ON THE EXISTENCE OF MULTIPLE PERIODIC SOLUTIONS FOR THE VECTOR p-LAPLACIAN VIA CRITICAL POINT THEORY*

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Abstract. We study the vector p-Laplacian

$$(*)\qquad \begin{cases} -(|u'|^{p-2}u')' = \nabla F(t,u) \quad \text{a.e. } t\in[0,T], \\ u(0)=u(T), \quad u'(0)=u'(T), \quad 1<p<\infty. \end{cases}$$

We prove that there exists a sequence (u_n) of solutions of $(*)$ such that u_n is a critical point of φ and another sequence (u_n^*) of solutions of $(*)$ such that u_n^* is a local minimum point of φ, where φ is a functional defined below.

Keywords: p-Laplacian equation, periodic solution, critical point theory

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1. Introduction and main results

Consider the second order system

$$(1.1)\qquad \begin{cases} -(|u'|^{p-2}u')' = \nabla F(t,u) \quad \text{a.e. } t\in[0,T], \\ u(0)=u(T), \quad u'(0)=u'(T), \quad 1<p<\infty, \end{cases}$$

where $T>0$ and $F\colon [0,T]\times\mathbb{R}^N\to\mathbb{R}$ satisfies the following assumption:

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(A) $F(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0,T;\mathbb{R}^+)$ such that

$$|F(t,x)| \leqslant a(|x|)b(t), \quad |\nabla F(t,x)| \leqslant a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$; here $\mathbb{R}^+ = [0,\infty)$.

Recently some papers have appeared discussing scalar periodic problems driven by the one-dimensional p-Laplacian. We refer the reader to the works of Del Pino-Manasevich-Murua [3], Fabry-Fayyad [4], Gao [5] and Dang-Oppenheimer [6]. In all of these works the approach is degree theoretical and the existence of one solution is established. In [2], J. Mawhin generalized the Hartman-Knobloch results to perturbations of a vector p-Laplacian ordinary operator.

For $p = 2$, Mawhin-Willem [1] proved the existence of solutions for the problem (1.1) under the conditions

(M1) there exists $g \in L^1(0,T;\mathbb{R}^+)$ such that $|\nabla F(t,x)| \leqslant g(t)$ for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, and

(M2) $\int_0^T F(t,x)\,\mathrm{d}t \to +\infty$ as $|x| \to \infty$ or $\int_0^T F(t,x)\,\mathrm{d}t \to -\infty$ as $|x| \to \infty$.

In this paper we study the existence of periodic solutions of a vector p-Laplacian with potential oscillating around the first eigenvalue. Our arguments are based on a result by Habets-Manasevich-Zanolin [8] related to the two-point boundary value problem

$$u'' + u + f(t,u) = 0, \quad u(0) = u(1) = 0.$$

Our main result is the following theorem.

Theorem 1.1. *Suppose*

(H1) *there exists* $g \in L^q(0,T;\mathbb{R}^+)$ *(here* q *is the conjugate of* p*) such that* $|\nabla F(t,x)| \leqslant g(t)$ *for all* $x \in \mathbb{R}^N$ *and a.e.* $t \in [0,T]$,

(H2)

$$\limsup_{R\to+\infty} \inf_{a\in\mathbb{R}^N,\,|a|=R} \int_0^T F(s,a)\,\mathrm{d}s = +\infty \tag{1.2}$$

and

$$\liminf_{r\to+\infty} \sup_{b\in\mathbb{R}^N,\,|b|=r} \int_0^T F(s,b)\,\mathrm{d}s = -\infty \tag{1.3}$$

hold. Then

i) *there exists a sequence* (u_n) *of solutions of* (1.1) *such that* u_n *is a critical point of* φ *and* $\lim_{n\to\infty} \varphi(u_n) = +\infty$;

ii) *there exists a sequence* (u_n^*) *of solutions of* (1.1) *such that* u_n^* *is a local minimum point of* φ *and* $\lim\limits_{n\to\infty}\varphi(u_n^*)=-\infty$*; here* φ *is a functional defined below.*

