boldsymbol{v}^N) \cdot \boldsymbol{v}^N \,\mathrm{d}\boldsymbol{x} - \frac{1}{2} \int_\Omega (\operatorname{div} \boldsymbol{v}^N) |\boldsymbol{v}^N|^2 \,\mathrm{d}\boldsymbol{x} = \frac{1}{2} \int_\Gamma (\boldsymbol{v}^N \cdot \boldsymbol{n}) |\boldsymbol{v}^N|^2 \,\mathrm{d}\boldsymbol{x}.
$$

Moreover, from (2.13) and (2.15) it follows that

$$
\frac{1}{2} \int_\Gamma (\boldsymbol{v}^N \cdot \boldsymbol{n}) |\boldsymbol{v}^N|^2 \,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{b}(\boldsymbol{v}^N), \boldsymbol{v}^N \rangle \geqslant \underline{\beta_2} \|\boldsymbol{v}^N\|_{\gamma_2, \Gamma_2}^{\gamma_2} - \beta_1 \|\boldsymbol{v}^N\|_{\gamma_1, \Gamma_1} - \beta_2,
$$

and thus

$$
\begin{aligned}
\varepsilon \|p^N\|_{1,2}^2 + \underline{\beta_2} \|\boldsymbol{v}^N\|_{\gamma_2, \Gamma_2}^{\gamma_2} &+ \int_\Omega \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) : \mathbf{D}(\boldsymbol{v}^N) \,\mathrm{d}\boldsymbol{x} \\
&\leqslant \|\boldsymbol{f}\|_{W^{1,r}_{\mathrm{b.c.}}(\Omega)^*} \|\boldsymbol{v}^N\|_{1,r} + \beta_1 \|\boldsymbol{v}^N\|_{\gamma_1, \Gamma_1} + \beta_2.
\end{aligned}
$$

Using (2.11), Korn's inequality, and the embedding $\mathbf{W}^{1,r}(\Omega) \hookrightarrow \mathbf{L}^{\gamma_1}(\Gamma_1)$ we finally arrive at

$$
(3.6) \qquad \varepsilon \|p^N\|_{1,2}^2 + \underline{\beta_2} \|\boldsymbol{v}^N\|_{\gamma_2, \Gamma_2}^{\gamma_2} + C_4 \min\{\|\mathbf{D}(\boldsymbol{v}^N)\|_r^2, \|\mathbf{D}(\boldsymbol{v}^N)\|_r^r\} \leqslant K.
$$

Here and in what follows, $C > 0$ and $K > 0$ stand for generic constants, independent of N and ε. In addition, $K \searrow 0$ whenever the problem data $\|\boldsymbol{f}\|_{W^{1,r}_{\mathrm{b.c.}}(\Omega)^*}$, β_1, and β_2 tend to zero (while the other data are fixed). From (3.6) it directly follows that

$$
(3.7) \qquad \|\boldsymbol{v}^N\|_{1,r} \leqslant K.
$$

Estimates (3.6) and (3.7) imply, with help of the Brouwer fixed point theorem, the solvability of (3.3). Using (2.5) we obtain the estimate

$$
\|\mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N))\|_{r'} \leqslant C.
$$

Due to this and the boundedness of b_2, there is a subsequence of $\{(\boldsymbol{v}^N, p^N)\}$ (denoted by the same symbol) and a pair $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$ such that

$$
(3.8) \qquad \begin{cases}
\boldsymbol{v}^N \rightharpoonup \boldsymbol{v}^\varepsilon & \text{weakly in } \mathbf{W}^{1,r}(\Omega) \text{ and in } \mathbf{L}^{\gamma_2}(\Gamma_2), \\
p^N \rightharpoonup p^\varepsilon & \text{weakly in } \mathrm{W}^{1,2}(\Omega), \\
\mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) \rightharpoonup \overline{\mathbf{S}^\varepsilon} & \text{weakly in } \mathrm{L}^{r'}(\Omega)^{d \times d}, \\
b_2(\boldsymbol{v}^N) \rightharpoonup \overline{b_2^\varepsilon} & \text{weakly in } \mathrm{L}^{\gamma_2'}(\Gamma_2).
\end{cases}
$$

Moreover, the compact embeddings yield

$$
(3.9)\qquad \begin{cases} p^N \to p^\varepsilon & \text{strongly in } \mathrm{L}^2(\Omega), \\ \boldsymbol{v}^N \to \boldsymbol{v}^\varepsilon & \text{strongly in } \mathbf{L}^s(\Omega) \text{ for all } s\colon 1 \leqslant s < \dfrac{rd}{d-r}, \\ \boldsymbol{v}^N \to \boldsymbol{v}^\varepsilon & \text{strongly in } \mathbf{L}^{\gamma_1}(\Gamma_1). \end{cases}
$$

The fact that $r > 3d/(d+2)$, $(3.8)_1$, and (3.9) are sufficient to show that

$$
\int_\Omega \operatorname{div}(\boldsymbol{v}^N \otimes \boldsymbol{v}^N)\cdot \boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_\Omega (\operatorname{div} \boldsymbol{v}^N)(\boldsymbol{v}^N\cdot\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} \longrightarrow \int_\Omega \operatorname{div}(\boldsymbol{v}^\varepsilon \otimes \boldsymbol{v}^\varepsilon)\cdot \boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_\Omega (\operatorname{div} \boldsymbol{v}^\varepsilon)(\boldsymbol{v}^\varepsilon\cdot\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x}
$$

for all $\boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega)$. Thus, we can pass to the limit in (3.3) and obtain (3.1) together with

$$
(3.10)\qquad \int_\Omega \operatorname{div}(\boldsymbol{v}^\varepsilon \otimes \boldsymbol{v}^\varepsilon)\cdot \boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_\Omega (\operatorname{div} \boldsymbol{v}^\varepsilon)(\boldsymbol{v}^\varepsilon\cdot\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} - \int_\Omega p^\varepsilon \operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \overline{\mathbf{S}^\varepsilon} : \mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{b}_1(\boldsymbol{v}^\varepsilon), \boldsymbol{\varphi}\rangle_{\Gamma_1} + \langle \overline{b_2^\varepsilon}, \boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_{\Gamma_2} = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle \quad \text{for all } \boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega).
$$

