APPLICATION OF RELAXATION SCHEME TO DEGENERATE VARIATIONAL INEQUALITIES

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Abstract. In this paper we are concerned with the solution of degenerate variational inequalities. To solve this problem numerically, we propose a numerical scheme which is based on the relaxation scheme using non-standard time discretization. The approximate solution on each time level is obtained in the iterative way by solving the corresponding elliptic variational inequalities. The convergence of the method is proved.

Keywords: degenerate variational inequalities, numerical solution of variational inequalities, free boundary problem, oxygen diffusion problem

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1. Introduction

The aim of this paper is to propose an efficient numerical scheme for solving a degenerate variational inequality of the form

$$
\begin{aligned}
(1)\qquad & \int_I \big(\partial_t b(u), v-u\big) + \int_I \big(k\nabla u, \nabla(v-u)\big) + \int_I (g, v-u)_{\Gamma_2} \\
& \geqslant \int_I \big(f(b(u)), v-u\big) \qquad u \in L_2(I,\mathcal{K}), \quad \forall v \in L_2(I,\mathcal{K}), \\
& \qquad u(x,0) = u_0(x) \quad \text{in } \Omega.
\end{aligned}
$$

Here $\Omega \subset \mathbb{R}^{\mathbb{N}}$ is a bounded domain with Lipschitz continuous boundary $\partial\Omega = \overline{\Gamma_1 \cup \Gamma_2}$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_1 \neq \emptyset$ is measurable, $I = (0,T)$ and

$$
\mathcal{K} = \{v \in V \colon\ v\big|_{\Gamma_1} = 0, \quad v \geqslant 0 \quad \text{a.e. in } \Omega\}
$$

is closed, convex, nonempty. The scheme is based on relaxation schemes developed in [11], [12] for equations.

We use the standard function spaces $V = W_2^1(\Omega)$ (Sobolev space), $L_2(\Omega)$, $L_2(I, L_2(\Omega))$. By $\|.\|, |.|_2, |.|_{\Gamma_2}$ we denote the norms in the function spaces V, $L_2(\Omega)$, $L_2(\Gamma_2)$ and $(u, v) = \int_\Omega uv \, \mathrm{d}x$.

We consider b, k, g, f, u_0 such that

1. $b\colon \mathbb{R} \to \mathbb{R}$ is a nondecreasing, Lipschitz continuous function with a constant L_b such that $0 \leqslant b'(s) \leqslant L_b$,
2. $k = (k^{l,m}(t))$ is a symmetric and uniformly positive definite matrix in I with a constant c_k, Lipschitz continuous (with a constant L_k),
3. $g = g(t)$ is a Lipschitz continuous function,
4. $f = f(t, s)$ is a Lipschitz continuous in s, moreover $\exists c_f > 0\colon\ |f_t(t, s)| \leqslant c_f(1 + |s|)$,
5. $u_0 \in L_2(\Omega)$ is such that $u_0\big|_{\Gamma_1} = 0$ and $u_0 \geqslant 0$ a.e. in Ω.

This mathematical model includes a large scale of problems from physics, mechanics, biology and chemistry. As an example we present an oxygen diffusion problem. The diffusion with the absorption process is represented by the partial differential equation

$$\begin{aligned} &\partial_t b(u) = \nabla(k(t).\nabla u) + f(u) && \text{in } \Omega, \quad t \in I, \\ &u(x, t) \geqslant 0 && \text{in } \Omega \end{aligned}$$

where $u(x, t)$ denotes the concentration of oxygen free to diffuse at a point x at time t, b can express the storativity of oxygen in the domain Ω which depends on the concentration of oxygen in Ω, k is the diffusion tensor, $f(u)$ is the rate of consumption of oxygen per unit volume of the medium such that $f(u) = -m$ for $u > 0$ and $f(u) = 0$ for $u \leqslant 0$. The boundary conditions are in the form

$$\begin{aligned} u &= 0 && \text{on } \Gamma_1, \quad t \in I, \\ -\nu^T k(t) \nabla u &= g(t) && \text{on } \Gamma_2, \quad t \in I \end{aligned}$$

and the initial condition

$$u(0, x) = u_0(x) \qquad \text{in } \Omega.$$

Since the concentration of oxygen is non-negative, we can formulate this problem in the form of a variational inequality (1).

R e m a r k 1. We can interpret the variational inequality (1) as a free boundary problem (see Glowinsky [8]). Considering the solution u of (1) we define

$$\begin{aligned}\Omega^{+} &= \{x \mid x\in\Omega\colon\ u(x,t)>0\},\\ \Omega^{0} &= \{x \mid x\in\Omega\colon\ u(x,t)=0\},\\ \gamma_t = \partial\Omega^{+}\cap\partial\Omega^{0},\quad u^{+}(x,t) &= u(x,t)\big|_{\Omega^{+}},\quad u^{0}(x,t)=u(x,t)\big|\Omega^{0}.\end{aligned}$$

Problem (1) can be formulated as a problem of finding γ_t (the free boundary) and u such that

$$\begin{aligned}\partial_t b(u(x,t)) - \nabla\big(k(t)\nabla u(x,t)\big) &= f\big(t,b(u(x,t))\big), && x\in\Omega^{+}, && t\in I,\\ u(x,t) &= 0, && x\in\Omega^{0}, && t\in I,\\ u(x,t) &= 0, && x\in\Gamma_1, && t\in I,\\ -\nu^{T}k(t)\nabla u(x,t) &= g(t), && x\in\Gamma_2, && t\in I,\\ u^{+}(x,t) &= u^{0}(x,t), && x\in\gamma_t, && t\in I,\\ \partial_\nu u^{+}(x,t) &= \partial_\nu u^{0}(x,t), && x\in\gamma_t, && t\in I,\\ u(x,0) &= u_0(x), && x\in\Omega.\end{aligned}$$

2. Relaxation scheme

Definition 1. A function $u(t)\in L_2(I,\mathcal{K})$ with $u\in L_\infty(I,\mathcal{K})$, $b(u(t))\in L_2(I,L_2(\Omega))$, $\partial_t b(u)\in L_2(I,L_2(\Omega))$ and $u(x,0)=u_0(x)$ satisfying (1) is called a variational solution.

Our goal is to solve numerically the variational inequality (1). An existence and uniqueness result for problem (1) was proved by Hornung [10] and in a more general form by Alt and Luckhaus [1].

