DISCONTINUOUS WAVE EQUATIONS AND A TOPOLOGICAL DEGREE FOR SOME CLASSES OF MULTI-VALUED MAPPINGS

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Abstract. The Leray-Schauder degree is extended to certain multi-valued mappings on separable Hilbert spaces with applications to the existence of weak periodic solutions of discontinuous semilinear wave equations with fixed ends.

Keywords: discontinuous wave equations, topological degree, multi-valued mappings

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1. Introduction

In this paper we study the existence of weak 2π-periodic solutions to the discontinuous semilinear wave equation

$$\text{(Prob)} \qquad \begin{aligned} u_{tt} - u_{xx} + g(u) + f(x,t,u) &= h(x,t),\\ u(0,\cdot) = u(\pi,\cdot) &= 0, \end{aligned}$$

where $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$, $\Omega=(0,\pi)\times(0,2\pi)$ is Carathéodory continuous and nondecreasing in u, $g\colon\mathbb{R}\to\mathbb{R}$ is bounded nondecreasing and $h\in L^2(\Omega)$. Moreover, we suppose

$$|f(x,t,u)| \leqslant c_0|u| + h_0(x,t) \quad \forall u\in\mathbb{R},\ \forall\,(x,t)\in\Omega$$

for a constant $c_0>0$ and $h_0\in L^2(\Omega)$. (Prob) with $g=0$, i.e. for the continuous case, was studied in [2], where a construction of a topological degree is introduced for a class of monotone single-valued mappings. The main purpose of this paper is to extend that method to monotone multi-valued mappings. Like in [2], we use a Galerkin projection method, which is however not at all the standard Galerkin approximation method of [2]. We construct continuous one-parametric generalized

Galerkin projections which we use for the derivation of a one-parametric family of multi-valued mappings possessing the Leray-Schauder degree [7]. The basic Lemma 6 below is the stabilization of this degree for large parameters. In this way, we can define a topological degree for our multi-valued mappings. The rest of the paper is the extension of the main results of [2] to the multi-valued case. We end the paper with a discontinuous equation of a vibrating string presenting a problem at resonance with an infinite-dimensional kernel. This result is motivated by [4]. Other topological degrees for single and multi-valued mappings have been introduced in [1], [3], [5], [6] and [9].

2. Standard classes of mappings

Let H be a real separable Hilbert space with an inner product $(\cdot,\cdot)$. In what follows we use the following notation:

- $\{a_n\}$ means $\{a_n\}_{n=1}^{\infty}$;
- $\lim a_n$ means $\lim\limits_{n\to\infty} a_n$ (similarly $\limsup$ and $\liminf$);
- $|\cdot|$ always denotes the norm in H;
- $\|\cdot\|$ denotes the norm in the space of continuous linear mappings $\mathscr{L}(H)$.

A multi-valued mapping $F\colon\ H \to 2^H$ is

- *monotone* (denote $F \in (mMON)$), if

$$(f_u^* - f_v^*, u - v) \geqslant 0$$

  for all $u, v \in H$ and all selections $f_u^* \in F(u)$, $f_v^* \in F(v)$;
- *quasimonotone* ($F \in (mQM)$), if for any sequence $\{u_n\}$ in H with $u_n \rightharpoonup u$ and for all selections $f_n^* \in F(u_n)$ we have

$$\liminf (f_n^*, u_n^* - u) \geqslant 0;$$

- *pseudomonotone* ($F \in (mPM)$), if for any sequence $\{u_n\}$ in H with $u_n \rightharpoonup u$, the existence of selections $f_n^* \in F(u_n)$ and a point $f^* \in H$ with $f_n^* \rightharpoonup f^*$ and $\limsup(f_n^*, u_n - u) \leqslant 0$ implies that $f^* \in F(u)$ and $(f_n^*, u_n) \to (f^*, u)$;
- *of class* (mS_+) ($F \in (mS_+)$), if for any sequence $\{u_n\}$ in H with $u_n \rightharpoonup u$, the existence of selections $f_n^* \in F(u_n)$ with $\limsup(f_n^*, u_n - u) \leqslant 0$ implies that $u_n \to u$;
- *compact* ($F \in (mCOMP)$), if for any bounded sequence $\{u_n\}$ in H and for any $f_n^* \in F(u_n)$ the sequence $\{f_n^*\}$ has a convergent subsequence;
- *of Leray-Schauder type* ($F \in (mLS)$), if $F = I + C$, where I is the identity on H, for some $C \in (mCOMP)$;

- *bounded*, if for any bounded set $B \subset H$ the set $\bigcup_{u \in B} F(u)$ is bounded;
- *convex-valued*, if $F(u)$ is a non-empty convex set in H for any $u \in H$;
- *upper semicontinuous* (F is usc), if $F^{-1}(A)$ is closed in H whenever $A \subset H$ is closed;
- *weak upper semicontinuous* (F is w-usc), if for any sequence $\{u_n\} \in H$, $u_n \to u \in H$, the existence of selections $f_n^* \in F(u_n)$ with $f_n^* \rightharpoonup f^* \in H$ implies $f^* \in F(u)$.

In what follows, we assume that all mappings used are bounded, w-usc and convex-valued. When a mapping is defined only on a subset of H, the above definitions can be modified in an obvious way.

Proposition 1. *For the classes defined above the following inclusions hold:*

$$(mLS) \subset (mS_+) \subset (mPM) \subset (mQM),$$
$$(mMON) \subset (mPM), \qquad (mCOMP) \subset (mQM).$$

Proposition 2. *If $F \in (mQM)$ and $G \in (mS_+)$ then $F + G \in (mS_+)$.*

Proofs of Propositions 1 and 2 are similar to those for single-valued mappings [2].

3. Classes of admissible mappings

Let G be a bounded open subset in H, M a closed subspace of H and let Q and P stand for the orthogonal projections to M and $M^\perp$, respectively. The family

$$\mathscr{F}_G = \{F\colon \overline{G} \to 2^H \mid F = Qg + Pf \text{ for some } g \in (mLS) \text{ and } f \in (mS_+)\}$$

is called *the class of admissible mappings*. Other very important and more general classes are

$$\mathscr{F}_G(mQM) = \{F\colon \overline{G} \to 2^H \mid F = Qg + Pf \text{ for some } g \in (mLS) \text{ and } f \in (mQM)\}$$

and $\mathscr{F}_G(mPM)$ defined accordingly. Obviously $\mathscr{F}_G = \mathscr{F}_G(mS_+) \subset \mathscr{F}_G(mPM) \subset \mathscr{F}_G(mQM)$.