Next, from inequality (2.8) with $p^1 := p^N$ and $p^2 := p^\varepsilon$ (and analogously for $\boldsymbol{v}^1$, $\boldsymbol{v}^2$), (2.10) and $\|\mathbf{D}(\boldsymbol{v}^\varepsilon)\|_r \leqslant \liminf_{N\to\infty} \|\mathbf{D}(\boldsymbol{v}^N)\|_r \leqslant C$ it follows that

$$
(3.11)\qquad C\|\mathbf{D}(\boldsymbol{v}^N) - \mathbf{D}(v^\varepsilon)\|_r^2 \leqslant \int_\Omega [\mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) - \mathbf{S}(p^\varepsilon, \mathbf{D}(\boldsymbol{v}^\varepsilon))] : (\mathbf{D}(\boldsymbol{v}^N) - \mathbf{D}(\boldsymbol{v}^\varepsilon))\,\mathrm{d}\boldsymbol{x} + \frac{\gamma_0^2}{2C_1}\|p^N - p^\varepsilon\|_2^2.
$$

Similarly to [16], we prove the strong convergence of $\mathbf{D}(\boldsymbol{v}^N)$. Using (3.11), (2.16), and letting $N \to \infty$ we observe (due to (3.8)) that

$$
\limsup_{N\to\infty}(\|\mathbf{D}(\boldsymbol{v}^N) - \mathbf{D}(\boldsymbol{v}^\varepsilon)\|_r^2 + m(\|\boldsymbol{v}^N - \boldsymbol{v}^\varepsilon\|_{\gamma_2,\Gamma_2})) \leqslant \limsup_{N\to\infty}\Big(\int_\Omega \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) : \mathbf{D}(\boldsymbol{v}^N)\,\mathrm{d}\boldsymbol{x} + \langle b_2(\boldsymbol{v}^N\cdot\boldsymbol{n}), \boldsymbol{v}^N\cdot\boldsymbol{n}\rangle_{\Gamma_2}\Big) - \int_\Omega \overline{\mathbf{S}^\varepsilon} : \mathbf{D}(\boldsymbol{v}^\varepsilon)\,\mathrm{d}\boldsymbol{x} - \langle \overline{b_2^\varepsilon}, \boldsymbol{v}^\varepsilon\cdot\boldsymbol{n}\rangle_{\Gamma_2}.
$$

This can be further estimated from above, with help of (3.4), (3.9), $\liminf_{N\to\infty} \|p^N\|_{1,2} \geqslant \|p^\varepsilon\|_{1,2}$, (3.1), and (3.10), by

$$\langle \boldsymbol{f}, \boldsymbol{v}^\varepsilon\rangle - \langle \boldsymbol{b}_1(\boldsymbol{v}^\varepsilon), \boldsymbol{v}^\varepsilon\rangle_{\Gamma_1} - \varepsilon\|p^\varepsilon\|_{1,2}^2 - \int_\Omega \operatorname{div}(\boldsymbol{v}^\varepsilon \otimes \boldsymbol{v}^\varepsilon)\cdot \boldsymbol{v}^\varepsilon \, \mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_\Omega (\operatorname{div} \boldsymbol{v}^\varepsilon)|\boldsymbol{v}^\varepsilon|^2 \, \mathrm{d}\boldsymbol{x} - \int_\Omega \overline{\mathbf{S}^\varepsilon} : \mathbf{D}(\boldsymbol{v}^\varepsilon)\, \mathrm{d}\boldsymbol{x} - \left\langle \overline{b_2^\varepsilon}, \boldsymbol{v}^\varepsilon \cdot \boldsymbol{n}\right\rangle_{\Gamma_2} = 0.$$

Therefore, and due to $(3.9)_1$, we have the almost everywhere convergence

$$\mathbf{D}(\boldsymbol{v}^N) \to \mathbf{D}(\boldsymbol{v}^\varepsilon) \text{ a.e. in } \Omega, \quad \boldsymbol{v}^N \to \boldsymbol{v}^\varepsilon \text{ a.e. on } \Gamma_2 \quad \text{and} \quad p^N \to p^\varepsilon \text{ a.e. in } \Omega.$$

Vitali's theorem and the continuity (2.14) of $b_2(\cdot)$ allow us to identify the limits as

$$\int_\Omega \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) : \mathbf{D}(\boldsymbol{\varphi})\, \mathrm{d}\boldsymbol{x} \to \int_\Omega \mathbf{S}(p^\varepsilon, \mathbf{D}(\boldsymbol{v}^\varepsilon)) : \mathbf{D}(\boldsymbol{\varphi})\, \mathrm{d}\boldsymbol{x} = \int_\Omega \overline{\mathbf{S}^\varepsilon} : \mathbf{D}(\boldsymbol{\varphi})\, \mathrm{d}\boldsymbol{x},$$
$$\left\langle b_2(\boldsymbol{v}^N \cdot \boldsymbol{n}), \boldsymbol{\varphi}\cdot\boldsymbol{n}\right\rangle_{\Gamma_2} \to \langle b_2(\boldsymbol{v}^\varepsilon \cdot \boldsymbol{n}), \boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_{\Gamma_2} = \left\langle \overline{b_2^\varepsilon}, \boldsymbol{\varphi}\cdot\boldsymbol{n}\right\rangle_{\Gamma_2}$$

for every $\boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\text{b.c.}}(\Omega)$.

