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A UNIQUENESS RESULT FOR A MODEL FOR MIXTURES IN THE ABSENCE OF EXTERNAL FORCES AND INTERACTION MOMENTUM*

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Abstract. We consider a continuum model describing steady flows of a miscible mixture of two fluids. The densities ϱ_i of the fluids and their velocity fields $u^{(i)}$ are prescribed at infinity: $\varrho_i|_\infty = \varrho_{i\infty} > 0$, $u^{(i)}|_\infty = 0$. Neglecting the convective terms, we have proved earlier that weak solutions to such a reduced system exist. Here we establish a uniqueness type result: in the absence of the external forces and interaction terms, there is only one such solution, namely $\varrho_i \equiv \varrho_{i\infty}$, $u^{(i)} \equiv 0$, $i = 1, 2$.

Keywords: miscible mixture, compressible fluid, uniqueness, zero force

MSC 2000: 35Q30, 76N10

1. Introduction

The aim of this introductory section is to describe the main result in a simplified form, to outline the scheme of the paper, to fix the notation and to give some insight into the difficulty of the problem.

For given positive numbers c_1, c_2, $\varrho_{1,\mathrm{ref}}$, $\varrho_{2,\mathrm{ref}}$ and for $\gamma > 1$ we define

$$P_i(\varrho) = P_i(\varrho_1, \varrho_2) = c_i \varrho_i \Big(\frac{\varrho_1}{\varrho_{1,\mathrm{ref}}} + \frac{\varrho_2}{\varrho_{2,\mathrm{ref}}} \Big)^{\gamma - 1} \quad (i = 1, 2). \tag{1}$$

Note that no summation convention over repeated indices i, $i = 1, 2$, is used in this paper.

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Further, for μ_{ij}, ν_{ij}, $i,j=1,2$, we set

$$\sigma^{(i)} = \sigma^{(i)}\big(\nabla u^{(1)}, \nabla u^{(2)}\big) = \mu_{i1}\mathbb{D}\big(u^{(1)}\big) + \mu_{i2}\mathbb{D}\big(u^{(2)}\big) + \nu_{i1}\operatorname{div} u^{(1)}\mathbb{I} + \nu_{i2}\operatorname{div} u^{(2)}\mathbb{I} \tag{2}$$

requiring that there is a $c_0>0$ such that μ_{ij}, ν_{ij} fulfil

$$\sum_{i=1}^{2}\int_{\mathbb{R}^3}\sigma^{(i)}\big(\nabla u^{(1)}, \nabla u^{(2)}\big) \colon \nabla u^{(i)}\,\mathrm{d}x \geqslant c_0\|\nabla u\|_{L^2}^2. \tag{3}$$

We use the notation $\mathbb{D}\big(u^{(i)}\big) = \frac{1}{2}\big(\nabla u^{(i)} + \nabla u^{(i),T}\big)$, $\mathbb{I}$ being the identity tensor.

Let $\varrho_{1\infty}$ and $\varrho_{2\infty}$ be positive numbers. We consider the following problem: find

$$\varrho = \begin{pmatrix}\varrho_1\\ \varrho_2\end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix}u^{(1)}\\ u^{(2)}\end{pmatrix} = \begin{pmatrix}\big(u_1^{(1)}, u_2^{(1)}, u_3^{(1)}\big)^T\\ \big(u_1^{(2)}, u_2^{(2)}, u_3^{(2)}\big)^T\end{pmatrix} \tag{4}$$

solving

$$\operatorname{div}\big(\varrho_i u^{(i)}\big) = 0 \quad \text{in } \mathbb{R}^3 \quad (i=1,2), \tag{5}$$

$$-\operatorname{div}\big(\sigma^{(i)}\big) = -\nabla P_i(\varrho) \quad \text{in } \mathbb{R}^3 \quad (i=1,2), \tag{6}$$

$$u^{(i)} \to 0 \quad \text{and} \quad \varrho_i \to \varrho_{i\infty} \text{ as } |x|\to\infty \quad (i=1,2). \tag{7}$$

We formulate the main result of this paper.

Theorem 1. *Assume that $P(\varrho) = (P_1(\varrho),\ P_2(\varrho))^T$ is of the form* (1) *and $\sigma^{(i)}$, $i=1,2$, of the form* (2) *fulfil the condition* (3).

Then there is (ϱ,u), a weak solution to (5)–(7), *such that*

$$\varrho_i - \varrho_{i\infty} \in L^2(\mathbb{R}^3)\cap L^{2\gamma}(\mathbb{R}^3) \quad (i=1,2), \tag{8}$$

$$u^{(i)} \in H_0^1(\mathbb{R}^3;\mathbb{R}^3) \quad (i=1,2). \tag{9}$$

Even more, if (ϱ,u) is a weak solution to (5)–(7) *satisfying* (8) *and* (9), *then necessarily*

$$\varrho \equiv \varrho_\infty \quad \textit{and} \quad u\equiv 0. \tag{10}$$

In other words, $\varrho\equiv\varrho_\infty$, $u\equiv 0$ is the unique solution to (5)–(7) within the class (8)–(9). Since the existence part of Theorem 1 follows from a more general result proved in [7] (see also [6]), the statement concerning the uniqueness is the main achievement herein.

The problem (1)–(6) arises from the continuum theory of mixtures. We provide a brief derivation of the model in Section 2 and motivate the boundary conditions (7) we have chosen.

In Section 3, we first formulate more general assumptions on the structure of P_i under which the statement of Theorem 1 is still valid. (The form (1) serves then as an example satisfying these assumptions.) We also discuss the interest in proving the result presented in Theorem 1. The core of Section 3 is the proof of Theorem 1 and its generalization (Theorem 3).

We complete this section by discussing the remarkable difference between the model (1)–(7) describing the steady flow of two miscible fluids on the one hand and the analogous model for a one-constituent compressible fluid on the other.