Throughout the paper, for $N\geqslant 1$ and $I=[0,T]$, we will set $C=C(I,\mathbb{R}^N)$, $L^p=L^p(I,\mathbb{R}^N)$, $W^{1,p}=W^{1,p}(I,\mathbb{R}^N)$ and $X=W_T^{1,p}=\{u\in W^{1,p};\, u(0)=u(T)\}$. The norm in C will be defined by $\|u\|_\infty=\max\limits_{t\in[0,T]}|u(t)|$, the norm in L^p by

$$\|u\|_p=\left(\int_0^T|u(t)|^p\,\mathrm{d}t\right)^{1/p},$$

and the norm in X by $\|u\|_X=\|u\|_p+\|u'\|_p$. Note that for each $p>1$, X is compact embedded in C. Let $B_r=\{x\in\mathbb{R}^N:\ |x|\leqslant r\}$.

Each $u\in L^1$ can be written as $u(t)=\overline{u}+\tilde{u}(t)$ with

$$\overline{u}:=\frac{1}{T}\int_0^T u(t)\,\mathrm{d}t,\qquad \int_0^T\tilde{u}(t)\,\mathrm{d}t=0.$$

We will use the Sobolev inequality

$$\|\tilde{u}\|_\infty\leqslant T^{1/q}\|u'\|_p\quad\text{for each } u\in X \text{ (here } q \text{ is the conjugate of } p)$$

and Wirtinger's inequality

$$\|\tilde{u}\|_p^p\leqslant T^p\|\tilde{u}'\|_p^p\quad\text{for each } u\in X.$$

The proof of the theorem is given in Section 3. In Section 2 we present some preliminary results on the variational setting of p-Laplacian equations in X and the related Palais-Smale compactness.

2. Preliminaries

We first recall some facts about the eigenvalue problem for the p-Laplacian, see [9]. A $\lambda\in\mathbb{R}$ is said to be an "eigenvalue" of the p-Laplacian with periodic boundary conditions, if the problem

$$\begin{cases}(|u'|^{p-2}u')'+\lambda|u|^{p-2}u=0 & \text{a.e. on } [0,T],\\ u(0)-u(T)=u'(0)-u'(T)=0\end{cases}$$

has a nontrivial solution $u\in C^1(I,\mathbb{R}^N)$, known as the corresponding one to λ "eigenfunction". Let S denote the set of these eigenvalues. Evidently $0\in S$, each element of S is nonnegative and 0 is the smallest (first) eigenvalue. If $N = 1$ (scalar case), by direct integration of the equation we see that all the eigenvalues are

$$\lambda_n = \left(\frac{2\pi_p n}{T}\right)^p, \quad n = 0, 1, 2, 3, \ldots$$

where

$$\pi_p = 2(p-1)^{1/p} \int_0^1 (1 - t^p)^{-1/p} \, \mathrm{d}t.$$

In the case $N > 1$ (vector case), $\{\lambda_n\}_{n \geqslant 1} \subseteq S$ but S contains more elements.

It follows from assumption (A) that the functional φ on X given by

$$\varphi(u) = \frac{1}{p} \int_0^T |u'(t)|^p \, \mathrm{d}t - \int_0^T F(t, u(t)) \, \mathrm{d}t$$

is continuously differentiable and weakly lower semicontinuous on X (the proof is similar to the case $p = 2$, see [1]). Moreover, one has

$$\langle \varphi'(u), v \rangle = \int_0^T |u'(t)|^{p-2} (u'(t), v'(t)) \, \mathrm{d}t - \int_0^T (\nabla F(t, u(t)), v(t)) \, \mathrm{d}t$$

for all $u, v \in X$. It is easy to see that the solutions of problem (1.1) correspond to the critical points of φ.