3.2. Uniform estimates for the pressure p^ε and the weak convergence

For any pair $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$ which solves (3.1) and (3.2) we can obtain the same energy estimates as in 3.1:

$$\varepsilon\|p^\varepsilon\|_{1,2}^2 + \|\boldsymbol{v}^\varepsilon\|_{\gamma_2,\Gamma_2}^{\gamma_2} + \|\boldsymbol{v}^\varepsilon\|_{1,r} \leqslant K \quad \text{and} \quad \|\mathbf{S}(p^\varepsilon, \mathbf{D}(\boldsymbol{v}^\varepsilon))\|_{r'} \leqslant C. \tag{3.12}$$

Let us recall Lemma 2.4 and test (3.2) with $\boldsymbol{\varphi}^\varepsilon := \tilde{\mathcal{B}}(|p^\varepsilon|^{r'-2}p^\varepsilon)$. Note that $\|\boldsymbol{\varphi}^\varepsilon\|_{1,r} \leqslant \tilde{C}_{\text{div}}(\Omega, \Gamma_1, \Gamma_2, r)\|p^\varepsilon\|_{r'}^{r'/r}$ and $\|\boldsymbol{\varphi}^\varepsilon\|_{\gamma_2,\Gamma_2} \leqslant C'_{\text{div}}(\Omega, \Gamma_1, \Gamma_2\gamma_2)\|p^\varepsilon\|_{r'/r}^{r'/r}$. Then, using (2.5), Hölder's inequality, (2.12), (2.14), the embedding $\mathbf{W}^{1,r}(\Omega) \hookrightarrow \mathbf{L}^{\gamma_1}(\Gamma_1)$, and at last the estimate (3.12), we get

$$\begin{aligned}
\|p^\varepsilon\|_{r'}^{r'} &= \int_\Omega \operatorname{div}(\boldsymbol{v}^\varepsilon \otimes \boldsymbol{v}^\varepsilon)\cdot \boldsymbol{\varphi}^\varepsilon \, \mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_\Omega (\operatorname{div} \boldsymbol{v}^\varepsilon)(\boldsymbol{v}^\varepsilon\cdot\boldsymbol{\varphi}^\varepsilon)\, \mathrm{d}\boldsymbol{x} \\
&\quad + \int_\Omega \mathbf{S}(p^\varepsilon, \mathbf{D}(\boldsymbol{v}^\varepsilon)) : \mathbf{D}(\boldsymbol{\varphi}^\varepsilon)\, \mathrm{d}\boldsymbol{x} + \langle \boldsymbol{b}(\boldsymbol{v}^\varepsilon), \boldsymbol{\varphi}^\varepsilon\rangle - \langle \boldsymbol{f}, \boldsymbol{\varphi}^\varepsilon\rangle \\
&\leqslant C\|\boldsymbol{v}^\varepsilon\|_{1,r}^2\|\boldsymbol{\varphi}^\varepsilon\|_{1,r} + \frac{C_2}{r-1}\|1 + |\mathbf{D}(\boldsymbol{v}^\varepsilon)|\|_r^{r-1}\|\boldsymbol{\varphi}^\varepsilon\|_{1,r} + \|\boldsymbol{f}\|_{W^{1,r}_{\text{b.c.}}(\Omega)^*}\|\boldsymbol{\varphi}^\varepsilon\|_{1,r} \\
&\quad + C\|\boldsymbol{b}_1(\boldsymbol{v}^\varepsilon)\|_{\gamma_1',\Gamma_1}\|\boldsymbol{\varphi}^\varepsilon\|_{1,r} + \|b_2(\boldsymbol{v}^\varepsilon\cdot\boldsymbol{n})\|_{\gamma_2',\Gamma_2}\|\boldsymbol{\varphi}^\varepsilon\|_{\gamma_2,\Gamma_2} \\
&\leqslant C\|p^\varepsilon\|_{r'}^{r'/r}.
\end{aligned}$$

Since $r > 1$, this implies

$$\|p^\varepsilon\|_{r'} \leqslant C. \tag{3.13}$$

Again, we find a sequence $\varepsilon_n \searrow 0$ and a pair $(\boldsymbol{v}, p)$ such that

$$
\begin{cases}
\boldsymbol{v}^{\varepsilon_n} \rightharpoonup \boldsymbol{v} & \text{weakly in } \mathbf{W}^{1,r}(\Omega) \text{ and in } \mathbf{L}^{\gamma_2}(\Gamma_2), \\
p^{\varepsilon_n} \rightharpoonup p & \text{weakly in } \mathrm{L}^{r'}(\Omega), \\
\mathbf{S}(p^{\varepsilon_n}, \mathbf{D}(\boldsymbol{v}^{\varepsilon_n})) \rightharpoonup \overline{\mathbf{S}} & \text{weakly in } \mathrm{L}^{r'}(\Omega)^{d\times d}, \\
b_2(\boldsymbol{v}^{\varepsilon}) \rightharpoonup \overline{b_2} & \text{weakly in } \mathrm{L}^{\gamma_2'}(\Gamma_2), \\
\boldsymbol{v}^{\varepsilon_n} \to \boldsymbol{v} & \text{strongly in } \mathbf{L}^{\gamma_1}(\Gamma_1), \\
\boldsymbol{v}^{\varepsilon_n} \to \boldsymbol{v} & \text{strongly in } \mathbf{L}^{s}(\Omega) \text{ for all } s\colon 1 \leqslant s < \dfrac{dr}{d-r}.
\end{cases}
\tag{3.14}
$$

Clearly, due to (3.12), $\boldsymbol{v}$ satisfies (ii) of Theorem 2.5. Note that $(3.14)_1$ and (3.12) together with (3.1) yield

$$
\operatorname{div} \boldsymbol{v} = 0 \quad \text{a.e. in } \Omega. \tag{3.15}
$$