The linear relaxation scheme corresponding to (1) reads

$$\begin{gathered}\big(\lambda_i(\theta_i-\theta_{i-1}),v-\theta_i\big)+\tau\big(k_i\nabla\theta_i,\nabla(v-\theta_i)\big)+\tau(g_i,v-\theta_i)_{\Gamma_2}\geqslant\tau(f_i,v-\theta_i)\\ \theta_i\in\mathcal{K},\quad\forall v\in\mathcal{K}\end{gathered}\tag{2}$$

where $\tau=\frac{T}{n}$ $(n\in\mathbb{N})$, $t_i=i\tau$, $i=1,\ldots,n$, $k_i:=k(t_i)$, $f_i:=f(t_i,b(u_{i-1}))$, $g_i:=g(t_i)$ and λ_i has to satisfy the convergence condition

$$\left|\lambda_i-\frac{b_n(u_{i-1}+\theta_i-\theta_{i-1})-b_n(u_{i-1})}{\theta_i-\theta_{i-1}}\right|<\tau^{\alpha}\tag{3}$$

with

$$b_n(s) = b(s) + \tau^d s \quad (0 < d < \alpha < 1) \tag{4}$$

and α and d constants independent of n. In the case $\theta_i = \theta_{i-1}$ we take $\lambda_i = b'(u_{i-1})$.

By means of θ_i we define u_i by the algebraic equality

$$u_i = u_{i-1} + \theta_i - \theta_{i-1} \tag{5}$$

with

$$\theta_0 = u_{0,n} \quad \text{for} \quad |u_0 - u_{0,n}|_2 = \mathcal{O}(\tau) \quad \text{and} \quad u_{0,n} \in W_2^1(\Omega). \tag{6}$$

Remark 2. The existence of the solution of the inequality (2) follows from Duvaut, Lions [5, Theorem 7.1] and from $\lambda_i > 0$ because of

$$\frac{b_n(u_i) - b_n(u_{i-1})}{\theta_i - \theta_{i-1}} \geqslant \tau^d,$$
$$\left|\lambda_i - \frac{b_n(u_i) - b_n(u_{i-1})}{\theta_i - \theta_{i-1}}\right| < \tau^\alpha,$$

and the assumption $0 < d < \alpha < 1$.

Remark 3. Now we introduce a constructive way of finding the couple λ_i, θ_i satisfying (2) and (3). We use an iteration scheme similar to that for variational equations by Jäger, Kačur [11], Kačur [12], which reads

$$\begin{aligned}\big(\lambda_{i,k-1}(\theta_{i,k} - \theta_{i-1}), v - \theta_{i,k}\big) + \tau\big(k_i \nabla\theta_{i,k}, \nabla(v - \theta_{i,k})\big) + \tau(g_i, v - \theta_{i,k})_{\Gamma_2} \\ \geqslant \tau(f_i, v - \theta_{i,k}) \quad \forall v \in \mathcal{K}, \quad \theta_{i,k} \in \mathcal{K}\end{aligned}$$

and

$$\lambda_{i,k} = \frac{b_n(u_{i-1} + \theta_{i,k} - \theta_{i-1}) - b_n(u_{i-1})}{\theta_{i,k} - \theta_{i-1}}, \qquad k \geqslant 1$$

starting with

$$\lambda_{i,0} = b'(u_{i-1}).$$

Note that if $\theta_{i,k} = \theta_{i-1}$, then we take $\lambda_{i,k} = b'(u_{i-1})$.

The convergence $\lambda_{i,k} \to \lambda_i$ and $\theta_{i,k} \to \theta_i$ was proved for variational equations (which are special cases of variational inequalities) in Kačur [12]. In the most practical implementations we can observe the convergence of iterations in k.

3. Convergence of the method (2)–(6)

We shall construct approximate solutions θ^n, u^n (Rothe's functions) by means of θ_i, u_i from (2) and (5) in the following way:

$$\theta^n(t) := \theta_{i-1} + \frac{t-t_{i-1}}{\tau}(\theta_i - \theta_{i-1}) \quad t \in \langle t_{i-1}, t_i\rangle\,, \quad i = 1, \dots, n,$$

and the step function

$$\overline{\theta}^n(t) := \theta_i \quad t \in (t_{i-1}, t_i\rangle, \quad i = 1, \dots, n$$

with

$$\overline{\theta}^n(0) := \theta_0.$$

Analogously we define u^n and $\overline{u}^n$. Here θ^n, $\overline{\theta}^n$, u^n, $\overline{u}^n$ are functions from $L_2(I, \mathcal{K})$, thus $\theta^n(t) := \theta^n(t,x)$, $\overline{\theta}^n(t) := \overline{\theta}^n(t,x)$, $u^n(t) := u^n(t,x)$, $\overline{u}^n(t) := \overline{u}^n(t,x)$.

We rewrite (2) in terms of the step function in the form

$$\begin{aligned}(7)\qquad (\partial_t \hat{b}_n(\overline{u}^n), v - \overline{\theta}^n) + (\overline{\omega}^n\tau^{\alpha-1}(\overline{\theta}^n - \overline{\theta}^n_\tau), v - \overline{\theta}^n) + (\overline{k}^n\nabla\overline{\theta}^n, \nabla(v - \overline{\theta}^n))\\ + (\overline{g}^n, v - \overline{\theta}^n)_{\Gamma_2} \geqslant (\overline{f}^n, v - \overline{\theta}^n) \quad \forall v \in \mathcal{K}\end{aligned}$$

where the convergence condition (3) was used for the function λ_i in the form

$$(8)\qquad \lambda_i = \frac{b_n(u_{i-1} + \theta_i - \theta_{i-1}) - b_n(u_{i-1})}{\theta_i - \theta_{i-1}} + \omega_i\tau^\alpha \quad \text{with} \quad |\omega_i| \leqslant 1.$$

Here $\overline{k}^n$, $\overline{g}^n$, $\overline{f}^n$, ω^n, θ^n, $\hat{b}_n$ are defined as $\overline{k}^n := k_i$, $\overline{g}^n := g_i$, $\overline{f}^n := f_i = f(t_i, b(u_{i-1}))$, $\omega^n := \omega_i$ for $t \in (t_{i-1}, t_i\rangle$, $\overline{\theta}^n_\tau := \overline{\theta}^n(t-\tau)$ and

$$(9)\qquad \begin{aligned}\hat{b}_n(\overline{u}^n) := b_n(u_{i-1}) + \frac{t-t_{i-1}}{\tau}\big(b_n(u_i) - b_n(u_{i-1})\big)\\ \text{for } t \in \langle t_{i-1}, t_i\rangle\,, \quad i = 1, \dots, n.\end{aligned}$$

Theorem 1. *Let assumptions 1–5 be fulfilled. Then there exists $u \in L_2(I, \mathcal{K})$ with $b(u) \in L_2(I, L_2(\Omega))$, $\partial_t b(u) \in L_2(I, L_2(\Omega))$ such that*

$$b_{\overline{n}}(\overline{u}^{\overline{n}}) \to b(u), \quad \partial_t \hat{b}_{\overline{n}}(\overline{u}^{\overline{n}}) \rightharpoonup \partial_t b(u) \quad \text{and} \quad \overline{\theta}^{\overline{n}} \rightharpoonup u \quad \text{in} \quad L_2(I, \mathcal{K})$$

where $\{\overline{n}\}$ is a suitable subsequence of $\{n\}$. If the solution u is unique then the original sequences $\{\theta^n\}, \{u^n\}$ are convergent.