Let us write

$$I(u) = \frac{1}{p} \int_0^T |u'|^p \, \mathrm{d}t, \quad G(u) = \int_0^T F(t, u(t)) \, \mathrm{d}t, \quad \forall \, u \in X.$$

Proposition 2.1. *The mapping $I' \colon X \to X^*$ is of type (S_+) (see [7]), i.e. any sequence $\{u_n\}$ in X satisfying $u_n \rightharpoonup u$ in X and*

$$\overline{\lim_{n \to \infty}} \, \langle I'(u_n), u_n - u \rangle \leqslant 0$$

contains a convergent subsequence.

Proof. Assume that $u_n \rightharpoonup u$ in X and $\overline{\lim\limits_{n \to \infty}} \, \langle I'(u_n), u_n - u \rangle \leqslant 0$. Then we get

$$\overline{\lim_{n \to \infty}} \, \langle I'(u_n) - I'(u), u_n - u \rangle \leqslant 0,$$

and together with the monotonicity property of I' this implies

$$\lim_{n \to \infty} \langle I'(u_n) - I'(u), u_n - u \rangle = 0,$$

i.e.,

$$\int_0^T \langle |u_n'|^{p-2}u_n' - |u'|^{p-2}u', u_n' - u'\rangle \,\mathrm{d}t \to 0 \quad \text{as } n\to\infty. \tag{2.1}$$

Recall that for all $x, y \in \mathbb{R}^N$ the following inequalities hold:

$$\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle \geqslant \Big(\frac{1}{2}\Big)^{p-1}|x-y|^p \quad \text{when } p\geqslant 2,$$

and

$$(|x|+|y|)^{2-p}\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle \geqslant (p-1)|x-y|^2 \quad \text{when } 1<p\leqslant 2.$$

Hence, by (2.1), one has that u_n' converges to u' in $[0,T]$ in measure. Thus there exists a subsequence (without loss of generality assume it to be the whole sequence) with

$$u_n'(t)\to u'(t) \quad \text{for a.e. } t\in[0,T], \quad n\to\infty,$$

and so

$$\frac{1}{p}|u_n'(t)-u'(t)|^p \to 0 \quad \text{for a.e. } t\in[0,T],\ \ n\to\infty. \tag{2.2}$$

One also has

$$\langle I'(u_n), u_n - u\rangle \to 0 \quad \text{as } n\to\infty.$$

Moreover,

$$\begin{aligned}\langle I'(u_n), u_n-u\rangle &= \int_0^T |u_n'|^p\,\mathrm{d}t - \int_0^T |u_n'|^{p-2}u_n'u'\,\mathrm{d}t\\ &\geqslant \int_0^T |u_n'|^p\,\mathrm{d}t - \int_0^T |u_n'|^{p-1}|u'|\,\mathrm{d}t\\ &\geqslant \int_0^T |u_n'|^p\,\mathrm{d}t - \int_0^T \Big[\frac{p-1}{p}|u_n'|^p + \frac{1}{p}|u'|^p\Big]\,\mathrm{d}t\\ &= \frac{1}{p}\int_0^T (|u_n'|^p - |u'|^p)\,\mathrm{d}t.\end{aligned}$$

Now the lower semi-continuity of I yields

$$\varliminf_{n\to\infty} \frac{1}{p}\int_0^T |u_n'|^p\,\mathrm{d}t \geqslant \frac{1}{p}\int_0^T |u'|^p\,\mathrm{d}t.$$

Thus

$$\frac{1}{p}\int_0^T |u_n'|^p\,\mathrm{d}t \to \frac{1}{p}\int_0^T |u'|^p\,\mathrm{d}t \quad \text{as } n\to\infty.$$

Consequently, the sequence $\left\{p^{-1}\int_0^t |u_n'|^p \,\mathrm{d}t\right\}$ is equi-absolutely continuous on $[0,T]$. From the inequality

$$\frac{1}{p}|u_n'(t)-u'(t)|^p \leqslant \frac{2^p}{p}(|u_n'|^p+|u'|^p)$$

we obtain that

$$\left\{\frac{1}{p}\int_0^t |u_n'(t)-u'(t)|^p \,\mathrm{d}t\right\} \text{ is equi-absolutely continuous on } [0,T]. \tag{2.3}$$

Now (2.2), (2.3) and the convergence theorem of Vitali (see [7], p. 1017) guarantee that

$$\int_0^T \frac{1}{p}|u_n'(t)-u'(t)|^p \,\mathrm{d}t \to 0 \quad \text{as } n\to\infty.$$

Thus $u_n' \to u'$ in L^p.