We can then pass to the limit in (3.2), obtaining

$$
\begin{aligned}
\int_\Omega \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \overline{\mathbf{S}} : \mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} - \int_\Omega p \operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} \\
+ \langle \boldsymbol{b}_1(\boldsymbol{v}), \boldsymbol{\varphi}\rangle_{\Gamma_1} + \left\langle \overline{b_2}, \boldsymbol{\varphi}\cdot\boldsymbol{n}\right\rangle_{\Gamma_2} = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle \quad \text{for all } \boldsymbol{\varphi}\in \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega).
\end{aligned}
\tag{3.16}
$$

Finally, we use Vitali's theorem and the continuity of $b_2(\cdot)$ again, to show that

$$
\int_\Omega \mathbf{S}(p^{\varepsilon_n}, \mathbf{D}(\boldsymbol{v}^{\varepsilon_n})) : \mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} \to \int_\Omega \mathbf{S}(p, \mathbf{D}(\boldsymbol{v})) : \mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} = \int \overline{\mathbf{S}} : \mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x},
$$
$$
\langle b_2(\boldsymbol{v}^{\varepsilon}\cdot\boldsymbol{n}), \boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_{\Gamma_2} \to \langle b_2(\boldsymbol{v}\cdot\boldsymbol{n}), \boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_{\Gamma_2} = \left\langle \overline{b_2}, \boldsymbol{\varphi}\cdot\boldsymbol{n}\right\rangle_{\Gamma_2}
$$

for all $\boldsymbol{\varphi}\in \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega)$. In order to do so, we prove the convergences

$$
\begin{aligned}
\mathbf{D}(\boldsymbol{v}^{\varepsilon_n}) \to \mathbf{D}(\boldsymbol{v}) \text{ a.e. in } \Omega, \quad \boldsymbol{v}^{\varepsilon_n} \to \boldsymbol{v} \text{ a.e. on } \Gamma_2, \\
\text{and} \quad p^{\varepsilon_n} \to p \text{ a.e. in } \Omega,
\end{aligned}
\tag{3.17}
$$

in the next subsection.

3.3. The almost everywhere convergence

Let us rewrite inequality (2.8) in the form

$$
Y^n := \int_\Omega\int_0^1 (1 + |\mathbf{D}(\boldsymbol{v}^{\varepsilon_n}) + s(\mathbf{D}(\boldsymbol{v}) - \mathbf{D}(\boldsymbol{v}^{\varepsilon_n}))|^2)^{(r-2)/2} |\mathbf{D}(\boldsymbol{v}^{\varepsilon_n}) - \mathbf{D}(\boldsymbol{v})|^2\,\mathrm{d}s\,\mathrm{d}\boldsymbol{x},
$$
$$
\frac{C_1}{2} Y^n \leqslant \int_\Omega [\mathbf{S}(p^{\varepsilon_n}, \mathbf{D}(\boldsymbol{v}^{\varepsilon_n})) - \mathbf{S}(p, \mathbf{D}(\boldsymbol{v}))] : (\mathbf{D}(\boldsymbol{v}^{\varepsilon_n} - \mathbf{D}(\boldsymbol{v}))\,\mathrm{d}\boldsymbol{x} + \frac{\gamma_0^2}{2C_1}\|p^{\varepsilon_n} - p\|_2^2.
$$

Taking $\boldsymbol{\varphi} := \boldsymbol{v}^{\varepsilon_n} - \boldsymbol{v}$ in (3.2), $\xi := p^{\varepsilon_n}$ in (3.1), using (3.14), (3.15), and taking $\boldsymbol{\varphi} := \boldsymbol{v}$ in (3.16), we observe that

$$\begin{aligned}
\limsup_{\varepsilon_n \searrow 0} \Big(\int_\Omega & [\mathbf{S}(p^{\varepsilon_n}, \mathbf{D}(\boldsymbol{v}^{\varepsilon_n})) - \mathbf{S}(p, \mathbf{D}(\boldsymbol{v}))] : (\mathbf{D}(\boldsymbol{v}^{\varepsilon_n}) - \mathbf{D}(\boldsymbol{v}))\,\mathrm{d}\boldsymbol{x} \\
&\qquad + \langle b_2(\boldsymbol{v}^{\varepsilon_n} \cdot \boldsymbol{n}) - b_2(\boldsymbol{v} \cdot \boldsymbol{n}), (\boldsymbol{v}^{\varepsilon_n} - \boldsymbol{v}) \cdot \boldsymbol{n} \rangle_{\Gamma_2} \Big) \\
&= \limsup_{\varepsilon_n \searrow 0} \Big(\int_\Omega \mathbf{S}(p^{\varepsilon_n}, \mathbf{D}(\boldsymbol{v}^{\varepsilon_n})) : \mathbf{D}(\boldsymbol{v}^{\varepsilon_n})\,\mathrm{d}\boldsymbol{x} + \langle b_2(\boldsymbol{v}^{\varepsilon_n} \cdot \boldsymbol{n}), \boldsymbol{v}^{\varepsilon_n} \cdot \boldsymbol{n} \rangle_{\Gamma_2} \Big) \\
&\qquad - \int_\Omega \overline{\mathbf{S}} : \mathbf{D}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} - \langle \overline{b_2}, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\Gamma_2} \leqslant 0,
\end{aligned}$$

which together with (2.16) yields (denoting by $o(1)$ a sequence vanishing as $\varepsilon_n \searrow 0$)

$$m(\|\boldsymbol{v}^{\varepsilon_n} - \boldsymbol{v}\|_{\gamma_2, \Gamma_2}) + \frac{C_1}{2} Y^n \leqslant \frac{\gamma_0^2}{2C_1} \|p^{\varepsilon_n} - p\|_2^2 + o(1). \tag{3.18}$$