To prove Theorem 1, we derive some a priori estimates, in which assumptions 1–5 will be employed. Denote $\delta\theta_i = \frac{\theta_i - \theta_{i-1}}{\tau}$ and $\delta b_i = \frac{b(u_i) - b(u_{i-1})}{\tau}$.

Lemma 1. *The estimate*

$$\tau\sum_{i=1}^{j}\int_\Omega \lambda_i|\delta\theta_i|^2 + \tau\sum_{i=1}^{j}|\delta b(u_i)|_2^2 + \|\theta_j\|^2 + \sum_{i=1}^{j}|\nabla(\theta_i-\theta_{i-1})|_2^2 \leqslant c \tag{10}$$

holds uniformly for $n \geqslant n_0 > 0$, where c is a generic positive constant independent of j, n.

P r o o f. We take the test function $v = \theta_{i-1}$ in (2). We sum it for $i = 1, \dots, j$ and write the corresponding inequality in terms $J_1 + J_2 + J_3 \leqslant J_4$.

We rearrange the term J_1 to the form

$$\begin{aligned} J_1 &= \sum_{i=1}^{j}\tau(\lambda_i\delta\theta_i, \delta\theta_i) \\ &= \frac{1}{2}\tau\Big(\sum_{i=1}^{j}\int_\Omega \lambda_i|\delta\theta_i|^2 + \sum_{i=1}^{j}\Big(\frac{b_n(u_i)-b_n(u_{i-1})}{\tau}, \frac{\theta_i-\theta_{i-1}}{\tau}\Big) \\ &\quad + \sum_{i=1}^{j}\tau^\alpha(\omega_i\delta\theta_i, \delta\theta_i)\Big) \\ &\geqslant \frac{1}{2}\tau\Big(\sum_{i=1}^{j}\int_\Omega \lambda_i|\delta\theta_i|^2 + \sum_{i=1}^{j}(\delta b(u_i), \delta\theta_i) + \sum_{i=1}^{j}\tau^d(\delta\theta_i, \delta\theta_i) \\ &\quad + \sum_{i=1}^{j}\tau^\alpha(\omega_i\delta\theta_i, \delta\theta_i)\Big) \\ &\geqslant \frac{1}{2}\tau\Big(\sum_{i=1}^{j}\int_\Omega \lambda_i|\delta\theta_i|^2 + \frac{1}{L_b}\Big(1 + \frac{1}{L_b}(\tau^d - \tau^\alpha)\Big)\sum_{i=1}^{j}|\delta b(u_i)|_2^2\Big) \\ &\geqslant \frac{1}{2}\tau\Big(\sum_{i=1}^{j}\int_\Omega \lambda_i|\delta\theta_i|^2 + \frac{1}{L_b}\sum_{i=1}^{j}|\delta b(u_i)|_2^2\Big), \end{aligned} \tag{11}$$

where we have used assumption 1, $\theta_i - \theta_{i-1} = u_i - u_{i-1}$ and the convergence condition (3) in the form (8).

For terms J_2 and J_3 we conclude

$$\begin{aligned}
(12)\qquad J_2 &= \frac{1}{2}\Big\{(k_j\nabla\theta_j,\nabla\theta_j)-(k_0\nabla\theta_0,\nabla\theta_0)\\
&\quad+\sum_{i=1}^{j}\big(k_i\nabla(\theta_i-\theta_{i-1}),\nabla(\theta_i-\theta_{i-1})\big)\\
&\quad-\sum_{i=1}^{j}\big((k_i-k_{i-1})\nabla\theta_{i-1},\nabla\theta_{i-1}\big)\Big\}\\
&\geqslant \frac{1}{2}\Big\{c_k|\nabla\theta_j|_2^2-c+c_k\sum_{i=1}^{j}|\nabla(\theta_i-\theta_{i-1})|_2^2-L_k\tau\sum_{i=1}^{j}|\nabla\theta_i|_2^2\Big\},
\end{aligned}$$

$$\begin{aligned}
(13)\qquad J_3 &\geqslant (g_j,\theta_j)_{\Gamma_2}-(g_0,\theta_0)_{\Gamma_2}-\sum_{i=1}^{j}(g_i-g_{i-1},\theta_{i-1})_{\Gamma_2}\\
&\geqslant (g_j,\theta_j)_{\Gamma_2}-c-\sum_{i=1}^{j}\tau c\Big\{\varepsilon|\nabla\theta_i|_2^2+\frac{1}{\varepsilon}|\theta_i|_2^2\Big\},
\end{aligned}$$

where the inequality

$$|\theta_i|_{\Gamma_2}^2 \leqslant c\Big(\varepsilon|\nabla\theta_i|_2^2+\frac{1}{\varepsilon}|\theta_i|_2^2\Big)$$

has been used.

We estimate

$$(14)\qquad (g_j,\theta_j)_{\Gamma_2}\leqslant |g_j|\,|\theta_j|_{\Gamma_2}\leqslant \frac{c}{\delta_1^2}+\delta_1^2 c\Big\{\varepsilon|\nabla\theta_j|_2^2+\frac{1}{\varepsilon}|\theta_i|_2^2\Big\}.$$

The right-hand side can be estimated (using Abel's summation) in the following way:

$$J_4 = K_2-K_3-K_1 = K_2-K_3-\sum_{i=2}^{j}(K_{1i},\theta_{i-1}).$$

Using assumption 4 we have

$$\begin{aligned}
|K_{1i}| &\leqslant |f(t_i,b(u_{i-1}))-f(t_{i-1},b(u_{i-1}))|+|f(t_{i-1},b(u_{i-1}))-f(t_{i-1},b(u_{i-2}))|\\
&\leqslant \tau c_f(1+|b(u_{i-1})|)+\tau c|\delta b(u_{i-1})|.
\end{aligned}$$

For K_1 we get

$$\begin{aligned}
(15)\quad |K_1| &\leqslant \tau c_f\sum_{i=2}^{j}|\theta_{i-1}|_2+\tau c_f\sum_{i=2}^{j}|b(u_{i-1})|_2|\theta_{i-1}|_2+\tau c\sum_{i=2}^{j}|\delta b(u_{i-1})|_2|\theta_{i-1}|_2\\
&\leqslant c+\tau\frac{c}{\delta_2^2}\sum_{i=1}^{j}|\theta_i|_2^2+\tau c\delta_2^2\sum_{i=1}^{j}|\delta b(u_i)|_2^2.
\end{aligned}$$