Let $L\colon H \supset D(L) \to H$ be a closed densely defined linear operator with $\operatorname{Im} L = (\operatorname{Ker} L)^\perp$. Let $L_0 = L/\operatorname{Im} L$ and assume that the right inverse $L_0^{-1}\colon \operatorname{Im} L \to \operatorname{Im} L$ is compact. We choose $M = \operatorname{Im} L$ and $M^\perp = \operatorname{Ker} L$. Let $N\colon H \to 2^H$. Then similarly to [2], we consider the mapping

$$F = Q(I + L_0^{-1}QN) + PN.$$

Clearly, $F \in \mathscr{F}_G$ for $N \in (mS_+)$.

Lemma 3. *Let F and N be defined as above. Then for $h \in H$*

$$\text{(SInc)} \qquad h \in Lu + N(u) \quad \text{with } u \in D(L) \cap \overline{G}$$

if and only if

$$y \in F(u) \text{ with } u \in \overline{G} \text{ for } y = (L_0^{-1}Q + P)h.$$

The proof is the same as for single-valued mappings [2]. The inclusion $h \in Lu - N(u)$ can be treated analogously.

4. Construction of the degree for $\mathscr{F}_G(mS_+)$

We construct a topological degree function for $\mathscr{F}_G$. First, we define a class of admissible homotopies. A mapping: $(t,u) \to f_t(u)$ from $[0,1] \times \overline{G}$ to 2^H is *a (multi-) homotopy of the class* (mS_+), if for any sequences $\{u_n\}$ in $\overline{G}$, $\{t_n\}$ in $[0,1]$, $f_n^* \in f_{t_n}(u_n)$ with $u_n \rightharpoonup u$, $t_n \to t$ and $\limsup(f_n^*, u_n - u) \leqslant 0$ we have $u_n \to u$. We also assume that the mapping $(t,u) \to f_t(u)$ satisfies all the necessary conditions.

Similarly, a mapping: $(t,u) \to g_t(u) = (I + C_t)(u)$ from $[0,1] \times \overline{G}$ to 2^H is called *a (multi-)homotopy of the Leray-Schauder type*, if the mapping $(t,u) \to C_t(u)$ is compact. We denote

$$\mathscr{H}_G = \{F_t \mid F_t = Qg_t + Pf_t\}$$

where g_t and f_t are homotopies of the Leray-Schauder type and of the class (mS_+), respectively. The set $\mathscr{H}_G$ is called *the class of admissible homotopies*. Obviously $F_t = (1-t)F_1 + tF_2 \in \mathscr{H}_G$, $0 \leqslant t \leqslant 1$ for any $F_1, F_2 \in \mathscr{F}_G$.

We use Galerkin approximations [8] with respect to the subspace $M^\perp$. The space H ($M^\perp \subset H$) is separable so there exists a sequence $\{N_n\}$ of finite dimensional subspaces of $M^\perp$ with $N_n \subset N_{n+1}$ for all n, and $\cup_{n=1}^{\infty} N_n$ is dense in $M^\perp$. We denote by P_n the orthogonal projection from H to N_n. We extend this to generalized Galerkin approximations defined by

$$P_\lambda = (\lambda - n)P_{n+1} + (n+1-\lambda)P_n \quad \text{for any } \lambda \in [n, n+1].$$

We have the following obvious result.

Proposition 4. *The generalized Galerkin approximations satisfy*

i) $(P_\lambda u, v) = (u, P_\lambda v)$ *for every* $\lambda \geqslant 1$, $u, v \in H$;

ii) $\|P_\lambda\| \leqslant 1$ *for all* $\lambda \geqslant 1$;

iii) $P_\lambda v \to Pv$ *for every* $v \in H$ *as* $\lambda \to \infty$*;*
iv) $P_n P_\lambda = P_n$ *for every* $\lambda \geqslant n \in \mathbb{N}$*;*
v) $(z, P_\lambda z) \geqslant 0$ *for every* $z \in H$ *and* $\lambda \geqslant 1$.

For each $F = Q(I + C) + Pf \in \mathscr{F}_G$, we define the approximations $\{F_\lambda \mid \lambda \geqslant 1\}$ by

$$F_\lambda = I + QC + \lambda P_\lambda f.$$

We note that $QC + \lambda P_\lambda f$ is compact, convex-valued and usc for each $\lambda \geqslant 1$.

Similarly, for each admissible homotopy $F_t = Q(I + C_t) + Pf_t$, $0 \leqslant t \leqslant 1$, we have

$$(F_t)_\lambda = I + QC_t + \lambda P_\lambda f_t,$$

which is obviously a homotopy of the Leray-Schauder type for any $\lambda \geqslant 1$. Finally, if $y \in H$ is given, we denote

$$y_\lambda = Qy + \lambda P_\lambda y \qquad \text{for each } \lambda \geqslant 1.$$

It is clear that $(F - y)_\lambda = F_\lambda - y_\lambda$.

Proposition 5. *Let* $\{u_k\}$, $u_k \in H$ *and let* $\{P_\lambda\}$ *be the projections defined as above. Then*

a) if $u_k \rightharpoonup u$ *and* $\lambda_k \to \infty$ *then* $P_{\lambda_k} u_k \rightharpoonup Pu$*;*
b) if $u_k \to u$ *and* $\lambda_k \to \infty$ *then* $P_{\lambda_k} u_k \to Pu$.

Now we can formulate the basic lemma.

Lemma 6. *Let* $F_t = Q(I + C_t) + Pf_t$ *be an admissible homotopy,* y_t $(0 \leqslant t \leqslant 1)$ *a continuous curve in* H *and let* A *be a closed subset in* $\overline{G}$*. If* $y_t \notin F_t(A)$ *for all* $t \in [0, 1]$*, then there exists* $n_0 \in \mathbb{N}$ *such that*

$$(y_t)_\lambda \notin (F_t)_\lambda(A) \qquad \textit{for all } t \in [0,1] \textit{ and } \lambda \geqslant n_0.$$

P r o o f. Since also $F_t - y_t$ defines an admissible homotopy and $(F_t - y_t)_\lambda = (F_t)_\lambda - (y_t)_\lambda$, we may assume, without loss of generality, that $y_t \equiv 0$.