Next, we set $\boldsymbol{\varphi}^n := \tilde{\mathcal{B}}(p^{\varepsilon_n} - p)$, $\|\boldsymbol{\varphi}^n\|_{1,2} \leqslant \tilde{C}_{\mathrm{div}}(\Omega, \Gamma_1, \Gamma_2, 2)\|p^{\varepsilon_n} - p\|_2$. Note that since $(p^{\varepsilon_n} - p) \rightharpoonup 0$ weakly in $\mathrm{L}^{r'}(\Omega)$, it follows that $\boldsymbol{\varphi}^n \rightharpoonup 0$ weakly in $\mathbf{W}^{1,r}(\Omega)$ and $\boldsymbol{\varphi}^n \to 0$ strongly in $\mathbf{L}^{\gamma_i}(\Gamma_i)$, $i = 1, 2$. Testing (3.2) with $\boldsymbol{\varphi}^n$, we obtain

$$\begin{aligned}
\int_\Omega p^{\varepsilon_n}(p^{\varepsilon_n} - p)\,\mathrm{d}\boldsymbol{x} &= \int_\Omega \operatorname{div}(\boldsymbol{v}^{\varepsilon_n} \otimes \boldsymbol{v}^{\varepsilon_n}) \cdot \boldsymbol{\varphi}^n\,\mathrm{d}\boldsymbol{x} - \frac{1}{2} \int_\Omega (\operatorname{div} \boldsymbol{v}^{\varepsilon_n})(\boldsymbol{v}^{\varepsilon_n} \cdot \boldsymbol{\varphi}^n)\,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_\Omega \mathbf{S}(p^{\varepsilon_n}, \mathbf{D}(\boldsymbol{v}^{\varepsilon_n})) : \mathbf{D}(\boldsymbol{\varphi}^n)\,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{b}(\boldsymbol{v}^{\varepsilon_n}), \boldsymbol{\varphi}^n \rangle - \langle \boldsymbol{f}, \boldsymbol{\varphi}^n \rangle,
\end{aligned}$$

from which it follows that

$$\|p^{\varepsilon_n} - p\|_2^2 = \int_\Omega [\mathbf{S}(p^{\varepsilon_n}, \mathbf{D}(\boldsymbol{v}^{\varepsilon_n})) - \mathbf{S}(p, \mathbf{D}(\boldsymbol{v}))] : \mathbf{D}(\boldsymbol{\varphi}^n)\,\mathrm{d}\boldsymbol{x} + o(1).$$

This implies, by virtue of (2.9), (3.14), and (3.18), that

$$\begin{aligned}
\|p^{\varepsilon_n} - p\|_2^2 &\leqslant C_2 \sqrt{Y^n} \|\mathbf{D}(\boldsymbol{\varphi}^n)\|_2 + \gamma_0 \|p^{\varepsilon_n} - p\|_2 \|\mathbf{D}(\boldsymbol{\varphi}^n)\|_2 + o(1) \\
&\leqslant \gamma_0 \tilde{C}_{\mathrm{div}}(\Omega, \Gamma_1, \Gamma_2, 2) \Big(1 + \frac{C_2}{C_1}\Big) \|p^{\varepsilon_n} - p\|_2^2 + o(1)\|p^{\varepsilon_n} - p\|_2 + o(1),
\end{aligned}$$

which leads to

$$\Big(1 - \gamma_0 \tilde{C}_{\mathrm{div}}(\Omega, \Gamma_1, \Gamma_2, 2) \Big(1 + \frac{C_2}{C_1}\Big)\Big) \|p^{\varepsilon_n} - p\|_2^2 \leqslant o(1)\|p^{\varepsilon_n} - p\|_2 + o(1).$$

Due to the assumption $(2.22)_2$, (3.18), and (2.10), we finally observe that

$$\|p^{\varepsilon_n} - p\|_2 \to 0, \quad \|\mathbf{D}(\boldsymbol{v}^{\varepsilon_n}) - \mathbf{D}(\boldsymbol{v})\|_r \to 0, \quad \text{and} \quad \|\boldsymbol{v}^{\varepsilon_n} - \boldsymbol{v}\|_{\gamma_2, \Gamma_2} \to 0,$$

which implies (3.17) and completes the proof of (i) of Theorem 2.5.

4. Uniqueness considerations

Take two possible weak solutions $(\boldsymbol{v}^1, p^1)$, $(\boldsymbol{v}^2, p^2)$. Subtracting (2.19) and denoting $\mathbf{S}^i := \mathbf{S}(p^i, \mathbf{D}(\boldsymbol{v}^i))$, $i = 1, 2$, we obtain (for every $\boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega)$)

$$\tag{4.1} \begin{aligned} \int_\Omega (\mathbf{S}^1 - \mathbf{S}^2) : \mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} = {}& \int_\Omega (p^1 - p^2) \operatorname{div} \boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \big\langle \boldsymbol{b}(\boldsymbol{v}^1) - \boldsymbol{b}(\boldsymbol{v}^2), \boldsymbol{\varphi} \big\rangle \\ & - \int_\Omega \operatorname{div}(\boldsymbol{v}^1 \otimes \boldsymbol{v}^1 - \boldsymbol{v}^2 \otimes \boldsymbol{v}^2) \cdot \boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Setting $\boldsymbol{\varphi} := \boldsymbol{v}^1 - \boldsymbol{v}^2$, we get (as $\operatorname{div} \boldsymbol{v}^i = 0$, $i = 1, 2$)