If we consider a fluid constituted of one homogeneous compressible liquid or gas, then the equations analogous to (1)–(7) are

$$\operatorname{div}(\varrho v) = 0 \quad \text{in } \mathbb{R}^3, \tag{11}$$

$$-\mu\Delta v - (\lambda+\mu)\nabla \operatorname{div} v = -\nabla \varrho^{\gamma} \quad \text{in } \mathbb{R}^3, \tag{12}$$

$$v \to 0 \text{ and } \varrho \to \varrho_\infty \quad \text{as } |x| \to \infty, \tag{13}$$

with $\varrho\colon \mathbb{R}^3 \to \mathbb{R}$, $v\colon \mathbb{R}^3 \to \mathbb{R}^3$, and a given $\varrho_\infty > 0$.

Assuming that $\varrho \in L^{2\gamma}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ and $v \in H^1_0(\mathbb{R}^3;\mathbb{R}^3)$ solve weakly (11)–(12), we can set $\varphi = v$ in the weak formulation of (12) and obtain

$$\begin{aligned} \mu\|\nabla v\|^2_{L^2} + (\lambda+\mu)\|\operatorname{div} v\|^2_{L^2} &= \int_{\mathbb{R}^3} \varrho^{\gamma} \operatorname{div} v \,\mathrm{d}x = -\gamma \int_{\mathbb{R}^3} \varrho^{\gamma-1} v\cdot\nabla\varrho \,\mathrm{d}x \\ &= -\frac{\gamma}{\gamma-1}\int_{\mathbb{R}^3} \varrho v \cdot \nabla \varrho^{\gamma-1}\,\mathrm{d}x \\ &= \frac{\gamma}{\gamma-1}\int_{\mathbb{R}^3} \operatorname{div}(\varrho v)\varrho^{\gamma-1}\,\mathrm{d}x = 0. \end{aligned} \tag{14}$$

If $\mu > 0$ and $\lambda + 2\mu > 0$, then it follows from (14) and (13) that

$$v \equiv 0 \quad \text{in } \mathbb{R}^3, \tag{15}$$

and (12) simplifies (in the weak formulation) to

$$\int_{\mathbb{R}^3} \left(\varrho^{\gamma} - \varrho^{\gamma}_{\infty}\right) \operatorname{div}\varphi \,\mathrm{d}x = 0. \tag{16}$$

Taking (at least formally) in (16) φ such that $\operatorname{div}\varphi = \varrho - \varrho_\infty$, the strict monotonicity of ϱ^{γ} then implies that

$$\varrho \equiv \varrho_\infty \quad \text{in } \mathbb{R}^3. \tag{17}$$

This argument however fails if we try to imitate the same procedure for (1)–(6). Indeed, multiplying the ith equation in (6) by $u^{(i)}$ and summing the resulting equations over $i = 1, 2$, we obtain with the aid of (3)

$$c_0\|\nabla u\|^2_{L^2} \leqslant \sum_{i=1}^{2} c_i \int_{\mathbb{R}^3} (\varrho_1 + \varrho_2)^{\gamma-1} \varrho_i \operatorname{div} u^{(i)} \,\mathrm{d}x, \tag{18}$$

and for any $\gamma > 1$ the right-hand side of (18) is not vanishing. This means that *no energy inequality is available for the model* (1)–(7).

In the analysis of the classical compressible fluid model, a crucial role is played by a quantity which is called the *effective viscous flux* (see [11], [12], [3], [4]). This tool also helps when studying mixture models of the type (1)–(7).

Just for clarity and simplicity, we explain this idea within the context of (11)–(13). Taking formally the divergence of equation (12) and applying $(-\Delta)^{-1}$, we obtain

$$(\lambda + 2\mu) \operatorname{div} v = \varrho^\gamma - \varrho_\infty^\gamma. \tag{19}$$

Multiplying (19) by $\varrho - \varrho_\infty$ and integrating over $\mathbb{R}^3$ leads (due to the fact that $\int_{\mathbb{R}^3} \varrho^\gamma \operatorname{div} u \,\mathrm{d}x = 0$, see (14) for a proof) to

$$\int_{\mathbb{R}^3} (\varrho^\gamma - \varrho_\infty^\gamma)(\varrho - \varrho_\infty) \,\mathrm{d}x = 0,$$

which implies that $\varrho \equiv \varrho_\infty$. Then (12) implies $v \equiv 0$.

The advantage of this approach consists in its applicability to the system (1)–(7), see Section 3 for details.

On the other hand, while the approach described between (14) and (17) can be extended easily to the model where the inertial (convective) terms are included, it is not clear whether one can conclude that $\varrho \equiv \varrho_\infty$ if one “replaces” (19) by

$$(2\mu + \lambda) \operatorname{div} v + (-\Delta)^{-1} \operatorname{div} \operatorname{div}(\varrho v \otimes v) = \varrho^\gamma - \varrho_\infty^\gamma. \tag{20}$$

Another drawback of the approach based on the effective viscous flux equation is its sensibility to the boundary conditions. In particular, it is not obvious how to extend this technique to the case of a bounded domain with homogeneous Dirichlet boundary conditions.

2. A continuum mixture theory

A continuum mechanics approach to model mixtures starts with the assumption of co-occupancy requiring that at each point $x \in \Omega \subset \mathbb{R}^3$ all components of the mixture (so-called constituents) coexist (see [14], [13], [8], [9]).

Here, for simplicity, we restrict ourselves to the mixture of two fluid constituents (labelled by i, $i = 1$ or 2).

We also assume that the motion of the mixture takes place at constant temperature and that there is no mass conversion between the two constituents (as it could be if chemical reactions between the constituents took place). Under these circumstances, the motion of the mixture can be described in terms of the balance of mass for each constituent, the balance of linear momentum for each constituent and the balance of entropy for the whole mixture. Since the considered flow is isothermal, the balance of energy is not even written below.