Also $u_n \rightharpoonup u$ in X, which implies that $u_n \to u$ in L^p. Hence $u_n \to u$ in X, and the proof is complete. □

Since the sum of a mapping of type (S_+) with a weakly-strongly continuous mapping is still a mapping of type (S_+), we obtain the following result.

Proposition 2.2. *$\varphi'\colon X \to X^*$ is a mapping of type (S_+).*

Proposition 2.3. *Suppose that if a sequence $\{u_n\}$ in X is such that*

$$\{\varphi(u_n)\} \text{ is bounded and } \varphi'(u_n)\to 0$$

as $n\to\infty$, then $\{u_n\}$ has a bounded subsequence. Then φ satisfies the PS-*condition.*

P r o o f. Assume that $\{u_n\}\subset X$, $\{\varphi(u_n)\}$ is bounded and $\varphi'(u_n)\to 0$. Now we know that $\{u_n\}$ contains a bounded subsequence and for simplicity we denote it again by $\{u_n\}$. Since X is reflexive we can extract a subsequence (again denoting it by $\{u_n\}$) such that $u_n \rightharpoonup u$ in X. Since $\varphi'(u_n)\to 0$, one has

$$\langle \varphi'(u_n), u_n-u\rangle \to 0 \quad \text{as } n\to\infty.$$

Since φ' is a mapping of type (S_+) we have that $u_n\to u$ in X. Hence φ satisfies the PS-condition. □

3. Proofs of theorems

We shall prove Theorem 1.1 from a sequence of claims.

Consider the direct sum decomposition $X=\mathbb{R}^N\oplus V$ with $V=\left\{v\in X\colon \int_0^T v(t)\,\mathrm{d}t=0\right\}$. So for all $u\in X$ we can write $u=\bar{u}+\tilde{u}$ with $\bar{u}\in\mathbb{R}^N$ and $\tilde{u}\in V$.

Claim 1. *φ is coercive on V.*

Proof. Combining hypotheses (H1) with the Mean Value Theorem, we see that for almost all $t \in I$ and all $x \in \mathbb{R}^N$, $|F(t,x)| \leqslant g(t)|x|$. If $\tilde{u} \in V$, then

$$\varphi(\tilde{u}) \geqslant \frac{1}{p}\|\tilde{u}'\|_p^p - \int_0^T g(t)|\tilde{u}(t)|\,\mathrm{d}t \geqslant \frac{1}{p}\|\tilde{u}'\|_p^p - \|g\|_q\|\tilde{u}\|_p \geqslant \frac{1}{p}\|\tilde{u}'\|_p^p - T\|g\|_q\|\tilde{u}'\|_p$$

by Wirtinger's inequality (here q is the conjugate of p). The above inequality and $\tilde{u} \in V$ imply that $\varphi(\tilde{u}) \to \infty$ as $\|\tilde{u}'\|_p \to \infty$. Now it follows that $\varphi(\tilde{u}) \to \infty$, as $\|\tilde{u}\|_X \to 0$. □

Claim 2. *There exists a sequence (R_n) such that*

$$\lim_{n\to\infty} R_n = +\infty, \quad \lim_{n\to\infty}\Big[\sup_{a\in\mathbb{R}^N,\,|a|=R_n} \varphi(a)\Big] = -\infty.$$