$$\tag{4.2} \begin{aligned} \int_\Omega (\mathbf{S}^1 - \mathbf{S}^2) : \mathbf{D}(\boldsymbol{v}^1 - \boldsymbol{v}^2)\,\mathrm{d}\boldsymbol{x} = {}& - \big\langle \boldsymbol{b}(\boldsymbol{v}^1) - \boldsymbol{b}(\boldsymbol{v}^2), \boldsymbol{v}^1 - \boldsymbol{v}^2 \big\rangle \\ & - \int_\Omega \operatorname{div}(\boldsymbol{v}^1 \otimes \boldsymbol{v}^1 - \boldsymbol{v}^2 \otimes \boldsymbol{v}^2) \cdot (\boldsymbol{v}^1 - \boldsymbol{v}^2)\,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Let us assume that (2.23) holds with $C_I K \leqslant K_1$, where C_I comes from the embedding inequality $\|\boldsymbol{u}\|_{\gamma_1, \Gamma_1} \leqslant C_I \|\boldsymbol{u}\|_{1,r}$. Then the right-hand side of (4.2) can be estimated using the embeddings $\mathbf{W}^{1,r}(\Omega) \hookrightarrow \mathbf{L}^{2r'}(\Omega)$, $\mathbf{W}^{1,r}(\Omega) \hookrightarrow \mathbf{L}^{\gamma_1}(\Gamma_1)$, (2.17), and the monotonicity of b_2, as follows:

$$\tag{4.3a} \left| \int_\Omega \operatorname{div}(\boldsymbol{v}^1 \otimes \boldsymbol{v}^1 - \boldsymbol{v}^2 \otimes \boldsymbol{v}^2) \cdot (\boldsymbol{v}^1 - \boldsymbol{v}^2)\,\mathrm{d}\boldsymbol{x} \right| \leqslant CK \|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{1,r}^2,$$

$$\tag{4.3b} - \big\langle \boldsymbol{b}(\boldsymbol{v}^1) - \boldsymbol{b}(\boldsymbol{v}^2), \boldsymbol{v}^1 - \boldsymbol{v}^2 \big\rangle \leqslant C\lambda_1 \|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{1,r}^2.$$

Again, in what follows, $C, K > 0$ stand for generic constants determined by the problem data. Here and later in this section, C is independent of $\boldsymbol{f}$, β_1, and β_2, i.e. it is not correlated to K. Applying this back to (4.2) and using (2.8), we thus obtain

$$\tag{4.4} \frac{C_1}{2} \int_\Omega I^{1,2}\,\mathrm{d}\boldsymbol{x} \leqslant \frac{\gamma_0^2}{2C_1} \|p^1 - p^2\|_2^2 + C(K + \lambda_1) \|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{1,r}^2.$$

This together with (2.10), Korn's and Friedrichs' inequalities yields that for λ_1 and K small enough

$$\tag{4.5} \|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{1,r} \leqslant C \|p^1 - p^2\|_2.$$

Next, using (2.9) and Hölder's inequality, we obtain for any $\boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega)$

$$\tag{4.6} \begin{aligned} & \left| \int_\Omega (\mathbf{S}^1 - \mathbf{S}^2) : \mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} \right| \\ & \quad \leqslant C_2 \Big(\int_\Omega I^{1,2}\,\mathrm{d}\boldsymbol{x} \Big)^{1/2} \|\mathbf{D}(\boldsymbol{\varphi})\|_2 + \gamma_0 \|p^1 - p^2\|_2 \|\mathbf{D}(\boldsymbol{\varphi})\|_2 \\ & \quad \overset{(4.4)}{\leqslant} \Big(\gamma_0 \Big(1 + \frac{C_2}{C_1} \Big) + C\sqrt{K + \lambda_1} \Big) \|p^1 - p^2\|_2 \|\mathbf{D}(\boldsymbol{\varphi})\|_2. \end{aligned}$$

Let us set $\boldsymbol{\varphi} := \tilde{\mathcal{B}}(p^1 - p^2)$ in (4.1). Note that $\|\boldsymbol{\varphi}\|_{1,2} \leqslant \tilde{C}_{\rm div}(\Omega, \Gamma_1, \Gamma_2, 2)\|p^1 - p^2\|_2$ and also that $\|\boldsymbol{\varphi}\|_{\gamma_1,\Gamma_1}, \|\boldsymbol{\varphi}\|_{\infty,\Gamma_2} \leqslant C\|p^1 - p^2\|_2$. We arrive at

$$\int_\Omega (\mathbf{S}^1 - \mathbf{S}^2) : \mathbf{D}(\boldsymbol{\varphi})\,{\rm d}\boldsymbol{x} = \|p^1 - p^2\|_2^2 - \big\langle \boldsymbol{b}(\boldsymbol{v}^1) - \boldsymbol{b}(\boldsymbol{v}^2), \boldsymbol{\varphi}\big\rangle \\ - \int_\Omega {\rm div}(\boldsymbol{v}^1 \otimes \boldsymbol{v}^1 - \boldsymbol{v}^2 \otimes \boldsymbol{v}^2) \cdot \boldsymbol{\varphi}\,{\rm d}\boldsymbol{x},$$

which in combination with (4.6) gives

$$\text{(4.7)}\quad \|p^1 - p^2\|_2^2 \leqslant \Big(\gamma_0\Big(1 + \frac{C_2}{C_1}\Big) + C\sqrt{K + \lambda_1}\Big)\|p^1 - p^2\|_2\|\mathbf{D}(\boldsymbol{\varphi})\|_2 \\ + \big\langle \boldsymbol{b}(\boldsymbol{v}^1) - \boldsymbol{b}(\boldsymbol{v}^2), \boldsymbol{\varphi}\big\rangle + \int_\Omega {\rm div}(\boldsymbol{v}^1 \otimes \boldsymbol{v}^1 - \boldsymbol{v}^2 \otimes \boldsymbol{v}^2) \cdot \boldsymbol{\varphi}\,{\rm d}\boldsymbol{x}.$$