Let ϱ_i, $u^{(i)}$, $T^{(i)}$, ψ_i, $f^{(i)}$, $I^{(i)}$ be the density, the velocity field, the Cauchy stress tensor, the Helmholtz potential, the density of the external forces and the momentum source associated to the ith constituent, $i = 1, 2$. Then the balance of mass takes the form

$$(\varrho_i)_t + \operatorname{div}\bigl(\varrho_i u^{(i)}\bigr) = 0 \tag{21}$$

(summation over repeated indices i never takes place unless it is explicitly mentioned), the balance of linear momentum leads to

$$\bigl(\varrho_i u^{(i)}\bigr)_t + \operatorname{div}\bigl(\varrho_i u^{(i)} \otimes u^{(i)}\bigr) = \operatorname{div} T^{(i)} + I^{(i)} + \varrho_i f^{(i)}, \tag{22}$$

and the entropy inequality (rewritten in terms of Helmholtz potentials) gives

$$\sum_{i=1}^{2} (\varrho_i \psi_i)_t + \operatorname{div}\Bigl(\sum_{i=1}^{2} \varrho_i \psi_i u^{(i)}\Bigr) - \sum_{i=1}^{2} T^{(i)} \colon \nabla u^{(i)} - \sum_{i=1}^{2} I^{(i)} \cdot u^{(i)} \leqslant 0. \tag{23}$$

The balance of linear momentum for the whole mixture implies

$$I^{(2)} = -I^{(1)}.$$

Thus, (23) can be rewritten as

$$\sum_{i=1}^{2} T^{(i)} \colon \nabla u^{(i)} + I^{(1)} \cdot \bigl(u^{(1)} - u^{(2)}\bigr) - \sum_{i=1}^{2} (\varrho_i \psi_i)_t - \operatorname{div}\Bigl(\sum_{i=1}^{2} \varrho_i \psi_i u^{(i)}\Bigr) \geqslant 0 \tag{24}$$

or

$$\sum_{i=1}^{2} T^{(i)} \colon \nabla u^{(i)} + I^{(1)} \cdot \bigl(u^{(1)} - u^{(2)}\bigr) - \sum_{i=1}^{2} \bigl[\varrho_i (\psi_i)_t + \varrho_i u_k^{(i)} (\psi_i)_{x_k}\bigr] \geqslant 0. \tag{25}$$

Next, we assume that the energy storage mechanism is the same for each constituent, i.e.,

$$\psi_1 = \psi_2, \tag{26}$$

and we require that

$$\psi_i = \Psi(\varrho_1 + \varrho_2) \tag{27}$$

or slightly more generally

$$\psi_i = \tilde{\Psi}\Big(\frac{\varrho_1}{\varrho_{1,\mathrm{ref}}} + \frac{\varrho_2}{\varrho_{2,\mathrm{ref}}}\Big). \tag{28}$$

Assuming (27), for simplicity, we compute

$$\begin{aligned}(\psi_i)_t + u_k^{(i)}(\psi_i)_{x_k} &= \Psi'(\varrho_1+\varrho_2)\big[(\varrho_1)_t + (\varrho_2)_t + u_k^{(i)}((\varrho_1)_{x_k} + (\varrho_2)_{x_k})\big] \\ &= \Psi'(\varrho_1+\varrho_2)\big[-\operatorname{div}\big(\varrho_1 u^{(1)}\big) - \operatorname{div}\big(\varrho_2 u^{(2)}\big) \\ &\qquad + u_k^{(i)}\big((\varrho_1)_{x_k} + (\varrho_2)_{x_k}\big)\big].\end{aligned}$$

Then

$$\begin{aligned}&\sum_{i=1}^{2}\big[\varrho_i(\psi_i)_t + \varrho_i u_k^{(i)}(\psi_i)_{x_k}\big] \\ &\quad= \Psi'(\varrho_1+\varrho_2)\big[-\varrho_1\operatorname{div}\big(\varrho_1 u^{(1)}\big) - \varrho_1\operatorname{div}\big(\varrho_2 u^{(2)}\big) + \varrho_1 u_k^{(1)}\big((\varrho_1)_{x_k} + (\varrho_2)_{x_k}\big) \\ &\qquad\quad - \varrho_2\operatorname{div}\big(\varrho_1 u^{(1)}\big) - \varrho_2\operatorname{div}\big(\varrho_2 u^{(2)}\big) + \varrho_2 u_k^{(2)}\big((\varrho_1)_{x_k} + (\varrho_2)_{x_k}\big)\big] \\ &\quad= \Psi'(\varrho_1+\varrho_2)\big[-\varrho_1^2\operatorname{div} u^{(1)} - \varrho_1\varrho_2\operatorname{div} u^{(2)} + \varrho_1(\varrho_2)_{x_k}\big(u_k^{(1)} - u_k^{(2)}\big) \\ &\qquad\quad - \varrho_2(\varrho_1)_{x_k}\big(u_k^{(1)} - u_k^{(2)}\big) - \varrho_1\varrho_2\operatorname{div} u^{(1)} - \varrho_2^2\operatorname{div} u^{(2)}\big] \\ &\quad= -\Psi'(\varrho_1+\varrho_2)(\varrho_1+\varrho_2)\big[\varrho_1\mathbb{I} : \nabla u^{(1)} + \varrho_2\mathbb{I} : \nabla u^{(2)}\big] \\ &\qquad + \Psi'(\varrho_1+\varrho_2)\big[(\varrho_1\nabla\varrho_2 - \varrho_2\nabla\varrho_1)\cdot\big(u^{(1)} - u^{(2)}\big)\big].\end{aligned}$$