If the assertion were false, there would exist sequences $\{u_k\}$ in A and $\{\lambda_k\}$ in $[1, \infty)$, $\lambda_k \to \infty$, and $\{t_k\}$ in $[0, 1]$ such that $0 \in (F_{t_k})_{\lambda_k}(u_k)$. This is equivalent to the existence of selections $g_k^* \in f_{t_k}(u_k)$ and $c_k^* \in C_{t_k}(u_k)$ for which

$$u_k + Qc_k^* + \lambda_k P_{\lambda_k} g_k^* = 0.$$

Writing this equation in both subspaces M and $M^{\perp}$ we get

$$Qu_k + Qc_k^* = 0, \tag{1}$$

$$Pu_k + \lambda_k P_{\lambda_k} g_k^* = 0. \tag{2}$$

The sequence u_k is bounded therefore we can (taking a subsequence, if necessary) assume that $u_k \rightharpoonup u$ for some $u \in H$. Similarly we can assume $t_k \to t, t \in [0,1]$. We have $c_k^* \in C_{t_k}(u_k)$. Since C is a compact mapping we can assume $c_k^* \to z^*$, $z^* \in H$ and, since the sequence g_k^* is bounded, $g_k^* \rightharpoonup g^*$, $g^* \in H$.

Hence we have $Qc_k^* \to Qz^*$ and by (1), $Qu_k \to Qu$. By (2),

$$\frac{1}{\lambda_k} Pu_k + P_{\lambda_k} g_k^* = 0.$$

The set $\{u_k \mid k = 1, 2, \ldots\}$ is bounded, thus $\dfrac{1}{\lambda_k} Pu_k \to 0$ for $k \to \infty$. This leads to $P_{\lambda_k} g_k^* \to 0$. On the other hand, for $g_k^* \rightharpoonup g^*$ we conclude $P_{\lambda_k} g_k^* \rightharpoonup Pg^*$, which yields $Pg^* = 0$. Hence we have $Pg_k^* \rightharpoonup Pg^* = 0$ followed by $\lim(g_k^*, Pu) = 0, \forall u \in H$.

We continue with calculating of $\limsup(g_k^*, u_k - u)$:

$$\limsup(g_k^*, u_k - u) = \limsup(g_k^*, Pu_k - Pu) = \limsup(g_k^*, Pu_k). \tag{3}$$

From (2) we obtain $Pu_k = -\lambda_k P_{\lambda_k} g_k^*$. Inserting it in to the last term of (3) we get

$$\limsup(g_k^*, u_k - u) = -\liminf \lambda_k (g_k^*, P_{\lambda_k} g_k^*).$$

By v) of Proposition 4, $\lambda_k (g_k^*, P_{\lambda_k} g_k^*) \geqslant 0$ and it immediately follows that

$$\limsup(g_k^*, u_k - u) \leqslant 0.$$

Hence, by the definition of the homotopy f_t of the class (mS_+) we have $u_k \to u$, $u \in A$. We have $t_n \to t$, $g_k^* \rightharpoonup g^*$ and $u_k \to u$. Since $(t, u) \to f_t(u)$ is w-usc, we get $g^* \in f_t(u)$. Similarly, we get $z^* \in C_t(u)$. From $Qu_k \to Qu$, $Qc_k^* \to Qz^*$ using (1), we obtain $Qu + Qz^* = 0$. Since $Pg^* = 0$ we have

$$0 = Qu + Qz^* + Pg^* \in Qu + QC_t(u) + Pf_t(u) = F_t(u),$$

a contradiction. The proof is complete. $\square$

If we choose a constant homotopy $F_t = F = Qg + Pf$, $y_t = y$, and $A = \partial G$, we obtain the stabilization of a degree. If $y \notin F(\partial G)$, then there exists $\lambda_0 \geqslant 1$ such that $y_\lambda \notin F_\lambda(\partial G)$ for all $\lambda \geqslant \lambda_0$.

Lemma 7. *Let $F \in \mathscr{F}_G$ and $y \notin F(\partial G)$. Then there exists $\lambda_1 \in [1, \infty)$ such that*

$$d(F_\lambda, G, y_\lambda) = constant \quad \text{for all } \lambda \geqslant \lambda_1.$$

Due to Lemma 7 we can define a degree function for the class $\mathscr{F}_G$. We put

$$d(F, G, y) = \lim_{\lambda \to \infty} d(F_\lambda, G, y_\lambda) \tag{4}$$

for any given $F \in \mathscr{F}_G$ and $y \in H$ with $y \notin F(\partial G)$. In the next lemma we show that the degree function defined by (4) has all the usual properties. The proof is the same as in [2] with the use of the degree theory for multi-valued mappings [7].

Lemma 8. *Let G be an open bounded subset of H and $F \in \mathscr{F}_G(mS_+)$. Then*

i) *$d(F, G, y) \neq 0$ implies that there exists $y \in F(\overline{G})$.*

ii) *$d(F, G, y) = d(F, G_1, y) + d(F, G_2, y)$ (thus $F \in \mathscr{F}_{G_1}$ and $F \in \mathscr{F}_{G_2}$), whenever G_1 and G_2 are disjoint open subsets of G such that $y \notin F(\overline{G} \setminus (G_1 \cup G_2))$.*

iii) *$d(F_t, G, y_t)$ is independent of $t \in J = [0, 1]$ if $F_t \in \mathscr{H}_G$, y_t is a continuous curve in H and $y_t \notin F_t(\partial G)$ for all $t \in J$.*

iv) *$d(I, G, w) = 1$ if and only if $w \in G$ (normalization).*

We can also simply prove the *Borsuk Theorem* for the class $\mathscr{F}_G(mS_+)$.

Proposition 9. *Let G be an open bounded symmetric subset of H containing the origin and let F be a multi-mapping $\overline{G} \to 2^H$ satisfying $F(-u) = -F(u)$ for all $u \in \partial G$. Then if $F \in \mathscr{F}_G(mS_+)$, the inclusion $0 \in F(u)$ admits a solution u in $\overline{G}$ and $d(F, G, 0)$ is odd whenever defined.*

We have already proved all the desired properties of the degree function for the class $\mathscr{F}_G$. We formulate our result in the next theorem.