We estimate the term K_2 by

$$|K_2| \leqslant |f(t_j, b(u_{j-1}))|_2 |\theta_j|_2 \leqslant c + \frac{c}{\delta_3^2} |b(u_{j-1})|_2^2 + c\delta_3^2 |\theta_j|_2^2$$

and from

$$\frac{1}{\delta_3^2} |b(u_{j-1})|_2^2 \leqslant c|b(u_0)|_2^2 + \frac{\delta_5^2}{\delta_3^2} \Bigl(\sum_{i=1}^{j-1} |\delta b(u_i)|_2 \tau \Bigr)^2 \leqslant c + c\frac{\delta_5^2}{\delta_3^2} T\tau \sum_{i=1}^{j} \|\delta b(u_i)\|_2^2$$

we get

$$|K_2| \leqslant c + \tau c \frac{\delta_5^2}{\delta_3^2} \sum_{i=1}^{j} |\delta b(u_i)|_2^2 + c\delta_3^2 |\theta_j|_2^2. \tag{16}$$

We can estimate the last term of J_4:

$$|K_3| \leqslant |f(t_1, b(u_0))|_2 |\theta_1|_2 \leqslant c_\varepsilon + \varepsilon |\theta_1 - \theta_0|^2 \leqslant c_\varepsilon + \varepsilon \sum_{i=1}^{j} |\nabla(\theta_i - \theta_{i-1})|_2^2, \tag{17}$$

where we have used $|v|_2 \leqslant c\|v\|$ from the embedding $W_1^2(\Omega)$ into $L_2(\Omega)$.

Finally, if we choose suitable parameters $\delta_1, \ldots, \delta_5, \varepsilon$, we can summarize (11)–(17) into

$$\tau \sum_{i=1}^{j} \int_\Omega \lambda_i |\delta\theta_i|^2 + \tau \sum_{i=1}^{j} |\delta b(u_i)|_2^2 + \sum_{i=1}^{j} |\nabla(\theta_i - \theta_{i-1})|_2^2 + \|\theta_j\|^2$$
$$\leqslant c + c\tau \sum_{i=1}^{j} |\theta_i|_2^2 + c\tau \sum_{i=1}^{j} |\nabla\theta_i|_2^2.$$

The Gronwall lemma enables us to obtain the estimate (10). □

Lemma 2. *The sequence of $\{b_n(\overline{u}^n)\}$ is compact in $L_2(I, L_2(\Omega))$.*

P r o o f. We use Kolmogorov's compactness criterion to prove the compactness of $\{b_n(\overline{u}^n)\}$.

We put (8) into (2), take $v = \theta_{i-1}$ and sum it up for $i = 1, \ldots, j$:

$$\sum_{i=1}^{j} \frac{1}{\tau} \bigl(b_n(u_i) - b_n(u_{i-1}), \theta_i - \theta_{i-1} \bigr) + \sum_{i=1}^{j} \bigl(\omega_i \tau^{\alpha-1} (\theta_i - \theta_{i-1}), \theta_i - \theta_{i-1} \bigr)$$
$$+ \sum_{i=1}^{j} \bigl(k_i \nabla\theta_i, \nabla(\theta_i - \theta_{i-1}) \bigr) + \sum_{i=1}^{j} (g_i, \theta_i - \theta_{i-1})_{\Gamma_2} \leqslant \sum_{i=1}^{j} (f_i, \theta_i - \theta_{i-1}).$$

The second term is estimated by

$$\sum_{i=1}^{j}\big(\omega_i\tau^{\alpha-1}(\theta_i-\theta_{i-1}),\theta_i-\theta_{i-1}\big)\leqslant\sum_{i=1}^{j}\tau^{\alpha+1}(\delta\theta_i,\delta\theta_i)$$
$$=\sum_{i=1}^{j}\tau^{\alpha}\Big(\tau\int_\Omega\frac{1}{\lambda_i}\lambda_i|\delta\theta_i|^2\Big)\leqslant\tau^{\alpha-d}c\to 0\quad\text{for}\quad\tau\to 0,$$

where we have used $\lambda_i\geqslant\frac{1}{2}\tau^d>0$ for $\tau\leqslant\tau_0$.

We have estimated the third, the fourth and the right-hand side terms in the previous part. So we have to estimate the first term only.

From (10) we get

$$\int_I|\partial_t\hat{b}_n(\overline{u}^n)|_2^2\leqslant c\tag{18}$$

and hence

$$\int_0^{T-z}|b_n(\bar{u}^n(t+z))-b_n(\bar{u}^n(t))|_2^2\,\mathrm{d}t\leqslant\int_0^{T-z}\int_\Omega\Big(\int_0^z\partial_t\hat{b}_n(\bar{u}^n(t+s))\,\mathrm{d}s\Big)^2\,\mathrm{d}\Omega\,\mathrm{d}t$$
$$\leqslant\int_0^z\mathrm{d}s\int_I\int_\Omega(\partial_t\hat{b}_n(\bar{u}^n(t)))^2\,\mathrm{d}\Omega\,\mathrm{d}t\leqslant z\int_I|\partial_t\hat{b}_n(\overline{u}^n)|_2^2\,\mathrm{d}t\leqslant cz.$$

The estimate

$$\begin{aligned}\int_I\int_\Omega|b_n(\overline{u}^n(t,x+y))-b_n(\overline{u}^n(t,x))|^2&\leqslant c\int_I\int_\Omega|\overline{u}^n(t,x+y)-\overline{u}^n(t,x)|^2\\&\leqslant c\Big\{\int_I\int_\Omega|\overline{u}^n(t,x+y)-\overline{\theta}^n(t,x+y)|^2+|\overline{\theta}^n(t,x+y)-\overline{\theta}^n(t,x)|^2\\&\qquad+|\overline{\theta}^n(t,x)-\overline{u}^n(t,x)|^2\Big\}\leqslant c(\tau^2+|y|^2)\end{aligned}\tag{19}$$

follows directly from (10) because

$$|\theta_i-u_i|^2\leqslant\tau^2\quad\text{implies}\quad\int_I\int_\Omega|\overline{u}^n(t,x)-\overline{\theta}^n(t,x)|^2\leqslant\tau^2c,$$
$$|\nabla\theta_j|^2\leqslant c\quad\text{implies}\quad\int_I\int_\Omega|\overline{\theta}^n(t,x+y)-\overline{\theta}^n(t,x)|^2\leqslant c|y|^2.$$

From (18) and (19) and Kolmogorov's compactness criterion we get the compactness of the sequence $\{b_n(\overline{u}^n)\}_{n=1}^\infty$. □

P r o o f of Theorem 1. From (5) we have $\overline{u}^n = \overline{\theta}^n + (u_0 - \theta_0)$. Because $\int_I |\overline{\theta}^n|_2^2 \leqslant c$ and $\theta_0 \to u_0$ in $L_2(\Omega)$ we have $\int_I |\overline{u}^n|_2^2 \leqslant c$ and there exists $u \in L_2(I, L_2(\Omega))$ such that

$$\overline{u}^n \rightharpoonup u$$

and hence

$$\overline{\theta}^n \rightharpoonup u \quad \text{in } L_2(I, L_2(\Omega)).$$

From (10) we have $\int_I |\nabla \overline{\theta}^n|_2^2 \leqslant c$, thus there exists $\chi \in L_2(I, L_2(\Omega))\colon\ \nabla \overline{\theta}^n \rightharpoonup \chi$ and from $\overline{u}^n \rightharpoonup u$ we get $\chi = \nabla u$. Thus $\overline{u}^n \rightharpoonup u$ in $L_2(I, V)$. The space $L_2(I, \mathcal{K})$ is convex and closed, so $u \in L_2(I, \mathcal{K})$.