Incorporating the last result into (25), we conclude that

$$\begin{aligned}&\sum_{i=1}^{2}\big[T^{(i)} + \Psi'(\varrho)\varrho\varrho_i\mathbb{I}\big] : \nabla u^{(i)} \\ &\qquad + \big[I^{(1)} - \psi'(\varrho)(\varrho_1\nabla\varrho_2 - \varrho_2\nabla\varrho_1)\big]\cdot\big(u^{(1)} - u^{(2)}\big) \geqslant 0.\end{aligned} \tag{29}$$

Setting

$$\sigma^{(i)} := T^{(i)} + \Psi'(\varrho)\varrho\varrho_i\mathbb{I} \quad \text{and} \quad J := I^{(1)} + \psi'(\varrho)(\varrho_2\nabla\varrho_1 - \varrho_1\nabla\varrho_2), \tag{30}$$

we observe that the inequality (29) will be fulfilled if we set

$$\sigma^{(i)} = \mu_{1i}\mathbb{D}\big(u^{(1)}\big) + \mu_{2i}\mathbb{D}\big(u^{(2)}\big) + \nu_{1i}\operatorname{div} u^{(1)}\mathbb{I} + \nu_{2i}\operatorname{div} u^{(2)}\mathbb{I},$$
$$J = a\big(\varrho_1, \varrho_2, \big|u^{(1)} - u^{(2)}\big|\big)\big(u^{(1)} - u^{(2)}\big)$$

and require for a certain $c_0 > 0$

$$\sum_{i=1}^{2} \sigma^{(i)} \colon \nabla u^{(i)} \geqslant c_0 |\nabla u|^2, \qquad a\big(\varrho_1, \varrho_2, \big|u^{(1)} - u^{(2)}\big|\big) \geqslant 0. \tag{31}$$

It follows from (30) that

$$T^{(i)} = -P_i(\varrho)\mathbb{I} + \sigma^{(i)} \quad \text{with } P_i(\varrho) = \Psi(\varrho)\varrho\varrho_i,$$
$$I^{(i)} = a\big(\varrho_1, \varrho_2, \big|u^{(1)} - u^{(2)}\big|\big)\big(u^{(1)} - u^{(2)}\big) + \psi'(\varrho)(\varrho_1\nabla\varrho_2 - \varrho_2\nabla\varrho_1). \tag{32}$$

For later use, we denote

$$L^{(i)}u = -\operatorname{div}\sigma^{(i)} = -\sum_{k=1}^{2}\big(\mu_{ik}\Delta u^{(k)} + (\mu_{ik} + \nu_{ik})\nabla\operatorname{div} u^{(k)}\big). \tag{33}$$

To summarize, we observe that the four unknown functions (ϱ_1, ϱ_2) and $\big(u^{(1)}, u^{(2)}\big)$ can be found as the solution of the system

$$(\varrho_i)_t + \operatorname{div}\big(\varrho_i u^{(i)}\big) = 0 \quad (i = 1, 2) \tag{34}$$

and

$$\big(\varrho_i u^{(i)}\big)_t + \operatorname{div}\big(\varrho_i u^{(i)} \otimes u^{(i)}\big) = -\nabla P_i(\varrho) - L^{(i)}\big(\big(u^{(1),T}, u^{(2),T}\big)^T\big) + J_0^{(i)} + \varrho_i f^{(i)} \tag{35}$$

with

$$J_0^{(1)} = -J_0^{(2)} = a\big(\varrho_1, \varrho_2, \big|u^{(1)} - u^{(2)}\big|\big)\big(u^{(1)} - u^{(2)}\big) + \psi'(\varrho)(\varrho_1\nabla\varrho_2 - \varrho_2\nabla\varrho_1), \tag{36}$$
$$P_i(\varrho) = \psi'(\varrho)\varrho\varrho_i \tag{37}$$

and $L^{(i)}\big(\big(u^{(1),T}, u^{(2),T}\big)^T\big)$ fulfils (33) and (31).

The system (34)–(37) satisfies an energy inequality.

We however consider the approximation of this system neglecting the second term in (36). This means that we replace $J_0^{(1)}$ and $J_0^{(2)}$, respectively, by

$$J^{(1)} = -J^{(2)} = a\big(\varrho_1, \varrho_2, \big|u^{(1)} - u^{(2)}\big|\big)\big(u^{(1)} - u^{(2)}\big). \tag{38}$$

This approximation, motivated by mathematical reasons (it is more or less hopeless to expect any information on $\nabla\varrho_1$ and/or $\nabla\varrho_2$ that would allow to pass to the limit in the term $\psi'(\varrho)(\varrho_1\nabla\varrho_2 - \varrho_2\nabla\varrho_1)$), can be justified physically as follows:

(i) (38) is a very good approximation of (36) provided that the variations of the densities are of lower order.

(ii) As it is very difficult to capture the term $\psi'(\varrho)(\varrho_1\nabla\varrho_2-\varrho_2\nabla\varrho_1)$ experimentally, it is always preferable to neglect it. In fact, it has been confirmed that the flows in simple geometries will not change significantly whether the second term in (36) is present or not.

However, once we accept (38) instead of (36), the basic energy identity is lost. In this paper, we use (38) instead of (36).

If we, in addition, restrict ourselves to steady flows, i.e. $(\varrho_i)_t = 0$ and $(\varrho_i u^{(i)})_t = 0$, then we end up with the system $(i = 1, 2)$

$$\operatorname{div}\bigl(\varrho_i u^{(i)}\bigr) = 0, \tag{39}$$

$$\operatorname{div}\bigl(\varrho_i u^{(i)} \otimes u^{(i)}\bigr) + L^{(i)}u = -\nabla P_i(\varrho) + \varrho_1 f^{(i)} + J^{(i)}, \tag{40}$$

$L^{(i)}u$ being of the form (33) and $J^{(i)}$ of the form (38), both satisfying (31).