Proof. This follows from (1.2) since

$$\varphi(a) = -\int_0^T F(t,a)\,\mathrm{d}t.$$

□

Claim 3. *There exists a sequence (r_m) such that*

$$\lim_{m\to\infty} r_m = +\infty, \quad \text{and} \quad \lim_{m\to\infty}\Big[\inf_{b\in\mathbb{R}^N,\,|b|=r_m,\,\tilde{u}\in V} \varphi(b+\tilde{u})\Big] = +\infty.$$

Proof. For any $b \in \mathbb{R}^N$, $|b| = r_m$, and $\tilde{u} \in V$, let $u = b + \tilde{u}$. Note that

$$\begin{aligned}
\varphi(u) &= \frac{1}{p}\|u'\|_p^p - \int_0^T F(t,u)\,\mathrm{d}t \\
&= \frac{1}{p}\|\tilde{u}'\|_p^p - \int_0^T [F(t,u) - F(t,b)]\,\mathrm{d}t - \int_0^T F(t,b)\,\mathrm{d}t \\
&= \frac{1}{p}\|\tilde{u}'\|_p^p - \int_0^T\int_0^1 (\nabla F(t, b + s\tilde{u}(t)), \tilde{u}(t))\,\mathrm{d}s\,\mathrm{d}t - \int_0^T F(t,b)\,\mathrm{d}t \\
&\geqslant \frac{1}{p}\|\tilde{u}'\|_p^p - \Big(\int_0^T g(t)\,\mathrm{d}t\Big)\|\tilde{u}\|_\infty - \int_0^T F(t,b)\,\mathrm{d}t \\
&\geqslant \frac{1}{p}\|\tilde{u}'\|_p^p - C_1\|\tilde{u}'\|_p - \int_0^T F(t,b)\,\mathrm{d}t
\end{aligned}$$

by Sobolev's inequality; here $C_1 = \left(\int_0^T g(t)\,\mathrm{d}t\right)T^{1/q}$. Thus

$$\inf_{b\in\mathbb{R}^N,\,|b|=r_m,\,\tilde{u}\in V} \varphi(b+\tilde{u}) \geqslant \inf_{\tilde{u}\in V}\left(\frac{1}{p}\|\tilde{u}'\|_p^p - C_1\|\tilde{u}'\|_p\right) + \inf_{b\in\mathbb{R}^N,\,|b|=r_m}\left(-\int_0^T F(t,b)\,\mathrm{d}t\right).$$

On the other hand, there exists $\beta\in\mathbb{R}$ such that $\inf_{\tilde{u}\in V}(p^{-1}\|\tilde{u}'\|_p^p - C_1\|\tilde{u}'\|_p) \geqslant \beta$. The claim now follows from (1.3). $\square$

Consider now the set

$$S_n = \left\{\gamma\in C(B_{R_n},X), \gamma|_{\partial B_{R_n}} = i|_{\partial B_{R_n}}\right\}$$

and define

$$c_n = \inf_{\gamma\in S_n}\left[\max_{x\in B_{R_n}} \varphi(\gamma(x))\right]. \tag{3.1}$$

We prove that each γ intersects the hyperplane V. Let $\pi\colon X\to\mathbb{R}^N$ be the (continuous) projection of X onto $\mathbb{R}^N$, defined by

$$\pi(u) = \frac{1}{T}\int_0^T u\,\mathrm{d}t \quad \text{for } u\in X.$$

Let γ be any continuous map such that $\gamma|_{\partial B_{R_n}} = i|_{\partial B_{R_n}}$. We have to show that $0\in\pi(\gamma(B_{R_n}))$.