Theorem 10. *Let H be a real separable Hilbert space, G a bounded open subset of H, $\mathscr{F}_G$ the class of admissible mappings and $\mathscr{H}_G$ the class of admissible homotopies defined above. Then there exists a classical topological degree function d on $\mathscr{F}_G$ satisfying the properties* i), ii), iii) *and* iv) *from Lemma* 8 *with respect to $\mathscr{H}_G$ and the normalizing mapping I.*

5. A degree theory for $\mathscr{F}_G(mQM)$ and $\mathscr{F}_G(mPM)$

The most common method of studying the existence of solutions of differential equations (inclusions) applying a degree theory is to use a homotopy between a mapping and the normalizing mapping—the identity. However, it is not always possible to use such a homotopy. Sometimes we can substitute the identity with an other mapping with a nonzero degree. A mapping $R \in \mathscr{F}_G(mS_+)$ with $d(R, G, y) \neq 0$ for all $y \in R(G)$ is called *a reference mapping.*

The next basic theorem immediately follows from the properties of the degree function d (see [2, Theorem 2]).

Theorem 11. *Let G be an open bounded set, $G \subset H$, $R \in \mathscr{F}_G$ a reference mapping and $F \in \mathscr{F}_G$. If for a given $y \in H$ there exists $w \in R(G)$ such that*

$$(1-t)w + ty \notin (1-t)R(u) + tF(u) \quad \text{for all } u \in \partial G \text{ and } t \in [0,1],$$

then $d(F, G, y) \neq 0$ and hence the inclusion $y \in F(u)$ admits a solution u in G.

We already have constructed a degree function for the class $\mathscr{F}_G = \mathscr{F}_G(mS_+)$. By using this we can also build a similar degree function for the larger class $\mathscr{F}_G(mQM)$. By Proposition 2 we can use an (mS_+) approximation $f_\varepsilon = f + \varepsilon I$ for any given $f \in (mQM)$ and $\varepsilon > 0$. In fact, we generalize Theorem 11 in the following form (see [2, Theorem 3]).

Theorem 12. *Let G be an open bounded subset of H, $R \in \mathscr{F}_G(mS_+)$ a reference mapping and $F \in \mathscr{F}_G(mQM)$. If for a given $y \in H$ there exists $w \in R(G)$ such that the condition*

$$(1-t)w + ty \notin (1-t)R(u) + tF(u) \quad \text{for all } u \in \partial G \text{ and } t \in [0,1] \tag{5}$$

holds, then the inclusion $y \in F(u)$ is almost solvable, i.e. $y \in \overline{F(\overline{G})}$.

P r o o f. Without loss of generality, we can assume that $y = 0$ and $w = 0 \in R(G)$. If $0 \in \overline{F(\partial G)} \subset \overline{F(\overline{G})}$, the assertion is true. Thus let $0 \notin \overline{F(\partial G)}$. Since $F \in \mathscr{F}_G(mQM)$, F has a representation $F = Q(I + C) + Pf$ for some $C \in (mCOMP)$ and $f \in (mQM)$. Similar argument leads to $R = Q(I + C_0) + Pf_0$ for some $C_0 \in (mCOMP)$ and $f_0 \in (mS_+)$. Now we introduce an $\mathscr{F}_G(mS_+)$ approximation of F, for each $\varepsilon > 0$ we denote

$$F_\varepsilon = F + \varepsilon PR = Q(I + C) + P(f + \varepsilon f_0).$$

By Proposition 2, $f + \varepsilon f_0 \in (mS_+)$, therefore $F_\varepsilon \in \mathscr{F}_G(mS_+)$ for all $\varepsilon > 0$. For applying Theorem 11 to the mapping F_ε we show there exists ε_0 such that

$$0 \notin (1-t)R(u) + tF_\varepsilon(u) \quad \text{for all } u \in \partial G \text{ and } t \in [0,1] \text{ and } 0 < \varepsilon < \varepsilon_0. \tag{6}$$

If (6) were not true, we could find sequences $\{\varepsilon_n\}$, $\{t_n\} \subset [0,1]$ and $\{u_n\} \subset \partial G$ for which

$$0 \in (1-t_n)R(u_n) + t_n F_{\varepsilon_n}(u_n) \quad \text{for all } n \in \mathbb{N}. \tag{7}$$

Taking subsequences, if necessary, we can assume that $\varepsilon_n \to 0^+$, $t_n \to t$ and $u_n \rightharpoonup u$. Clearly $t \neq 1$, since otherwise $0 \in \overline{F(\partial G)}$ which contradicts the assumption. Thus $t \in [0,1)$. Writing (6) in the components and taking selections $c_n^* \in C(u_n)$, $d_n^* \in C_0(u_n)$, $f_n^* \in f(u_n)$ and $g_{n,1}^*, g_{n,2}^* \in f_0(u_n)$ we have

$$Qu_n + (1-t_n)Qd_n^* + t_n Qc_n^* = 0, \tag{8}$$
$$(1-t_n)Pg_{n,1}^* + t_n\varepsilon_n Pg_{n,2}^* + t_n Pf_n^* = 0 \tag{9}$$

with $r_n^* = Q(u_n + d_n^*) + Pg_{n,1}^* \in R(u_n)$ and $h_n^* = Q(u_n + c_n^*) + P(f_n^* + \varepsilon_n g_{n,2}^*) \in F_{\varepsilon_n}(u_n)$. Without loss of generality, we can assume that $c_n^* \to c^*$ and $d_n^* \to d^*$ for some $c^*, d^* \in H$, which yields $r_n^* \rightharpoonup r^*$ and $h_n^* \rightharpoonup h^*$. Therefore from (8) we have $Qu_n \to Qu$ ($u_n \rightharpoonup u$). By (9)

$$Pg_{n,1}^* = -\frac{t_n}{1-t_n}Pf_n^* - \frac{t_n\varepsilon_n}{1-t_n}Pg_{n,2}^*.$$

By using this and $Qu_n \to Qu$ we can calculate

$$\begin{aligned}
\limsup(g_{n,1}^*, u_n - u) &= \limsup(Pg_{n,1}^*, u_n - u) \\
&= \limsup\Big\{-\frac{t_n}{1-t_n}(Pf_n^*, u_n - u) - \frac{t_n\varepsilon_n}{1-t_n}(Pg_{n,2}^*, u_n - u)\Big\} \\
&= -\frac{t}{1-t}\liminf(f_n^*, u_n - u).
\end{aligned}$$