One can imagine that the mixture occupies a large domain (e.g. $B_R(0) \subset \mathbb{R}^3$, $R \gg 1$) and it is at rest, i.e. $u^{(i)} \equiv 0$ and $\varrho_i \equiv \varrho_{i\infty}$. Then if the motion is initiated near the origin, it can take some time t^* till the motion reaches the boundary of the domain. Before t^*, it is reasonable to assume that

$$\varrho_i = \varrho_{i\infty} \quad \text{and} \quad u^{(i)} = 0 \ \text{ on } \ [0, t^*) \times \partial B_R(0). \tag{41}$$

Of course, this type of boundary condition is not acceptable if we consider steady flows in a bounded domain. On the other hand, it does not seem too awful to assume that

$$\varrho_i \to \varrho_{i\infty} \quad \text{and} \quad u^{(i)} \to 0 \ \text{ as } \ |x| \to \infty$$

if the mixture fills the whole space $\mathbb{R}^3$.

Finally, we neglect the convective term in (40), which may be acceptable for special flows or under the conditions that $u^{(i)} \otimes u^{(i)}$ are of smaller order. Since the full system is very complex, our motivation was rather technical: to start with the investigation of a simpler system first and thus, in analogy to the mathematical theory for incompressible fluids, we study first the Stokes-like system for mixtures.

3. Mathematical theory for the Stokes-like system for mixtures

Neglecting the convective term in (40), the system reduces to

$$\operatorname{div}\big(\varrho_i u^{(i)}\big) = 0 \quad (i = 1, 2), \tag{42}$$

$$L^{(i)} u = -\nabla P_i(\varrho) + \varrho_i f^{(i)} + J^{(i)} \quad (i = 1, 2). \tag{43}$$

We consider (42) and (43) in $\mathbb{R}^3$ and assume that for given $\varrho_{1\infty}, \varrho_{2\infty} > 0$

$$\varrho_i \to \varrho_{i\infty} \quad \text{and} \quad u^{(i)} \to 0 \ \text{ as } |x| \to \infty. \tag{44}$$

Further, $L^{(i)}$ are of the form (33) and fulfil for a $c_0 > 0$

$$\sum_{i=1}^{2} \int_{\mathbb{R}^3} \sigma^{(i)} \colon \nabla u^{(i)} \,\mathrm{d}x \geqslant c_0 \|\nabla u\|^2_{L^2}, \tag{45}$$

which is a weaker condition in comparison to (31). We also assume that

$$J^{(1)} = -J^{(2)} \quad \text{and} \quad J^{(1)} = a\big(\varrho_1, \varrho_2, \big|u^{(1)} - u^{(2)}\big|\big)\big(u^{(1)} - u^{(2)}\big) \tag{46}$$

and

$$a \ \text{ is a sublinear continuous function of } \ \big|u^{(1)} - u^{(2)}\big|. \tag{47}$$

We also require that $P(\varrho) = (P_1(\varrho), P_2(\varrho))^T \in C(\mathbb{R}) \times C(\mathbb{R})$ satisfies the condition

$$\begin{gathered} \text{there are } \gamma > 1,\ C > 0 \text{ and } \beta_0 \neq 0 \text{ such that} \\ (\varrho - \tilde\varrho)^T A_0 (P(\varrho) - P(\tilde\varrho)) \geqslant C\big(|\varrho|^{\gamma-1} + |\tilde\varrho|^{\gamma-1}\big)|\varrho - \tilde\varrho|^2, \\ \text{for all } \varrho = (\varrho_1, \varrho_2)^T \quad \text{and} \quad \tilde\varrho = (\tilde\varrho_1, \tilde\varrho_2)^T \text{ with } \varrho_i \geqslant 0,\ \tilde\varrho_i \geqslant 0 \end{gathered} \tag{48}$$

where

$$A_0 = \begin{pmatrix} \beta_0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2\mu_{11} + \nu_{11} & 2\mu_{12} + \nu_{12} \\ 2\mu_{21} + \nu_{21} & 2\mu_{22} + \nu_{22} \end{pmatrix}^{-1}. \tag{49}$$

Note that β_0 is imposed in order to include more general forms for $P(\varrho)$. We remark that the pressure given in (1) satisfies the condition (48), as proved in [6].

Finally, we require that

$$f^{(i)} \in L^\infty(\mathbb{R}^3; \mathbb{R}^3) \quad \text{and} \quad f^{(i)} \text{ and } a \text{ have compact support in } \mathbb{R}^3. \tag{50}$$

Theorem 2. *Let the assumptions* (45)–(50) *be fulfilled. Then there is* (ϱ, u) *such that*

$$\varrho_i - \varrho_{i\infty} \in L^{2\gamma}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3),$$
$$u^{(i)} \in H_0^1(\mathbb{R}^3; \mathbb{R}^3),$$

solving (42), (43) *in distributional sense.*

Proof. See [6], [7]. Note that this proves the existence part of Theorem 1 since $a \equiv 0$ and $f^{(i)} \equiv 0$ certainly satisfy the required assumptions. $\square$

Once having the existence of a weak solution to such a system, it is natural to ask whether such a solution is unique.

In general, this is a difficult task even for a one-component fluid, as the reader can check in the case of steady isothermal flows in a recent survey article by J. Heywood and M. Padula, where the uniqueness is established under a severe restriction on the data (cf. [10]). Even more, as observed by E. Feireisl and H. Petzeltová (see [5]), even if $u \equiv 0$ and $f = \nabla F$, the equations

$$\nabla P(\varrho) = \varrho \nabla F$$

can have multiple solutions. For a more detailed and complete exposition in this direction see also [2].

Thus we ask herein if the uniqueness of the solution can be deduced provided the external body forces $f^{(i)}$ and the interaction terms vanish. The following theorem gives an affirmative answer to this issue.