For $t\in[0,1]$, $u\in\mathbb{R}^N$ define

$$\gamma_t(u) = t\pi(\gamma(u)) + (1-t)u.$$

Note that $\gamma_t\in C^0(\mathbb{R}^N;\mathbb{R}^N)$ defines a homotopy of $\gamma_0 = \mathrm{id}$ with $\gamma_1 = \pi\circ\gamma$. Moreover, $\gamma_t|_{\partial B_{R_n}} = \mathrm{id}$ for all t. By homotopy invariance and normalization of the degree (see for instance Deimling [10, Theorem 1.3.1]) we have

$$\deg(\pi\circ\gamma, B_{R_n}, 0) = \deg(\mathrm{id}, B_{R_n}, 0) = 1.$$

Hence $0\in\pi(\gamma(B_{R_n}))$. Thus each γ intersects the hyperplane V.

Now since φ is coercive on V, there is a constant K such that

$$\max_{x\in B_{R_n}} \varphi(\gamma(x)) \geqslant \inf_{\tilde{u}\in V} \varphi(\tilde{u}) \geqslant K.$$

Hence $c_n \geqslant K$ and, for all large values of n,

$$c_n > \sup_{a \in \mathbb{R}^N,\, |a| = R_n} \varphi(a).$$

Let us now fix such n and apply Theorem 4.3 in [1]. This proves the next claim (see also [1, Corollary 4.3]).

Claim 4. *If n is large enough there exist sequences (γ_k) in S_n and (v_n) in X such that*

$$\max_{x \in B_{R_n}} \varphi(\gamma_k(x)) \to c_n \quad (k \to \infty).$$

Then there exists a sequence (v_k) in X such that

$$\varphi(v_k) \to c_n, \quad \operatorname{dist}(v_k, \gamma_k(B_{R_n})) \to 0, \quad |\varphi'(v_k)| \to 0$$

when $k \to \infty$.

Claim 5. *The sequence (v_k) is bounded in X.*

P r o o f. For any k large enough,

$$c_n \leqslant \max_{x \in B_{R_n}} \varphi(\gamma_k(x)) \leqslant c_n + 1$$

and we can find $w_k \in \gamma_k(B_{R_n})$ such that

$$\|v_k - w_k\|_X \leqslant 1. \tag{3.2}$$

Using Claim 3, for a fixed $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $r_m > R_n$. As in the proof above, let $\pi \colon X \to \mathbb{R}^N$ be the (continuous) projection of X onto $\mathbb{R}^N$. Then $|\pi(\gamma_k(B_{R_n}))| \leqslant R_n$. Let $H_{r_m} = \{b \in \mathbb{R}^N \colon |b| = r_m\} + V$. Then $|\pi(H_{r_m})| = r_m$, so $\gamma_k(B_{R_n})$ cannot intersect the hyperplanes H_{r_m}. Hence, if we write $w_k = \overline{w}_k + \tilde{w}_k$, where $\overline{w}_k \in \gamma_k(B_{R_n})$ and $\tilde{w}_k \in V$, we have

$$|\overline{w}_k| < r_m.$$

We also have

$$
\begin{aligned}
c_n + 1 \geqslant \varphi(w_k) &= \frac{1}{p}\|(\overline{w}_k + \tilde{w}_k)'\|_p^p - \int_0^T F(t, w_k(t))\,\mathrm{d}t \\
&= \frac{1}{p}\|\tilde{w}_k'\|_p^p - \int_0^T [F(t, w_k(t)) - F(t, \overline{w}_k)]\,\mathrm{d}t - \int_0^T F(t, \overline{w}_k)\,\mathrm{d}t \\
&= \frac{1}{p}\|\tilde{w}_k'\|_p^p - \int_0^T \int_0^1 (\nabla F(t, \overline{w}_k + s\tilde{w}_k(t)), \tilde{w}_k(t))\,\mathrm{d}s\,\mathrm{d}t - \int_0^T F(t, \overline{w}_k)\,\mathrm{d}t \\
&\geqslant \frac{1}{p}\|\tilde{w}_k'\|_p^p - \left(\int_0^T g(t)\,\mathrm{d}t\right)\|\tilde{w}_k\|_\infty - \int_0^T F(t, \overline{w}_k)\,\mathrm{d}t \\
&\geqslant \frac{1}{p}\|\tilde{w}_k'\|_p^p - C_2\|\tilde{w}_k'\|_p - \int_0^T F(t, \overline{w}_k)\,\mathrm{d}t
\end{aligned}
$$