Due to the compactness of $\{b_n(\overline{u}^n)\}_{n=1}^{\infty}$ in $L_2(I, L_2(\Omega))$ there exists a subsequence $\{b_n(\overline{u}^{n_k})\}_{k=1}^{\infty}$ which converges in $L_2(I, L_2(\Omega))$ to a function χ, i.e.

$$\exists \chi \in L_2(I, L_2(\Omega))\colon\ b_n(\overline{u}^{n_k}) \to \chi.$$

Let us denote this subsequence again by $\{b_n(\overline{u}^n)\}_{n=1}^{\infty}$ and say that the original sequence converges to χ in the sense of subsequence.

Using the Minty-Browder argument and the monotonicity of b we obtain that $b(u) = \chi$. Indeed, the monotonicity of b_n implies

$$\int_I (b_n(\overline{u}^n) - b_n(v), \overline{u}^n - v) \geqslant 0 \quad \forall v \in L_2(I, L_2(\Omega))$$

and then for $n \to \infty$ we have $\int_I (\chi - b(v), u - v) \geqslant 0$. We set $v = u + \varepsilon w$, then $\int_I (\chi - b(u + \varepsilon w), w) \geqslant 0$. Now we set $v = u - \varepsilon w$; then $\int_I (\chi - b(u - \varepsilon w), w) \leqslant 0$. For $\varepsilon \to 0$ we obtain that $\chi = b(u)$.

Since $\int_I |\partial_t \hat{b}_n(\overline{u}^n(t))|_2^2 \leqslant c$, there exists a function $\chi \in L_2(I, L_2(\Omega))$ such that $\partial_t \hat{b}_n(\overline{u}^n(t)) \rightharpoonup \chi$. The definition of the function $\hat{b}_n$ implies

$$\begin{aligned} \int_I |\hat{b}_n(\overline{u}^n(t)) - b_n(\overline{u}^n(t))|_2^2 &\leqslant \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} 2|b_n(u_i) - b_n(u_{i-1})|_2^2 \\ &\leqslant 2\tau \sum_{i=1}^{j} |b_n(u_i) - b_n(u_{i-1})|_2^2 \leqslant c\tau \end{aligned}$$

and hence $\chi = \partial_t b(u)$ because $b_n(\overline{u}^n(t)) \to b(u)$ and $\hat{b}_n(\bar{u}^n) \to b(u)$ in $L_2(I, L_2(\Omega))$. $\square$

In order to pass to the limit for $n \to \infty$ in (7) we need two lemmas:

Lemma 3. *Let u be as in Theorem* 1. *Then*

$$\int_0^t (\partial_t b(u), u) = \int_\Omega B(u(t)) - \int_\Omega B(u_0),$$

where

$$B(s) := b(s)s - \int_0^s b(z)\,\mathrm{d}z.$$

P r o o f. For the proof see, e.g. Alt, Luckhaus [1], the proof of Lemma 1.3. □

Lemma 4. *Let u be as in Theorem* 1. *Then*

$$\liminf_{n\to\infty} \int_0^t (\partial_t \hat{b}_n(\overline{u}^n), \overline{u}^n) \geqslant \int_\Omega B(u) - \int_\Omega B(u_0).$$

P r o o f. The proof of this lemma follows from [9]. We have $\overline{u}^n \rightharpoonup u$ in $L_2(I, L_2(\Omega))$ by virtue of Theorem 1. There exists a convex combination $v^l = \sum\limits_{n=l}^{N(l)} \alpha_n^l \overline{u}^n$ with $\sum\limits_{n=l}^{N(l)} \alpha_n^l = 1$, $\alpha_n^l \geqslant 0$ of $\{\overline{u}^n\}_{n=l}^{N(l)}$ such that $v^l \to u$ in $L_2(I, L_2(\Omega))$ (see Ekeland, Temam [6]).

Let us denote

$$r_n(t) = \int_0^t (\partial_t \hat{b}_n(\overline{u}^n), \overline{u}^n) \quad \text{for } t \in I.$$

Since $\{r_n\}$ is bounded in $L_1(I)$ and $\{\partial_t r_n\}$ is bounded in $L_1(I)$, $\{r_n\}$ is compact in $L_1(I)$. Choosing a suitable subsequence we can assume $r_n(t) \to r(t)$ with $n \to \infty$ for a.e. t in I. Hence we deduce

$$\begin{aligned}
r(t) &\geqslant \liminf_{l\to\infty} \sum_{n=l}^{N(l)} \alpha_n^l r_n(t) = \liminf_{l\to\infty} \sum_{n=l}^{N(l)} \int_0^t \alpha_n^l (\partial_t \hat{b}_n(\overline{u}^n), \overline{u}^n) \\
&\geqslant \liminf_{l\to\infty} \sum_{n=l}^{N(l)} \alpha_n^l \Big(\int_\Omega B_n(\overline{u}^n(t)) - \int_\Omega B_n(u_0) \Big) \\
&= \liminf_{l\to\infty} \sum_{n=l}^{N(l)} \alpha_n^l \Big(\int_\Omega B(\overline{u}^n(t)) - \Big(\int_\Omega B(u_0) + \frac{1}{2}\tau^d (|\overline{u}^n|_2^2 - |u_0|_2^2) \Big) \Big) \\
&\geqslant \liminf_{l\to\infty} \int_\Omega B(v^l(t)) - \int_\Omega B(u_0) = \int_\Omega B(u(t)) - \int_\Omega B(u_0)
\end{aligned}$$

because of the convexity of B. Thus, the proof is complete. □

Theorem 2. *Let assumptions* 1–5 *be fulfilled. Then* u *from Theorem* 1 *is a solution of the variational inequality* (1).

Proof. We integrate the inequality (7) over $(0,t)$ and rewrite it to

$$\begin{aligned}(20)\qquad &\int_0^t (\partial_t \hat{b}_n(\overline{u}^n), v) + \int_0^t (\overline{k}^n \nabla\overline{\theta}^n, \nabla v) + \int_0^t (\overline{g}^n, v)_{\Gamma_2} \\ &\geqslant \int_0^t (\partial_t \hat{b}_n(\overline{u}^n), \overline{\theta}^n) + \int_0^t (\overline{\omega}^n \tau^{\alpha-1}(\overline{\theta}^n - \overline{\theta}^n_\tau), \overline{\theta}^n - v) \\ &\quad + \int_0^t (\overline{k}^n \nabla\overline{\theta}^n, \nabla\overline{\theta}^n) + \int_0^t (\overline{g}^n, \overline{\theta}^n)_{\Gamma_2} + \int_0^t (\overline{f}^n, v - \overline{\theta}^n).\end{aligned}$$

Now we can take the limit for $n \to \infty$.