Theorem 3. *Let* $(\varrho, u) = \big((\varrho_1, \varrho_2)^T, (u^{(1),T}, u^{(2),T})^T\big)$, *satisfying*

$$\varrho_i - \varrho_{i\infty} \in L^2(\mathbb{R}^3) \cap L^{2\gamma}(\mathbb{R}^3), \tag{51}$$
$$u^{(i)} \in H_0^1(\mathbb{R}^3; \mathbb{R}^3), \tag{52}$$

be a weak solution to

$$\operatorname{div}\big(\varrho_i u^{(i)}\big) = 0 \quad \text{in } \mathbb{R}^3, \tag{53}$$
$$L^{(i)} u = -\nabla P_i(\varrho) \quad \text{in } \mathbb{R}^3, \tag{54}$$
$$u^{(i)} \to 0 \quad \text{and} \quad \varrho_i \to \varrho_{i\infty} \quad \text{as } |x| \to \infty \tag{55}$$

with $L^{(i)}$ *of the form* (33) *fulfilling* (45), $P(\varrho)$ *fulfilling the condition* (48).

Then $\varrho \equiv \varrho_\infty$ *and* $u \equiv 0$.

P r o o f. We start with the equation[1]

$$\begin{pmatrix} \beta_0 \operatorname{div} u^{(1)} \\ \operatorname{div} u^{(2)} \end{pmatrix} = A_0(P(\varrho) - P(\varrho_\infty)), \tag{56}$$

where A_0 is introduced in (49), and multiply this equation by $\tau(\varrho * \omega_h - \varrho_\infty)$, where τ is the usual localization function with support in B_{2R}, being equal to 1 in B_R. Integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned} \int A_0(P(\varrho) - P(\varrho_\infty)) \cdot (\varrho * \omega_h - \varrho_\infty)\tau \, \mathrm{d}x = \beta_0 \int \operatorname{div} u^{(1)} \tau(\varrho_1 * \omega_h - \varrho_{1\infty}) \, \mathrm{d}x \\ + \int \operatorname{div} u^{(2)} \tau(\varrho_2 * \omega_h - \varrho_{2\infty}) \, \mathrm{d}x. \end{aligned} \tag{57}$$

Analyzing the terms on the right-hand side, we arrive by integration by parts at

$$\begin{aligned} \int \operatorname{div} u^{(i)} \tau \left(\varrho_i * \omega_h - \varrho_{i\infty}\right) \mathrm{d}x = & - \int u^{(i)} \nabla\tau(\varrho_i * \omega_h - \varrho_{i\infty}) \, \mathrm{d}x \\ & - \int u^{(i)} \tau \nabla(\varrho_i * \omega_h) \, \mathrm{d}x =: A_{i1} + A_{i2}. \end{aligned}$$

First we observe that

$$\begin{aligned} |A_{i1}| &\leqslant \|(\varrho - \varrho_\infty) * \omega_h\|_{L^2(B_{2R}\setminus B_R)} \|u\|_{L^6(B_{2R}\setminus B_R)} \|\nabla\tau\|_{L^3(B_{2R}\setminus B_R)} \\ &\leqslant C\|(\varrho - \varrho_\infty) * \omega_h\|_{L^2(B_{2R}\setminus B_R)} \|\nabla u\|_{L^2(B_{2R}\setminus B_R)}. \end{aligned}$$

We see that, thanks to (51)–(52), the term A_{i1} will vanish when the parameter R tends to infinity (after h has gone to 0). We study now the term A_{i2}:

$$\begin{aligned} A_{i2} &= - \int (\varrho_i * \omega_h + \delta) u^{(i)} \tau \nabla \log(\varrho_i * \omega_h + \delta) \, \mathrm{d}x \\ &= \int \operatorname{div}\big((\varrho_i * \omega_h + \delta)\, u^{(i)}\big) \tau \log(\varrho_i * \omega_h + \delta) \, \mathrm{d}x \\ &\quad + \int (\varrho_i * \omega_h + \delta) u^{(i)} \nabla\tau \log\left(\varrho_i * \omega_h + \delta\right) \, \mathrm{d}x \end{aligned}$$

[1] Eq. (56), modulo multiplication by constant matrices, can be formally obtained by taking the divergence of (54) followed by applying $(-\Delta)^{-1}$ to the result. The proof of (56), even with additional terms due to nonzero $f^{(i)}$ and $I^{(i)}$ can be found in [7], see the sketch of the same in [6].

$$
\begin{aligned}
&= \int \operatorname{div}\bigl((\varrho_i * \omega_h)u^{(i)}\bigr)\tau \log(\varrho_i * \omega_h + \delta)\,\mathrm{d}x \\
&\quad + \delta \int \operatorname{div} u^{(i)} \log(\varrho_i * \omega_h + \delta)\tau\,\mathrm{d}x \\
&\quad + \delta \int u^{(i)} \nabla\tau \log(\varrho_i * \omega_h + \delta)\,\mathrm{d}x \\
&\quad + \int (\varrho_i * \omega_h)u^{(i)} \log(\varrho_i * \omega_h + \delta)\nabla\tau\,\mathrm{d}x \\
&=: D_{i1} + D_{i2} + D_{i3} + D_{i4}.
\end{aligned}
$$

The term D_{i1} can be treated with the aid of the lemma of DiPerna and Lions (cf. [1]), which asserts that under our assumptions

$$\tau\bigl(\operatorname{div}\bigl[(\varrho_i * \omega_h)u^{(i)}\bigr] - \operatorname{div}\bigl[\omega_h * \bigl(\varrho_i u^{(i)}\bigr)\bigr]\bigr) \rightharpoonup 0 \text{ weakly in } L^{2\gamma/(\gamma+1)} \text{ as } h \to 0.$$