Since $f \in (mQM)$, we conclude that

$$\limsup(g_{n,1}^*, u_n - u) = -\frac{t}{1-t}\liminf(f_n^*, u_n - u) \leqslant 0.$$

By using the (mS_+) property of f_0 we obtain $u_n \to u$. Clearly $u \in \partial G$. From the w-usc property of the mapping R we get $r_n^* \rightharpoonup r^* \in R(u)$ and similarly $h_n^* \rightharpoonup h^* \in F(u)$. Consequently,

$$0 = (1-t_n)r_n^* + t_n h_n^* \rightharpoonup (1-t)r^* + th^*$$

giving $0 = (1-t)r^* + th^*$ and hence

$$0 \in (1-t)R(u) + tF(u) \quad \text{for some } u \in \partial G.$$

That gives a contradiction to (5) ($y = w = 0$). Hence we have proved (6). Thus applying Theorem 11 we find $u_\varepsilon \in G$ such that

$$0 \in F_\varepsilon(u_\varepsilon) \quad \text{for every } \varepsilon \in (0, \varepsilon_0). \tag{10}$$

Choosing an arbitrary sequence $\{\varepsilon_i\}$ with $\varepsilon_i \to 0^+$ along with solutions u_{ε_i} from (10) we get

$$F(u_{\varepsilon_i}) = F_{\varepsilon_i}(u_{\varepsilon_i}) - \varepsilon_i PR(u_{\varepsilon_i})$$

where the set $\cup_i PR(u_{\varepsilon_i})$ is bounded. Thus $0 \in \overline{F(\overline{G})}$. □

Because the condition of quasimonotonicity is in applications the easiest to verify (e.g. it is implied by the monotonicity), we need a result for $\mathscr{F}_G(mPM)$. By using similar argument as in Theorem 12, we obtain the following result (see [2, Theorem 4]).

Theorem 13. *Let the assumptions of Theorem 12 be satisfied and let, in addition, G be convex and $F \in \mathscr{F}_G(mPM)$. If for a given $y \in H$ there exists $w \in R(G)$ such that the condition (5) holds, then the inclusion $y \in F(u)$ does admit a solution u in G.*

Now we can generalize the *Borsuk Theorem* to the class $\mathscr{F}_G(mQM)$ (see [2, Theorem 5]).

Theorem 14. *Let G be an open bounded symmetric subset of H containing the origin and let F be a multi-mapping $\overline{G} \to 2^H$ satisfying $F(-u) = -F(u)$ for all $u \in \partial G$. Then if $F \in \mathscr{F}_G(mQM)$, the inclusion $0 \in F(u)$ is almost solvable in $\overline{G}$, i.e., $0 \in \overline{F(\overline{G})}$.*

For the class $\mathscr{F}_G(mQM)$ we have the following surjectivity theorem (see [2, Theorem 6]).

Theorem 15. *Let $F\colon\ H \to 2^H$ be a mapping of the class $\mathscr{F}_H(mQM)$ satisfying the following condition:*

for any $K \in \mathbb{R}^+$ there exists $R_0 \in \mathbb{R}$ such that

$$|f^*| > K \quad \text{for all selections } f^* \in F(u),\ |u| \geqslant R_0. \tag{A}$$

Assume moreover that there exists a positive real constant r_0 such that one of the following condition holds:

$$\text{(B)} \qquad \frac{(f^*, u)}{|u|} + |f^*| > 0 \quad \textit{for all } |u| \geqslant r_0 \textit{ and for all } f^* \in F(u);$$

$$\text{(C)} \qquad F(-u) = -F(u) \quad \textit{for all } |u| \geqslant r_0.$$

Then $\overline{F(H)} = H$.

Proof. Let $y \in H$. By (A), there exists $r_1 > r_0$ such that $0 \notin \overline{F(\partial B_r)}$ for all $r \geqslant r_1$, where B_r is the ball $B_r = \{u \in H \mid |u| < r\}$.

Assume first that (B) holds. It gives

$$0 \notin (1-t)u + tF(u) \quad \text{for all } t \in [0,1] \text{ and } u \in \partial B_r.$$

By using the same argument as in the proof of Theorem 12 (with $R = I$) we can conclude (see (6)) that there exists a positive constant $\varepsilon_0 = \varepsilon_0(r)$ such that

$$0 \notin (1-t)u + t(F(u) + \varepsilon P u) \quad \text{for all } |u| = r,\ r > r_1,\ t \in [0,1] \text{ and } 0 < \varepsilon < \varepsilon_0.$$

Hence

$$d(F + \varepsilon P, B_r, 0) = +1 \quad \text{for all } r > r_1 \text{ and } 0 < \varepsilon < \varepsilon_0.$$

We take $r_2 > r_1$ such that $|f^*| \geqslant (3|y|+1)$ for all $f^* \in F(u)$, $|u| \geqslant r_2$. Let $0 < \varepsilon_1 < \varepsilon_0$ be such that $\varepsilon_1 r_2 < |y| + 1$. Consequently,

$$|f^* + \varepsilon P u| \geqslant 3|y| - |y| = 2|y|$$

for all $f^* \in F(u)$, $u \in \partial B_{r_2}$ and $0 < \varepsilon < \varepsilon_1$. Since $|ty| \leqslant |y|$ we have

$$ty \notin (F + \varepsilon P)(\partial B_{r_2})$$

and

$$d(F + \varepsilon P, B_{r_2}, y) = d(F + \varepsilon P, B_{r_2}, 0) = +1 \quad \text{for all } 0 < \varepsilon < \varepsilon_1.$$

Hence $y \in (F + \varepsilon P)(B_{r_2})$ for all $0 < \varepsilon < \varepsilon_1$. It follows that $y \in \overline{F(B_{r_2})} \subset \overline{F(H)}$.

Now we assume that (C) holds. We can use similar argument that in the first case. By applying Theorem 14, from (C) we obtain

$$d(F + \varepsilon P, B_{r_2}, y) = d(F + \varepsilon P, B_{r_2}, 0) = \text{odd}$$

for all $0 < \varepsilon < \varepsilon_1$. Hence $y \in (F + \varepsilon P)(B_{r_2})$ for all $0 < \varepsilon < \varepsilon_1$ and the conclusion $y \in \overline{F(H)}$ follows. Thus the proof is completed. $\square$

It is not easy to check whether a multi-mapping F is of the class $\mathscr{F}_G(mQM)$ or $\mathscr{F}_G(mS_+)$. The simplest condition to check is the monotonicity and therefore the most useful theorems concern the class $\mathscr{F}_G(mPM)$. We prove the following existence result (see [2, Theorem 9]).