The convergence of the terms on the left-hand side of (20) follows directly from $\partial_t \hat{b}_n(\overline{u}^n) \rightharpoonup \partial_t b(u)$ in $L_2(I, L_2(\Omega))$, $\nabla\overline{\theta}^n \rightharpoonup \nabla u$ in $L_2(I, L_2(\Omega))$ and assumptions 2, 3. So we have

$$(21)\qquad \int_0^t (\partial_t \hat{b}_n(\overline{u}^n), v) \to \int_0^t (\partial_t b(u), v),$$

$$(22)\qquad \int_0^t (\overline{k}^n \nabla\overline{\theta}^n, \nabla v) \to \int_0^t (k\nabla u, \nabla v),$$

$$(23)\qquad \int_0^t (\overline{g}^n, v)_{\Gamma_2} \to \int_0^t (g, v)_{\Gamma_2}.$$

We pass to the limit for $n \to \infty$ with the first three terms on the right-hand side of (20) separately:

1. Due to Lemmas 4, 3 and by virtue of $\overline{\theta}^n - \overline{u}^n \to 0$ in $L_2(I, L_2(\Omega))$ and $\partial_t \hat{b}_n(\overline{u}^n) \rightharpoonup \partial_t b(u)$ we obtain

$$(24)\qquad \liminf_{n\to\infty} \int_0^t (\partial_t \hat{b}_n(\overline{u}^n), \overline{\theta}^n) \geqslant \int_0^t (\partial_t b(u), u).$$

2. For the second term we get

$$(25)\qquad \left| \int_0^t (\overline{\omega}^n \tau^{\alpha-1}(\overline{\theta}^n - \overline{\theta}^n_\tau), \overline{\theta}^n - v) \right|$$

$$\leqslant \int_0^t \tau^\alpha |(\delta\overline{\theta}^n, \overline{\theta}^n - v)| \leqslant \int_0^t c\tau^{\alpha - \frac{d}{2}} \left(\int_\Omega \overline{\lambda}^n |\delta\overline{\theta}^n|^2 \int_\Omega (\overline{\theta}^n - v)^2 \right)^{1/2} \to 0$$

because of (10) and $\overline{\theta}^n \rightharpoonup u$.

3. We know that

$$\int_0^t (\overline{k}^n \nabla\overline{\theta}^n, \nabla\overline{\theta}^n) = \int_0^t (k\nabla\overline{\theta}^n, \nabla\overline{\theta}^n) + \mathcal{O}(\tau).$$

We define an equivalent norm in $V=\{v\in W_1^2(\Omega)\colon\ v|_{\Gamma_1}=0\}$ by

$$\|v\|_- := \left(\int_\Omega k(\nabla v)^2\,\mathrm{d}x\right)^{1/2},$$

which is a weakly lower semicontinuous function; then

$$\liminf_{n\to\infty}\int_0^t(\overline{k}^n\nabla\overline{\theta}^n,\nabla\overline{\theta}^n)\geqslant\liminf_{n\to\infty}\int_0^t\|\bar\theta^n\|_-^2\geqslant\int_0^t(k\nabla u,\nabla u). \tag{26}$$

The convergence of the last two terms on the right-hand side of (20) follows from assumptions 3, 4 and from $\overline{\theta}^n\rightharpoonup u$ in $L_2(\Gamma_2)$, or $\overline{\theta}^n\rightharpoonup u$ in $L_2(\Omega)$, respectively. Thus

$$\int_0^t(\overline{g}^n,\overline{\theta}^n)_{\Gamma_2}\to\int_0^t(g,u)_{\Gamma_2}, \tag{27}$$

$$\int_0^t(\overline{f}^n,v-\overline{\theta}^n)\to\int_0^t(f,v-u). \tag{28}$$

So, taking the limit $n\to\infty$ in (20) and exploiting (21)–(28) we get $\forall v\in L_2(I,\mathcal{K})$

$$\int_0^t\big(\partial_t b(u),v-u\big)+\int_0^t\big(k\nabla u,\nabla(v-u)\big)+\int_0^t(g,v-u)_{\Gamma_2}\geqslant\int_0^t(f,v-u).$$

Thus u is a variational solution of (1). □

It is also possible to obtain a stronger convergence result:

Theorem 3. *Let assumptions* 1–5 *be fulfilled. Then*

$$\overline{\theta}^n\to u\quad\text{in } L_2(I,\mathcal{K}).$$

P r o o f. To prove the stronger convergence result we put (8) into (2) and integrate it over $(0,t)$. When taking $v=u$ we obtain

$$\begin{aligned}\int_0^t(\partial_t\hat b_n(\overline{u}^n),\overline{\theta}^n-u)&+\int_0^t\big(\overline{\omega}^n\tau^{\alpha-1}(\overline{\theta}^n-\overline{\theta}^n_\tau),\overline{\theta}^n-u\big)\\&+\int_0^t\big(\overline{k}^n\nabla\overline{\theta}^n,\nabla(\overline{\theta}^n-u)\big)+\int_0^t(\overline{g}^n,\overline{\theta}^n-u)_{\Gamma_2}\\&\leqslant\int_0^t(\overline{f}^n,\overline{\theta}^n-u)\quad u\in L_2(I,\mathcal{K}),\end{aligned}\tag{29}$$

where $\overline{\theta}^n_\tau:=\overline{\theta}^n(t-\tau,x)$.

We take into account that

$$\liminf_{n\to\infty} \int_0^t (\partial_t \hat{b}_n(\overline{u}^n), \overline{\theta}^n - u) \geqslant 0,$$

which is a consequence of

$$\int_0^t (\partial_t \hat{b}_n(\overline{u}^n), \overline{\theta}^n) = \int_0^t (\partial_t \hat{b}_n(\overline{u}^n), \overline{u}^n) + \int_0^t (\partial_t \hat{b}_n(\overline{u}^n), \overline{\theta}^n - \overline{u}^n),$$

$$\liminf_{n\to\infty} \int_0^t (\partial_t \hat{b}_n(\overline{u}^n), \overline{u}^n) \geqslant \int_0^t (\partial_t b(u), u)$$

and $\partial_t \hat{b}_n(\overline{u}^n) \rightharpoonup \partial_t b(u)$ in $L_2(I, L_2(\Omega))$ since $\bar{\theta}^n - \bar{u}^n \to 0$ in $L_2(I, L_2(\Omega))$. We rearrange the elliptic term of (29) to the form