From (2.18), $(2.21)_3$, the embedding and (4.5) it follows that

$$\text{(4.8)}\quad \big\langle b_2(\boldsymbol{v}^1 \cdot \boldsymbol{n}) - b_2(\boldsymbol{v}^2 \cdot \boldsymbol{n}), \boldsymbol{\varphi} \cdot \boldsymbol{n}\big\rangle \leqslant C\lambda_2\|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{1,r}\|p^1 - p^2\|_2 \\ \leqslant C\lambda_2\|p^1 - p^2\|_2^2,$$

provided that $C_I K \leqslant K_2$, with C_I from $\|\boldsymbol{u}\|_{r^*,\Gamma_2} \leqslant C_I\|\boldsymbol{u}\|_{1,r}$. Applying the same technique as in (4.3), namely the embeddings and (2.17), then using (4.5) and (4.8), we can collectively estimate the boundary and the convective term on the right-hand side of (4.7) by the expression $C(\lambda_1 + \lambda_2 + K)\|p^1 - p^2\|_2^2$ and obtain

$$\text{(4.9)}\quad \Big(1 - \tilde{C}_{\rm div}(\Omega, \Gamma_1, \Gamma_2, 2)\Big(\gamma_0\Big(1 + \frac{C_2}{C_1}\Big)\Big) \\ - C(\sqrt{K + \lambda_1} + \lambda_1 + \lambda_2 + K)\Big) \cdot \|p^1 - p^2\|_2^2 \leqslant 0.$$

Due to $(2.22)_2$, for λ_1, λ_2 and K small enough the coefficient on the left-hand side is positive and thus $(\boldsymbol{v}^1, p^1) = (\boldsymbol{v}^2, p^2)$.

R e m a r k 4.1 (pressure is fixed by velocity). Let $(\boldsymbol{v}, p^1)$ and $(\boldsymbol{v}, p^2)$ be weak solutions to (P). Then, under the assumptions of Theorem 2.5, $p^1 = p^2$.

P r o o f. From (2.9) we observe that

$$\Big|\int_\Omega (\mathbf{S}^1 - \mathbf{S}^2) : \mathbf{D}(\boldsymbol{\varphi})\,{\rm d}\boldsymbol{x}\Big| \leqslant \gamma_0\|p^1 - p^2\|_2\|\mathbf{D}(\boldsymbol{\varphi})\|_2 \quad \text{for all } \boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\rm b.c.}(\Omega).$$

Then we subtract (2.19), take a test function $\boldsymbol{\varphi} := \tilde{\mathcal{B}}(p^1 - p^2)$ and obtain

$$\|p^1 - p^2\|_2^2 \leqslant \gamma_0\tilde{C}_{\rm div}(\Omega, \Gamma_1, \Gamma_2, 2)\|p^1 - p^2\|_2^2.$$

Since by assumption $\gamma_0\tilde{C}_{\rm div}(\Omega, \Gamma_1, \Gamma_2, 2) < 1$, we conclude that $p^1 = p^2$. □

R e m a r k 4.2. Note that the additional assumptions—namely the requirement of small data $\boldsymbol{f}$, β_1, β_2—stated in (iii) of Theorem 2.5, are due to the presence of the convective term and the nonlinear boundary terms, not due to the nonlinear viscosity.

Indeed, one can consider a Stokes-like system ($\mathrm{P_S}$)

$$-\operatorname{div}\mathbf{S} + \nabla p = \boldsymbol{f}, \quad \operatorname{div}\boldsymbol{v} = 0 \quad \text{in } \Omega$$

and the boundary terms

$$\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{x}) \quad \text{on } \Gamma.$$

The readers can verify themselves that the weak solution to ($\mathrm{P_S}$) exists and is unique even for large data.

5. Boundary conditions in applications

Although the assumptions (B1)–(B7) seem to be motivated mainly by PDE analysis, they cover important engineering applications; we mention three types of them in the sequel.

Artificial boundary. In numerical simulations, large or even unbounded domains arising from the physical model must be truncated and the boundary condition for artificial boundaries has to be provided. For example in [13], an application to the flow through a cascade of profiles with the outflow condition

$$-\mathbf{T}\boldsymbol{n} = \boldsymbol{h}(\boldsymbol{x}) + \frac{1}{2}(\boldsymbol{v}\cdot\boldsymbol{n})^-\boldsymbol{v} \tag{5.1}$$

is considered (see also Section 1). In [6], several b.c. including (5.1) were proposed (for unsteady incompressible Navier-Stokes equations) in order to perform long-time simulations at high Reynolds numbers. See also [4], [5], [7].

Note that $\boldsymbol{b}_1$ given by (5.1) meets (B1), (B2) with $\gamma_1 = 3$ and $\beta_1 = \|\boldsymbol{h}\|_{3/2,\Gamma_1}$. Note also that $\|\boldsymbol{b}_1(\boldsymbol{v}^1) - \boldsymbol{b}_1(\boldsymbol{v}^2)\|_{3/2,\Gamma_1} \leqslant \frac{1}{2}\|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{3,\Gamma_1}(\|\boldsymbol{v}^1\|_{3,\Gamma_1} + \|\boldsymbol{v}^2\|_{3,\Gamma_1})$ allows to establish (B6) with any $\lambda_1 > 0$, provided $K_1 > 0$ is chosen sufficiently small.

Conditions involving Bernoulli's pressure. In some applications, the quantity $p + \frac{1}{2}|\boldsymbol{v}|^2$, referred to as the *total pressure* or the *Bernoulli pressure*, is used for prescribing the inflow/outflow boundary conditions on artificial boundaries (see e.g. [12], [14], [20], [33]). Note that this class of conditions

$$\Big(p + \frac{1}{2}|\boldsymbol{v}|^2\Big)\boldsymbol{n} - \mathbf{S}\boldsymbol{n} = \boldsymbol{h}(\boldsymbol{x}) \tag{5.2}$$

is covered by our theory. Similarly to (5.1), $\boldsymbol{b}_1$ given by (5.2) satisfies (B1), (B2) with $\gamma_1 = 3$ and $\beta_1 = \|\boldsymbol{h}\|_{3/2,\Gamma_1}$, and (B6) with any $\lambda_1 > 0$, provided that $K_1 > 0$ is sufficiently small.