by Sobolev's inequality; here $C_2 = \left(\int_0^T g(t)\,\mathrm{d}t\right)T^{1/q}$. Notice also that since $|\overline{w}_k| < r_m$, the integral $\int_0^T F(t, \overline{w}_k)\,\mathrm{d}t$ is bounded. Consequently, $\tilde{w}_k$ is bounded in X. Thus the sequence w_k is bounded and the claim follows from 3.2. □

Claim 6. *c_n is a critical value.*

Proof. From Proposition 2.3 and the last claim (recall also Claim 4) it follows that (v_k) contains a convergent subsequence, which we rename as (v_k). Let $u_n = \lim\limits_{k\to\infty} v_k$, then (see Claim 4)

$$
\varphi'(u_n) = \lim_{k\to\infty} \varphi'(v_k) = 0 \quad \text{and} \quad \varphi(u_n) = \lim_{k\to\infty} \varphi(v_k) = c_n.
$$

□

Proof of Theorem 1.1. (a) Claim 6 proves that, for each n large enough, there exists at least one solution u_n of (1.1) such that $\varphi(u_n) = c_n$, where c_n is given by (3.1). If $0 < r_k \leqslant R_n$ and $H_{r_k} = \{b \in \mathbb{R}^N : |b| = r_k\} + V$, then for any $\gamma \in S_n$ we have that γ intersects the hyperplane H_{r_k} (the proof is similar to the above). Then

$$
\max_{x\in B_{R_n}} \varphi(\gamma(x)) \geqslant \inf_{b\in\mathbb{R}^N,\, |b|=r_k,\, \tilde{u}\in V} \varphi(b + \tilde{u}).
$$

Thus, using Claim 3, we obtain that

$$
\lim_{n\to\infty} c_n = +\infty,
$$

and (a) follows.

(b) For $n \in \{1, 2, \ldots\}$, define a subset P_n of X by

$$P_n = \{u \in X\colon\ u = \overline{u} + \tilde{u},\ \overline{u} \in \mathbb{R}^N,\ |\overline{u}| \leqslant r_n,\ \tilde{u} \in V\}.$$

We note that for $u \in P_n$ we have, proceeding as in Claim 3,

$$\begin{aligned} \varphi(u) &= \frac{1}{p}\|u'\|_p^p - \int_0^T F(t,u)\,\mathrm{d}t \\ &\geqslant \frac{1}{p}\|\tilde{u}'\|_p^p - C_1\|\tilde{u}'\|_p - \int_0^T F(t,\overline{u})\,\mathrm{d}t \end{aligned} \tag{3.3}$$

by Sobolev's inequality; here $C_1 = \bigl(\int_0^T g(t)\,\mathrm{d}t\bigr)T^{1/q}$. Notice also that $\int_0^T F(t,\overline{u})\,\mathrm{d}t$ is bounded. Thus φ is bounded below on P_n.

Let us set

$$\mu_n = \inf_{u \in P_n} \varphi(u) \tag{3.4}$$

and let (u_k) be a sequence in P_n such that

$$\varphi(u_k) \to \mu_n \quad \text{as } k \to \infty. \tag{3.5}$$

We have

$$u_k = \overline{u}_k + \tilde{u}_k, \quad \overline{u}_k \in \mathbb{R}^N \quad \text{and } |\overline{u}_k| \leqslant r_n.$$

Without loss of generality we assume that $\overline{u}_k \to \overline{u} \in B_{r_n}$. From (3.3) and (3.5) we obtain that (u_k) is a bounded sequence in X and thus passing to a subsequence, which we rename (u_k), we have that

$$u_k \rightharpoonup u_n^* \quad \text{in } X.$$

Since P_n is a convex closed subset of X, $u_n^* \in P_n$.