$$\int_0^t \big(\overline{k}^n \nabla \overline{\theta}^n, \nabla(\overline{\theta}^n - u)\big) = \int_0^t \big(\overline{k}^n \nabla(\overline{\theta}^n - u), \nabla(\overline{\theta}^n - u)\big) + \int_0^t \big(\overline{k}^n \nabla u, \nabla(\overline{\theta}^n - u)\big)$$
$$\geqslant c \int_0^t \|\overline{\theta}^n - u\|^2 + \mathcal{O}(1)$$

since

$$\overline{\theta}^n \rightharpoonup u \quad \text{in } L_2(I, V) \quad \text{and} \quad |\overline{k}^n - k| \to 0$$

and $|\nabla v|_2$ is a norm equivalent to $\|v\|$ because of $\operatorname{mess} \Gamma_1 > 0$. Due to the convergence properties of $\overline{\theta}^n$ we find out

$$\int_0^t \big(\overline{\omega}^n \tau^{\alpha-1}(\overline{\theta}^n - \overline{\theta}^n_\tau), \overline{\theta}^n - u\big) \to 0,$$

$$\int_0^t (\overline{g}^n, \overline{\theta}^n - u)_{\Gamma_2} \to 0 \quad \text{for} \quad n \to \infty,$$

$$\int_0^t (\overline{f}^n, \overline{\theta}^n - u) \to 0$$

because of Theorem 1, 3 and 4. Thus the proof is complete. □

Similar convergence results can be obtained when elliptic variational inequality (2) is projected to a finite dimensional space V_h (e.g. by FEM) provided $V_h \to V$ for $h \to 0$ in the canonical sense. Then instead of $\bar{u}^n$ we obtain $\bar{u}^{(\mu)}$ with $\mu = (\tau, h)$, $\mu \to 0$.

4. Numerical experiments

To illustrate the efficiency of the approximation scheme (2) we use a model of the oxygen diffusion problem in the form (1). Applying the approximation scheme (2) to the variational inequality (1) we get a sequence of linear elliptic variational inequalities, which we have solved by a standard method for solving such problems (modification of Gauss-Seidel method, see, e.g. Cea [2]).

E x a m p l e 1. We consider a one-dimensional problem (1). We consider $\Omega = (0,1)$, $\Gamma_1 = \{1\}$, $\Gamma_2 = \{0\}$,

$$b(s) = \begin{cases} u^r, & u > 0, \\ 0, & u < 0, \end{cases}$$

$k(t) \equiv 1$, $g \equiv 0$,

$$f(s) = \begin{cases} -m, & u > 0, \\ 0, & u \leqslant 0, \end{cases}$$

$m = 1$ and $u_0(x) = \frac{1}{2}(1-x)^2$ for $x \in (0,1)$. We present the results of this problem for $r = 1$ (Tab. 1, Fig. 1), $r = 2$ (Tab. 2, Fig. 2) and $r = 3$ (Tab. 3, Fig. 3). Our figures show the evolution of a) the concentration of oxygen for various times and b) the moving free boundary. In the case of $r = 1$ we compare our numerical solution with those obtained by Crank and Gupta [3] (C-G columns in Tab. 1), Donat, Marquina and Martínez [4] (D-M-M columns in Tab. 1) and Furzeland [7] (F columns in Tab. 1). We have arrived at the "total absorption time" $T = 0.1977$, i.e., at the point where there is no oxygen in the domain Ω. The total absorption time for the analytical solution derived by Crank and Gupta [3] is $T = 0.196731$.

time	$u(0,t)$			
	81 points	C-G	D-M-M	F
0.04	0.27084	0.274324	0.276975	0.2745
0.10	0.143658	0.143177	0.144939	0.1433
0.18	0.022048	0.0215383	0.023538	0.0219
0.19	0.009279	0.00853796	0.010913	0.0091

time	$s(t)$			
	81 points	C-G	D-M-M	F
0.04	0.983	0.998271	0.998273	0.9992
0.10	0.924	0.892989	0.932287	0.9358
0.18	0.507	0.400949	0.531313	0.5028
0.19	0.341	0.255895	0.400626	0.3477

Table 1. Values of $u(0,t)$ and free boundary position $x = s(t)$ for $r = 1$.

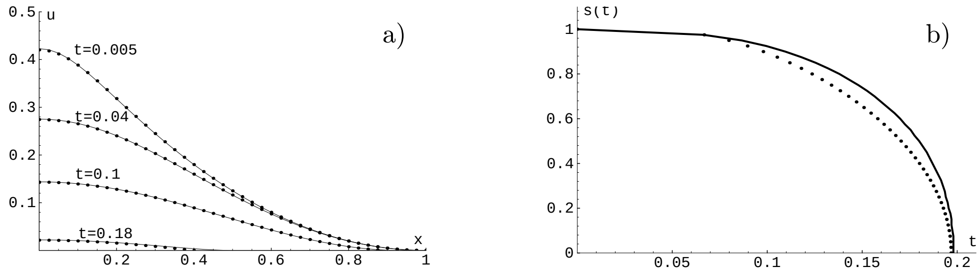

Figure 1. Evolution of a) the concentration of oxygen and b) the moving boundary $x = s(t)$ (full line—numerical solution, dotted line—solution obtained by Crank and Gupta).

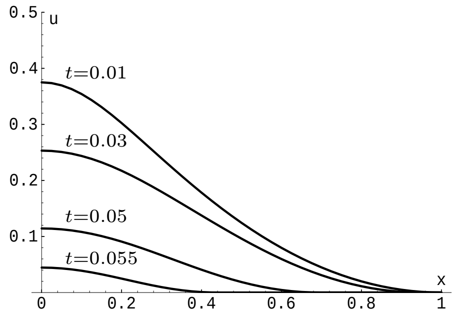

Figure 2. Concentration of oxygen ($r = 2$).

time	$u(0,t)$	s(t)
0.01	0.375009	0.983
0.03	0.253189	0.935
0.05	0.114358	0.682
0.055	0.0445203	0.447

Table 2. Values of $u(0,t)$ and free boundary position $x = s(t)$ for $r = 2$.

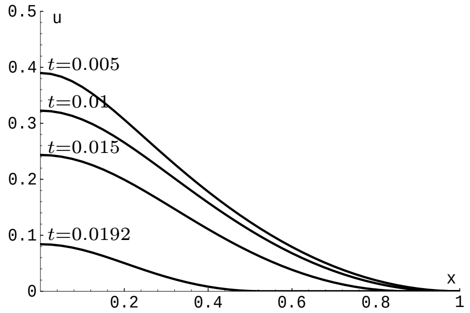

Figure 3. Concentration of oxygen ($r = 3$).

time	$u(0,t)$	s(t)
0.005	0.389865	0.983
0.01	0.322587	0.95
0.015	0.243483	0.864
0.0192	0.0840281	0.52

Table 3. Values of $u(0,t)$ and free boundary position $x = s(t)$ for $r = 3$.