Theorem 16. *Let $N \in (mPM)$, let $L\colon D(L) \subset H \to H$ be the linear mapping defined as above and let $c > 0$, $-c \notin \sigma(L)$, where $\sigma(L)$ is the spectrum of L. If there exist real positive constants τ, $\tau < \operatorname{dist}(-c, \sigma(L))$ and K such that*

$$|f^* - cu| \leqslant (\tau|u| + K) \quad \text{for all } u \in H,\ f^* \in N(u),$$

then for all $h \in H$ the inclusion (SInc) admits a solution in H, i.e. there exists $u \in D(L) \cap G$ such that $h \in Lu + N(u)$.

P r o o f. We consider the homotopy

$$F_t(u) = Lu + cu + t(N(u) - cu) \quad \text{for all } t \in [0, 1]$$

and $y_t = th$. Clearly $F_0(u) = Lu + cu$ and $F_1(u) = Lu + N(u)$. Since $-c \notin \sigma(L)$ the operator $(L + cI)^{-1}$ exists and is bounded. Thus we can put $r = \|(L + cI)\|^{-1}(K + |h| + 1)/(1 - \tau\|(L + cI)^{-1}\|)$ and $G = B_r$. If for some $u \in \partial G \cap D(L)$ we have

$$th \in Lu + cu + t(N(u) - cu)$$

for some $t \in [0, 1]$ then

$$u = (L + cI)^{-1}(-t(f^* - cu) + th), \quad f^* \in N(u).$$

Hence

$$r = |u| \leqslant \|(L + cI)^{-1}\|(\tau r + K + |h|) < r,$$

a contradiction. Thus

$$th \notin Lu + cu + t(N(u) - cu) \quad \text{for all } t \in [0, 1] \text{ and } u \in \partial G \cap D(L).$$

For $t = 0$ we have $Lu + cu = 0$ not possessing a non-zero solution. The mapping $L + cI$ is linear, odd and $cI \in (mS_+)$, therefore using Lemma 3, Proposition 9 and then applying Theorem 13, we conclude $h \in F_1(G)$, i.e. $h \in Lu + N(u)$ for some $u \in D(L) \cap G$. $\square$

A function $p\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is called *superpositionally measurable* [6] if $p(x,t,u(x,t))$ is measurable for any measurable function $u\colon \Omega\to\mathbb{R}$. A multi-function $S\colon \Omega\times\mathbb{R}\to 2^{\mathbb{R}}$ is called *measurable-bounded* if there exist two superpositionally measurable functions $q_-(x,t,u)$ and $q_+(x,t,u)$ such that

$$q_-(x,t,u)\leqslant q_+(x,t,u) \quad \text{and} \quad S(x,t,u)=[q_-(x,t,u),q_+(x,t,u)]$$

for any $(x,t,u)\in\Omega\times\mathbb{R}$ where the function $q_-(x,t,u)$ is lsc in u, the function $q_+(x,t,u)$ is usc in u and there exist positive constants d_1,d_2 and $c_1,c_2\in L^2(\Omega)$ such that

$$|q_-(x,t,u)|\leqslant c_1(x,t)+d_1|u| \qquad \text{and} \qquad |q_+(x,t,u)|\leqslant c_2(x,t)+d_2|u|$$

for any $(x,t,u)\in\Omega\times\mathbb{R}$. We denote by (mMB) the set of all measurable-bounded multi-functions. By using a multi-function $S\in(mMB)$, for any $u\in L^2(\Omega)$ we put

$$\begin{aligned} N(u)&=\{v\in L^2(\Omega)\mid v(x,t)\in S(x,t,u(x,t))\}\\ &=\{v\in L^2(\Omega)\mid q_-(x,t,u(x,t))\leqslant v(x,t)\leqslant q_+(x,t,u(x,t))\}\end{aligned}$$

and call it *an M-regular multi-function*. We denote the set of all such multi-functions by (mMr). $N(u)$ is nonempty because $q_\pm(x,t,u(x,t))\in N(u)$. The degree theory we have constructed works just for multi-mappings that are w-usc.

Lemma 17. *Let $N\in(mMr)$ and $u_n\to u$ in $L^2(\Omega)$. If a sequence $\{w_n^*\}$ satisfies $w_n^*\in N(u_n)$ and $w_n^*\rightharpoonup w^*$ in $L^2(\Omega)$ then $w^*\in N(u)$, i.e. $N\colon L^2(\Omega)\to 2^{L^2(\Omega)}$ is w-usc.*

P r o o f. First, we have

$$q_-(x,t,u_n(x,t))\leqslant w_n^*(x,t) \tag{11}$$

for every $n\in\mathbb{N}$. Since $w_n^*\rightharpoonup w^*$, using the Mazur theorem we can choose a sequence $\{v_n\}$, $v_n\in\operatorname{conv}\{w_n^*,w_{n+1}^*,\ldots\}$ such that $v_n\to w^*$ almost everywhere in $L^2(\Omega)$. Thus we have

$$v_n=\sum_{k=n}^{m_n}\lambda_{n,k}w_k^*;\quad 0\leqslant\lambda_{n,k}\leqslant 1;\quad \sum_{k=n}^{m_n}\lambda_{n,k}=1 \quad \text{for } n<m_n\in\mathbb{N},\ n\leqslant k\leqslant m_n.$$

From (11) we have

$$\sum_{k=n}^{m_n} \lambda_{n,k} q_-(x,t,u_k(x,t)) \leqslant \sum_{k=n}^{m_n} \lambda_{n,k} w_k^*(x,t) = v_n(x,t).$$