Since $\operatorname{div}\bigl(\varrho_i u^{(i)}\bigr) = 0$ weakly, we conclude that

$$D_{i1} \to 0 \quad \text{as} \quad h \to 0.$$

Moreover,

$$
\begin{aligned}
D_{i2} &\to \delta \int \operatorname{div} u^{(i)} \log(\varrho_i + \delta)\tau\,\mathrm{d}x \quad \text{as } h \to 0, \\
D_{i3} &\to \delta \int u^{(i)} \nabla\tau \log(\varrho_i + \delta)\,\mathrm{d}x \qquad \text{as } h \to 0.
\end{aligned}
$$

As the integration takes place over B_{2R}, both the integrals are finite for fixed R and consequently

$$D_{i2} \to 0 \quad \text{and} \quad D_{i3} \to 0 \quad \text{as } h, \delta \to 0.$$

Finally, we deal with D_{i4} observing first that

$$D_{i4} \to \int \varrho_i u^{(i)} \log(\varrho_i + \delta)\nabla\tau\,\mathrm{d}x \quad \text{as } h \to 0.$$

Using again the fact that $\operatorname{div}\bigl(\varrho_i u^{(i)}\bigr) = 0$ weakly, we see that

$$\int \varrho_i u^{(i)} \log(\varrho_i + \delta)\nabla\tau\,\mathrm{d}x = \int \varrho_i u^{(i)} (\log(\varrho_i + \delta) - \log(\varrho_{i\infty} + \delta))\nabla\tau\,\mathrm{d}x. \tag{58}$$

Setting $\varepsilon_0 = \frac{1}{2}\varrho_{i\infty}$ and $L = 2\varrho_{i\infty}$ we see that

$$
\begin{aligned}
|\{x \in \mathbb{R}^3;\, \varrho_i \leqslant \varepsilon_0\}| \Bigl(\frac{\varrho_{i\infty}}{2}\Bigr)^2 &\leqslant \int_{\{x \in \mathbb{R}^3;\, \varrho_i \leqslant \varepsilon_0\}} |\varrho_i - \varrho_{i\infty}|^2\,\mathrm{d}x \leqslant K, \\
|\{x \in \mathbb{R}^3;\, \varrho_i > L\}| 4\varrho_{i\infty}^2 &\leqslant \int_{\{x \in \mathbb{R}^3;\, \varrho_i > L\}} |\varrho_i - \varrho_{i\infty}|^2\,\mathrm{d}x \leqslant K.
\end{aligned}
$$

Consequently,

$$(59)\qquad |\{x \in \mathbb{R}^3;\ \varrho_i \leqslant \varepsilon_0\}| \leqslant C < \infty \ \text{ and } \ |\{x \in \mathbb{R}^3;\ \varrho_i > L\}| \leqslant C < \infty.$$

Thus, with the aid of (59),

$$\begin{aligned} &\Big|\int_{\{\varrho_i \leqslant \varepsilon_0\}} \varrho_i u^{(i)} (\log(\varrho_i + \delta) - \log(\varrho_{i\infty} + \delta)) \nabla \tau \,\mathrm{d}x\Big| \\ &\qquad \leqslant K(\varrho_{i\infty}) \|\nabla \tau\|_{L^\infty} \|u\|_{L^6} |\{\varrho_i \leqslant \varepsilon_0\}|^{6/5} \leqslant \frac{K}{R}, \end{aligned}$$

and the integral vanishes as $R \to \infty$. Next,

$$\begin{aligned} &\int_{\{\varrho_i > \varepsilon_0\}} \varrho_i u^{(i)} (\log(\varrho_i + \delta) - \log(\varrho_{i\infty} + \delta)) \nabla \tau \,\mathrm{d}x \\ &\quad = \int_{\{\varrho_i > \varepsilon_0\}} (\varrho_i - \varrho_{i\infty}) u^{(i)} (\log(\varrho_i + \delta) - \log(\varrho_{i\infty} + \delta)) \nabla \tau \,\mathrm{d}x \\ &\qquad + \varrho_{i\infty} \int_{\{\varrho_i > \varepsilon_0\}} u^{(i)} (\log(\varrho_i + \delta) - \log(\varrho_{i\infty} + \delta)) \nabla \tau \,\mathrm{d}x =: E_{i1} + E_{i2}. \end{aligned}$$

Using the fact that on $\{\varrho_i > \varepsilon_0\}$ the function $\log(\varrho_i + \delta)$ is Lipschitz and on $\{\varepsilon_0 \leqslant \varrho_i < L\}$ the function $\varrho_i \log(\varrho_i + \delta)$ is Lipschitz, we obtain

$$\begin{aligned} |E_{i1}| &= \Big|\int_{\{\varepsilon_0 < \varrho_i \leqslant L\}} \ldots + \int_{\{\varrho_i > L\}} \ldots\Big| \\ &\leqslant C(\varrho_{i\infty}) \Big[\int_{\{\varepsilon_0 < \varrho_i \leqslant L\}} |\varrho_i - \varrho_{i\infty}|\, |u|\, |\nabla \tau| \,\mathrm{d}x + \int_{\{\varrho_i > L\}} |\varrho_i - \varrho_{i\infty}|^2 |u|\, |\nabla \tau| \,\mathrm{d}x\Big] \\ &\leqslant C(\varrho_{i\infty}) \|\varrho_i - \varrho_{i\infty}\|_{L^2(B_{2R} \setminus B_R)} \|u\|_{L^6(B_{2R} \setminus B_R)} \\ &\quad + \|\varrho_i - \varrho_{i\infty}\|_{L^{2\gamma}} \|u\|_{L^6} \|\nabla \tau\|_{L^\infty} |\{\varrho_i > L\}|^{6\gamma/(5\gamma - 6)}. \end{aligned}$$