Finally, Figure 4 presents the evolution of the moving boundary for different exponents r in the function b ($r = 1$—dotted line, $r = 2$—dashed line and $r = 3$—full line).

E x a m p l e 2. We consider the following two-dimensional situation. Let $\Omega = (0,1) \times (0,1)$. Each boundary side of Ω is divided into three parts in the ratio $\frac{1-q}{2}$: q : $\frac{1-q}{2}$ as depicted in Figure 5.

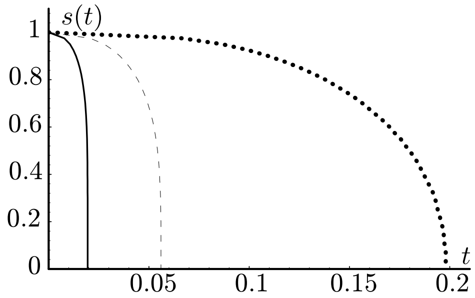

Figure 4. Evolution of the moving boundary for $r = 1$ (dotted line), $r = 2$ (dashed line), $r = 3$ (full line).

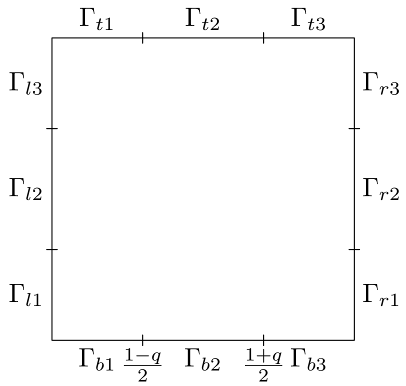

Figure 5. Domain Ω with boundary Γ.

Here we take $q = 0.4$. We denote $\Gamma_1 = \Gamma_{b1} \cup \Gamma_{b3} \cup \Gamma_{r1} \cup \Gamma_{r3} \cup \Gamma_{t1} \cup \Gamma_{t3} \cup \Gamma_{l1} \cup \Gamma_{l3}$ and $\Gamma_2 = \Gamma_{b2} \cup \Gamma_{r2} \cup \Gamma_{t2} \cup \Gamma_{l2}$. The function g is defined as follows:

$$g(x,t) = \begin{cases} g_{b2} & \text{on } \Gamma_{b2} \times (0,T), \\ g_{r2} & \text{on } \Gamma_{r2} \times (0,T), \\ g_{t2} & \text{on } \Gamma_{t2} \times (0,T), \\ g_{l2} & \text{on } \Gamma_{l2} \times (0,T) \end{cases}$$

and the initial condition is $u_0((x,y)) = 0$. Functions b and f are the same as in the previous one-dimensional example.

We are interested in the stationary state of the concentration of oxygen in the domain Ω where the consumption of oxygen and the flow of oxygen into the domain through the boundary Γ_2 are in balance. The stationary solution appears when $q(g_{b2} + g_{r2} + g_{t2} + g_{l2}) = m|\Omega^+|$, where Ω^+ is the subdomain of Ω in which oxygen is present. Figures 6–7 represent equilibrium for different exponents in the function b, constants g_{b2}, g_{r2}, g_{t2}, g_{l2} and the constant of consumption m (see Tab. 4).

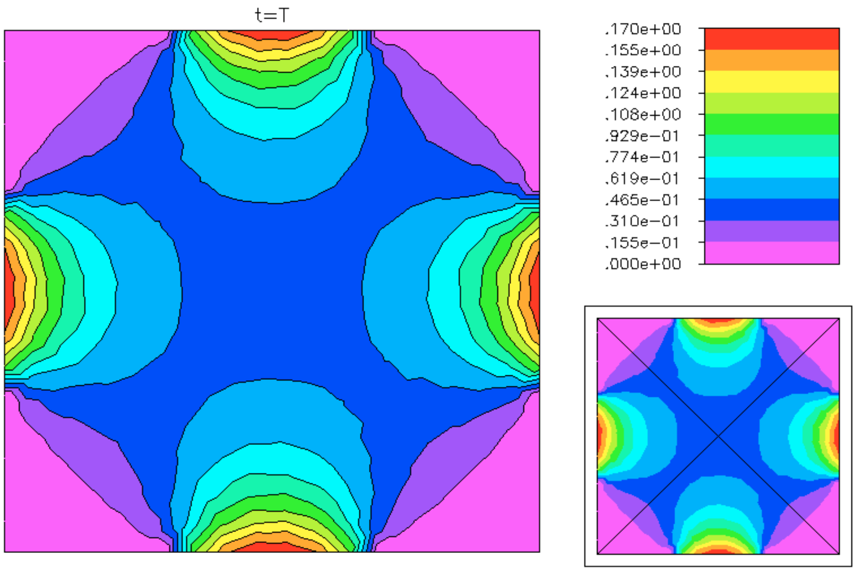

Figure 6. Equilibrium $u((x, y))$.

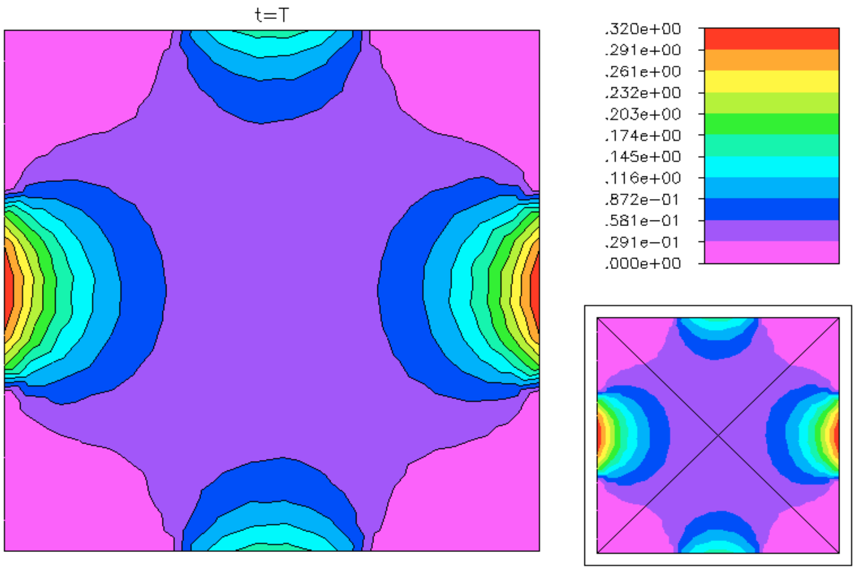

Figure 7. Equilibrium $u((x, y))$.

	g_{b2}	g_{r2}	g_{t2}	g_{l2}	r	m
Figure 6	1	1	1	1	2	-0.5
Figure 7	1	2	1	2	3	-1

Table 4. Data for Figures 6–7.

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