By virtue of the convergence in the measure we can assume that $v_n(x,t) \to w^*(x,t)$ almost everywhere in (x,t) [10]. Let $(x_0,t_0) \in \Omega$ be such an element. The mapping $s \to q_-(x_0,t_0,s)$ is lsc and so for every $\varepsilon > 0$ there exists a positive integer n_0 such that for every $k \geqslant n_0$ we have

$$q_-(x_0,t_0,u(x_0,t_0)) - \varepsilon \leqslant q_-(x_0,t_0,u_k(x_0,t_0)).$$

Summing this inequality for $k = n, n+1, \ldots, m_n$ with weights $\lambda_{n,k}$ we get

$$\sum_{k=n}^{m_n} \lambda_{n,k}(q_-(x_0,t_0,u(x_0,t_0)) - \varepsilon) \leqslant \sum_{k=n}^{m_n} \lambda_{n,k} q_-(x_0,t_0,u_k(x_0,t_0)),$$
$$q_-(x_0,t_0,u(x_0,t_0)) - \varepsilon \leqslant v_n(x_0,t_0)$$

for all $n \geqslant n_0$. Hence by the convergence $v_n(x_0,t_0) \to w^*(x_0,t_0)$ we have

$$q_-(x_0,t_0,u(x_0,t_0)) - \varepsilon \leqslant w^*(x_0,t_0) \text{ for every } \varepsilon > 0.$$

Finally, we get

$$q_-(x_0,t_0,u(x_0,t_0)) \leqslant w^*(x_0,t_0)$$

as we need. Similar argument leads to

$$w^*(x_0,t_0) \leqslant q_+(x_0,t_0,u(x_0,t_0)).$$

Thus $w^* \in N(u)$. □

7. Semilinear wave equations

We show how the previous results can be applied to the semilinear wave equation (Prob). We state the precise setting of (Prob) by putting

$$q_-(x,t,u) = g_-(u) + f(x,t,u), \quad q_+(x,t,u) = g_+(u) + f(x,t,u),$$
$$g_+(u) = \limsup_{s \to u} g(u), \quad g_-(u) = \liminf_{s \to u} g(u).$$

We note that $g_\pm$ are Borel measurable. By Lemma 17, the Nemytskij operator $N\colon H \to 2^H$, $H = L^2(\Omega)$ defined by

$$N(u) = \{v \in L^2(\Omega) \mid q_-(x,t,u(x,t)) \leqslant v(x,t) \leqslant q_+(x,t,u(x,t))\}$$

is bounded and w-usc. $N \in (mMON)$ and hence by Proposition 1, $N \in (mPM)$. Let C^2 be the set of twice continuously differentiable functions $v\colon [0,\pi] \times \mathbb{R} \to \mathbb{R}$ satisfying $v(0,\cdot) = v(\pi,\cdot) = 0$ and 2π-periodic in $t \in \mathbb{R}$.

A weak 2π-periodic solution of (Prob) for $h \in H$ is any $u \in H$ satisfying

$$\text{(wProb)} \quad (u, v_{tt} - v_{xx}) + (u^*, v) = (h, v) \quad \text{for some } u^* \in N(u) \text{ and for all } v \in C^2.$$

Let $\varphi_{m,n}(x,t) = \pi^{-1} e^{imt} \sin nx$ for all $m \in \mathbb{Z}, n \in \mathbb{Z}^+$. Each $u \in L^2(\Omega)$ has a representation

$$u = \sum_{m\in\mathbb{Z}, n\in\mathbb{Z}^+} (n^2 - m^2) u_{m,n} \varphi_{m,n},$$

where $u_{m,n} = (u, \varphi_{m,n})$ and $\overline{u}_{m,n} = u_{m,n}$, since u is a real function. The abstract realization of the wave operator $\dfrac{\partial^2}{\partial t^2} - \dfrac{\partial^2}{\partial x^2}$ in $L^2(\Omega)$ is the linear operator $L\colon D(L) \to L^2(\Omega)$ defined by

$$Lu = \sum_{m\in\mathbb{Z}, n\in\mathbb{Z}^+} (n^2 - m^2) u_{m,n} \varphi_{m,n},$$

where

$$D(L) = \Big\{ u \in L^2(\Omega) \mid \sum_{m\in\mathbb{Z}, n\in\mathbb{Z}^+} |n^2 - m^2|^2 |u_{m,n}|^2 < \infty \Big\}.$$

It can be shown that $u \in L^2(\Omega)$ is a weak solution of (wProb) if and only if $h \in Lu + N(u)$ with $u \in D(L)$. Moreover, L is densely defined, self-adjoint, closed, $\operatorname{Im} L = (\operatorname{Ker} L)^\perp$ and L has a pure point spectrum of eigenvalues $\sigma(L) = \{n^2 - m^2 \mid m \in \mathbb{Z}, n \in \mathbb{Z}^+\}$ with the corresponding eigenfunctions $\{\varphi_{m,n}\}$. Clearly $\sigma(L)$ is unbounded both from above and from below, any eigenvalue $\lambda \neq 0$ has a finite multiplicity, but $\operatorname{Ker} L$ is infinite dimensional. The right inverse of L_0, $L_0^{-1}\colon \operatorname{Im} L \to \operatorname{Im} L$ is compact. Hence (Prob) fulfils all conditions for the inclusion (SInc) of Lemma 3. Theorem 16 immediately implies the following result (see [2, pp. 961–962]).

Theorem 18. *The semilinear wave equation (Prob) under the conditions mentioned above admits a weak 2π-periodic solution for any $h \in L^2(\Omega)$ provided that in addition*

$$a \leqslant f(x,t,u)/u \leqslant b \quad \forall\, (x,t) \in \Omega \text{ and } \forall\, u \in \mathbb{R},\ |u| > R$$

for positive constants a, b, R such that $m^2 - n^2 \notin [a,b]$ for all $n \in \mathbb{Z}^+$ and $m \in \mathbb{Z}$.

We leave to the reader other extensions of results on pp. 960–962 from [2] to (Prob). In the conclusion, we consider the nonlinear discontinuous vibrating string equation

$$\begin{gathered} u_{tt} - u_{xx} + g(u) = f(x,t) \\ u(0,\cdot) = u(\pi,\cdot) = 0, \end{gathered}$$

where $f \in L^\infty(\Omega)$, $g\colon \mathbb{R} \to \mathbb{R}$ is nondecreasing and bounded on $\mathbb{R}$. This equation is considered in the form

$$\begin{gathered} u_{xx} - u_{tt} + f(x,t) \in \big[g_-(u), g_+(u)\big] \\ u(0,\cdot) = u(\pi,\cdot) = 0. \end{gathered} \tag{12}$$

We put $g(+\infty) = \sup\limits_{\mathbb{R}} g$, $g(-\infty) = \inf\limits_{\mathbb{R}} g$. We extend Theorem 1 of [4] to (12).