Note that φ is weakly lower semi-continuous, so

$$\mu_n = \lim_{k\to\infty} \varphi(u_k) \geqslant \varphi(u_n^*)$$

and, since $u_n^* \in P_n$, we must have

$$\mu_n = \varphi(u_n^*).$$

Next we want to show that $u_n^* \in \operatorname{Int} P_n$ for large n, where

$$\operatorname{Int} P_n := \{u \in X\colon\ u = \overline{u} + \tilde{u}, |\overline{u}| < r_n\}.$$

Indeed, taking

$$0 < R_n < r_n,$$

it follows from Claim 2 and Claim 3 that $u_n^* \notin \{b + \tilde{u}, b \in Y, |b| = r_n\}$ for large n and hence $u_n^* \in \operatorname{Int} P_n$. This fact and (3.4) imply that

$$\varphi'(u_n^*) = 0$$

and u_n^* is a solution of (1.1).

Finally, from the fact that

$$\varphi(u_n^*) \leqslant \inf_{a \in \mathbb{R}^N,\, |a| = R_n} \varphi(a)$$

and from Claim 2 we obtain that

$$\lim_{n \to \infty} \varphi(u_n^*) = -\infty.$$

$\square$

R e m a r k 1. Going through the above proof, it is easy to see that the existence of critical points u_n still holds if assumption (1.2) is weakened to

$$\limsup_{R \to +\infty} \left[\inf_{a \in \mathbb{R}^N,\, |a| = R} \int_0^T F(s, a)\, \mathrm{d}s \right] \quad \text{is bounded from below.}$$

In that case, however, the existence of the minima u_n^* is no longer guaranteed.

E x a m p l e 1. Let us consider the scalar problem

$$\begin{gathered} -(|u'|^{p-2}u')' = a \sin(\ln(u^2 + 1)) + \frac{2au^2}{1 + u^2} \cos(\ln(u^2 + 1)) + e(t), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{gathered} \tag{3.6}$$

where $1 < p < \infty$, $a > 0$, $e \in L^1(0, T)$ and $\int_0^T e(t)\, \mathrm{d}t = 0$. In this case

$$F(t, u) = au \sin(\ln(u^2 + 1)) + e(t)u$$

and hence

$$\int_0^T F(t, x)\, \mathrm{d}t = ax \sin(\ln(x^2 + 1)).$$

Let $R_k = \sqrt{\mathrm{e}^{2k\pi+1/2\pi} - 1}$, $r_k = R_k = \sqrt{\mathrm{e}^{2k\pi+3/2\pi} - 1}$ for $k = 1, 2, \ldots$ Then

$$\begin{aligned}\lim_{k\to+\infty} \int_0^T F(t, R_k)\,\mathrm{d}t &= \lim_{k\to+\infty} aR_k \sin(\ln(R_k^2+1)) \\ &= \lim_{k\to+\infty} a\sqrt{\mathrm{e}^{2k\pi+1/2\pi}-1} = +\infty\end{aligned}$$

and

$$\begin{aligned}\lim_{k\to+\infty} \int_0^T F(t, r_k)\,\mathrm{d}t &= \lim_{k\to+\infty} ar_k \sin(\ln(r_k^2+1)) \\ &= \lim_{k\to+\infty} -a\sqrt{\mathrm{e}^{2k\pi+3/2\pi}-1} = -\infty.\end{aligned}$$

As a result, (1.2) and (1.3) are satisfied. Moreover, $|F_u'(t,u)|$ is clearly bounded by $3a + |e(t)|$ and so we have the following result:

(i) there exists a sequence (u_n) of solutions of (3.6) such that u_n is a critical point of φ and $\lim\limits_{n\to\infty} \varphi(u_n) = +\infty$;

(ii) there exists a sequence (u_n^*) of solutions of (3.6) such that u_n^* is a local minimum point of φ and $\lim\limits_{n\to\infty} \varphi(u_n^*) = -\infty$.

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