However, it is questionable whether the *total pressure* is generally applicable, when seeking after proper boundary conditions for viscous flows. The authors of [20] note: "The total pressure is constant along streamlines in Euler flow and therefore is an important quantity in some high-Reynolds-number situations", but later they correctly point out that these conditions[3] "... are not satisfied by Poiseuille flow. Thus their poor performance is to be expected." In other words, we do not recommend (5.2) as a suitable outflow condition for artificial boundaries. At the same time, this emphasizes that (5.1) is intended to be used for outflow—not inflow—boundaries.

Porous wall. Boundary conditions of the type (1.3) are applicable to the flows where an inflow/outflow is possible through a porous wall (*filtration* boundary conditions). In most studies, for the flow through an isotropic porous medium the linear law of Darcy

$$-\nabla p = \frac{\mu}{k}\,\boldsymbol{v}$$

is considered (with k the permeability of the medium, $\boldsymbol{v}$ the volumetric velocity, μ the viscosity and p the pressure; body forces such as gravity are neglected here). As an analogy, when studying the flow where a part of the boundary is a thin porous wall (or membrane), one can prescribe the condition

$$-\mathbf{T}\boldsymbol{n}\cdot\boldsymbol{n} = p_{\text{out}} + c_1\boldsymbol{v}\cdot\boldsymbol{n} \quad \text{with } c_1 \geqslant 0 \tag{5.3}$$

for the normal part of the velocity, see e.g. [34]. However, Darcy's law is valid only for slow flows. It can be in fact derived from the Stokes equation, i.e. neglecting the inertia of the fluid, see e.g. [31]. For higher Reynolds numbers, the experimental observations "did not allow to find a universally accepted formula" [31]. Nevertheless, the relation

$$-\nabla p = \frac{\mu}{k}\,\boldsymbol{v} + d_2|\boldsymbol{v}|\boldsymbol{v} + d_3|\boldsymbol{v}|^2\boldsymbol{v}, \quad \text{with } d_2, d_3 > 0, \tag{5.4}$$

was proposed more than a century ago in [15]. Here, the last two terms were added to make the equation fit the experimental results. Formula (5.4) with $d_3 = 0$ is well established as the Forchheimer equation; see e.g. [2] for a survey of both experimental and theoretical results prior to 1972, or [19], [31] for more recent references. The authors are not aware of any reference concerning the porous wall boundary condition

[3] considering the intuitive setting of $\boldsymbol{h}(\boldsymbol{x})$ constant across the channel, analogously to (1.2)

which would involve both the high velocity effects and the non-Newtonian fluids with pressure and/or shear rate dependent viscosities.

As an analogy of (5.4), the boundary condition of the type

$$(5.5)\qquad -\mathbf{T}\boldsymbol{n}\cdot\boldsymbol{n} = p_{\text{out}} + (c_1 + c_2|\boldsymbol{v}\cdot\boldsymbol{n}| + c_3|\boldsymbol{v}\cdot\boldsymbol{n}|^2)\boldsymbol{v}\cdot\boldsymbol{n} \quad \text{with } c_1, c_2, c_3 \geqslant 0$$

seems to correspond to the physics better than (5.3). If $c_3 > 0$ then b_2 given by (5.5) meets (B3)–(B5) with $\gamma_2 = 4$ and e.g. with $\underline{\beta_2} = c_3/2$ and $\beta_2 = |\Gamma_2|(1/c_3)^3 + \|p_{\text{out}}\|_{4/3,\Gamma_2}^{4/3}(1/c_3)^{1/3}$. Considering (5.5) with $c_3 = 0$, one has to assume $c_2 > \frac{1}{2}$ and verify (B3)–(B5) e.g. by setting $\underline{\beta_2} = \frac{1}{2}(c_2 - \frac{1}{2})$ and $\beta_2 = (c_2 - \frac{1}{2})^{-1/2}\|p_{\text{out}}\|_{3/2,\Gamma_2}^{3/2}$. From Hölder's inequality we have

$$(5.6)\quad \|b_2(w) - b_2(z)\|_{1,\Gamma_2} \leqslant c_1|\Gamma_2|^{1/r^{*\prime}}\|w - z\|_{r^*} + c_2(\|w\|_{r^{*\prime}} + \|z\|_{r^{*\prime}})\|w - z\|_{r^*} + \frac{3}{2}c_3(\|w\|_{2r^{*\prime}}^2 + \|z\|_{2r^{*\prime}}^2)\|w - z\|_{r^*}.$$

Note that $2r^{*\prime} < r^*$, since $r > 3d/(d+2)$. Thus, (B7) can be achieved for any $\lambda_2 > c_1|\Gamma_2|^{(r^*-1)/r^*}$, choosing $K_2 > 0$ sufficiently small.

Concerning the boundary conditions given on the tangential part of the velocity on a porous wall, the no-slip condition $(2.4)_1$ is chosen here as one of several possible choices. It was preferred mainly in order to keep the ideas simple, even though from the physical point of view there is no particular preference over kinds of the slip condition. Nevertheless, the no-slip condition can be reasonable either as an approximation or in cases justified by the particular application, see for instance [17], [18], [34].

6. Conclusion

The class of fluids with pressure and shear rate dependent viscosities together with mixed boundary conditions involving the pressure was studied. Under certain assumptions, it was shown that a weak solution exists and that this weak solution is unique if the data are small. In contrast to previous studies, no constraint on the pressure mean value is present in the formulation of the problem. The proof follows the ideas of [16], except for the treatment of the inflow/outflow boundary conditions. Finally, a brief survey on these boundary conditions fitting to our theory is presented together with their physical application.

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