Theorem 19. *Let f have a decomposition*

$$f(x,t) = f_1(x,t) + f_2(x,t), \quad f_1 \in (\ker L)^\perp, \quad f_2 \in \ker L$$

together with

$$g(-\infty) + \delta \leqslant f_2(x,t) \leqslant g(+\infty) - \delta \quad \forall\ (x,t) \in \Omega$$

for some $\delta > 0$. Then (12) has a weak 2π-periodic solution.

P r o o f. For $1 > \varepsilon > 0$ fixed, we consider the problem

$$\begin{gathered} u_{xx} - u_{tt} + f(x,t) \in \big[g_-(u), g_+(u)\big] + \varepsilon u \\ u(0,\cdot) = u(\pi,\cdot) = 0. \end{gathered} \tag{13}$$

Since $1 > \varepsilon > 0$ and g is nondecreasing and bounded on $\mathbb{R}$, Theorem 16 implies the existence of a weak 2π-periodic solution u of (13). Now we show that u is bounded in $L^\infty(\Omega)$ uniformly for $\varepsilon > 0$ small. From (13) we have

$$-Lu + f = z + \varepsilon u, \quad z \in \big[g_-(u), g_+(u)\big]. \tag{14}$$

We decompose $u = u_1 + u_2$, $z = z_1 + z_2$, $u_1, z_1 \in (\ker L)^\perp$, $u_2, z_2 \in \ker L$. Then (14) implies

$$Lu_1 + \varepsilon u_1 + z_1 = f_1 \tag{15}$$

$$f_2 = z_2 + \varepsilon u_2. \tag{16}$$

Since $f_1, z \in L^\infty(\Omega)$ for $\varepsilon > 0$ small, we have $|u_1|_{L^\infty} < \tilde{C}$ (see [4, p. 417]). In what follows $\tilde{C}$ denotes positive constants independent of ε. By (16) also

$$\int_0^{2\pi}\!\!\int_0^{\pi} f_2 u_2 \, \mathrm{d}x \, \mathrm{d}t = \int_0^{2\pi}\!\!\int_0^{\pi} z_2 u_2 \, \mathrm{d}x \, \mathrm{d}t + \varepsilon \int_0^{2\pi}\!\!\int_0^{\pi} u_2^2 \, \mathrm{d}x \, \mathrm{d}t \geqslant \int_0^{2\pi}\!\!\int_0^{\pi} z_2 u_2 \, \mathrm{d}x \, \mathrm{d}t. \tag{17}$$

We take $M > 0$ such that

$$g(v) \geqslant g(+\infty) - \frac{\delta}{2} \quad \forall\, v \geqslant M, \quad g(v) \leqslant g(-\infty) + \frac{\delta}{2} \quad \forall\, v \leqslant -M.$$

Let us define

$$\Sigma_+ = \big\{(x,t) \mid u_2(x,t) \geqslant M + \tilde{C}\big\}, \quad \Sigma_- = \big\{(x,t) \mid u_2(x,t) \leqslant -M - \tilde{C}\big\}.$$

Then we have

$$z(x,t) \geqslant g(+\infty) - \frac{\delta}{2} \quad \forall\, (x,t) \in \Sigma_+, \quad z(x,t) \leqslant g(-\infty) + \frac{\delta}{2} \quad \forall\, (x,t) \in \Sigma_-.$$

Now (17) gives

$$\begin{aligned}
&(g(+\infty) - \delta) \int_{\Sigma_+} u_2 \, \mathrm{d}x \, \mathrm{d}t + (g(-\infty) + \delta) \int_{\Sigma_-} u_2 \, \mathrm{d}x \, \mathrm{d}t + \int_{(\Sigma_+ \cup \Sigma_-)'} f_2 u_2 \, \mathrm{d}x \, \mathrm{d}t \\
&\geqslant \int_0^{2\pi}\!\!\int_0^{\pi} f_2 u_2 \, \mathrm{d}x \, \mathrm{d}t \\
&\geqslant \int_{(\Sigma_+ \cup \Sigma_-)'} z_2 u_2 \, \mathrm{d}x \, \mathrm{d}t + \Big(g(+\infty) - \frac{\delta}{2}\Big) \int_{\Sigma_+} u_2 \, \mathrm{d}x \, \mathrm{d}t + \Big(g(-\infty) + \frac{\delta}{2}\Big) \int_{\Sigma_-} u_2 \, \mathrm{d}x \, \mathrm{d}t,
\end{aligned}$$

hence we arrive at

$$\int_{\Sigma_+ \cup \Sigma_-} |u_2| \, \mathrm{d}x \, \mathrm{d}t \leqslant \tilde{C}.$$

Consequently, we obtain that $|u_2|_{L^1} < \tilde{C}$. Since $\dim \ker L = \infty$, this estimate is not enough. Now following the same argument like on p. 419 of [4], we obtain that $|u_2|_{L^\infty} < \tilde{C}$. Hence for any $\varepsilon > 0$ small, any solution of (14) satisfies $|u|_{L^\infty} < \tilde{C}$. Then clearly $|u|_{L^2} < \tilde{C}$. By passing to the limit $\varepsilon \to 0_+$ in (14) like for Theorem 13, we obtain a weak 2π-periodic solution of (12). □

We remark according to Theorem 19: If $-g(-\infty) = g(+\infty) > 0$ then the condition

$$\left|\int_0^{2\pi}\int_0^{\pi} f_2(x,t)z(x,t)\,\mathrm{d}x\,\mathrm{d}t\right| \leqslant g(+\infty)\int_0^{2\pi}\int_0^{\pi} |z(x,t)|\,\mathrm{d}x\,\mathrm{d}t, \quad \forall\, z \in \ker L$$

is necessary, and the condition

$$\left|\int_0^{2\pi}\int_0^{\pi} f_2(x,t)z(x,t)\,\mathrm{d}x\,\mathrm{d}t\right| \leqslant (g(+\infty) - \delta)\int_0^{2\pi}\int_0^{\pi} |z(x,t)|\,\mathrm{d}x\,\mathrm{d}t, \quad \forall\, z \in \ker L$$

for some $0 < \delta < g(+\infty)$ is sufficient for the weak 2π-periodic solvability of (12). Finally, we note that similar results hold for (Prob) with nonincreasing f and g in u.

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