From (59), (51)–(52) and $\|\nabla \tau\|_{L^\infty} \leqslant C/R$, we conclude that

$$E_{i1} \to 0 \quad \text{as} \quad R \to \infty.$$

Similarly, we have

$$\begin{aligned} |E_{i2}| &\leqslant C(\varrho_{i\infty}) \int_{\{\varrho_i > \varepsilon_0\}} \big|u^{(i)}\big|\, |\varrho_i - \varrho_{i\infty}|\, |\nabla \tau| \,\mathrm{d}x \\ &\leqslant C(\varrho_{i\infty}) \|u\|_{L^6(B_{2R} \setminus B_R)} \|\varrho_i - \varrho_{i\infty}\|_{L^2(B_{2R} \setminus B_R)} \|\nabla \tau\|_{L^3}, \end{aligned}$$

and as above

$$E_{i2} \to 0 \quad \text{as} \quad R \to \infty.$$

To summarize: letting in (57) $h \to 0$, then $\delta \to 0$ and finally $R \to \infty$, we obtain

$$\int A_0(P(\varrho) - P(\varrho_\infty)) \cdot (\varrho - \varrho_\infty)\,\mathrm{d}x = 0. \tag{60}$$

It then follows from the monotonicity condition (48) that

$$\varrho \equiv \varrho_\infty.$$

Thus, it follows that $\nabla P(\varrho) = 0$, and Eq. (54) simplifies to

$$Lu = 0.$$

With the aid of Cacciopoli's inequality one can conclude from this equation that the velocity u vanishes everywhere. We briefly mention the proof. Denoting by $\overline{u}$ the mean value of u taken over the support of τ and taking the scalar product of the equation with $(u - \overline{u})\tau^2$, we obtain

$$(Lu, (u - \overline{u})\tau^2) = 0.$$

Using then (45) implies

$$\begin{aligned}\int |\nabla u|^2 \tau^2\,\mathrm{d}x &\leqslant C \int \tau |\nabla u|\,|\nabla \tau| (u - \overline{u})\,\mathrm{d}x \\ &\leqslant \frac{1}{2} \int |\nabla u|^2 \tau^2\,\mathrm{d}x + C \int |\nabla \tau|^2 |u - \overline{u}|^2\,\mathrm{d}x.\end{aligned}$$

The first integral can be absorbed into the left-hand side. The second can be estimated by Poincaré's inequality because $|\nabla \tau|^2$ behaves like $1/R^2$. Thus, we have

$$\int |\nabla u|^2 \tau^2\,\mathrm{d}x \leqslant K \int_{B_{2R} \setminus B_R} |\nabla u|^2\,\mathrm{d}x,$$

and the last term vanishes, as $R \to \infty$, since $\nabla u \in L^2$. This implies $u \equiv 0$. $\square$

Remark. The statement of the theorem also holds if

$$\varrho_i f^{(i)} = \operatorname{curl} g^{(i)} \tag{61}$$

(i.e. f need not be necessarily vanishing), and also some small interaction term is allowed (where a having compact support is used):

$$a \text{ small enough.} \tag{62}$$

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References

[1] *R. J. DiPerna, P.-L. Lions*: Ordinary differential equations, transport theory and Sobolev spaces. Invent. Math. *98* (1989), 511–547.

[2] *R. Erban*: On the static-limit solutions to the Navier-Stokes equations of compressible flow. J. Math. Fluid Mech. *3* (2001), 393–408.

[3] *E. Feireisl*: On compactness of solutions to the compressible isentropic Navier-Stokes equations when the density is not square integrable. Commentat. Math. Univ. Carol. *42* (2001), 83–98.

[4] *E. Feireisl*: Dynamics of Viscous Compressible Fluids. Oxford University Press, New York, 2004.

[5] *E. Feireisl, H. Petzeltová*: On the zero-velocity-limit solutions to the Navier-Stokes equations of compressible flow. Manuscr. Math. *97* (1998), 109–116.

[6] *J. Frehse, S. Goj, and J. Málek*: A Stokes-like system for mixtures. In: Nonlinear Problems in Mathematical Physics and Related Topics II. International Mathematical Series (M. Sh. Birman, S. Hildebrandt, V. A. Solonnikov, and N. N. Uraltseva, eds.). Kluwer, New York, 2002, pp. 119–136.

[7] *J. Frehse, S. Goj, and J. Málek*: Existence of solutions to a Stokes-like system for mixtures. SIAM J. Math. Anal. *36* (2005), 1259–1281.

[8] *A. E. Green, P. M. Naghdi*: A dynamical theory of interacting continua. Internat. J. Engrg. Sci. *3* (1965), 231–241.

[9] *A. E. Green, P. M. Naghdi*: A unified procedure for construction of theories of deformable media. III. Mixtures of interacting continua. Proc. Roy. Soc. London Ser. A *448* (1995), 379–388.

[10] *J. G. Heywood, M. Padula*: On the existence and uniqueness theory for steady compressible viscous flow. In: Fundamental Directions in Mathematical Fluid Mechanics (G. P. Galdi et al., eds.). Birkhäuser-Verlag, Basel, 2000, pp. 171–189.

[11] *P.-L. Lions*: Mathematical Topics in Fluid Mechanics. Vol. 2: Compressible Models. Oxford University Press, New York, 1998.

[12] *A. Novotný*: Compactness of steady compressible isentropic Navier-Stokes equations via the decomposition method (the whole 3-D space). In: Theory of the Navier-Stokes Equations (J. G. Heywood et al., eds.). World Sci. Publishing, River Edge, 1998, pp. 106–120.

[13] *K. R. Rajagopal, L. Tao*: Mechanics of Mixtures. World Scientific Publishing, River Edge, 1995.

[14] *C. Truesdell*: Mechanical basis of diffusion. J. Chemical Physics *37* (1962), 2337.

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